Fine aspects of pluripotential theory
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Chapter 5

Finely Holomorphic Functions and Pluripolar Hulls

In this chapter we discuss the connection between the theory of finely holomorphic functions and the pluripolar hulls. The results presented here are selected from the papers [41, 49, 50]. We keep the notations of Subsection 3.1.1.

5.1 Introduction

A subset $E \subset \mathbb{C}^n$ is said to be pluripolar if for each point $a \in E$ there is an open neighborhood $\Omega$ of $a$ and a function $\varphi$ ($\neq -\infty$) plurisubharmonic in $\Omega$, $(\varphi \in \text{PSH}(\Omega))$ such that

$$E \cap \Omega \subset \{z \in \Omega : \varphi(z) = -\infty\}.$$ 

It is a fundamental result of Josefson [78] that this local definition is equivalent to the global one, i.e., in this definition one can assume $\varphi$ to be plurisubharmonic in all of $\mathbb{C}^n$ with

$$E \subset \{z \in \mathbb{C}^n : \varphi(z) = -\infty\}.$$ 

$E$ is called complete pluripolar (in $\mathbb{C}^n$) if for some plurisubharmonic function $\varphi \in \text{PSH}(\mathbb{C}^n)$, we have $E = \{z \in \mathbb{C}^n : \varphi(z) = -\infty\}$. Unlike the situation in classical potential theory, pluripolar sets often "propagate": it may happen that any PSH function $\varphi$ which is $-\infty$ on a pluripolar set $E$ is automatically $-\infty$ on a larger set. For example, if the $-\infty$ locus of a PSH function $\varphi$ contains a non-polar piece of a complex analytic variety $A$, then the set $\{z \in \mathbb{C}^n : \varphi(z) = -\infty\}$ must contain all the points of $A$. However, the structure of pluripolar sets may be much more complicated, cf. [25, 91]. Completeness of pluripolar sets has received growing attention, and in particular cases many results were obtained,
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see [25, 38, 40, 43, 44, 91, 125, 134, 140]. But our knowledge and understanding of the general situation is fragmentary, and a good characterization of complete pluripolar sets is still lacking, even in the case of the graph of an analytic function.

Recently, in [44], Edlund and Jöricke have connected the propagation of the graph of a holomorphic function as a pluripolar set to finely holomorphic continuation of the function.

**Theorem 5.1.1 (Edlund and Jöricke, [44] Theorem 1).** Let \( f \) be holomorphic in the unit disk \( \mathbb{D} \subset \mathbb{C} \) and let \( p \in \partial \mathbb{D} \). Suppose that \( f \) has a finely holomorphic continuation \( F \) at \( p \) to a closed \( \mathcal{F} \)-neighborhood \( V \) of \( p \). Then there exists another closed \( \mathcal{F} \)-neighborhood \( V_1 \subset V \) of \( p \), such that the graph \( \Gamma_F(V_1) \) is contained in the pluripolar hull of \( \Gamma_f(\mathbb{D}) \).

The definitions of the pluripolar hull is given in Section 5.2. A detailed discussion of the theory of finely holomorphic functions is included in Chapter 2.

In view of this result, it is reasonable to try and investigate the connection between finely holomorphic functions and pluripolar sets. Indeed, using Fuglede’s fundamental work, both in fine potential theory and fine holomorphy, we proved in [41] that graphs of finely holomorphic maps are pluripolar. Accordingly, a generalized and precise version of Theorem 5.1.1 was obtained. The main argument used in [41] was a kind of integral representation of finely holomorphic functions, cf. Theorem 2.4.8, together with Theorem 2.3.14. In order to deal with some technical difficulties we appealed to the quasi-Lindelöf property, cf. Theorem 2.2.20, and the local connectedness property of the classical fine topology.

Nevertheless, we will leave here the above arguments aside and rely instead on Theorem 4.4.1 from the preceding Chapter. This allows us to recover the main results in [41] and even obtain stronger ones. This is done in Sections 5.3, and 5.4. In Section 5.5 we introduce the concept of a finely analytic curve, and relate this to pluripolar hulls. Finally, we discuss some open problems.

### 5.2 Preliminaries

#### 5.2.1 Pluripolar Hulls

Let \( E \) be pluripolar set in \( \mathbb{C}^n \). The **pluripolar hull** of \( E \) relative to an open subset \( \Omega \) of \( \mathbb{C}^n \) is the set

\[
E^\ast_{\Omega} = \{ z \in \Omega : \text{for all } \varphi \in \text{PSH}(\Omega) : \varphi|_E = -\infty \implies \varphi(z) = -\infty \}.
\]

The notion of pluripolar hull was first introduced by Zeriahi in [139]. The paper [93] of Levenberg and Poletsky contains a more detailed study of this notion.

Let \( f \) be a holomorphic function in an open set \( \Omega \subseteq \mathbb{C}^n \). We denote by \( \Gamma_f(\Omega) \) the graph of \( f \) over \( \Omega \),

\[
\Gamma_f(\Omega) = \{(z, f(z)) : z \in \Omega \}.
\]

It is immediate that \( \Gamma_f(\Omega) \) is a pluripolar subset of \( \mathbb{C}^{n+1} \). The pluripolar hull of the graph of a holomorphic function was studied in e.g. [38, 40, 43, 44, 125, 134, 140].
Of particular interest for our present considerations is the following (see [38, 40]).

**Theorem 5.2.1 (Edigarian and Wiegerinck).** Let $D$ be a domain in $\mathbb{C}$ and let $A$ be a closed polar subset of $D$. Suppose that $f \in \mathcal{O}(D \setminus A)$ and that $z_0 \in A$. Then the following conditions are equivalent:

1. $\{(z_0) \times \mathbb{C}) \cap (\Gamma_f)^{D \times \mathbb{C}} \not= \emptyset$.
2. The set $\{z \in D \setminus A : |f(z)| \geq R\}$ is thin at $z_0$ for some $R > 0$.

### 5.2.2 $\mathcal{F}$-Holomorphic Functions

In this subsection we define $\mathcal{F}$-holomorphic functions of several complex variables. The one variable case was treated in Section 2.4

**Definition 5.2.2.** Let $U \subseteq \mathbb{C}^n$ be $\mathcal{F}$-open. A function $f : U \rightarrow \mathbb{C}$ is said to be $\mathcal{F}$-holomorphic if every point of $U$ has a compact $\mathcal{F}$-neighborhood $K \subseteq U$ such that the restriction $f|_K$ belongs to $H(K)$.

Here $H(K)$ denotes the uniform closure on $K$ of the algebra of holomorphic functions in a neighborhood of $K$.

**Definition 5.2.3.** Let $f_1$ and $f_2$ be $\mathcal{F}$-holomorphic functions on $\mathcal{F}$-domains $U_1$ and $U_2$, respectively, and suppose that the intersection $U_1 \cap U_2$ is non empty and that $f_1 = f_2$ on $U_1 \cap U_2$. Then $f_2$ is called a direct $\mathcal{F}$-holomorphic continuation of $f_1$ to $U_2$, and vice versa.

A fundamental result of Fuglede, cf. [61, Theorem 15], asserts that a finely holomorphic function of one variable has at most countably many zeros (if $\neq 0$). Accordingly, the direct $\mathcal{F}$-holomorphic continuation is unique in view of Theorem 3.4.10.

### 5.3 Graphs of Finely Holomorphic Functions

The main result in this section is Theorem 5.3.1. It generalizes an earlier result which was obtained in collaboration with Edigarian and Wiegerinck, cf. [41]. The proof will be an easy consequence of Theorem 4.4.1.

**Theorem 5.3.1.** [50] Let $h : U \rightarrow \mathbb{C}$ be an $\mathcal{F}$-holomorphic function on an $\mathcal{F}$-open subset $U$ of $\mathbb{C}^n$. Then the zero set of $h$ is pluripolar. In particular, the graph $\Gamma_h(U)$ of $h$ is also pluripolar.

**Proof of Theorem 5.3.1.** Let $a \in U$. Definition 5.2.2 gives us a compact (in the usual topology) $\mathcal{F}$-neighborhood $K$ of $a$ in $U$, and a sequences $(h_n)_{n \geq 0}$, of holomorphic functions defined in Euclidean neighborhoods of $K$ such that

\[ h_n|_K \rightarrow h|_K, \text{ uniformly.} \]
For $k \in \mathbb{N}$ we define $v_{n,k} = \max(\log |h_n|, -k)$ and $v_k = \max(\log |h|, -k)$. Clearly, $v_{n,k}$ converges uniformly on $K$ to $v_k$ as $n$ goes to infinity. Accordingly, $v_k$ is $\mathcal{F}$-plurisubharmonic on the $\mathcal{F}$-interior $K'$ of $K$. Since $v_k$ is decreasing, the limit function $\log |h|$ is $\mathcal{F}$-plurisubharmonic in $K'$. Theorem 4.4.1 shows that the set $K' \cap \{ h = 0 \}$ is pluripolar. The corollary follows now by application of the quasi-Lindelöf property, cf. Theorem 3.2.5.

As an easy consequence, we recover the following result which was obtained in [41] first.

**Corollary 5.3.2.** Let $f : U \rightarrow \mathbb{C}^n$, $f(z) = (f_1(z), \ldots, f_n(z))$, be a finitely holomorphic map on an $\mathcal{F}$-open $U \subseteq \mathbb{C}$. Then the graph $\Gamma_f(U)$ of $f$ over $U$ is a pluripolar subset of $\mathbb{C}^{n+1}$.

**Proof.** It is enough to apply Theorem 5.3.1 to the function $h(z, w_1, \ldots, w_n) = \sum_{k=1}^{k=n} f_k(z) - w_k$, which is clearly $\mathcal{F}$-holomorphic on the $\mathcal{F}$-domain $U \times \mathbb{C}^n$.

**Example 5.3.3.** Let $K \subset \mathbb{C}$ be a compact set with non-empty $\mathcal{F}$-interior $K'$. Every function $f \in R(K)$ (the uniform closure of the algebra of restrictions to $K$ of holomorphic functions in open sets containing $K$) is finitely holomorphic in $K'$, see Definition 2.4.1. Hence, by the above corollary, the graph $\Gamma_f(K') = \{(z, f(z)) : z \in K'\}$ is a pluripolar subset of $\mathbb{C}^2$. Note however that in general $K$ may not have any Euclidean interior points. See also the examples in Chapter 7.

**Remark 5.3.4.** For $n = 1$, a partial converse of Corollary 5.3.2 was obtained by Edlund in his thesis [43]. Namely, he proved that if $f$ is a function of class $\mathcal{C}^2$ on an $\mathcal{F}$-open set $V \subset \mathbb{C}$, and the graph $\Gamma_f(V)$ of $f$ is pluripolar subset of $\mathbb{C}^2$, then $f$ is finitely holomorphic in $V$. Edlund’s result together with our Corollary 5.3.2 give actually (in the particular case $n = 1$) a partial ”fine” analog of a deep theorem of N. V. Shcherbina that was obtained shortly before, cf. [124]. Shcherbina’s results asserts that the graph $\Gamma_f(\Omega)$ of a continuous function $f$ on an open set $\Omega \subset \mathbb{C}^n$ is pluripolar subset of $\mathbb{C}^{n+1}$ if and only if $f$ is holomorphic. It is therefore a natural question to ask whether the $\mathcal{C}^2$-regularity in Edlund’s theorem can be weakened to just fine continuity. See also Section 7.3.

The inversion theorem for finitely holomorphic functions, Theorem 2.4.14, allows us to strengthen Corollary 5.3.2.

**Corollary 5.3.5.** Let $f : U \rightarrow \mathbb{C}^n$, $f(z) = (f_1(z), \ldots, f_n(z))$, be a finitely holomorphic map on an $\mathcal{F}$-domain $U \subseteq \mathbb{C}$. Then the image $f(U)$ of $U$ is a pluripolar subset of $\mathbb{C}^n$.

The proof of Corollary 5.3.5 relies on Theorem 4.4.1 and the following lemma which appeared in [41].
Lemma 5.3.6. Let $U \subseteq \mathbb{C}$ be an $\mathcal{F}$-domain, and let $f : U \to \mathbb{C}^n$, $f(z) = (f_1(z), \ldots, f_n(z))$, be a finely holomorphic map. If $h$ is a plurisubharmonic function in $\mathbb{C}^n$, then the function $h \circ f$ is either finely subharmonic on $U$ or $\equiv -\infty$.

Proof. First, we assume that $h$ is everywhere finite and continuous. Let $a \in U$. Definition 2.4.1 gives us a compact (in the usual topology) finely neighborhood $K$ of $a$ in $U$, and $n$ sequences $(f_k^j)_{k \geq 0}$, $j = 1, \ldots, n$, of holomorphic functions defined in Euclidean neighborhoods of $K$ such that

$$f_j^k|_{K} \to f_j|_{K}, \quad j = 1, \ldots, n$$

Clearly, $(f_1^1, \ldots, f_n^k)$ converges uniformly on $K$ to $(f_1, \ldots, f_n)$. Since $h$ is continuous, the sequence $h(f_1^k, \ldots, f_n^k)$, of finite continuous subharmonic functions, converges uniformly to $h(f_1, \ldots, f_n)$ on $K$. According to 4) in Theorem 2.3.7, $h(f_1, \ldots, f_n)$ is finely subharmonic in the fine interior of $K$.

For the general case. We can assume that the fine interior of $K$ is finely connected. Let $(h_m)_{m \geq 0}$ be a decreasing sequence of continuous plurisubharmonic functions which converges (pointwise) to $h$. By the first part of the proof, $h_m(f_1, \ldots, f_n)$ is a decreasing sequence of finely subharmonic functions in the fine interior of $K$. The limit function $h(f_1, \ldots, f_n)$ is, by 3) in Theorem 2.3.7, finely subharmonic or identically $-\infty$ in the fine interior of $K$. The sheaf property of finely subharmonic function, cf. [56, page 70] implies that $h(f_1, \ldots, f_n)$ is indeed finely subharmonic in all of $U$ or is identically equal to $-\infty$.

Proof of Corollary 5.3.5. Without loss of generality we may assume that $f_1$ is not constant and $U$ is an $\mathcal{F}$-domain. Let $a \in U$ be such that the fine derivative $f'(a)$ is not equal to zero. By Theorem 2.4.14, there exists non-empty $\mathcal{F}$-open subset $W_a \subseteq U$ of $U$ such that $a \in W_a$ and $f_1|_{W_a} : W_a \to f_1(W_a)$ is bijective and the inverse function $f_1^{-1}$ is finely holomorphic in the $\mathcal{F}$-open set $f_1(W_a)$. Now, observe that

$$f(W_a) = \{(w, f_2(f_1^{-1}(w)), \ldots, f_n(f_1^{-1}(w))) : w \in f_1(W_a)\},$$

where $w = f_1(z)$. Since the composition of two finely holomorphic functions is finely holomorphic, cf. Corollary 2.4.10, the map

$$w \mapsto (f_2(f_1^{-1}(w)), \ldots, f_n(f_1^{-1}(w)))$$

is finely holomorphic in $f_1(W_a)$. By Corollary 5.3.2, the graph

$$\{(w, f_2(f_1^{-1}(w)), \ldots, f_n(f_1^{-1}(w))) : w \in f_1(W_a)\} = f(W_a)$$

is a pluripolar subset of $\mathbb{C}^n$. Josefson’s theorem ensures the existence of a plurisubharmonic function $h$ in $\text{PSH}(\mathbb{C}^n)$ such that

$$h(f_1(z), \ldots, f_n(z)) = -\infty, \forall z \in W_a.$$

By Lemma 5.3.6, the function $z \mapsto h(f_1(z), \ldots, f_n(z))$ is either finely subharmonic or identically equal to $-\infty$. Since it equals $-\infty$ on the non-polar subset $W_a \subset U$, we must have $h(f(U)) = -\infty$, by Theorem 2.3.14 (or Theorem 4.4.1). This completes the proof.
5.4 Application to Pluripolar Hulls

**Theorem 5.4.1.** Let \( f : U \rightarrow \mathbb{C}^n \), \( f(z) = (f_1(z), \ldots, f_n(z)) \), be a finely holomorphic map on an \( \mathcal{F} \)-domain \( U \subseteq \mathbb{C} \). If \( E \) is a non-polar subset of \( U \), then the pluripolar hull of \( f(E) \) contains \( f(U) \).

**Proof.** By Corollary 5.3.5, the set \( f(E) \) is a pluripolar subset of \( \mathbb{C}^n \). Let \( \varphi \in \text{PSH}(\mathbb{C}^n) \) be such that \( \varphi \not\equiv -\infty \), and \( \varphi(f(E)) = -\infty \). By Lemma 5.3.6, the function \( \varphi \circ f \) is finely subharmonic on \( U \) or identical to \( -\infty \). Since it assumes the value \( -\infty \) on the non-polar set \( E \), we must have \( \varphi \circ f \equiv -\infty \) on \( U \), by Theorem 2.3.14 (or Theorem 4.4.1). Hence \( f(U) \subseteq (f(E))_{-\infty}^\mathbb{C} \). \( \square \)

As a particular case of Theorem 5.4.1, we formulate here a generalized and precise version of Theorem 5.1.1.

**Corollary 5.4.2.** Let \( f \) be holomorphic in a connected open set \( U \subseteq \mathbb{C} \) and let \( p \in \partial U \). Suppose that \( f \) has a finely holomorphic continuation \( F \) at \( p \) to an \( \mathcal{F} \)-open and \( \mathcal{F} \)-connected neighborhood \( V \) of \( p \). Then \( \Gamma_F(V) \subseteq (\Gamma_f(U))_{-\infty}^\mathbb{C} \). Moreover, if \( E \) is a non-polar subset of \( V \cap \overline{U} \), then \( \Gamma_f(U) \cup \Gamma_F(V) \subseteq (\Gamma_f(E))_{-\infty}^\mathbb{C} \).

**Proof.** Denote by \( g \) the finely holomorphic function which is equal to \( f \) on \( U \) and to \( F \) on \( V \). Let \( h \in \text{PSH}(\mathbb{C}^2) \) be a plurisubharmonic function such that \( h(z, f(z)) = -\infty \), \( \forall z \in U \). According to Lemma 5.3.6, the function \( z \mapsto h(z, g(z)) \) is finely subharmonic on the \( \mathcal{F} \)-domain \( U \cup V \) or \( \equiv -\infty \). Since it assumes \( -\infty \) on the non-polar set \( U \), it must be identically equal to \( -\infty \) in view of Theorem 2.3.14 (or Theorem 4.4.1). Hence \( \Gamma_F(V) \subseteq (\Gamma_f(U))_{-\infty}^\mathbb{C} \). The second statement can be proved similarly. See Proposition 5.5.1 below for a more general results. \( \square \)

The next theorem is a higher dimensional analog of Theorem 5.4.1, cf. [50].

**Theorem 5.4.3.** Let \( U \subseteq \mathbb{C}^n \) be an \( \mathcal{F} \)-domain, and let \( h \) be \( \mathcal{F} \)-holomorphic in \( U \). Denote by \( \Gamma_h(U) \) the graph of \( h \) over \( U \), and let \( E \) be a non-pluripolar subset of \( U \). Then \( \Gamma_h(U) \subseteq (\Gamma_h(E))_{-\infty}^{\mathbb{C}^n} \).

To prove Theorem 5.4.3 we need the following lemma, which is the higher dimensional version of Lemma 5.3.6 with a similar proof.

**Lemma 5.4.4.** Let \( U \subseteq \mathbb{C}^n \) be an \( \mathcal{F} \)-domain, and let \( f : U \rightarrow \mathbb{C} \) be an \( \mathcal{F} \)-holomorphic function. Suppose that \( h : \mathbb{C}^{n+1} \rightarrow [-\infty, +\infty[ \) is a plurisubharmonic function. Then the function \( z \mapsto h(z, f(z)) \) is \( \mathcal{F} \)-plurisubharmonic on \( U \).

**Proof.** First, we assume that \( h \) is continuous and finite everywhere. Let \( a \in U \). By Definition 5.2.2 there is a compact \( \mathcal{F} \)-neighborhood \( K \) of \( a \) in \( U \) and a sequence \( f_k \) of finely holomorphic functions defined in usual neighborhoods of \( K \) that converges uniformly to \( f|_K \). Since \( h(z, f(z)) \) is plurisubharmonic and converges uniformly to \( h(z, f(z)) \) on \( K \), \( h(z, f(z)) \) is \( \mathcal{F} \)-plurisubharmonic in the \( \mathcal{F} \)-interior of \( K \). Now
by the sheaf property of $\mathcal{F}$-plurisubharmonic function we conclude that $h(z, f(z))$ is $\mathcal{F}$-plurisubharmonic on $U$.

Suppose that $h$ is arbitrary. Then $h$ is the limit of some decreasing sequence of continuous plurisubharmonic functions $h_n \in \text{PSH}(\mathbb{C}^{n+1})$. By the first part of the proof, \( \{h_n(z, f(z))\}_n \) is a decreasing sequence of $\mathcal{F}$-plurisubharmonic functions in the $\mathcal{F}$-interior of $K$. The limit $h(z, f(z))$ is therefore $\mathcal{F}$-plurisubharmonic in the $\mathcal{F}$-interior of $K$. \qed

**Proof of Theorem 5.4.3.** By Theorem 5.3.1, the set $\Gamma_h(E)$ is pluripolar subset of $\mathbb{C}^{n+1}$. Let $\varphi \in \text{PSH}(\mathbb{C}^{n+1})$ with $\varphi \neq -\infty$ and $\varphi(z, h(z)) = -\infty$, for every $z \in E$. Lemma 5.4.4 shows that the function $z \mapsto \varphi(z, h(z))$ is $\mathcal{F}$-plurisubharmonic in $U$.

Since $E$ is not pluripolar, it follows from Theorem 4.4.1 that $\varphi(z, h(z)) = -\infty$ everywhere in $U$. Hence $\Gamma_h(U) \subseteq (\Gamma_h(E))_{\mathbb{C}^{n+1}}$.

**Corollary 5.4.5.** Let $f$ be holomorphic in a connected open set $U \subset \mathbb{C}^n$ and let $p \in \partial U$. Suppose that $f$ has an $\mathcal{F}$-holomorphic continuation $\tilde{F}$ at $p$ to an $\mathcal{F}$-open and $\mathcal{F}$-connected neighborhood $V$ of $p$. Then $\Gamma_{\mathcal{F}}(V) \subset (\Gamma_{\mathcal{F}}(U))_{\mathbb{C}^{n+1}}$. Moreover, if $E$ is a non-polar subset of $V \cap \overline{U}$ then $\Gamma_{\mathcal{F}}(U) \cup \Gamma_{\mathcal{F}}(V) \subset (\Gamma_{\mathcal{F}}(E))_{\mathbb{C}^{n+1}}$.

**Remark 5.4.6.** Corollary 5.4.5 only explains for a small part the propagation of pluripolar hulls. E.g., take $U$ the unit ball in $\mathbb{C}^n$ and consider the function $g(z) = f(z)(z_1 - z_2^2)$. Then, whatever the extendibility properties of $f$ may be, the pluripolar hull of the graph of $g$ will contain the set $\{z_1 = z_2^2\}$.

The next theorem is a simple, precise, and complete interpretation of recent results of Edigarian and Wiegerinck, cf. [38, 40].

**Theorem 5.4.7.** Let $D$ be a domain in $\mathbb{C}$ and let $A$ be a closed polar subset of $D$. Suppose that $f \in \mathcal{O}(D\setminus A)$ and that $z_0 \in A$. Then the following conditions are equivalent:

1. $\{(z_0) \times \mathbb{C} \cap (\Gamma_{\mathcal{F}})_D \neq \emptyset$.
2. $f$ has a finely holomorphic extension $\tilde{f}$ at $z_0$.

Moreover, if one of these conditions is met, then $\{(z_0) \times \mathbb{C} \cap (\Gamma_{\mathcal{F}})_D \neq \emptyset$.

**Proof.** (1) $\Rightarrow$ (2). According to Theorem 5.2.1, there exists $R > 0$ such that the set $\{z \in D\setminus A : |f(z)| \geq R\}$ is thin at $z_0$. Clearly, the set $U = \{z \in D\setminus A : |f(z)| < R\} \cup \{z_0\}$ is an $\mathcal{F}$-open neighborhood of $z_0$. Since $f$ is bounded in $U \setminus \{z_0\}$ and finely holomorphic in $U \setminus \{z_0\}$, Corollary 2.4.5 shows that $f$ has a finely holomorphic extension at $z_0$.

(2) $\Rightarrow$ (1). Suppose that $f$ has a finely holomorphic extension $\tilde{f}$ at $z_0$. Clearly, $(D\setminus A) \cup \{z_0\}$ is an $\mathcal{F}$-open neighborhood of $z_0$. Since polar sets do not separate $\mathcal{F}$-domains, cf. Theorem 2.2.29, the set $(D\setminus A) \cup \{z_0\}$ is $\mathcal{F}$-connected. Let $h \in \text{PSH}(D \times \mathbb{C})$ be a plurisubharmonic function such that $h(z, f(z)) = -\infty$, $\forall z \in D\setminus A$. According to Lemma 5.3.6, the function $z \mapsto h(z, \tilde{f}(z))$ is either finely
subharmonic on \((D \setminus A) \cup \{z_0\}\) or \(\equiv -\infty\). As it assumes \(-\infty\) on \(D \setminus A\), it must be identically equal to \(-\infty\) in view of Theorem 2.3.14. Consequently, \((z_0, f(z_0)) \in (\Gamma_f)_{A \times C}\). The last assertion follows from Theorem 5.10 in [40]. □

5.5 Concluding Remarks and Open Questions

A finely analytic curve is a pair \((U, f)\), where \(U\) is an \(F\)-domain in \(\mathbb{C}\) and \(f = (f_1, \ldots, f_n) : U \to \mathbb{C}^n\) is a finely holomorphic map. As usual we will identify a curve with its image. Let \(E \subset \mathbb{C}^n\) be a pluripolar set and \(E_{\mathbb{C}^n}\) its pluripolar hull. It follows from the arguments used before that if \(E\) hits a finely analytic curve \(f(U)\) in some non "small" set, then \(E_{\mathbb{C}^n}\) contains all the points of \(f(U)\). Namely, we have the following.

**Proposition 5.5.1.** Let \(f : U \to \mathbb{C}^n\) be a finely holomorphic map on a bounded \(F\)-domain \(U \subset \mathbb{C}\) and let \(E \subset \mathbb{C}^n\) be a pluripolar set. If \(f(U) \cap E \neq \emptyset\) and \(f^{-1}(f(U) \cap E)\) is non-polar, then \(f(U) \subset E_{\mathbb{C}^n}\).

**Proof.** Let \(h \in \text{PSH}(\mathbb{C}^n)\) be a plurisubharmonic function such that \(h(z) = -\infty, \forall z \in E\). By Lemma 5.3.6, \(f \circ f\) is either finely subharmonic on \(U\) or \(\equiv -\infty\). As it assumes \(-\infty\) on \(f^{-1}(f(U) \cap E)\), it must be, by Theorem 2.3.14, identically \(-\infty\) on \(U\). This proves that \(f(U) \subset E_{\mathbb{C}^n}\). □

The conclusion of the above proposition remains valid if one assumes that \(E\) contains merely the "boundary of a finely analytic curve".

**Proposition 5.5.2.** Let \(f\) and \(E\) be as above. If \(f\) extends by fine continuity to the \(F\)-boundary \(\partial_f U\) of \(U\) and \(f(\partial_f U) \subset E\), then \(f(U) \subset E_{\mathbb{C}^n}\).

**Proof.** Let \(h \in \text{PSH}(\mathbb{C}^n)\) be plurisubharmonic function such that \(h(z) = -\infty, \forall z \in E\). Let \(a \in \partial_f U\). By assumption, \(f\) has a fine limit at \(a\). Using Cartan’s theorem (cf. [76], Theorem 10.15), one can easily find a finely open neighborhood \(V_a\) of \(a\) such that the usual limit, \(\lim_{z \to a, z \in V_a \cap U} f(z)\), exists and is equal to \(f(a)\). Let \(M > 0\). Since \(h\) is upper semicontinuous, the set \(\{z \in \mathbb{C}^n : h(z) < -M\}\) is open. As \(f(a) \in \{z \in \mathbb{C}^n : h(z) < -M\}\), one can find a positive number \(\delta_a > 0\) such that

\[
 f(w) \in \{z \in \mathbb{C}^n : h(z) < -M\}, \forall w \in D(a, \delta_a) \cap V_a \cap U,
\]

where \(D(a, \delta_a)\) is the disk with center \(a\) and radius \(\delta_a\). Consequently

\[
 f \text{-}\limsup_{z \to a, z \in U} h(f(z)) \leq f \text{-}\limsup_{z \to a, z \in V_a \cap U} h(f(z)) < -M, \forall a \in \partial_f U
\]

where \(f\text{-}\limsup\) denotes the limit with respect to the fine topology.

By Lemma 5.3.6, the function \(h \circ f\) is a finely subharmonic function on \(U\), or is identically equal to \(-\infty\), the fine boundary maximum principle, cf. Theorem 2.3.11, shows that \(h \circ f(z) < -M, \forall z \in U\). Since \(M\) was arbitrary, we conclude that \(h \circ f(U) = -\infty\). This proves the proposition. □
Our results reveal a very close relationship between the pluripolar hull of the graph of a holomorphic function and the theory of finely holomorphic functions (see also [44]). This leads naturally to the following fundamental problem.

**Problem 1.** Let \( f : \Omega \rightarrow \mathbb{C} \) be a holomorphic function on a simply connected open subset \( \Omega \subset \mathbb{C} \). Suppose that the graph \( \Gamma_f(\Omega) \) of \( f \) over \( \Omega \) is not complete pluripolar. Must then \( (\Gamma_f(\Omega))^*_{\mathbb{C}^2} \backslash \Gamma_f(\Omega) \) have a fine analytic structure? i.e., let \( z \in (\Gamma_f(\Omega))^*_{\mathbb{C}^2} \backslash \Gamma_f(\Omega) \). Must there exist a finely analytic curve passing through \( z \) and contained in \( (\Gamma_f(\Omega))^*_{\mathbb{C}^2} \backslash \Gamma_f(\Omega) \)?

Obviously, a positive answer to the above problem would, in particular, solve the following problem posed in [44].

**Problem 2.** Let \( f \) be a holomorphic function in the unit disk \( D \). Suppose that \( (\Gamma_f(D))^*_{\mathbb{C}^2} \) is the graph of some function \( F \). Is \( F \) a finely holomorphic continuation of \( f \)?

It was proved in [25] that one cannot detect "pluripolarity" via intersection with one dimensional complex analytic varieties. Since there are, roughly speaking, much more finely analytic curves in \( \mathbb{C}^n \) than analytic varieties, one can naturally pose the following

**Problem 3.** Let \( K \) be a compact set in \( \mathbb{C}^n \) and suppose that \( f^{-1}(K \cap f(U)) \) is a polar subset of \( U \) (or empty) for any finely analytic curve \( f : U \rightarrow \mathbb{C}^n \). Must \( K \) be a pluripolar subset of \( \mathbb{C}^n \)?