Chapter 3

The Pluri-fine Topology

In this chapter we study the connectedness properties of the pluri-fine topology. The contents is based on the two papers [48, 49] that were jointly written with Jan Wiegerinck.

3.1 Introduction

The pluri-fine topology on an open set $\Omega$ in $\mathbb{C}^n$ is the coarsest topology on $\Omega$ making all the plurisubharmonic functions on $\Omega$ continuous. Almost all the results concerning the classical fine topology, discussed in Chapter 2 remain valid, and even with the same proofs. For example, Theorem 2.2.5 and Corollary 2.2.8 clearly extend to the plurisubharmonic case. In other words, the pluri-fine topology is Baire and completely regular. It was observed by Bedford and Taylor [5], that it has the quasi-Lindelöf property in the sense of Theorem 3.2.5.

In view of the interesting connection between pluripolar hulls and finely holomorphic functions, cf. [41, 44], and the increasing range of application of fine potential theory, see e.g. [58, 101], it seems quite natural to try to extend these “fine” theories to $\mathbb{C}^n$, $n > 1$. In fact the paper of Fuglede [65] already contains an attempt to introduce fine holomorphy in $\mathbb{C}^n$. Fuglede compares three possible fine topologies on $\mathbb{C}^n$: the fine topology on $\mathbb{R}^{2n}$, the pluri-fine topology, and the $n$-fold product topology induced by the fine topology on $\mathbb{C}$. He makes it clear that the pluri-fine topology is the right one to use. Then he notes that local connectedness needs to be established before fine holomorphy, or “fine pluripotential theory”, can be developed at all. Problems of pluri-fine-topological nature are also easily encountered in complex analysis and other related areas. See for example [6, 54, 109, 122]

It is the purpose of this chapter to show that the pluri-fine topology enjoys pleasant connectedness properties similar to those in Section 2.2.3. The proof given by Fuglede in [55] of the local connectedness of the fine topology in $\mathbb{R}^n$ does not carry over to the plurisubharmonic case. In fact, Fuglede’s proof was strongly
based on the theory of balayage of measures, especially the balayage of the unit
Dirac measure, cf. Theorem 2.3.2. In particular, the strong subadditivity of the
harmonic measure was used in an essential way. As far as we know there is no
analog to Theorem 2.3.2 for plurisubharmonic functions. Moreover, the strong
subadditivity of the relative extremal plurisubharmonic function fails to hold in
higher dimensions, as was proved by Thorbiörnson [129].

Moreover, unlike the situation in classical potential theory, the notions of thin-
ness and pluripolarity are not equivalent, cf. Example 1.4.1. This means that
pluri-thin sets can not be characterized in terms of capacity, which accounts for a
big difference between the pluri-fine and fine topology.

Nevertheless, using elementary properties of finely subharmonic functions, that
were found by Fuglede [56, 57], we give in Section 3.3 a surprisingly simple proof
of the local connectedness of the pluri-fine topology, cf. Theorem 3.3.4.

In Section 3.4 we study the structure of open sets in the pluri-fine topology
in terms of slices. The main result here is proposition 3.4.9. Its proof proceeds
by “slicing” and using estimates on subharmonic functions, relying ultimately on
a classical harmonic measure estimate of A. Beurling and R. Nevanlinna. The
results obtained in Section 3.4 will be applied to study finely plurisubharmonic
functions in the next chapter.

3.1.1 Notation

In order to avoid cumbersome expressions like “locally pluri-finely connected sets”,
we adopt the following convention: Topological notions referring to the pluri-
fine topology will be qualified by the prefix “F” to distinguish them from those
pertaining to the Euclidean topology. For example, F-open, F-domain (it means
F-open and F-connected), F-component..... In view of the fact that the pluri-fine
topology restricted to a complex line coincides with the fine topology on that line,
this convention can be used in the one dimensional setting, where we will work
with the fine topology, at the same time.

3.2 Pluri-thin Sets

The first discussion of pluri-thinness appears perhaps in [122]. The definition of
thinness in classical potential theory can be translated to the plurisubharmonic
case.

Definition 3.2.1. A set $E \subset \mathbb{C}^n$ is pluri-thin at a point $a \in \mathbb{C}^n$ if and only if
either $a$ is not a limit point of $E$ or there exist $r > 0$ and a plurisubharmonic
function $\varphi$ on $B(0,r)$ such that

$$\limsup_{z \to a, z \in E \setminus \{a\}} \varphi(z) < \varphi(a). \quad (3.2.1)$$

The fundamental result of H. Cartan, cf. Theorem 2.2.18, connecting the
notion of thinness and the fine topology extends to the plurisubharmonic case.
Theorem 3.2.2. The \( F \)-neighborhoods of a point \( a \in \mathbb{C}^n \) are precisely the sets of the form \( \mathbb{C}^n \setminus E \), where \( E \) is pluri-thin at \( a \) and \( a \not\in E \).

The following result (see e.g. [80]) and its corollary assert, that being an \( F \)-open set is a local property.

Theorem 3.2.3. Finite intersections of sets of the form
\[
B_{\Omega}^\varphi = \{ z \in \Omega : \varphi(z) > 0 \},
\]
where \( \Omega \subseteq \mathbb{C}^n \) is open, and \( \varphi \in \text{PSH}(\Omega) \), constitute a base of the pluri-fine topology on \( \mathbb{C}^n \).

Corollary 3.2.4. If \( \Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}^n \) are open subsets, then the pluri-fine topology on \( \Omega_1 \) is the same as the topology on \( \Omega_1 \) induced by the pluri-fine topology on \( \Omega_2 \).

For later reference we recall the following result, cf. [5], which is analogue to Theorem 2.2.20.

Theorem 3.2.5. (Quasi-Lindel"of property) An arbitrary union of \( F \)-open subsets of \( \mathbb{C}^n \) differs from a suitable countable subunion by at most a pluripolar set.

### 3.3 Local Connectedness

The main result in this section is Theorem 3.3.4, which gives a positive answer to a problem raised by Fuglede in [65]. See also [6, page 62]. Its proof relies on the two lemmas below.

Lemma 3.3.1. Sets of the form
\[
B_{\Omega}^\varphi = \{ z \in \Omega : \varphi(z) > 0 \},
\]
where \( \Omega \subseteq \mathbb{C}^n \) is open, and \( \varphi \in \text{PSH}(\Omega) \), constitute a base of the pluri-fine topology on \( \mathbb{C}^n \).

Lemma 3.3.1 was stated by Bedford and Taylor in [5, Theorem 2.3] without proof. Since we could not find one in the literature, we prove it here.

Proof. By Theorem 3.2.3, \( B_{\Omega}^\varphi \) is an \( F \)-open set. Let \( U \subseteq \mathbb{C}^n \) be an \( F \)-open set, and let \( a \in U \). By Theorem 3.2.2, the complement \( E \) of \( U \) is pluri-thin at \( a \). We will prove that \( U \) contains an \( F \)-open neighborhood of \( a \) of the form stated in the lemma. This is trivial if \( a \) belongs to the Euclidean interior of \( U \). Consider the case when \( a \) is an accumulation point of \( E \). There exist then \( \delta > 0 \) and a plurisubharmonic function \( \varphi \) on \( B(a, \delta) \) such that
\[
\limsup_{z \to a, z \in E \setminus \{a\}} \varphi(z) < \varphi(a).
\]
Without loss of generality we may suppose that \( \varphi(E \cap B(a, \delta)) \leq 0 < \varphi(a) \). Since \( \mathbb{C}^n \setminus (E \cap B(a, \delta)) = U \cap B(a, \delta) \cup \mathbb{C}^n \setminus B(a, \delta) \), we get \( \{ z \in B(a, \delta) : \varphi(z) > 0 \} \subset U \cap B(a, \delta) \subset U \). This proves the lemma. \( \square \)
Denote by \( B = B(0,1) \) the open unit ball in \( \mathbb{C}^n \), and let \( \varphi \in \text{PSH}(B(0,1)) \) such that \( 0 \leq \varphi \leq 1 \) on \( B(0,1) \).

The next lemma is the second ingredient in the proof of Theorem 3.3.4.

**Lemma 3.3.2.** Let \( U = \{ \varphi > 0 \} \cap B(0,1) \). If \( U = V \cup W \), where \( V \) and \( W \) are non-empty \( \mathcal{F} \)-open sets such that \( V \cap W = \emptyset \), then the following function is plurisubharmonic:

\[
\varphi_V(z) = \begin{cases} 
\varphi(z) & \text{if } z \in B \setminus W, \\
0 & \text{if } z \in W.
\end{cases}
\]

**Proof.** Let \( L \) be a complex line passing through \( B \). We will prove that the function

\[
\varphi_{V,L}(z) = \begin{cases} 
\varphi(z) & \text{if } z \in L \cap B \setminus W_L, \\
0 & \text{if } z \in W_L,
\end{cases}
\]

is subharmonic on \( L \cap B \). Here \( \varphi \) denotes the restriction of \( \varphi \) to \( L \cap B \), and \( W_L := W \cap L \). The set \( W_L \) is an \( \mathcal{F} \)-open subset of \( L \cap B \) which may be empty.

Note that \( \varphi_{V,L} \) is the restriction of \( \varphi_V \) to \( L \cap B \). Denote by \( \partial_f W_L \) the \( \mathcal{F} \)-boundary of \( W_L \) relative to \( L \cap B \). We claim that \( \tilde{\varphi} = 0 \) on \( \partial_f W_L \). To prove the claim observe first that \( \partial_f W_L \subset \partial_f \{ \tilde{\varphi} > 0 \} \) and \( \partial_f \{ \tilde{\varphi} > 0 \} = \partial_f \{ \tilde{\varphi} = 0 \} \). Moreover, the set \( \{ \tilde{\varphi} = 0 \} \) is \( \mathcal{F} \)-closed, which means that \( \partial_f \{ \tilde{\varphi} = 0 \} \) is a subset of \( \{ \tilde{\varphi} = 0 \} \), and hence \( \partial_f W_L \subset \{ \tilde{\varphi} = 0 \} \).

Next, we can assume that \( L \cap B \setminus W_L \) is nonempty, for otherwise \( \varphi_{V,L} \equiv 0 \) hence subharmonic. Using the claim and the fact that \( \varphi \) is a non-negative upper-semicontinuous function, we get the following:

\[
\limsup_{z \to a, z \in B \cap L \setminus W_L} \tilde{\varphi}(z) \leq \tilde{\varphi}(a) = 0, \ \forall a \in \partial_f W_L,
\]

and clearly,

\[
\limsup_{z \to a, z \in B \cap L \setminus W_L} \tilde{\varphi}(z) \leq 0, \ \forall a \in \partial_f W_L,
\]

because the ordinary \( \limsup \) majorizes the \( f \)-\( \limsup \).

In view of the claim, the definition of \( \varphi_{V,L} \) does not change if we replace \( W_L \) by its fine closure \( \hat{W}_L \). Since \( \partial_f (L \cap B \setminus W_L) \cap B = \partial_f W_L \), Lemma 2.3.13 applies and \( \varphi_{V,L} \) is therefore finely subharmonic. It is clearly bounded, and hence subharmonic by Theorem 2.3.9.

It is a well known result that a bounded function, which is subharmonic on each complex line where it is defined, is plurisubharmonic. (see e.g. [85]).

**Remark 3.3.3.** The proof of Lemma 3.3.2 uses Lemma 2.3.13 and Theorem 2.3.9. These are rather deep results in fine potential theory. However, we do not use their full strength. Lemma 3.4.12 below, which has a short direct proof, could be used instead.

**Theorem 3.3.4.** The pluri-fine topology on an open set \( \Omega \) in \( \mathbb{C}^n \) is locally connected.
Proof of Theorem 3.3.4. Let $z_0 \in \mathbb{C}^n$ and let $D$ be an $\mathcal{F}$-open neighborhood of $z_0$. By Lemma 3.3.1 there exist an open set $\Omega$ in $\mathbb{C}^n$ and a plurisubharmonic function $\varphi \in \text{PSH}(\Omega)$, such that the set $\{z \in \Omega : \varphi(z) > 0\}$ is an $\mathcal{F}$-open neighborhood of $z_0$ contained in $D$. In view of Corollary 3.2.4 and the fact that the pluri-fine topology is biholomorphically invariant, there is no loss of generality if we assume that $z_0 = 0$, $\Omega$ is the unit ball $B(0,1)$, and that $0 \leq \varphi \leq 1$ on $B(0,1)$. To prove the theorem we will find an $\mathcal{F}$-open neighborhood of 0 which is $\mathcal{F}$-connected and contained in $B^\mathcal{F}_0 := \{ z \in B(0,1) : \varphi(z) > 0 \}$. 

Denote by $\mathcal{P}$ the set of all $\mathcal{F}$-open sets $V$ which contain 0 and for which there exists a non empty $\mathcal{F}$-open set $W_V$ such that $B^\mathcal{F}_{B(0,1)} = V \cup W_V$ and $V \cap W_V = \emptyset$. It follows from Lemma 3.3.2 that the function $\varphi_V$ is plurisubharmonic for all $V \in \mathcal{P}$. Moreover, the family $(\varphi_V)_{V \in \mathcal{P}}$ is left directed and lower bounded by 0. It is a classical result, see e.g. [76, Theorem 4.15], that the infimum $\psi$ of such a family exists and is plurisubharmonic on $B(0,1)$. Now we claim that the set $U = \{ z \in B^\mathcal{F}_{B(0,1)} : \psi(z) > 0 \}$ is $\mathcal{F}$-open and $\mathcal{F}$-connected neighborhood of 0.

To prove the claim observe first that for every $V$, $\psi(0) = \varphi_V(0) = \varphi(0) > 0$, which means that $U$ is a non-empty $\mathcal{F}$- open neighborhood of 0. Next, from the definition of $\varphi_V$ in Lemma 3.3.2 we see that $\varphi_V(z) = \varphi(z) > 0$ on $V$ and $\varphi_V(z) = 0$ on $B(0,1) \setminus V$. Consequently, $U = \bigcap_{V \in \mathcal{P}} \{ \varphi_V > 0 \} = \bigcap_{V \in \mathcal{P}} V$ and $B^\mathcal{F}_{B(0,1)} = \bigcap_{V \in \mathcal{P}} V \cup \bigcup_{V \in \mathcal{P}} W_V$, where $\bigcap_{V \in \mathcal{P}} V \cap \bigcup_{V \in \mathcal{P}} W_V = \emptyset$. Therefore $U$ is an element of $\mathcal{P}$. It is minimal in the sense that it can not be split into two disjoint non-empty $\mathcal{F}$-open sets, which proves the claim and the theorem.

Corollary 3.3.5. Every $\mathcal{F}$-connected component of a $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ is $\mathcal{F}$-open. Moreover the set of these components is at most countable.

The proof of this corollary is the same as the proof given by Fuglede in the classical case. It uses the quasi-Lindelöf property. See Fuglede [55].

Let us mention the following theorem, which is due to Fuglede in the classical case, see Theorem 2.2.23.

Theorem 3.3.6. A Euclidean open set $U \subseteq \mathbb{C}^n$ is $\mathcal{F}$-connected if and only if $U$ is connected in the Euclidean topology.

Thanks to the fact that the fine topology on $\mathbb{R}^{2n}$ is finer than the pluri-fine topology on $\mathbb{C}^n$, Theorem 3.3.6 is an immediate consequence of Theorem 2.2.23 (with $\mathbb{C}$ replaced by $\mathbb{R}^{2n}$).

Remark 3.3.7. In Subsection 3.4.3 we will give an alternative, more constructive, proof of Theorem 3.3.4. However, it is important to observe that the proof presented above provides an explicit neighborhood basis of $\mathcal{F}$-connected sets: Each point $z$ in an $\mathcal{F}$-open set $\Omega$ has a neighborhood basis consisting of $\mathcal{F}$-domains of the form $B^\mathcal{F}_{z}$.
3.4 Further Results on Connectedness

In this section we will investigate connectedness in terms of slices. The main result is proposition 3.4.9, which leads to an alternative proof of the local connectedness of the pluri-fine topology. The results in Subsection 3.4.2 will be vital for the theory of finely plurisubharmonic functions, which is developed in the next chapter.

3.4.1 Harmonic Measure

We fix the following notations: \( D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \} \), \( D = D(0, 1) \), \( C(a, r) = \{ |z - a| = r \} \), while \( B(a, r) = \{ |z - a| < r \} \subset \mathbb{C}^n \). For \( z \in \mathbb{C} \) we denote by \( \Re z \) (resp. \( \Im z \)) the real (resp. imaginary) part of \( z \). The usual distance from a point \( z \) to a set \( A \) will be denoted by \( d(z, A) \).

Let \( \Omega \) be an open set in the complex plane \( \mathbb{C} \) and let \( E \subset \overline{\Omega} \). Subharmonic functions on \( \Omega \) are denoted by \( \text{SH}(\Omega) \), while \( \text{SH}^-(\Omega) = \{ u \in \text{SH}(\Omega) : u \leq 0 \} \). The harmonic measure (or the relative extremal function) of \( E \) (relative to \( \Omega \)) at \( z \in \Omega \) is defined as follows (see, e.g. [117])

\[
\omega(z, E, \Omega) = \sup\{ u(z) : u \in \text{SH}^-(\Omega), \limsup_{v \to \zeta} u(v) \leq -1 \text{ for } \zeta \in E \}.
\]

This function need not be subharmonic in \( \Omega \), but its upper semi-continuous regularization

\[
\omega(z, E, \Omega)^* = \limsup_{\Omega \ni v \to z} \omega(v, E, \Omega) \quad (\geq \omega(z, E, \Omega))
\]

is subharmonic, in view of Theorem 1.1.5. If \( E \) is a closed subset of \( \Omega \), then \( \omega(\cdot, E, \Omega) \) coincides with the Perron solution of the Dirichlet problem in \( \Omega \setminus E \) with boundary values \(-1\) on \( \partial E \cap \Omega \) and \( 0 \) on \( \partial \Omega \setminus \partial E \).

Recall the following result, cf. [17] and [23].

**Theorem 3.4.1.** Let \( \Omega \) be a bounded open subset of \( \mathbb{C} \). If \( E \subset \Omega \) is a Borel set, then there exists an increasing sequence of compact sets \( K_j \subset E \) such that \( \omega(z, K_j, \Omega)^* \downarrow \omega(z, E, \Omega)^* \).

Let \( E \subset \mathbb{D} \). We associate to \( E \) its circular projection

\[
E^\circ = \{ |z| : z \in E \}.
\]

There is extensive literature on harmonic measure and its behavior under geometric transformations such as projection, symmetrization, and polarization. We refer to [117] and the survey article [7] where a complete bibliography on this subject is given.

Our main tool in what follows is the following classical theorem of A. Beurling and R. Nevanlinna related to the Carleman-Milloux problem, cf. [8] and [104]. See also [7].

**Theorem 3.4.2.** Let \( F \subset \mathbb{D} \) be compact. Let \( F^\circ \) be its circular projection. Then

\[
\omega(z, F, \mathbb{D}) \leq \omega(-|z|, F^\circ, \mathbb{D}), \text{ for all } z \in \mathbb{D} \setminus F.
\]
Let us also recall the following formula for the harmonic measure of an interval [3].

**Proposition 3.4.3.** Let $\mathbb{D}(0, R) \subset \mathbb{C}$ be a disk with radius $R$, and let $0 < \kappa < r < R$. Then

$$\omega(0, [\kappa, r], \mathbb{D}(0, R)) = \frac{2}{\pi} \arctan\left(\frac{R + \kappa}{R + r}\sqrt{\frac{r}{\kappa} - 1}\right).$$

### 3.4.2 Estimates for Subharmonic Functions

**Lemma 3.4.4.** For every $d < c < 0$ there exists $\kappa > 0$ such that for every $\varphi \in \text{SH}^{-}(\mathbb{D})$ with $\varphi(0) > c$ and for every point $a$ in the $\mathcal{F}$-open set

$$V = \{z \in \mathbb{D}(0, 1/8) : \varphi(z) > c\},$$

the set

$$\Omega = \{z \in \mathbb{D} : \varphi(z) \geq d\}$$

contains a circle $C(a, \delta_{\varphi, a})$ with radius $\delta_{\varphi, a} > \kappa$.

**Proof.** After multiplying $\varphi$ by a constant we can assume that $d = -1$. Moreover, we may assume that the set $E = \{z \in \mathbb{D} : \varphi(z) < d\}$ is non empty since otherwise the lemma trivially holds.

Let $a \in V$ be fixed. We will first prove the following estimate

$$\varphi(a) \leq \omega(a, E^\circ_a, \mathbb{D}(a, 3/4))^*, \quad (3.4.1)$$

where $E^\circ_a = \{a + |z - a| : z \in E \cap \mathbb{D}(a, 3/4)\}$.

Let $f$ be the function $f(z) = z + a$. Note that the circular projection commutes with $f^{-1}$, i.e., $f^{-1}(E^\circ_a) = (f^{-1}(E \cap \mathbb{D}(a, 3/4)))^\circ$. Hence, to prove (3.4.1) it is enough, in view of the conformal invariance of the harmonic measure, to prove that the estimate (3.4.1) holds for the particular point $a = 0$, i.e.,

$$\varphi(0) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^*. \quad (3.4.2)$$

By Theorem 3.4.1, there is an increasing sequence of compact subset $K_j$ of $E^\circ$ such that

$$\omega(0, K_j, \mathbb{D}(0, 3/4))^* = \omega(0, K_j, \mathbb{D}(0, 3/4)) \downarrow \omega(0, E^\circ, \mathbb{D}(0, 3/4))^*. \quad (3.4.3)$$

The equality in (3.4.3) holds because $K_j$ is compact and $0 \notin K_j$. Let $\varepsilon > 0$. It follows from (3.4.3) that there exists a natural number $j_0$ such that

$$\omega(0, K_{j_0}, \mathbb{D}(0, 3/4)) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^* + \varepsilon. \quad (3.4.4)$$

Because $E$ is open, we can find a compact set $L \subset E$ such that $L^\circ = K_{j_0}$. By Theorem 3.4.2 together with inequality (3.4.4) we get

$$\omega(0, L, \mathbb{D}(0, 3/4)) \leq \omega(0, K_{j_0}, \mathbb{D}(0, 3/4)) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^* + \varepsilon. \quad (3.4.5)$$
Since $L \subset E$, and $\varphi(z) < -1$, for all $z \in E$, inequality (3.4.5) implies the following estimate
\[ \varphi(0) \leq \omega(0, E^\circ, \mathbb{D}(0, 3/4))^* + \varepsilon. \] (3.4.6)
As $\varepsilon$ is arbitrary, the estimate (3.4.2), and therefore also (3.4.1), follows.

Let now $\alpha \in ]0, 1/4[$ be a constant such that
\[ I = \{ z \in \mathbb{D}(a, 3/4) : \exists z = \Im a, \text{ and } \Re a + \alpha \leq \Re z \leq 1/2 \} \subset E^\circ_a. \]
Then by (3.4.1)
\[ \varphi(a) \leq \omega(a, E^\circ_a, \mathbb{D}(a, 3/4))^* \leq \omega(a, I, \mathbb{D}(a, 3/4)). \] (3.4.7)
Again by the conformal invariance, (3.4.7) yields
\[ \varphi(a) \leq \omega(0, f^{-1}(I), \mathbb{D}(0, 3/4)). \] (3.4.8)
Since $f^{-1}(I) = [\alpha, 1/2 - \Re a]$, it follows that
\[ \varphi(a) \leq \omega(0, [\alpha, 3/8], \mathbb{D}(0, 3/4)). \] (3.4.9)
Let $\beta_j \downarrow 0$ be a sequence decreasing to 0. Since $j \mapsto \omega(0, [\beta_j, 3/8], \mathbb{D}(0, 3/4))$ decreases to $-1$ (see e.g [76], Theorem 8.38), there exists a constant $0 < \kappa < 3/8$ depending only on $c$ but not on the function $\varphi$ such that
\[ \omega(0, [\kappa, 3/8], \mathbb{D}(0, 3/4)) < c. \] (3.4.10)
The last inequality, together with (3.4.9), hence shows that for all $a \in V$, the interval $\{ z \in \mathbb{D}(a, 3/4) : \exists z = \Im a, \text{ and } \kappa + \Re a \leq \Re z \leq 3/8 \}$ can not be a subset of $E^\circ_a$. We conclude that there exists a $\delta_{\varphi, a} \in [\kappa, 1/2]$ such that
\[ \{ z : |z - a| = \delta_{\varphi, a} \} \subset \Omega = \{ z \in \mathbb{D} : \varphi(z) \geq d \}. \] (3.4.11)

\[ \square \]

For our purposes we don’t need precise estimates for $\kappa$, but these can be easily obtained using the formula of the harmonic measure of an interval, cf. Proposition 3.4.3.

**Lemma 3.4.5.** Every interval in $\mathbb{C}$ is $\mathcal{F}$-connected.

**Proof.** It suffices to prove that the interval $[0, 1]$ is $\mathcal{F}$-connected. Suppose that $E$ and $F$ are non-empty disjoint $\mathcal{F}$-open subset of $[0, 1]$ with $[0, 1] = E \cup F$. Denote by $1_E$ the characteristic function of $E$, and observe that it is Lebesgue measurable in view of Corollary 2.2.21. For $0 \leq x \leq 1$, define $f(x) = \int_0^x 1_E d\lambda$, where $d\lambda$ is the Lebesgue measure of the real line. Invoking the Wiener’s criterion, it is an easy exercise, using the classical Polya’s inequality between Logarithmic capacity and Lebesgue measure, to prove that
\[ f'(x) = \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} 1_E d\lambda = 1_E. \] (3.4.12)
So $f$ is differentiable, and in particular $E$ and $F$ have positive measure. By the mean value theorem, for every $x \in [0,1]$, $f'(x)$ equals 0 or 1. Thus $f(x) \equiv x$ or $f(x) \equiv 0$. This is impossible since $E$ and $F$ have both positive measure. \hfill \Box

**Lemma 3.4.6.** Let $\varphi \in \text{SH}(\mathbb{D})$ such that $0 \leq \varphi \leq 1$. Let $U$ be the $F$-open subset of $\mathbb{D}$ where $\varphi > 0$. Suppose that there exists a piecewise-$C^1$ Jordan curve $\gamma \subset \mathbb{D}$ such that $\gamma \subset U$. Let $\Gamma$ be the bounded component of the complement of $\gamma$. Then $W = U \cap \Gamma$ is polygonally connected, and hence $F$-connected.

**Proof.** We follow Fuglede’s ideas in [58, Section 5]. A square shall be an open square with sides parallel to the coordinate axes. The square centered at $z$ with diameter $d$ will be denoted by $Q(z, d)$, its boundary by $S(z, d)$.

Let $n \geq 1$ be a natural number. For every $z \in \gamma$ there exists $0 < d_z < 1/n$ such that $\varphi > \varphi(z)/2$ on $S(z, d_z) \subset \mathbb{D}$. This may be proved similarly as the corresponding well-known statement for circles, cf. Theorem 2.2.15. The squares $Q(z, d_z)$ cover $\gamma$. By compactness we can select a finite subcover $\{Q(z_j, d_j) : j = 1, \ldots, m_n\}$, that is minimal in the sense that no square can be removed without losing the covering property. Now $\Omega_n = \bigcup_{j=1}^{m_n} Q(z_j, d_j)$ is an open neighborhood of $\gamma$, the boundary of which is contained in

$$\{\varphi > \frac{1}{2} \min_{1 \leq j \leq m_n} \{\varphi(z_j)\}\}.$$  

Since $\gamma$ is locally connected, the boundary of $\Omega_n$ will consist of two polygonal curves if $n$ is sufficiently large. One of these components, say, $\gamma_n$ is contained in $\Gamma$. Denote by $\Gamma_n$ the bounded component of the complement of $\gamma_n$.

Let $0 < \varepsilon < \frac{1}{2} \min_{1 \leq j \leq m_n} \{\varphi(z_j)\}$ and $K_\varepsilon^n = \{\varphi \geq \varepsilon\} \cap \Gamma_n$. Then $K_\varepsilon^n$ is a compact subset of $\mathbb{D}$. Since $\partial \Gamma_n(= \gamma_n)$ is contained in $K_\varepsilon^n$, an easy application of the maximum principle shows that $K_\varepsilon^n$ is connected. Let $z_1, z_2$ be points in $K_\varepsilon^n$. Repeating the above argument, we find for every $\delta > 0$ a polygonal curve $C$ contained in $K_{\varepsilon/2}^n$, such that $d(z_1, C), d(z_2, C) < \delta$. A well known result, cf. [117] states that for $z \in \mathbb{D}$ and almost all $\theta \in [0, 2\pi]$

$$\lim_{r \to 0} \varphi(z + re^{i\theta}) = \varphi(z).$$

The conclusion is that there exists a polygonal line in $K_{\varepsilon/2}^n$ that connects $z_1$ with $z_2$. Letting $\varepsilon \to 0$ we conclude that $U \cap \Gamma_n = \bigcup_{\varepsilon>0} K_\varepsilon^n$ is polygonally connected. Since every interval is $F$-connected, cf. Lemma 3.4.5, so is $U \cap \Gamma_n$. Finally, since $W = \bigcup_{n \geq 1} U \cap \Gamma_n$, we conclude that $W$ is $F$-connected. \hfill \Box

As an easy consequence of Lemma 3.4.6 we recover the following known result, cf. Theorem 2.2.23.

**Corollary 3.4.7.** The fine topology on $\mathbb{C}$ is locally connected.

**Proof.** Let $z \in \mathbb{C}$ and let $U \subset \mathbb{C}$ be an $F$-neighborhood of $z$. By Lemma 3.3.1 there exists an $F$-open $F$-neighborhood $V = B^*_1 \subset U$ of $z$. Without loss of generality we may assume that

$$V = \mathbb{D}(z, 1) \cap \{\varphi > 0\},$$
as noted before $V$ contains arbitrarily small circles about $z$. Let $\partial D(z, \delta)$ be one of them. Then by Lemma 3.4.6, $V \cap D(z, \delta)$ is an $F$-neighborhood of $z$ which is an $F$-domain. 

Remark 3.4.8. Besides the elementary proof that we presented here there are at least three proofs of this corollary. The first one was found by Fuglede [55], who gave a second proof in [56], page 92. Fuglede [70] observed, furthermore, that since our proof of the local connectedness in Section 3.3 does not use the fact that the fine topology on $C = \mathbb{R}^2$ is locally connected, it provides of course (for $n = 1$) a third proof of that fact.

3.4.3 Structure of $F$-Open Sets

We start this Subsection with the technical result that was alluded to at the beginning of Section 3.4.

Proposition 3.4.9. Let $U \subseteq C^n$ be an $F$-open subset and let $a \in U$. Then there exist a constant $\kappa = \kappa(U, a)$ and an $F$-neighborhood $V \subset U$ of $a$ with the property that for any complex line $L$ through $v \in V$ the $F$-component of the $F$-open set $U \cap L$ contains $v$, contains a circle about $v$ with radius at least $\kappa$.

Proof. Let $a \in U$. By Lemma 3.3.1 there is an open set $\Omega \subset C^n$, and a plurisubharmonic function $\varphi \in \text{PSH}(\Omega)$ such that $B^0_\Omega$ is an $F$-neighborhood of $a$ contained in $U$. Replacing $\varphi$ by $\varphi - 1$, we may assume that $B^0_\Omega = \{ z \in \Omega : \varphi(z) > -1 \}$. Moreover, since the pluri-fine topology is biholomorphically invariant, there is no loss of generality if we assume that $a = 0$, $\Omega = B(0, 2)$, $\varphi \leq 0$, and $\varphi(0) = -1/4$. Let

$$V = \{ z \in B(0, 1/8) : \varphi(z) > -1/2 \}.$$ 

Let $v \in V$ and let $L$ be a complex line through $v$. The restriction $\varphi_L$ of $\varphi$ to $B(v, 1) \cap L$ is subharmonic and satisfies the conditions of Lemma 3.4.4 with $c = -1/2$. Consequently, there exists a constant $\kappa$ (not depending on $\varphi_L$), such that the set $\{ z \in B(v, 1) : \varphi(z) \geq -1 \} \cap L$, and therefore $U \cap L$, contains a circle with radius $\delta_{\varphi_L, v} \in [\kappa, 1]$ about $v$. It follows from Lemma 3.4.6 that the set $U \cap L \cap B(v, \delta_{\varphi_L, v})$ is $F$-connected. This completes the proof of the proposition. 

A slightly weaker but easy formulation is as follows. For a point $z$ in an $F$-open subset $\Omega \subset C^n$ and $L$ a complex line passing through $z$, denote by $C_L$ the $F$-component of $z$ in the $F$-open set $\Omega \cap L$.

Theorem 3.4.10. Let $\Omega$ be an $F$-open subset of $C^n$ and let $z \in \Omega$. Then $\bigcup_{L \ni z} C_L$ is an $F$-neighborhood of $z$ which is $F$-connected.

Note that $C_L$ is $F$-open in $L$, because the fine topology is locally connected, in view of Corollary 3.4.7 (or Theorem 2.2.23).
3.4. Further Results on Connectedness

Proof of Theorem 3.4.10. Let $V \subseteq \Omega$ be an $\mathcal{F}$-neighborhood of $z$ provided by Lemma 3.3.1. Without loss of generality we may assume

$$V = B(z, 1) \cap \{\varphi > 0\},$$

for some $\varphi \in \text{PSH}(B(z, 1))$. Recall that for a complex line $L$ through $z$, $C_L$ is the $\mathcal{F}$-component of $\Omega \cap L$ that contains $z$. It is immediate that $\bigcup_{L \ni z} C_L$ is $\mathcal{F}$-connected. We denote by $C_L$ the $\mathcal{F}$-component of $V \cap L$ that contains $z$. By Lemma 3.4.4 together with Lemma 3.4.6 we can find a constant $\varepsilon > 0$ such that $V \cap B(z, \varepsilon) \subseteq C_L$, for all complex lines $L$ through $z$. As $C_L$ is clearly contained in $C_L$, $V \cap B(z, \varepsilon)$ is a subset of $\bigcup_{L \ni z} C_L$. This proves that $\bigcup_{L \ni z} C_L$ is an $\mathcal{F}$-neighborhood of $z$.

As an immediate corollary to Theorem 3.4.10 we recover the main result of the Section 3.3:

Corollary 3.4.11. The pluri-fine topology on an open set $\Omega$ in $\mathbb{C}^n$ is locally connected.

The next gluing lemma was used in the proof of Lemma 3.3.2. It is actually an easy consequence of Lemma 2.3.13 and Theorem 2.3.9. But, as promised in Remark 3.3.3, we give here a direct proof that avoids heavy use of the fine potential theory machinery.

Lemma 3.4.12. Let $v \in \text{SH}(D)$ for some domain $D \subset \mathbb{C}$. Suppose that $v \geq 0$ and that there exist nonempty, disjoint $\mathcal{F}$-open sets $D_1, D_2 \subset D$ such that

$$\{v > 0\} = D_1 \cup D_2.$$

Then the function $v_1$ defined by

$$v_1(z) = \begin{cases} 0 & \text{if } z \in D \setminus D_1, \\ v(z) & \text{if } z \in D_1, \end{cases}$$

is subharmonic in $D$.

Proof. For $\varepsilon > 0$ let $D_i(\varepsilon) = D_i \cap \{v \geq \varepsilon\}$, $(i = 1, 2)$. We claim that $D_1(\varepsilon)$ is closed in $D$. Indeed, take a sequence $\{x_n\}$ in $D_1(\varepsilon)$ that converges to $y \in D$. Since $\{v \geq \varepsilon\}$ is closed in $D$, $v(y) \geq \varepsilon$. Thus $y \in D_1 \cup D_2$. Suppose that $y \in D_2$. Again there exists $r > 0$ such that $C(y, r) \subseteq D_2$. By Lemma 3.4.6 the set

$$U = \mathbb{D}(y, r) \cap \{v > \varepsilon/2\}$$

is an $\mathcal{F}$-connected subset of $D_1 \cup D_2$. Since $U \cap D_2$ is non-empty, $U \cap D_1 = \emptyset$. This contradicts the fact that $U$ contains $x_n$, for $n$ sufficiently large. Thus $y \in D_1$ and hence $y \in D_1(\varepsilon)$, which proves the claim. Similarly, $D_2(\varepsilon)$ is closed.

Now define

$$v_\varepsilon(z) = \begin{cases} \varepsilon & \text{if } z \in D \setminus D_1(\varepsilon), \\ v(z) & \text{if } z \in D_1(\varepsilon). \end{cases}$$

(3.4.14)
The function $v_\varepsilon$ is clearly upper semicontinuous in $D$ and it satisfies the mean value inequality in $D \setminus D_1(\varepsilon)$. Let $a \in D_1(\varepsilon)$ and denote by $\overline{\mathbb{D}}(a, r)$ the mean value of $v$ over the circle $C(a, r)$. Since $D_2(\varepsilon)$ is closed, the exists $0 < r_0 < d(a, \overline{\mathbb{D}})$ such that $D(a, r_0) \cap D_2(\varepsilon) = \emptyset$. Thus for $r < r_0$ we have $v \leq \varepsilon$ on the part of $C(a, r)$ that lies outside $D_1(\varepsilon)$. Hence $v \leq v_\varepsilon$ on $C(a, r)$ for $r < r_0$. Consequently,

$$v_\varepsilon(a) = v(a) \leq \overline{v}(a, r) \leq \overline{v}(a, r).$$

This proves that $v_\varepsilon$ is subharmonic in $D$. Finally, the sequence $\{v_{1/n}\}_n$ decreases to $v_1$, showing that $v_1$ is subharmonic. \hfill \Box

It was proved by Gamelin and Lyons in [71] that an $\mathcal{F}$-open subset of $\mathbb{C}$ is $\mathcal{F}$-connected if and only if it is connected with respect to the usual topology on $\mathbb{C}$. The next example shows that this result has no analog in $\mathbb{C}^n$ for $n > 1$.

**Example 3.4.13.** There exists an $\mathcal{F}$-open set $U \subset \mathbb{C}^2$, which is connected but not $\mathcal{F}$-connected. Indeed, consider the set

$$\Gamma = \{(x, y) \in \mathbb{C}^2 : y = e^{1/x}, -1 \leq x < 0\}.$$

As was proved by Wiegerinck [133], one can find a plurisubharmonic function $\varphi \in \text{PSH}(B(0, 2))$ such that $\varphi|_{\Gamma} = -\infty$ and $\varphi(0) = 0$. Let $V = \{\varphi > -1/2\}$ and $W = \{\varphi < -1/2\}$. Let $W_1$ be the connected component of $W$ that contains $\Gamma \cap B(0, 2)$, and let $V_1$ the $\mathcal{F}$-component of $V$ that contains $0$. Since the pluri-fine topology is locally connected, $V_1$ is $\mathcal{F}$-open. Of course $V_1$ is connected since it is already $\mathcal{F}$-connected. Observe now that $U = V_1 \cup W_1$ is $\mathcal{F}$-open and $\mathcal{F}$-disconnected. On the other hand, since $W_1 \cup \{0\}$ is clearly connected, the $\mathcal{F}$-open set $U = V_1 \cup W_1 = V_1 \cup (W_1 \cup \{0\})$ is connected.