Fine aspects of pluripotential theory

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Chapter 6

Fine Analytic Structure

This chapter is joint work with Tomas Edlund and presents, with no changes, the contents of the paper [46] which will appear in Indagationes Mathematicae.

We discuss the relation between pluripolar hulls and fine analytic structure. Our main result is the following. For each non polar subset $S$ of the complex plane $\mathbb{C}$ we prove that there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that the pluripolar hull of $E$ relative to $\mathbb{C}^2$ contains no fine analytic structure and its projection onto the first coordinate plane equals $\mathbb{C}$.

6.1 Introduction

Denote by $\Omega$ an open subset of $\mathbb{C}^n$ and let $E \subset \Omega$ be a pluripolar subset. It might be the case that any plurisubharmonic function $u(z)$ defined in $\Omega$ that is equal to $-\infty$ on the set $E$ is necessarily equal to $-\infty$ on a strictly larger set. For instance, if $E$ contains a non polar proper subset of a connected Riemann surface embedded into $\mathbb{C}^n$, then any plurisubharmonic function defined in a neighborhood of the Riemann surface which is equal to $-\infty$ on $E$ is automatically equal to $-\infty$ on the whole Riemann surface. In order to try to understand some aspect of the underlying mechanism of the described ”propagation” property of pluripolar sets, the pluripolar hull of graphs $\Gamma_f(D)$ of analytic functions $f$ in a domain $D \subset \mathbb{C}$ has been studied in a number of papers. (See for instance [38], [44], [93] and [134].)

The pluripolar hull $E^\ast_\Omega$ relative to $\Omega$ of a pluripolar set $E$ is defined as follows.

$$E^\ast_\Omega = \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over all plurisubharmonic functions defined in $\Omega$ which are equal to $-\infty$ on $E$. The set $E$ is called complete pluripolar in $\Omega$ if there exists a plurisubharmonic function on $\Omega$ which equals $-\infty$ precisely on $E$.

As remarked above a necessary condition for a pluripolar set $E$ to satisfy $E^\ast_\Omega = E$ is that $E \cap A$ is polar in $A$ (or $E \cap A = A$) for all one-dimensional complex analytic varieties $A \subset \Omega$. The fact that this is not a sufficient condition
was proved by Levenberg in [91]. By using a refinement of Wermer’s example of a polynomially convex compact set with no analytic structure (cf. [132]) Levenberg proved that there exists a compact set \( K \subset \mathbb{C}^2 \) satisfying \( K \neq K_{\mathbb{C}^2} \), and the intersection of \( K \) with any one dimensional analytic variety \( A \) is polar in \( A \). In this example it is not clear what the pluripolar hull \( K_{\mathbb{C}^2} \) equals.

We will say that a set \( S \subset \mathbb{C}^n \) contains fine analytic structure if there exists a non constant map \( \varphi : U \to S \) from a fine domain \( U \subset \mathbb{C} \) whose coordinate functions are finely holomorphic in \( U \) (see Definition 2.3 below). Such a map \( \varphi \) will be called a finely analytic curve.

Motivated by recent results of Jöricle and the first author (cf. [44]), the following result was proved in [41].

**Theorem 6.1.1.** Let \( \varphi : U \to \mathbb{C}^n \) be a finely holomorphic map on a fine domain \( U \subset \mathbb{C} \) and let \( E \subset \mathbb{C}^n \) be a pluripolar set. Then the following hold

1. \( \varphi(U) \) is a pluripolar subset of \( \mathbb{C}^n \)
2. If \( \varphi^{-1}(\varphi(U) \cap E) \) is a non polar subset of \( \mathbb{C} \) then \( \varphi(U) \subset E^*_{\mathbb{C}^n} \).

In view of this result one may expect to get more information on the pluripolar hull \( E^*_{\mathbb{C}^n} \) by examining the intersection of the pluripolar set \( E \) with finely analytic curves. Since many curves in \( \mathbb{C}^n \) are complete pluripolar (see [45]) one cannot expect that \( E^*_{\mathbb{C}^n} \) always contains fine analytic structure. However if we consider the non trivial part \( E^*_{\mathbb{C}^n} \setminus E \) the situation is up to now slightly different. In fact, all examples we have seen so far have the property that if \( E^*_{\mathbb{C}^n} \setminus E \) is nonempty then for each \( w \in E^*_{\mathbb{C}^n} \setminus E \) there exists a finely analytic curve \( \varphi \) contained in \( E^*_{\mathbb{C}^n} \) which passes through the point \( w \). (i.e. \( \varphi : U \to E^*_{\mathbb{C}^n} \) is a finely analytic curve and \( \varphi(z) = w \) for some \( z \in U \)). In this paper we prove that no such conclusion holds in general. We have the following main result.

**Theorem 6.1.2.** For each proper non polar subset \( S \subset \mathbb{C} \) there exists a pluripolar set \( E \subset (S \times \mathbb{C}) \) with the property that \( E^*_{\mathbb{C}^2} \) contains no fine analytic structure and the projection of \( E^*_{\mathbb{C}^2} \) onto the first coordinate plane equals \( \mathbb{C} \).

The set \( E \) will be a subset of a complete pluripolar set \( X \) which is constructed in the same spirit as Wermer’s polynomially convex compact set without analytic structure.

Let us describe more precisely the content of the paper. In Section 2 we briefly recall the construction of Wermer’s set and prove that it contains no fine analytic structure. This leads to Theorem 6.2.4 which slightly generalizes a result in [91]. The main result is proved in Section 3. Subsection 3.1 is devoted to construct the above mentioned set \( X \) and in Subsection 3.2 we show that \( X \) contains no fine analytic structure. In Subsection 3.3 we define the set \( E \) and describe \( E^*_{\mathbb{C}^2} \). Finally, in Section 4 we make some remarks and pose two open questions.

Readers who are not familiar with basic results on finely holomorphic functions and fine potential theory are referred to [56] and [61].

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6.2 Wermer’s Example

In this Section we sketch the details of Wermer’s construction given in [132]. Denote by \( D_r \), the open disk with center zero and radius \( r \) and by \( C_r \), the open cylinder \( D_r \times \mathbb{C} \). Let \( a_1, a_2, \ldots \) denote the points in the disk \( D_1 \) with rational real and imaginary part. For each \( j \) we denote by \( B_j(z) \) the algebraic (2-valued) function

\[
B_j(z) = (z - a_1)(z - a_2) \cdots (z - a_{j-1}) \sqrt{z - a_j}.
\]

To each \( n \)-tuple of positive constants \( c_1, c_2, \ldots, c_n \) we associate the algebraic (2-valued) function \( g_n(z) = \sum_{j=1}^{n} c_j B_j(z) \). Let \( \mathcal{P}(c_1, \ldots, c_n) \), \( n = 1, 2, \ldots \) be the subset of the Riemann surface of \( g_n(z) \) which lies in \( \mathbb{C}_2^\ast \).

**Lemma 6.2.1.** ([132], lemma 1) There exist positive constants \( \varepsilon_n \) and \( c_n, n = 1, 2, \ldots \), with \( c_1 = 1/10 \) and \( c_n+1 \leq (1/10) c_n \), \( n = 1, 2, \ldots \) and a sequence of polynomials \( \{p_n(z, w)\} \) such that:

1. \( \{p_n = 0\} \cap \{|z| \leq \frac{1}{2}\} = \sum(e_1, \ldots, e_n), n = 1, 2, \ldots \)
2. \( \{p_{n+1} \leq \varepsilon_{n+1}\} \cap \{|z| \leq \frac{1}{2}\} \subset \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}, n = 1, 2, \ldots \)
3. If \( |a| \leq \frac{1}{2} \) and \( |p_n(a, w)| \leq \varepsilon_n \), then there is a \( w_n \) with \( p_n(a, w_n) = 0 \) and \( |w-w_n| \leq \frac{1}{2^n} \), \( n = 1, 2, \ldots \).

With \( p_n, \varepsilon_n, n = 1, 2, \ldots \) chosen as in Lemma 6.2.1, we put

\[
Y = \bigcap_{n=1}^{\infty} \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}.
\]

Clearly, \( Y \) is a compact polynomially convex subset of \( \mathbb{C}^2 \). It was shown by Wermer that \( Y \) has no analytic structure i.e. \( Y \) contains no non-constant analytic disk. In fact he proves something stronger. The set \( Y \) defined above contains no graph of a continuous function defined on a circle in \( D_{1/2} \) which avoids all the branch points \( \{a_i\} \). Using this observation the following lemma follows.

**Lemma 6.2.2.** There is no finely analytic curve contained in \( Y \).

Before we prove Lemma 6.2.2 we recall the following definition (cf. [61], page 75):

**Definition 6.2.3.** Let \( U \) be a finely open set in \( \mathbb{C} \). A function \( f : U \rightarrow \mathbb{C} \) is said to be finely holomorphic if every point of \( U \) has a compact (in the usual topology) fine neighborhood \( K \subset U \) such that the restriction \( f|_K \) belongs to \( R(K) \).

Here \( R(K) \) denotes the uniform closure of the algebra of all restrictions to \( K \) of rational functions on \( \mathbb{C} \) with poles off \( K \).

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Proof of Lemma 2.2. Let \( \varphi : U \to Y, z \mapsto (\varphi_1(z), \varphi_2(z)) \) be a finely analytic curve contained in \( Y \). If \( \varphi_1(z) \) is constant on \( U \) then \( \varphi_2(z) \) must also be a constant since non constant finely holomorphic functions are finely open maps and by the construction of the set \( Y \) the fibre \( Y \cap \{z \times \mathbb{C} \} \) is a Cantor set or a finite set for any point \( z \in \overline{D}_{1/2} \). Assume therefore that \( \varphi_1(z) \) is non-constant. In particular, there is a point \( z_0 \in U \) where the fine derivative of \( \varphi_1(z) \) does not vanish. Hence \( \varphi_1(z) \) is one-to-one on some finely open neighborhood \( V \subset U \) of the point \( z_0 \). By considering the map \( z \mapsto (\varphi_1 \circ \varphi_1^{-1}(z), \varphi_2 \circ \varphi_1^{-1}(z)) \), defined on the finely open set \( \varphi_1(V) \) we may assume that \( \varphi \) is of the form \( z \mapsto (z, g(z)) \) where \( g(z) = \varphi_2 \circ \varphi_1^{-1}(z) \) is finely holomorphic in the finely open set \( V' = \varphi_1(V) \subset D_{1/2} \). By Definition 6.2.3 there exists a compact subset \( K \subset V' \) with non-empty fine interior such that \( g(z) \) is a continuous function on \( K \) (with respect to the Euclidean topology). Shrinking \( K \) if necessary we may assume that \( K \cap \{a_1, a_2, \ldots \} = \emptyset \). Let \( p \) be a point in the fine interior of \( K \). It is well known that there exists a sequence of circles \( \{C(p, r_j)\} \) contained in \( K \) with centers \( p \) and radii \( r_j \to 0 \) as \( j \to \infty \). Clearly, the circle \( C(p, r_j) \) avoids the branch points \( \{a_1, a_2, \ldots \} \) and its image under the continuous map \( z \mapsto (z, g(z)) \) is contained in \( Y \). By the above observation this is not possible. Hence \( Y \) contains no fine analytic structure.

Denote by \( d_n \) the degree of the one variable polynomial \( w \mapsto p_n(z, w) \) where \( p_n(z, w) \) is the polynomial given in Lemma 6.2.1. Assume that the set \( Y \) is constructed using the parameters \( \varepsilon_n \) satisfying the following condition

\[
\lim_{n \to \infty} (\varepsilon_n)^{1/d_n} = 0. \tag{6.2.1}
\]

It is shown in [91] that with this choice the set \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \). Using this result and Lemma 6.2.2 we are able to generalize a result in [91].

Theorem 6.2.4. Fix \( \delta \in (0, 1/2) \) and let \( Y_\delta = \bigcap_{n=1}^{\infty} \{ \{ |p_n| \leq \epsilon_n \} \cap \{|z| \leq \delta \} \} \) be constructed using the parameters \( \varepsilon_n \) satisfying (6.2.1). Then

(a) \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a polar subset of \( U \) for all finely analytic curves \( \varphi : U \to \mathbb{C}^2 \).

(b) \( Y_\delta \neq (Y_\delta)^\circ_{C^2} \).

Proof of Theorem 6.2.4. In order to prove (a) we argue by contradiction. Assume therefore that \( \varphi : U \to \mathbb{C}^2 \) is a finely analytic curve and \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a non polar subset of \( U \). Then there is a fine domain \( U_{k_0} \subseteq U \) such that \( \varphi(U_{k_0}) \subset C_{1/2} \) and \( \varphi^{-1}(\varphi(U_{k_0}) \cap Y_\delta) \) is non polar. Indeed, the set \( \varphi^{-1}(\varphi(U) \cap C_{1/2}) \) is a finely open subset of \( U \) and hence has at most countably many finely connected components \( \{ U_k \}_{k=1}^{\infty} \). Moreover, \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0} \) is non polar for some natural number \( k_0 \), since otherwise \( \bigcup_{k=1}^{\infty} \{ \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_k \} = \varphi^{-1}(\varphi(U) \cap Y_\delta) \) would be polar contrary to our assumption. Since \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \) there exists a plurisubharmonic function \( u \) defined in \( C_{1/2} \) which is equal to \(-\infty\) exactly on \( Y \cap C_{1/2} \). By Lemma 5.3.6, the function \( u \circ \varphi \) is either finely subharmonic on \( U_{k_0} \) or identically equal to \(-\infty\). Since \( u \) equals \(-\infty\) on the non polar subset \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0} \), it must be identically equal to \(-\infty\) on \( U_{k_0} \). Therefore \( \varphi(U_{k_0}) \subset \{ u = -\infty \} = Y \cap C_{1/2} \) contradicting Lemma 6.2.2 and (a) follows.
6.3 Proof of Theorem 6.1.2

In this Subsection we construct the set \( X \). Denote by \( \{a_k\}_{k=1}^{\infty} \) the points in the complex plane both of whose coordinates are rational numbers. Without loss of generality we may assume that \( a_k \in \mathcal{D}_k \). For any sequence of points \( \{a_i\}_{i=1}^{l} \) we denote by \( B_j(z) \) the algebraic function

\[
B_j(z) = (z - a_1) \ldots (z - a_j) \sqrt{z - a_j}.
\]

Denote by \( \gamma_j \) a simple smooth curve with endpoints \( a_j \) and \( \infty \). For each \( j \), \( B_j(z) \) has two single-valued analytic branches on \( \mathbb{C} \setminus \gamma_j \). Following the notation in [132] we choose one of the branches \( B_j(z) \) arbitrarily and denote it by \( \beta_j(z) \). Then \( |\beta_j(z)| = |B_j(z)| \) is continuous on \( \mathbb{C} \).

For each \( n + 1 \)-tuple of positive constants \( (c_1, c_2, \ldots, c_{n+1}) \) we denote by \( g_n(z) \) the algebraic function defined recursively in the following way. Put \( g_1(z) = c_1B_1(z) \) and \( g_2(z) = c_1B_1(z) + c_2B_2(z) \) and if \( g_n(z) \) has been chosen we will choose \( g_{n+1}(z) \) as described below. Put \( Z_1(z) = 1 \) and for \( n = 2, 3, \ldots \) define the function \( Z_n(z) \) as follows. Denote by \( z_1, z_2, \ldots, z_l \) all the zeros of all possible different differences \( h_j(z) - h_i(z) \) \( (i \neq j) \) of branches \( h_j(z), h_i(z) \) of the function \( g_n(z) \). Suppose \( z_k \) is a zero of \( h_j(z) - h_i(z) \) of order \( m_k \) and put \( Z_n(z) = \Pi_{i=1}^{l}(z - z_i)^{m_i} \). Note that the zeros of \( Z_n(z) \) are also zeros of the function \( Z_{n+1}(z) \) of the same or greater multiplicity. Define \( g_{n+1}(z) = g_n(z) + c_{n+1}Z_n(z)B_{n+1}(z) \).

By \( \Sigma(c_1, \ldots, c_n) \) we mean the Riemann surface of \( g_n(z) \) which lies in \( \mathbb{C}^2 \). In other words, \( \Sigma(c_1, \ldots, c_n) = \{(z, w) : z \in \mathbb{C}, w = w_j, j = 1, 2, \ldots, 2^n\} \), where \( w_j, j = 1, 2, \ldots, 2^n \) are the values of \( g_n(z) \) at \( z \).
We will choose positive constants \( c_n, \epsilon_n \) and polynomials \( p_n(z, w) \) recursively so that
\[
\{ p_n(z, w) = 0 \} \cap C_{n+1} = \Sigma(c_1, c_2, \ldots, c_n) \cap C_{n+1} \quad \text{and} \quad (6.3.1)
\]
\[
|p_{n+1}(z, w)| \leq \epsilon_{n+1} \cap C_{n+1} \subset \{ |p_n(z, w)| < \epsilon_n \} \cap C_{n+1} \quad \text{and} \quad (6.3.2)
\]
hold for \( n = 1, 2, \ldots \). The set \( X \) will be of the form
\[
X = \bigcap_{n=1}^{\infty} \left( \bigcap_{j=n}^{\infty} \{ |p_j(z, w)| \leq \epsilon_j \} \cap C_{n+1} \right). \quad (6.3.3)
\]

Put \( c_1 = 1 \) and let \( p_1(z, w) = w^2 - (z - a_1) \). It is clear that \( \Sigma(c_1) \cap C_2 = \{ p_1(z, w) = 0 \} \cap C_2 \). Choose \( \epsilon_1 > 0 \) so that if \( z_0 \in D_2 \) and \( |p_1(z_0, w)| \leq \epsilon_1 \) then there exists \( (z_0, w_1) \in \Sigma(c_1) \cap C_2 \) with \( |w - w_1| \leq 1 \). Let \( B_2 = D_2 \times D_{p_1} \) be a bidisk where \( p_1 \) is chosen so that
\[
\{ |p_1(z, w)| \leq \epsilon_1 \} \cap C_2 = \{ |p_1(z, w)| \leq \epsilon_1 \} \cap B_2.
\]

Assume that \( c_n, \epsilon_n \) and \( p_n(z, w) \) have been chosen so that (6.3.1) and (6.3.2) hold. We will now choose \( c_{n+1} \) and \( p_{n+1}(z, w) \). We denote by \( w_j(z) \), \( j = 1, 2, \ldots, 2^n \) the roots of \( p_n(z, \cdot) = 0 \) and to each positive constant \( c \) we assign a polynomial \( p_c(z, w) \) by putting
\[
p_c(z, w) = \Pi_{j=1}^{2^n} \left( (w - w_j(z))^2 - c^2 (Z_n(z)B_{n+1}(z))^2 \right). \quad (6.3.4)
\]

Then \( p_c(z, \cdot) = 0 \) has the roots \( w_j(z) \pm cZ_n(z)B_{n+1}(z), j = 1, 2, \ldots, 2^n \) and so
\[
\{ p_c(z, w) = 0 \} = \Sigma(c_1, c_2, \ldots, c_n, c).
\]

Note that from (6.3.4)
\[
p_c = p_n^2 + c^2 q_1 + \ldots + (c^2)^{2^n} q_{2^n},
\]
where the \( q_j \) are polynomials in \( z \) and \( w \), not depending on \( c \). Choose \( c > 0 \) so that the following hold for all \( z \in D_{n+1} \).
\[
\Sigma(c_1, c_2, \ldots, c_n, c) \cap C_{n+1} \subset \{ |p_n(z, w)| < \epsilon_n/2 \} \cap C_{n+1} \quad \text{and} \quad (6.3.5)
\]
\[
c \cdot |Z_n(z)B_{n+1}(z)| \leq (1/10)c_{n}|Z_{n-1}(z)B_n(z)|. \quad (6.3.6)
\]

Decreasing \( c \) if necessary we may assume that if \( h_i(z) \) and \( h_j(z) \) are any different branches of the function \( q_n(z) \) the estimate
\[
|h_j(z) - h_i(z)| \geq 2c|Z_n(z)B_{n+1}(z)| \quad (6.3.7)
\]
holds in \( D_{n+1} \) with equality exactly at the zeros of \( Z_n(z) \) which are contained in \( D_{n+1} \) and at the points \( a_1, \ldots, a_n \). This estimate will be needed later when we prove that \( X \) contains no fine analytic structure. Choose \( c_{n+1} = c \).
Let $B_{n+2} = D_{n+2} \times D_{n+2}$ be a bidisk where $\rho_{n+2}$ is chosen so that $\{|p_n(z,w)| \leq \epsilon_n\} \cap C_{n+2} = \{|p_n(z,w)| \leq \epsilon_n\} \cap B_{n+2}$ and $\rho_{n+2} > \rho_{n+1} + 1$. Let $\delta > 0$ be a constant such that $|\delta \cdot p_n(z,w)| < 1$ in $B_{n+2}$ and choose $p_{n+1}(z,w) = \delta \cdot p_n(z,w)$.

We now turn to the choice of $\epsilon_{n+1}$. Since the part of the zero set of $p_{n+1}(z,w)$ which is contained in $B_{n+1}$ is a subset of $\{|p_n(z,w)| < \epsilon_n/2\} \cap B_{n+1}$ it is possible to find a natural number $m_{n+1}$ so that

$$\frac{1}{m_{n+1}} \log |p_{n+1}(z,w)| \geq -\frac{1}{2^n} \text{ for all } (z,w) \in B_{n+1} \setminus \{|p_n(z,w)| \leq \epsilon_n\}. \tag{6.3.8}$$

Choose $\epsilon_{n+1} < \epsilon_n$ so that

$$\frac{1}{m_{n+1}} \log |p_{n+1}(z,w)| \leq -1 \text{ for all } (z,w) \in \{|p_{n+1}(z,w)| \leq \epsilon_{n+1}\} \cap C_{n+2}. \tag{6.3.9}$$

By decreasing $\epsilon_{n+1}$ we may assume that (6.3.2) and the following assumption hold.

If $(z_0,w) \in C_{n+2}$ and $|p_{n+1}(z_0,w)| \leq \epsilon_{n+1}$, then there exists $(z_0,w_n) \in C_{n+2}$ such that $|p_{n+1}(z_0,w_n)| = 0$ and $|w - w_n| \leq 1/n$. \tag{6.3.10}

This ends the recursion.

**Lemma 6.3.1.** The set $X$ defined by (6.3.3) is complete pluripolar in $\mathbb{C}^2$.

**Proof.** Define for $n \geq 2$ the plurisubharmonic function

$$u_n(z,w) = \max \left\{ \frac{1}{m_n} \log |p_n(z,w)|, -1 \right\}$$

and put $u(z,w) = \sum_{n \geq 2} u_n(z,w)$. Then $u(z,w)$ is plurisubharmonic in $\mathbb{C}^2$. Indeed, since the bidisks $B_n$ exhaust $\mathbb{C}^2$ and $|p_n(z,w)| < 1$ in $B_{n+1}$ the series $\sum_{n \geq 2} u_n(z,w)$ will be decreasing on each fixed bidisk $B_N$ after a finite number of terms and hence plurisubharmonic there. Since plurisubharmonicity is a local property $u(z,w)$ is plurisubharmonic in $\mathbb{C}^2$. If $(z_0,w_0) \in X$, then for some natural number $N$, $(z_0,w_0) \in \bigcap_{n=N}^{\infty} \{|p_n(z,w)| \leq \epsilon_n\} \cap C_{n+1}$. Condition (6.3.9) above implies that $u(z_0,w_0) = Const + \sum_{n \geq N} u_n(z_0,w_0) = -\infty$. Finally if $(z_0,w_0) \notin X$ then there exists a natural number $N$ such that $(z_0,w_0) \notin B_N$ and $(z_0,w_0) \notin \{|p_n(z,w)| \leq \epsilon_n\} \cap B_N$ for all $n \geq N$. By (6.3.8)

$$u(z,w) = Const + \sum_{n > N} \max \left\{ \frac{1}{m_n} \log |p_n(z,w)|, -1 \right\} \geq Const + \sum_{n > N} -\frac{1}{2^n} > -\infty.$$ 

The Lemma follows. \hfill $\square$

### 6.3.2 X Contains No Fine Analytic Structure

In this Section we show that $X$ contains no fine analytic structure. Suppose that $z \mapsto (\varphi_1(z), \varphi_2(z))$ is a finely analytic curve whose image is contained in $X$. If
\(\varphi_1(z)\) is constant then \(\varphi_2(z)\) must be constant since \(X \cap (\{z_0\} \times \mathbb{C})\) is a Cantor set or a finite set for any point \(z_0 \in \mathbb{C}\). On the other hand, if \(\varphi_1(z)\) is non-constant, then using the arguments given in the proof of Lemma 6.2.2 we may assume that the finely analytic curve contained in \(X\) is given by \(z \mapsto (z, m(z))\) where \(m(z)\) is a finely holomorphic function defined in \(U\) where \(U \subset \mathcal{D}_n\) for some natural number \(n\). Fix a point \(z' \in U \setminus \{a_1, \ldots, a_n\}\). By the definition of finely holomorphic functions we can find a compact (in the usual topology) fine neighborhood \(K\) of \(z'\) where \(m(z)\) is continuous. Shrinking \(K\) if necessary we may assume that \((K \setminus \{z'\}) \cap (\{a_j\}_{j=1}^n \cup \{Z_{k-1}(z) = 0\}_{k=2}^\infty) = \emptyset\). Since the complement of \(K\) is thin at \(z'\), one can find a sequence of circles \(\{C(z', r_i)\} \subset K\) with \(r_i \to 0\) as \(i \to \infty\). Choose one of the circles \(C(z', r_j)\) so that none of the points \(a_1, \ldots, a_n\) are contained in \(\{|z - z'| \leq r_j\}\). Let \(a_k\) be the first point in the sequence \(\{a_j\}_{j=1}^\infty\) which is contained in \(\{|z - z'| \leq r_j\}\). Note that \(a_k \in \{|z - z'| < r_j\}\) and \(m(z)\) is continuous on \(C(z', r_j)\) and the function \(Z_{k-1}(z)\beta_k(z) \neq 0\) when \(z \in C(z', r_j)\). The fact that the image of \(C(z', r_j)\) under the map \(z \mapsto (z, m(z))\) is a subset of \(X\) will lead us to a contradiction and hence \(X\) contains no fine analytic structure. In order to prove this fix a point \(z_1 \in C(z', r_j)\) and denote by \(\mathcal{R}\) the \(2^k\) branches of the algebraic function \(g_k(z)\) defined on \(C(z', r_j) \setminus \{z_1\}\).

**Lemma 6.3.2.** If \(h_1(z)\) and \(h_2(z)\) are any different functions from \(\mathcal{R}\), then

\[
|h_1(z) - h_2(z)| > (3/2)c_k|Z_{k-1}(z)\beta_k(z)|
\]

(6.3.11)

holds for all \(z \in C(z', r_j) \setminus \{z_1\}\).

**Proof.** This follows directly from (6.3.7) since \(C(z', r_j) \subset \mathcal{D}_n\) and \(C(z', r_j)\) does not intersect any of the branch points \(a_1, \ldots, a_k\) or the zeros of \(Z_{k-1}(z)\). \(\square\)

From now on the proof that \(X\) contains no fine analytic structure follows the arguments given in [132].

**Lemma 6.3.3.** Fix \(z_0\) in \(C(z', r_j) \setminus \{z_1\}\). There exists a function \(h_i(z) \in \mathcal{R}\), where \(h_i(z)\) depends on \(z_0\) such that

\[
|m(z_0) - h_i(z_0)| < (1/4)c_k|Z_{k-1}(z_0)\beta_k(z_0)|
\]

(6.3.12)

**Proof.** By (6.3.10) there exists \(N \geq k\) and \(w_N\) such that \((z_0, w_N)\) lies on the set \(\Sigma(c_1, \ldots, c_N)\) and \(m(z_0) = w_N + R(z_0)\) where \(|R(z_0)| \leq (1/10)c_k|Z_{k-1}(z_0)\beta_k(z_0)|\). Thus

\[
m(z_0) = \pm c_1 \beta_1(z_0) + \sum_{\nu=2}^{N} \pm c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0) = \\
def h_i(z_0) + \sum_{\nu=k+1}^{N} c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0).
\]
Since $C(z', r_j) \subset D_{n+1}$ and the constants $c_v$ are chosen so that (6.3.6) holds,

$$|m(z_0) - h_i(z_0)| \leq \sum_{\nu=k+1}^{N} c_\nu |Z_{\nu-1}(z_0)\beta_\nu(z_0)| + |R(z_0)| \leq$$

$$\leq c_k |Z_{k-1}(z_0)\beta_k(z_0)|\bigg(\frac{1}{10} + \frac{1}{10^2} + \ldots \bigg) + |R(z_0)| =$$

$$= \frac{1}{9} c_k |Z_{k-1}(z_0)\beta_k(z_0)| + \frac{1}{10} c_k |Z_{k-1}(z_0)\beta_k(z_0)| <$$

$$< (1/4)c_k |Z_{k-1}(z_0)\beta_k(z_0)|.$$  

Hence (6.3.12) holds and the Lemma is proved.  

**Lemma 6.3.4.** Fix $z_0 \in C(z', r_j) \setminus \{z_1\}$ and let $h_i(z) \in \mathbb{R}$ satisfy (6.3.12). Then for all $z$ in $C(z', r_j) \setminus \{z_1\}$

$$|m(z) - h_i(z)| < (1/3)c_k |Z_{k-1}(z)\beta_k(z)|.$$  

**Proof.** The set $\mathcal{O} = \{z \in C(z', r_j) \setminus \{z_1\} : (6.3.13) \text{ holds at } z\}$ is open in $C(z_0, r_j) \setminus \{z_1\}$ and contains $z_0$. If $\mathcal{O} \neq C(z', r_j) \setminus \{z_1\}$ then there is a boundary point $p$ of $\mathcal{O}$ on $C(z', r_j) \setminus \{z_1\}$ for which

$$|m(p) - h_i(p)| = (1/3)c_k |Z_{k-1}(p)\beta_k(p)|$$  

holds. By Lemma 6.3.3 there is some $h_j(z)$ in $\mathbb{R}$ such that

$$|m(p) - h_j(p)| < (1/4)c_k |Z_{k-1}(p)\beta_k(p)|.$$  

Thus $|h_i(p) - h_j(p)| \leq (7/12)c_k |Z_{k-1}(p)\beta_k(p)|$. Also $h_i(z) \neq h_j(z)$, in view of (6.3.14) and (6.3.15). This contradicts Lemma 6.3.2. Thus $\mathcal{O} = C(z', r_j) \setminus \{z_1\}$ and Lemma 6.3.4 follows.

For each continuous function $v(z)$ defined on $C(z', r_j) \setminus \{z_1\}$ which has a jump at $z_1$ we write $L^+(v)$ and $L^-(v)$ for the two limits of $v(z)$ as $z \to z_1$ along $C(z', r_j)$. Then, by (6.3.13),

$$|L^+(m) - L^+(h_i)| \leq (1/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)|$$

and

$$|L^-(m) - L^-(h_i)| \leq (1/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)|,$$

so

$$|(L^+(m) - L^+(h_i)) - (L^- (m) - L^- (h_i))| \leq (2/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)|.$$

Since $m(z)$ is continuous on $C(z', r_j)$ the jump of $h_i(z)$ at $z_1$ is in modulus less than or equal to $(2/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)| \neq 0$. But $h_i(z)$ is in $\mathbb{R}$, so its jump at $z_1$ has modulus at least $2c_k |Z_{k-1}(z_1)\beta_k(z_1)|$. This is a contradiction.
6.3.3 The Sets $E$ and $E^*_C$

Denote by $E$ the pluripolar set $E = (S \times \mathbb{C}) \cap X$ where $S$ is a non polar subset of $\mathbb{C}$. Since $X$ is complete pluripolar in $\mathbb{C}^2$ it follows that $E^*_C \subset X$. To prove that $X \subset E^*_C$ we argue as follows. First we claim that the set $X$ is pseudoconcave. Indeed, by the construction of the set $X$,

$$\mathbb{C}^2 \setminus X = \bigcup_{n=1}^{\infty} \{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1}. \quad (6.3.16)$$

By the choice of the polynomials $p_n(z, w)$ it follows that

$$\{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1} \subset \{|p_{n+1}(z, w)| > \epsilon_{n+1}\} \cap \mathcal{C}_{n+2}.$$ 

Moreover, for each natural number $n$ the set $\{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1}$ is a domain of holomorphy. Hence $\mathbb{C}^2 \setminus X$ is a countable union of increasing domains of holomorphy. By the Behnke-Stein Theorem $\mathbb{C}^2 \setminus X$ is pseudoconvex and the claim follows.

Denote by $u(z, w)$ a globally defined plurisubharmonic function which equals $-\infty$ on $E$. It is shown in [127] that the function $z \mapsto \max\{u(z, w) : (z, w) \in X\}$ is subharmonic in $\mathbb{C}$. Since the projection $S$ of $E$ onto the first coordinate plane is non polar the function $z \mapsto \max\{u(z, w) : (z, w) \in X\}$ will be identically equal to $-\infty$ on $\mathbb{C}$ hence $u(z, w) = -\infty$ on the whole of $X$ and consequently $E^*_C = X$. This ends the proof of Theorem 6.1.2.

6.4 Final Remarks and Open Problems

It follows immediately from Theorem 6.1.1 and the fact that $X$ contains no fine analytic structure that if $\varphi : U \to \mathbb{C}^2$ is a finely analytic curve, then the set $\varphi^{-1}(\varphi(U) \cap X)$ is polar in $\mathbb{C}$.

Despite the result of Theorem 6.1.2 it should be mentioned here that in the situation where one considers the pluripolar hull of the graph of a finely holomorphic function defined in a fine domain $D$, the following problem still remains open.

**Problem 1.** Let $z \in \Gamma_f(D)^C_\mathbb{C}$. Does this imply that there is a finely analytic curve contained in $\Gamma_f(D)^C_\mathbb{C}$ which passes through the point $z$?

It is proved in [40] that the pluripolar hull relative to $\mathbb{C}^n$ of a connected pluripolar $F_\sigma$ subset is a connected set. It is a fairly easy exercise to show that the set $X = E^*_C$ in Theorem 6.1.2 is path connected, but in general the pluripolar hull of a connected $(F_\sigma)$ pluripolar set is not path connected. Indeed, denote by $f(z)$ an entire function of order $1/3$. $f(1/z)$ has an essential singularity at $0$ and in [134] Wiegerinck proved that the graph $\Gamma_f(1/z)$ of $f(1/z)$ over $\mathbb{C} \setminus \{0\}$ is complete pluripolar in $\mathbb{C}^2$. Consequently, if we put $E = \Gamma_f(1/z) \cup \{0\} \times \mathbb{C}$ then $E$ is complete pluripolar in $\mathbb{C}^2$ and hence $E^*_C = E$. Moreover $E$ is a connected $F_\sigma$ subset of $\mathbb{C}^2$. By the famous Denjoy-Carleman-Ahlfors theorem (see e.g. [1]), entire functions of order $1/3$ do not have finite asymptotic values; i.e., there are no curves $\gamma$ ending at infinity such that $f(z)$ approaches a finite value as $z \to \infty$ along $\gamma$. Hence it is not possible to find a path in $E^*_C$ connecting a point on $\Gamma_f(1/z)$ with a point in...
the set \( \{0\} \times \mathbb{C} \). In view of this remark it would be interesting to know the answer to the following question.

**Problem 2.** Is \( \Gamma_{f}(D)_{\mathbb{C}^2} \) path connected?