Chapter 4

Finely Plurisubharmonic Functions

In this chapter we study finely plurisubharmonic functions. The contents is based on the two papers [49, 50] that were jointly written with Jan Wiegerinck. We will keep the notations used in Chapter 3 to distinguish the notions referring to the pluri-fine topology from those pertaining to the Euclidean one, see Subsection 3.1.1.

4.1 Introduction

In the previous chapter we showed that the pluri-fine topology of $\mathbb{C}^n$ shares many properties with the classical fine topology of $\mathbb{R}^n$, see Subsection 2.2.3. Here we will investigate to what extent the results in section 2.3.2 about finely subharmonic functions can be extended to the “pluri-fine” case. To do this one has first to choose a “good” definition of finely plurisubharmonic functions. Indeed, the definition of ordinary plurisubharmonic functions by complex lines provides one possibility and can be easily translated to the pluri-fine setting. But the approximation theorem in Chapter 2 (Theorem 2.3.20), and the third property in Theorem 2.3.7 suggest other choices. However in order to develop a rich theory, the first option, which is of course weaker than the other ones, seems more appropriate.

Since we are motivated by the application to pluripolar hulls, we will focus on the $\mathcal{F}$-pluripolar sets; that is, the sets where a finely plurisubharmonic functions can take the value $-\infty$. Using the results obtained in Section 3.4 we show that $\mathcal{F}$-pluripolar sets have no $\mathcal{F}$-interior points, cf. Theorem 4.2.3. This result has the interesting consequence that pluripolar sets do not separate $\mathcal{F}$-domains. Theorem 4.2.3 appeared in [49] and was already applied to questions about pluripolar hulls.

In Section 4.3 we prove that every bounded finely plurisubharmonic functions can be $\mathcal{F}$-locally written as differences of two ordinary plurisubharmonic functions. Accordingly, every finely plurisubharmonic functions is pluri-finely continuous (not
just pluri-finely upper semi-continuous by definition). These results together with Theorem 4.2.3 will be used in Section 4.4 to prove that $\mathcal{F}$-pluripolar sets are actually pluripolar in the usual sense. This will have interesting consequences in the next chapter.

### 4.2 Finely Plurisubharmonic Functions

As far as we know there is no generally accepted definition of $\mathcal{F}$-plurisubharmonic functions. The following, cf. [47], seems quite natural.

**Definition 4.2.1.** A function $f : \Omega \rightarrow [-\infty, \infty]$ ($\Omega$ is $\mathcal{F}$-open in $\mathbb{C}^n$) is called $\mathcal{F}$-plurisubharmonic if $f$ is $\mathcal{F}$-upper semicontinuous on $\Omega$ and if the restriction of $f$ to any complex line $L$ is finely subharmonic or $\equiv -\infty$ on any $\mathcal{F}$-component of $\Omega \cap L$.

We shall let $\mathcal{F}$-PSH($U$) designate the set of $\mathcal{F}$-plurisubharmonic functions on $U$.

It follows immediately from this definition, Theorem 2.3.9 and Remark 2.3.10 that any usual plurisubharmonic function is $\mathcal{F}$-plurisubharmonic where it is defined.

Clearly, an $\mathcal{F}$-plurisubharmonic function $f$ on an $\mathcal{F}$-open set $\Omega$ has an $\mathcal{F}$-pluri-subharmonic restriction to every $\mathcal{F}$-open subset of $\Omega$. Conversely, suppose that $f$ is $\mathcal{F}$-plurisubharmonic in some $\mathcal{F}$-neighborhood of each point of $\Omega$. Then $f$ is $\mathcal{F}$-plurisubharmonic in $\Omega$, see [56, page 70]. We shall refer to this by saying that the $\mathcal{F}$-plurisubharmonic functions have the *sheaf property*.

Using Theorem 2.3.7 one can easily check the following properties. See also [47].

**Theorem 4.2.2.**

1) $\mathcal{F}$-PSH($U$) is a convex cone, i.e $c_1 u_1 + c_2 u_2 \in \mathcal{F}$-PSH($U$) for any $c_1, c_2 \geq 0$ and any $u_1, u_2 \in \mathcal{F}$-PSH($U$).

2) If $u_1, ..., u_2 \in \mathcal{F}$-PSH($U$), then the function $\max(u_1(x), ..., u_2(x))$ belongs also to $\mathcal{F}$-PSH($U$).

3) The limit of a monotonically decreasing sequence of $\mathcal{F}$-plurisubharmonic functions is $\mathcal{F}$-plurisubharmonic.

4) $\mathcal{F}$-PSH($U$) is closed under finely locally uniform convergence.

A subset $E$ of $\mathbb{C}^n$ is called $\mathcal{F}$-pluripolar if for every point $z \in E$ there is an $\mathcal{F}$-open subset $U \subset \mathbb{C}^n$ containing $z$ and an $\mathcal{F}$-plurisubharmonic function ($\not\equiv -\infty$) $f$ on $U$ such that $E \cap U \subset \{ f = -\infty \}$.

The next theorem shows that $\mathcal{F}$-pluripolar have no $\mathcal{F}$-interior points. This will be used in Section 4.4 to show that these sets are actually pluripolar in the usual sense.

**Theorem 4.2.3.** Let $f$ be an $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega$. If $f = -\infty$ on an $\mathcal{F}$-open subset $U$ of $\Omega$, then $f \equiv -\infty$. 

Proof. Without loss of generality we can assume that $U$ is the $F$-interior of the set $\{f = -\infty\}$. Suppose there exists $z_0 \in \Omega$ which is an $F$-boundary point of $U$. After scaling we can assume that $z_0 = 0$ and that

$$V = B(0,1) \cap \{\varphi > 0\} \subset \Omega. \quad (4.2.1)$$

is an $F$-neighborhood of 0 defined by a function $\varphi \in \text{PSH}(B(0,1))$ with $\varphi(0) = 1$, see Lemma 3.3.1. Then

$$V_{1/2} = B(0,1/2) \cap \{\varphi > 1/2\} \quad (4.2.2)$$

is a smaller $F$-neighborhood of 0. Notice that $V_{1/2} \setminus U$ is non empty, because 0 is an $F$-boundary point of $U$. For every $z \in V_{1/2} \cap U$ the function $\varphi$ is defined on $B(z,1/2)$ and $B(z,1/2) \cap \{\varphi > 0\}$ is an $F$-neighborhood of $z$ contained in $\Omega$.

By Lemma 3.4.4 together with Lemma 3.4.6 there exists $\kappa > 0$ such that for every line $L$ passing through $z \in V_{1/2} \cap U$ there exists $\delta_{z,L} \in [\kappa,1/2]$ such that $C_{z,L} = \{\varphi > 0\} \cap B(z,\delta_{z,L}) \cap L$ is an $F$-connected $F$-neighborhood of $z$ in $L \cap V$. Because $z \in U$, $C_{z,L}$ meets $U$ in an $F$-open subset of $L$. Therefore $f \equiv -\infty$ on $C_{z,L}$ according to Theorem 2.3.14. It follows that $f \equiv -\infty$ on $V \cap B(z,\kappa)$.

Now if $|z| < \kappa$, then $0 \in V \cap B(z,\kappa)$. The conclusion is that $0 \in U$. This is a contradiction. Hence $U = \Omega$. 

Corollary 4.2.4. Let $U$ be an $F$-domain in $\mathbb{C}^n$, and let $E \subset \{f = -\infty\}$, where $f$ is $F$-plurisubharmonic on $U$ ($\not\equiv -\infty$). Then $U \setminus E$ is $F$-connected.

Proof. Suppose that $U \setminus E = (V \cup W) \setminus E$, where $V$ and $W$ are non-empty $F$-open subsets of $U$ such that $V \cap W \subset E$. Define $h : U \setminus E \to [-\infty,\infty]$ by

$$h(z) = \begin{cases} 0 & \text{if } z \in V \setminus E, \\ -\infty & \text{if } z \in W \setminus E, \end{cases} \quad (4.2.3)$$

and

$$\tilde{f}(z) = \begin{cases} f + h & \text{if } z \in V \cup W \setminus E, \\ -\infty & \text{if } z \in E. \end{cases} \quad (4.2.4)$$

Then $\tilde{f}$ is $F$-upper semi-continuous. If we restrict $\tilde{f}$ to a complex line $L$, it is finely hypoharmonic. Indeed, on $V \setminus \tilde{f}$ is $F$-plurisubharmonic because $V \cap L$ is $F$-open and $\tilde{f} = f \not\equiv -\infty$, and in $U \setminus V$ there is nothing to prove because there $\tilde{f} = -\infty$. By Theorem 4.2.3, $\tilde{f} \equiv -\infty$, a contradiction.

Since the restriction of a usual plurisubharmonic function $\varphi \in \text{PSH}(\mathbb{C}^n)$ to an $F$-domain $U \subset \mathbb{C}^n$ is $F$-plurisubharmonic on $U$, Corollary 4.2.4 has the following immediate consequence. See Theorem 2.2.29 for an analogue result.
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Theorem 4.2.5. Let \( U \) be an \( \mathcal{F} \)-domain in \( \mathbb{C}^n \). If \( E \) is a pluripolar set, then \( U \setminus E \) is \( \mathcal{F} \)-connected.

It should be mentioned that contrary to the situation in the classical fine topology, the set \( U \setminus E \) in Theorem 4.2.5 is not \( \mathcal{F} \)-open in general. This is of course due to the fact that the pluripolar set \( E \) might not be \( \mathcal{F} \)-closed as is the case in Example 1.4.1.

One more consequence of Theorem 4.2.3 is the following maximum principle for \( \mathcal{F} \)-plurisubharmonic functions.

Theorem 4.2.6. Let \( f \) be a bounded \( \mathcal{F} \)-plurisubharmonic function on a bounded \( \mathcal{F} \)-open subset \( U \) of \( \mathbb{C}^n \). Then either \( f < 0 \) or \( f \equiv 0 \).

Proof. Suppose that the \( \mathcal{F} \)-open set \( V = \{ z \in U : f(z) < 0 \} \) is not empty. The function \( g_n = nf \) is \( \mathcal{F} \)-plurisubharmonic. Since \( g_n \) decreases on \( V \), the limit function \( g \) is \( \mathcal{F} \)-plurisubharmonic. By Theorem 4.2.3, \( g \equiv -\infty \) since it equals \( -\infty \) in \( V \). Hence \( f < 0 \).

4.3 Continuity of Finely PSH Functions

Theorem 4.3.1. Let \( f \) be a bounded \( \mathcal{F} \)-plurisubharmonic function in a bounded \( \mathcal{F} \)-open subset \( U \) of \( \mathbb{C}^n \). Every point \( z \in U \) has then an \( \mathcal{F} \)-neighborhood \( O \subseteq U \) such that \( f \) is representable in \( O \) as the difference between two locally bounded plurisubharmonic functions defined on some usual neighborhood of \( z \).

Proof. We may assume that \(-1 < f < 0\). Let \( z_0 \in U \), and let \( V \subseteq U \) be a compact \( \mathcal{F} \)-neighborhood of \( z_0 \). Since the complement \( \mathbb{C}^n \setminus V \) of \( V \) is pluri-thin at \( z_0 \), there exist \( 0 < r < 1 \) and a plurisubharmonic function \( \varphi \) on \( B(z_0, r) \) such that

\[
\limsup_{z \rightarrow z_0, z \in \mathbb{C}^n \setminus V} \varphi(z) < \varphi(z_0).
\]

Without loss of generality we may suppose that \( \varphi \) is negative in \( B(z_0, r) \) and

\[
\varphi(z) = -1 \text{ if } z \in B(z_0, r) \setminus V \text{ and } \varphi(z_0) = -\frac{1}{2}.
\]

Define

\[
\Phi = \sup \{ \psi \in \text{PSH}^-(B(z_0, r)) : \psi \leq -1 \text{ on } B(z_0, r) \setminus V \}.
\]

Here \( \text{PSH}^-(B(z_0, r)) \) denotes the cone of negative plurisubharmonic functions in \( B(z_0, r) \).

It is well known that the upper semi-continuous regularization of \( \Phi \), i.e. \( \Phi^*(z) = \limsup_{B(z_0, r) \ni v \rightarrow z} \Phi(v) \), is plurisubharmonic in \( B(z_0, r) \). In view of (4.3.2), we get \( \varphi \leq \Phi \leq \Phi^* \). In particular \( -\frac{1}{2} \leq \Phi^*(z_0) \).

As \( \Phi^*(z) = -1 \) for every \( z \in B(z_0, r) \setminus V \), we get

\[
f(z) + \lambda \Phi^*(z) \leq -\lambda \text{ for any } z \in U \cap B(z_0, r) \setminus V \text{ and } \lambda > 0.
\]
Now define a function $u_\lambda$ on $B(z_0, r)$ as follows

$$u_\lambda(z) = \begin{cases} \max\{-\lambda, f(z) + \lambda \Phi^*(z)\} & \text{if } z \in U \cap B(z_0, r), \\ -\lambda & \text{if } z \in B(z_0, r) \setminus V. \end{cases}$$

(4.3.5)

This definition makes sense because $[U \cap B(z_0, r)] \cup [B(z_0, r) \setminus V] = B(z_0, r)$, and the two definitions agree on $U \cap B(z_0, r) \setminus V$ in view of (4.3.4).

Clearly, $u_\lambda$ is $\mathcal{F}$-plurisubharmonic in $U \cap B(z_0, r)$ and in $B(z_0, r) \setminus V$, hence in all $B(z_0, r)$ in view of the sheaf property, cf. [49]. Since $u_\lambda$ is bounded in $B(z_0, r)$, it follows from Theorem 2.3.9 that $u_\lambda$ is subharmonic on each complex line where it is defined. It is a well known result that a bounded function, which is subharmonic on each complex line where it is defined is a plurisubharmonic, cf. [85]. In other words $u_\lambda$ is plurisubharmonic in $B(z_0, r)$.

Since $-\frac{1}{2} \leq \Phi^*(z_0)$, the set $\mathcal{O} = \{-4 < -1 + 4\Phi^*\} \in \mathcal{F}$-neighborhood of $z_0$. Since $\Phi^* = -1$ on $B(z_0, r) \setminus V$, it is clear that $\mathcal{O} \subset V \subset U$.

Observe now that $-4 \leq f(z) + 4\Phi^*(z)$, for every $z \in \mathcal{O}$. Hence

$$f(z) = u_4(z) - 4\Phi^*(z), \text{ for every } z \in \mathcal{O}. \quad (4.3.6)$$

The proof is inspired by [56, page 88-90]. It also shows that $f$ is $\mathcal{F}$-continuous in the $\mathcal{F}$-open set $\mathcal{O}$. In fact, we have the following more precise result.

**Corollary 4.3.2.** Every $\mathcal{F}$-plurisubharmonic function is $\mathcal{F}$-continuous; that is, continuous with respect to the pluri-fine topology.

**Proof.** Let $f$ be $\mathcal{F}$-plurisubharmonic in an $\mathcal{F}$-open subset $\Omega \subset \mathbb{C}^n$. Since $f$ is $\mathcal{F}$-upper semicontinuous and $f < +\infty$, the set $\{f < c\}$ is $\mathcal{F}$-open for any $c \in \mathbb{R}$. It remains to prove that $\{f > c\}$ is $\mathcal{F}$-open. Let $M \in \mathbb{R}$, and suppose that there exists $z_0 \in \Omega$ with $f(z_0) > M$. The function $g = \max\{f, M\}$ is $\mathcal{F}$-plurisubharmonic and bounded in an $\mathcal{F}$-open neighborhood $U$ of $z_0$. Hence it follows from Theorem 4.3.1 that $g$ is $\mathcal{F}$-continuous in a possibly smaller $\mathcal{F}$-neighborhood $\mathcal{O} \subset U$ of $z_0$. Consequently, the set

$$\{z \in \mathcal{O} : f(z) > M\} = \{z \in \mathcal{O} : g(z) > M\}, \quad (4.3.7)$$

containing $z_0$, is $\mathcal{F}$-open and hence an $\mathcal{F}$-neighborhood of $z_0$; and so is therefore $\{f > M\}$. \hfill \Box

The following result gives a partial analogue to the Brelot property, cf. Theorem 2.3.16. See also [66, page 284] or [60, Lemma 1].

**Theorem 4.3.3.** (Quasi-Brelot property) Let $f$ be a plurisubharmonic function in the unit ball $B \subset \mathbb{C}^n$. Then there exists a pluripolar set $E \subset B$ such that for every $z \in B \setminus E$ we can find an $\mathcal{F}$-neighborhood $\mathcal{O}_z \subset B$ of $z$ such that $f$ is continuous in the usual sense in $\mathcal{O}_z$. 

In this section we prove that

\( E \)

Hence to the study of pluripolar hulls in the next chapter.

Theorem 4.4.1.

Let \( f : U \rightarrow [-\infty, +\infty] \) be an \( \mathcal{F} \)-plurisubharmonic function \((\neq -\infty)\) on an \( \mathcal{F} \)-open and \( \mathcal{F} \)-connected subset \( U \) of \( \mathbb{C}^n \). Then the set \( \{ z \in U : f(z) = -\infty \} \) is a pluripolar subset of \( \mathbb{C}^n \).

Proof. We may assume that \( f < 0 \). Let \( z_0 \in U \) such that \( f(z_0) = -\infty \). Let \( f_n = \frac{1}{n} \max(f, -n) \). Then \( -1 \leq f_n < 0 \). We keep the notations as in the proof of Theorem 4.3.1. In particular, \( V \subset U \) is an \( \mathcal{F} \)-neighborhood of \( z_0 \), and \( -\frac{1}{n} \leq \Phi^*(z_0) \). Define a function \( v_n(z) \) on \( B(z_0, r) \) as follows.

\[
v_n(z) = \begin{cases} 
\max\{-1, \frac{1}{n}f_n(z) + \Phi^*(z)\} & \text{if } z \in U \cap B(z_0, r), \\
-1 & \text{if } z \in B(z_0, r) \setminus V. 
\end{cases}
\]

(4.4.1)

Since \( v_n \) is analogous to the function \( u_\lambda \) in (4.3.5), a similar argument shows that \( v_n \in \text{PSH}(B(z_0, r)) \). Since \( f_n(z) \) increases to 0 for every \( z \in U \) such that \( f(z) \neq -\infty \), it is clear that \( \{ v_n(z) \} \) is increasing. Let \( \lim v_n = \psi \). It is well known that the upper semi-continuous regularization \( \psi^* \) of \( \psi \) is plurisubharmonic in \( B(z_0, r) \). Also, the set \( E = \{ \psi \neq \psi^* \} \) is a pluripolar subset of \( B(z_0, r) \), by Theorem 6.6.3 in [80].

We claim that \( \psi^* = \Phi^* \) on \( B(z_0, r) \). Indeed, Observe first that \( \psi = \Phi^* \) on \( B(z_0, r) \setminus \{ f = -\infty \} \), because \( v_n = \Phi^* = -1 \) on \( B(z_0, r) \setminus V \). Hence

\[
\psi^* = \Phi^* \text{ on } B(z_0, r) \setminus (\{ f = -\infty \} \cup E).
\]

(4.4.2)
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Next, suppose that there exists $a \in B(z_0, r)$ such that $\psi^*(a) \neq \Phi^*(a)$. Clearly, the set $\{\psi^* \neq \Phi^*\}$ is an $\mathcal{F}$-open neighborhood of $a$, which is contained in $\{f = -\infty\} \cup E$. But the set $\{f = -\infty\} \cup E$ is obviously $\mathcal{F}$-pluripolar. Hence, it has no $\mathcal{F}$-interior points, in view of Theorem 4.2.3. This contradiction proves the claim.

Fix $z \in \{\Phi^* > -\frac{4}{3}\} \cap \{f = -\infty\}$. Since $\Phi^* = -1$ on $B(z_0, r) \setminus V$, it follows immediately from the definition of $v_n$ that $\psi(z) = -\frac{1}{4} + \Phi^*(z)$. Hence $\psi(z) = -\frac{1}{4} + \psi^*(z)$, according to the above claim. This shows that $\{\Phi^* > -\frac{4}{3}\} \cap \{f = -\infty\}$ is a subset of $E$. Since $\{\Phi^* > -\frac{4}{3}\}$ is an $\mathcal{F}$-neighborhood of $z_0$, we conclude that every point $z \in \{f = -\infty\}$ has an $\mathcal{F}$-neighborhood $\mathcal{O}_z \subset U$ such that $\mathcal{O}_z \cap \{f = -\infty\}$ is a pluripolar set. If $f(z) \neq -\infty$ we choose $\mathcal{O}_z$ such that $\mathcal{O}_z \cap \{f = -\infty\} = \emptyset$. By the quasi-Lindelöf property, cf. Theorem 3.2.5, there is a sequence $\{z_n\}_{n \geq 1} \subset U$ and a pluripolar subset $P$ of $U$ such that

$$U = \bigcup_n \mathcal{O}_{z_n} \cup P.$$ (4.4.3)

Hence

$$\{f = -\infty\} \subset (\bigcup_n \mathcal{O}_{z_n} \cap \{f = -\infty\}) \cup P.$$ (4.4.4)

This completes the proof since a countable union of pluripolar sets is pluripolar.

\[\square\]

Remark 4.4.2. Corollary 4.3.2 and Theorem 4.4.1 give affirmative answers to two questions in [47].

A weaker formulation of Theorem 4.4.1, but perhaps more useful, is as follows.

**Corollary 4.4.3.** Let $f : U \to [-\infty, +\infty]$ be a function defined in an $\mathcal{F}$-domain $U \subset \mathbb{C}^n$. Suppose that every point $z \in U$ has a compact $\mathcal{F}$-neighborhood $K_z \subset U$ such that $f|_{K_z}$ is the decreasing limit of usual plurisubharmonic functions in Euclidean neighborhoods of $K_z$. Then either $f \equiv -\infty$ or the set $\{f = -\infty\}$ is pluripolar subset of $U$. 