Fine aspects of pluripotential theory
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Chapter 6
Fine Analytic Structure

This chapter is joint work with Tomas Edlund and presents, with no changes, the contents of the paper [46] which will appear in Indagationes Mathematicae.

We discuss the relation between pluripolar hulls and fine analytic structure. Our main result is the following. For each non-polar subset $S$ of the complex plane $\mathbb{C}$ we prove that there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that the pluripolar hull of $E$ relative to $\mathbb{C}^2$ contains no fine analytic structure and its projection onto the first coordinate plane equals $\mathbb{C}$.

6.1 Introduction

Denote by $\Omega$ an open subset of $\mathbb{C}^n$ and let $E \subset \Omega$ be a pluripolar subset. It might be the case that any plurisubharmonic function $u(z)$ defined in $\Omega$ that is equal to $-\infty$ on the set $E$ is necessarily equal to $-\infty$ on a strictly larger set. For instance, if $E$ contains a non-polar proper subset of a connected Riemann surface embedded into $\mathbb{C}^n$, then any plurisubharmonic function defined in a neighborhood of the Riemann surface which is equal to $-\infty$ on $E$ is automatically equal to $-\infty$ on the whole Riemann surface. In order to try to understand some aspect of the underlying mechanism of the described "propagation" property of pluripolar sets, the pluripolar hull of graphs $\Gamma_f(D)$ of analytic functions $f$ in a domain $D \subset \mathbb{C}$ has been studied in a number of papers. (See for instance [38], [44], [93] and [134].)

The pluripolar hull $E^*_\Omega$ relative to $\Omega$ of a pluripolar set $E$ is defined as follows.

$$E^*_\Omega = \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over all plurisubharmonic functions defined in $\Omega$ which are equal to $-\infty$ on $E$. The set $E$ is called complete pluripolar in $\Omega$ if there exists a plurisubharmonic function on $\Omega$ which equals $-\infty$ precisely on $E$.

As remarked above a necessary condition for a pluripolar set $E$ to satisfy $E^*_\Omega = E$ is that $E \cap A$ is polar in $A$ (or $E \cap A = A$) for all one-dimensional complex analytic varieties $A \subset \Omega$. The fact that this is not a sufficient condition

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was proved by Levenberg in [91]. By using a refinement of Wermer’s example of a polynomially convex compact set with no analytic structure (cf. [132]) Levenberg proved that there exists a compact set \( K \subset \mathbb{C}^2 \) satisfying \( K \neq K_{\mathbb{C}^2} \), and the intersection of \( K \) with any one dimensional analytic variety \( A \) is polar in \( A \). In this example it is not clear what the pluripolar hull \( K_{\mathbb{C}^2} \) equals.

We will say that a set \( S \subset \mathbb{C}^n \) contains fine analytic structure if there exists a non constant map \( \varphi : U \rightarrow S \) from a fine domain \( U \subset \mathbb{C} \) whose coordinate functions are finely holomorphic in \( U \) (see Definition 2.3 below). Such a map \( \varphi \) will be called a finely analytic curve.

Motivated by recent results of Jöricle and the first author (cf. [44]), the following result was proved in [41].

**Theorem 6.1.1.** Let \( \varphi : U \rightarrow \mathbb{C}^n \) be a finely holomorphic map on a fine domain \( U \subset \mathbb{C} \) and let \( E \subset \mathbb{C}^n \) be a pluripolar set. Then the following hold
1. \( \varphi(U) \) is a pluripolar subset of \( \mathbb{C}^n \)
2. If \( \varphi^{-1}(\varphi(U) \cap E) \) is a non polar subset of \( \mathbb{C} \) then \( \varphi(U) \subset E_{\mathbb{C}^n}^* \).

In view of this result one may expect to get more information on the pluripolar hull \( E_{\mathbb{C}^n}^* \) by examining the intersection of the pluripolar set \( E \) with finely analytic curves. Since many curves in \( \mathbb{C}^n \) are complete pluripolar (see [45]) one cannot expect that \( E_{\mathbb{C}^n}^* \) always contains fine analytic structure. However if we consider the non trivial part \( E_{\mathbb{C}^n} \setminus E \) the situation is up to now slightly different. In fact, all examples we have seen so far have the property that if \( E_{\mathbb{C}^n} \setminus E \) is non empty then for each \( w \in E_{\mathbb{C}^n} \setminus E \) there exists a finely analytic curve \( \varphi \) contained in \( E_{\mathbb{C}^n} \) which passes through the point \( w \). (i.e. \( \varphi : U \rightarrow E_{\mathbb{C}^n}^* \) is a finely analytic curve and \( \varphi(z) = w \) for some \( z \in U \)). In this paper we prove that no such conclusion holds in general. We have the following main result.

**Theorem 6.1.2.** For each proper non polar subset \( S \subset \mathbb{C} \) there exists a pluripolar set \( E \subset (S \times \mathbb{C}) \) with the property that \( E_{\mathbb{C}^2}^* \) contains no fine analytic structure and the projection of \( E_{\mathbb{C}^2}^* \) onto the first coordinate plane equals \( \mathbb{C} \).

The set \( E \) will be a subset of a complete pluripolar set \( X \) which is constructed in the same spirit as Wermer’s polynomially convex compact set without analytic structure.

Let us describe more precisely the content of the paper. In Section 2 we briefly recall the construction of Wermer’s set and prove that it contains no fine analytic structure. This leads to Theorem 6.2.4 which slightly generalizes a result in [91]. The main result is proved in Section 3. Subsection 3.1 is devoted to construct the above mentioned set \( X \) and in Subsection 3.2 we show that \( X \) contains no fine analytic structure. In Subsection 3.3 we define the set \( E \) and describe \( E_{\mathbb{C}^2}^* \). Finally, in Section 4 we make some remarks and pose two open questions.

Readers who are not familiar with basic results on finely holomorphic functions and fine potential theory are referred to [56] and [61].

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6.2 Wermer’s Example

In this Section we sketch the details of Wermer’s construction given in [132]. Denote by $D_r$, the open disk with center zero and radius $r$ and by $C_r$, the open cylinder $D_r \times \mathbb{C}$. Let $a_1, a_2, ...$ denote the points in the disk $D_1$ with rational real and imaginary part. For each $j$ we denote by $B_j(z)$ the algebraic (2-valued) function

$$B_j(z) = (z - a_1)(z - a_2)...(z - a_{j-1}) \sqrt{z - a_j}.$$ 

To each $n$-tuple of positive constants $c_1, c_2, ..., c_n$ we associate the algebraic ($2^n$-valued) function $g_n(z) = \sum_{j=1}^{n} c_j B_j(z)$. Let $g_n(z)$ be the subset of the Riemann surface of $g_n(z)$ which lies in $C_1$.  

**Lemma 6.2.1.** [[132], lemma 1] There exist positive constants $\varepsilon_n$ and $c_n$, $n = 1, 2, ...$ and a sequence of polynomials $\{p_n(z, w)\}$ such that:

1. $\{p_n = 0\} \cap \{|z| \leq \frac{1}{2}\} = \sum (c_1, ..., c_n)$, $n = 1, 2, ...$
2. $\{|p_{n+1}| \leq \varepsilon_{n+1}\} \cap \{|z| \leq \frac{1}{2}\} \subset \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}$, $n = 1, 2, ...$
3. If $|a| \leq \frac{1}{2}$ and $|p_n(a, w)| \leq \varepsilon_n$, then there is a $w_n$ with $p_n(a, w_n) = 0$ and $|w - w_n| \leq \frac{1}{n}$, $n = 1, 2, ...$.

With $p_n, \varepsilon_n, n = 1, 2, ...$ chosen as in Lemma 6.2.1, we put

$$Y = \bigcap_{n=1}^{\infty} \{ |p_n| \leq \varepsilon_n \} \cap \{ |z| \leq \frac{1}{2} \}.$$ 

Clearly, $Y$ is a compact polynomially convex subset of $\mathbb{C}^2$. It was shown by Wermer that $Y$ has no analytic structure i.e. $Y$ contains no non-constant analytic disk. In fact he proves something stronger. The set $Y$ defined above contains no graph of a continuous function defined on a circle in $D_\frac{1}{2}$ which avoids all the branch points $\{a_i\}$. Using this observation the following lemma follows.

**Lemma 6.2.2.** There is no finely analytic curve contained in $Y$.

Before we prove Lemma 6.2.2 we recall the following definition (cf. [61], page 75):

**Definition 6.2.3.** Let $U$ be a finely open set in $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is said to be finely holomorphic if every point of $U$ has a compact (in the usual topology) fine neighborhood $K \subset U$ such that the restriction $f |_K$ belongs to $R(K)$.

Here $R(K)$ denotes the uniform closure of the algebra of all restrictions to $K$ of rational functions on $\mathbb{C}$ with poles off $K$. 
Proof of Lemma 2.2. Let \( \varphi : U \to Y \), \( z \mapsto (\varphi_1(z), \varphi_2(z)) \) be a finely analytic curve contained in \( Y \). If \( \varphi_1(z) \) is constant on \( U \) then \( \varphi_2(z) \) must also be a constant since non constant finely holomorphic functions are finely open maps and by the construction of the set \( Y \) the fibre \( Y \cap \{ \{ z \} \times \mathbb{C} \} \) is a Cantor set or a finite set for any \( z \in \overline{D}_{1/2} \). Assume therefore that \( \varphi_1(z) \) is non-constant. In particular, there is a point \( z_0 \in U \) where the fine derivative of \( \varphi_1(z) \) does not vanish. Hence \( \varphi_1(z) \) is one-to-one on some finely open neighborhood \( V \subset U \) of the point \( z_0 \). By considering the map \( z \mapsto (\varphi_1 \circ \varphi_1^{-1}(z), \varphi_2 \circ \varphi_1^{-1}(z)) \), defined on the finely open set \( \varphi_1(V) \) we may assume that \( \varphi \) is of the form \( z \mapsto (z, g(z)) \) where \( g(z) = \varphi_2 \circ \varphi_1^{-1}(z) \) is finely holomorphic in the finely open set \( V' = \varphi_1(V) \subset D_{1/2} \). By Definition 6.2.3 there exists a compact subset \( K \subset V' \) with non-empty fine interior such that \( g(z) \) is a continuous function on \( K \) (with respect to the Euclidean topology). Shrinking \( K \) if necessary we may assume that \( K \cap \{ a_1, a_2, \ldots \} = \emptyset \). Let \( p \) be a point in the fine interior of \( K \). It is well known that there exists a sequence of circles \( \{ C(p, r_j) \} \) contained in \( K \) with centers \( p \) and radii \( r_j \to 0 \) as \( j \to \infty \). Clearly, the circle \( C(p, r_j) \) avoids the branch points \( \{ a_1, a_2, \ldots \} \) and its image under the continuous map \( z \mapsto (z, g(z)) \) is contained in \( Y \). By the above observation this is not possible. Hence \( Y \) contains no fine analytic structure. \( \square \)

Denote by \( d_n \) the degree of the one variable polynomial \( w \mapsto p_n(z, w) \) where \( p_n(z, w) \) is the polynomial given in Lemma 6.2.1. Assume that the set \( Y \) is constructed using the parameters \( \epsilon_n \) satisfying the following condition

\[
\lim_{n \to \infty} (\epsilon_n)^{1/d_n} = 0. \tag{6.2.1}
\]

It is shown in [92] that with this choice the set \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \). Using this result and Lemma 6.2.2 we are able to generalize a result in [91].

**Theorem 6.2.4.** Fix \( \delta \in (0, 1/2) \) and let \( Y_{\delta} = \bigcap_{n=1}^{\infty} \left( \{ |p_n| \leq \epsilon_n \} \cap \{|z| \leq \delta \} \right) \) be constructed using the parameters \( \epsilon_n \) satisfying (6.2.1). Then

(a) \( \varphi^{-1}(\varphi(U) \cap Y_{\delta}) \) is a polar subset of \( U \) for all finely analytic curves \( \varphi : U \to \mathbb{C}^2 \).

(b) \( Y_{\delta} \neq (Y_{\delta})^*_{\mathbb{C}^2} \).

**Proof of Theorem 6.2.4.** In order to prove (a) we argue by contradiction. Assume therefore that \( \varphi : U \to \mathbb{C}^2 \) is a finely analytic curve and \( \varphi^{-1}(\varphi(U) \cap Y_{\delta}) \) is a non polar subset of \( U \). Then there is a fine domain \( U_{k_0} \subset U \) such that \( \varphi(U_{k_0}) \subset C_{1/2} \) and \( \varphi^{-1}(\varphi(U_{k_0}) \cap Y_{\delta}) \) is non polar. Indeed, the set \( \varphi^{-1}(\varphi(U) \cap C_{1/2}) \) is a finely open subset of \( U \) and hence has at most countably many finely connected components \( \{ U_k \}_{k=1}^{\infty} \). Moreover, \( \varphi^{-1}(\varphi(U) \cap Y_{\delta}) \cap U_{k_0} \) is non polar for some natural number \( k_0 \), since otherwise \( \bigcup_{k=1}^{\infty} \left\{ \varphi^{-1}(\varphi(U) \cap Y_{\delta}) \cap U_k \right\} = \varphi^{-1}(\varphi(U) \cap Y_{\delta}) \) would be polar contrary to our assumption. Since \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \) there exists a plurisubharmonic function \( u \) defined in \( C_{1/2} \) which is equal to \( -\infty \) exactly on \( Y \cap C_{1/2} \). By Lemma 5.3.6, the function \( u \circ \varphi \) is either finely subharmonic on \( U_{k_0} \) or identically equal to \( -\infty \). Since \( u \) equals \( -\infty \) on the non polar subset \( \varphi^{-1}(\varphi(U) \cap Y_{\delta}) \cap U_{k_0} \), it must be identically equal to \( -\infty \) on \( U_{k_0} \). Therefore \( \varphi(U_{k_0}) \subset \{ u = -\infty \} = Y \cap C_{1/2} \) contradicting Lemma 6.2.2 and (a) follows.
The proof of assertion (b) follows immediately from the proof of Proposition 3.1 in [91]. Indeed, if \( u \) is a plurisubharmonic function defined in \( \mathbb{C}^2 \) which equals \(-\infty\) on \( Y \), then the function \( z \mapsto \max\{ u(z, w) : (z, w) \in Y \} \) is subharmonic in \( D_{1/2} \) and since it equals \(-\infty\) on \( D_8 \) it equals \(-\infty\) on \( D_{1/2} \). Consequently \( Y \cap C_{1/2} \subset (Y_\delta)^c_{\mathbb{C}^2} \) and hence \( Y_\delta \neq (Y_\delta)^c_{\mathbb{C}^2} \).

**Remark.** It follows from the argument used in the proof of assertion (b) in Theorem 6.2.4 that \( Y \cap C_{1/2} \subset (Y_\delta)^c_{C_{1/2}} \). Since the first set is complete pluripolar in \( C_{1/2} \) it follows that \( (Y_\delta)^c_{C_{1/2}} = Y \cap C_{1/2} \). Consequently, \( (Y_\delta)^c_{C_{1/2}} \) contains no fine analytic structure. It would be nice to determine what the set \( (Y_\delta)^c_{\mathbb{C}^2} \) equals and to figure out whether this set contains fine analytic structure. We are unable to do this. But by modifying Wermer’s construction, we will in the next Section construct a complete pluripolar Wermer-like set \( X \subset \mathbb{C}^2 \) with the property that \( (X \cap (S \times \mathbb{C}))^c_{\mathbb{C}^2} \) contains no fine analytic structure for all non polar subset \( S \subset \mathbb{C} \).

## 6.3 Proof of Theorem 6.1.2

### 6.3.1 Construction of the Set \( X \)

In this Subsection we construct the set \( X \). Denote by \( \{a_k\}_{k=1}^\infty \) the points in the complex plane both of whose coordinates are rational numbers. Without loss of generality we may assume that \( a_k \in D_6 \). For any sequence of points \( \{a_i\}_{i=1}^n \) we denote by \( B_j(z) \) the algebraic function

\[
B_j(z) = (z - a_1) \ldots (z - a_{j-1}) \sqrt[2]{z - a_j}.
\]

Denote by \( \gamma_j \) a simple smooth curve with endpoints \( a_j \) and \( \infty \). For each \( j \), \( B_j(z) \) has two single-valued analytic branches on \( \mathbb{C} \setminus \gamma_j \). Following the notation in [132] we choose one of the branches \( B_j(z) \) arbitrarily and denote it by \( \beta_j(z) \). Then \( |\beta_j(z)| = |B_j(z)| \) is continuous on \( \mathbb{C} \).

For each \( n+1 \)-tuple of positive constants \( (c_1, c_2, \ldots, c_{n+1}) \) we denote by \( g_n(z) \) the algebraic function defined recursively in the following way. Put \( g_1(z) = c_1 B_1(z) \) and \( g_2(z) = c_1 B_1(z) + c_2 B_2(z) \) and if \( g_n(z) \) has been chosen we will choose \( g_{n+1}(z) \) as described below. Put \( Z_1(z) = 1 \) and for \( n = 2, 3, \ldots \) define the function \( Z_n(z) \) as follows. Denote by \( z_1, z_2, \ldots, z_l \) all the zeros of all possible different differences \( h_j(z) - h_i(z) (i \neq j) \) of branches \( h_j(z), h_i(z) \) of the function \( g_n(z) \). Suppose \( z_k \) is a zero of \( h_j(z) - h_i(z) \) of order \( m_k \) and put \( Z_n(z) = \Pi_{i=1}^l (z - z_i)^{m_i} \). Note that the zeros of \( Z_n(z) \) are also zeros of the function \( Z_{n+1}(z) \) of the same or greater multiplicity. Define \( g_{n+1}(z) = g_n(z) + c_{n+1} Z_n(z) B_{n+1}(z) \).

By \( \Sigma(c_1, \ldots, c_n) \) we mean the Riemann surface of \( g_n(z) \) which lies in \( \mathbb{C}^2 \). In other words, \( \Sigma(c_1, \ldots, c_n) = \{(z, w) : z \in \mathbb{C}, w = w_j, j = 1, 2, \ldots, 2^n\} \), where \( w_j, j = 1, 2, \ldots, 2^n \) are the values of \( g_n(z) \) at \( z \).
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We will choose positive constants \( c_n \), \( \epsilon_n \) and polynomials \( p_n(z, w) \) recursively so that

\[
\begin{align*}
\{ p_n(z, w) = 0 \} \cap \mathcal{C}_{n+1} &= \Sigma(c_1, c_2, \ldots, c_n) \cap \mathcal{C}_{n+1} \quad (6.3.1) \\
\{|p_{n+1}(z, w)| \leq \epsilon_n+1 \} \cap \mathcal{C}_{n+1} &\subset \{ |p_n(z, w)| < \epsilon_n \} \cap \mathcal{C}_{n+1} \quad (6.3.2)
\end{align*}
\]

hold for \( n = 1, 2, \ldots \). The set \( X \) will be of the form

\[
X = \bigcup_{n=1}^{\infty} \left( \bigcap_{j=n}^{\infty} \{|p_j(z, w)| \leq \epsilon_j \} \cap \mathcal{C}_{n+1} \right). \quad (6.3.3)
\]

Put \( c_1 = 1 \) and let \( p_1(z, w) = w^2 - (z-a_1) \). It is clear that \( \Sigma(c_1) \cap \mathcal{C}_2 = \{ p_1(z, w) = 0 \} \cap \mathcal{C}_2 \). Choose \( \epsilon_1 > 0 \) so that if \( z_0 \in \mathcal{D}_2 \) and \( |p_1(z_0, w)| \leq \epsilon_1 \) then there exists \( (z_0, w_1) \in \Sigma(c_1) \cap \mathcal{C}_2 \) with \( |w-w_1| \leq 1 \). Let \( \mathcal{B}_2 = \mathcal{D}_2 \times \mathcal{D}_{p_1} \) be a bidisk where \( p_1 \) is chosen so that

\[
\{|p_1(z, w)| \leq \epsilon_1 \} \cap \mathcal{C}_2 = \{ |p_1(z, w)| \leq \epsilon_1 \} \cap \mathcal{B}_2.
\]

Assume that \( c_n, \epsilon_n \) and \( p_n(z, w) \) have been chosen so that (6.3.1) and (6.3.2) hold. We will now choose \( c_{n+1} \) and \( p_{n+1}(z, w) \). We denote by \( w_j(z), j = 1, 2, \ldots, 2^n \) the roots of \( p_n(z, \cdot) = 0 \) and to each positive constant \( c \) we assign a polynomial \( p_c(z, w) \) by putting

\[
p_c(z, w) = \prod_{j=1}^{2^n} \left( (w - w_j(z))^2 - c^2(Z_n(z)B_{n+1}(z))^2 \right). \quad (6.3.4)
\]

Then \( p_c(z, \cdot) = 0 \) has the roots \( w_j(z) \pm cZ_n(z)B_{n+1}(z), j = 1, 2, \ldots, 2^n \) and so

\[
\{ p_c(z, w) = 0 \} = \Sigma(c_1, c_2, \ldots, c_n, c).
\]

Note that from (6.3.4)

\[
p_c = p_n^2 + c^2q_1 + \ldots + (c^2)^{2^n}q_{2^n},
\]

where the \( q_j \) are polynomials in \( z \) and \( w \), not depending on \( c \). Choose \( c > 0 \) so that the following hold for all \( z \in \mathcal{D}_{n+1} \).

\[
\Sigma(c_1, c_2, \ldots, c_n, c) \cap \mathcal{C}_{n+1} \subset \{ |p_n(z, w)| < \epsilon_n/2 \} \cap \mathcal{C}_{n+1} \quad (6.3.5)
\]

\[
c \cdot |Z_n(z)B_{n+1}(z)| \leq (1/10)c_n|Z_{n-1}(z)B_n(z)|. \quad (6.3.6)
\]

Decreasing \( c \) if necessary we may assume that if \( h_i(z) \) and \( h_j(z) \) are any different branches of the function \( g_n(z) \) the estimate

\[
|h_j(z) - h_i(z)| \geq 2c|Z_n(z)B_{n+1}(z)| \quad (6.3.7)
\]

holds in \( \mathcal{D}_{n+1} \) with equality exactly at the zeros of \( Z_n(z) \) which are contained in \( \mathcal{D}_{n+1} \) and at the points \( a_1, \ldots, a_n \). This estimate will be needed later when we prove that \( X \) contains no fine analytic structure. Choose \( c_{n+1} = c \).
In this Section we show that (6.3.9) above implies that 
\[ \text{deed, since the bidisks} \]
and put 
\[ P \]
\[ \text{The Lemma follows.} \]
\[ \text{Lemma 6.3.1. The set } X \text{ defined by (6.3.3) is complete pluripolar in } \mathbb{C}^2. \]
\[ \text{Proof. Define for } n \geq 2 \text{ the plurisubharmonic function} \]
\[ u_n(z, w) = \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\} \]
\[ \text{and put } u(z, w) = \sum_{n \geq 2} u_n(z, w). \text{ Then } u(z, w) \text{ is plurisubharmonic in } \mathbb{C}^2. \]
\[ \text{Indeed, since the bidisks } B_n \text{ exhaust } \mathbb{C}^2 \text{ and } |p_n(z, w)| < 1 \text{ in } B_{n+1} \text{ the series} \]
\[ \sum_{n \geq 2} u_n(z, w) \text{ will be decreasing on each fixed bidisk } B_N \text{ after a finite number of terms and hence plurisubharmonic there. Since plurisubharmonicity is a local property } u(z, w) \text{ is plurisubharmonic in } \mathbb{C}^2. \]
\[ \text{If } (z_0, w_0) \in X, \text{ then for some natural number } N, \text{ } (z_0, w_0) \in \bigcap_{j=N}^{\infty} \{ |p_j(z, w)| \leq \epsilon_j \} \cap C_{n+1}. \text{ Condition} \]
\[ (6.3.9) \text{ above implies that } u(z_0, w_0) = \text{Const} + \sum_{n \geq N} u_n(z_0, w_0) = -\infty. \text{ Finally if} \]
\[ (z_0, w_0) \notin X \text{ then there exists a natural number } N \text{ such that } (z_0, w_0) \in B_N \text{ and} \]
\[ \text{By (6.3.8)} \]
\[ u(z, w) = \text{Const} + \sum_{n > N} \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\} \geq \text{Const} + \sum_{n > N} -\frac{1}{2^n} > -\infty. \]
\[ \text{The Lemma follows.} \]

6.3.2 X Contains No Fine Analytic Structure

In this Section we show that X contains no fine analytic structure. Suppose that 
\[ z \mapsto (\varphi_1(z), \varphi_2(z)) \text{ is a finely analytic curve whose image is contained in } X. \text{ If} \]
\( \varphi_1(z) \) is constant then \( \varphi_2(z) \) must be constant since \( X \cap (\{z_0\} \times \mathbb{C}) \) is a Cantor set or a finite set for any point \( z_0 \in \mathbb{C} \). On the other hand, if \( \varphi_1(z) \) is non-constant, then using the arguments given in the proof of Lemma 6.2.2 we may assume that the finely analytic curve contained in \( X \) is given by \( z \mapsto (z, m(z)) \) where \( m(z) \) is a finely holomorphic function defined in \( U \) where \( U \subset \mathcal{D}_n \) for some natural number \( n \). Fix a point \( z' \in U \setminus \{a_1, \ldots, a_n\} \). By the definition of finely holomorphic functions we can find a compact (in the usual topology) fine neighborhood \( K \subset U \) of \( z' \) where \( m(z) \) is continuous. Shrinking \( K \) if necessary we may assume that \( (K \setminus \{z'\}) \cap (\{a_j\}_{j=1}^\infty \cup \{Z_{k-1}(z) = 0\}_{k=2}^\infty) = \emptyset \). Since the complement of \( K \) is thin at \( z' \), one can find a sequence of circles \( \{C(z', r_j)\} \subset K \) with \( r_j \to 0 \) as \( j \to \infty \). Choose one of the circles \( C(z', r_j) \) so that none of the points \( a_1, \ldots, a_n \) are contained in \( \{|z - z'| \leq r_j\} \). Let \( a_k \) be the first point in the sequence \( \{a_j\}_{j=n+1}^\infty \) which is contained in \( \{|z - z'| < r_j\} \). Note that \( a_k \in \{|z - z'| < r_j\} \) is continuous on \( C(z', r_j) \) and the function \( Z_{k-1}(z) \beta_k(z) \neq 0 \) when \( z \in C(z', r_j) \). The fact that the image of \( C(z', r_j) \) under the map \( z \mapsto (z, m(z)) \) is a subset of \( X \) will lead us to a contradiction and hence \( X \) contains no fine analytic structure. In order to prove this fix a point \( z_1 \in C(z', r_j) \) and denote by \( \mathcal{R} \) the \( 2^k \) branches of the algebraic function \( g_k(z) \) defined on \( C(z', r_j) \setminus \{z_1\} \).

**Lemma 6.3.2.** If \( h_i(z) \) and \( h_j(z) \) are any different functions from \( \mathcal{R} \), then

\[
|h_i(z) - h_j(z)| > (3/2)c_k|Z_{k-1}(z)\beta_k(z)|
\]

holds for all \( z \in C(z', r_j) \setminus \{z_1\} \).

**Proof.** This follows directly from (6.3.7) since \( C(z', r_j) \subset \mathcal{D}_n \) and \( C(z', r_j) \) does not intersect any of the branch points \( a_1, \ldots, a_k \) or the zeros of \( Z_{k-1}(z) \).

From now on the proof that \( X \) contains no fine analytic structure follows the arguments given in [132].

**Lemma 6.3.3.** Fix \( z_0 \in C(z', r_j) \setminus \{z_1\} \). There exists a function \( h_i(z) \in \mathcal{R} \), where \( h_i(z) \) depends on \( z_0 \) such that

\[
|m(z_0) - h_i(z_0)| < (1/4)c_k|Z_{k-1}(z_0)\beta_k(z_0)|
\]

**Proof.** By (6.3.10) there exists \( N \geq k \) and \( w_N \) such that \( (z_0, w_N) \) lies on the set \( \Sigma(c_1, \ldots, c_N) \) and \( m(z_0) = w_N + R(z_0) \) where \( |R(z_0)| \leq (1/10)c_k|Z_{k-1}(z_0)\beta_k(z_0)| \). Thus

\[
m(z_0) = \pm c_1\beta_1(z_0) + \sum_{\nu=2}^N \pm c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0) = \defn h_i(z_0) + \sum_{\nu=k+1}^N c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0).
\]
Since $C(z', r_j) \subset \mathcal{D}_{n+1}$ and the constants $c_{\nu}$ are chosen so that (6.3.6) holds,

$$|m(z_0) - h_i(z_0)| \leq \sum_{\nu=k+1}^{N} c_{\nu}|Z_{\nu-1}(z_0)|\beta_{\nu}(z_0)| + |R(z_0)| \leq$$

$$\leq c_k|Z_{k-1}(z_0)|\beta_k(z_0)|\left(\frac{1}{10} + \frac{1}{10^2} + \ldots\right) + |R(z_0)| =$$

$$= \frac{1}{9}c_k|Z_{k-1}(z_0)|\beta_k(z_0)| + \frac{1}{10}c_k|Z_{k-1}(z_0)|\beta_k(z_0)| <$$

$$< (1/4)c_k|Z_{k-1}(z_0)|\beta_k(z_0)|.$$ 

Hence (6.3.12) holds and the Lemma is proved. \qed

**Lemma 6.3.4.** Fix $z_0 \in C(z', r_j) \setminus \{z_1\}$ and let $h_i(z) \in \mathbb{R}$ satisfy (6.3.12). Then for all $z$ in $C(z', r_j) \setminus \{z_1\}$

$$|m(z) - h_i(z)| < (1/3)c_k|Z_{k-1}(z)|\beta_k(z). \quad (6.3.13)$$

**Proof.** The set $\mathcal{O} = \{z \in C(z', r_j) \setminus \{z_1\} : (6.3.13) \text{ holds at } z\}$ is open in $C(z_0, r_j) \setminus \{z_1\}$ and contains $z_0$. If $\mathcal{O} \neq C(z', r_j) \setminus \{z_1\}$ then there is a boundary point $p$ of $\mathcal{O}$ on $C(z', r_j) \setminus \{z_1\}$ for which

$$|m(p) - h_i(p)| = (1/3)c_k|Z_{k-1}(p)|\beta_k(p) \quad (6.3.14)$$

holds. By Lemma 6.3.3 there is some $h_j(z)$ in $\mathbb{R}$ such that

$$|m(p) - h_j(p)| < (1/4)c_k|Z_{k-1}(p)|\beta_k(p). \quad (6.3.15)$$

Thus $|h_i(p) - h_j(p)| \leq (7/12)c_k|Z_{k-1}(p)|\beta_k(p)$. Also $h_i(z) \neq h_j(z)$, in view of (6.3.14) and (6.3.15). This contradicts Lemma 6.3.2. Thus $\mathcal{O} = C(z', r_j) \setminus \{z_1\}$ and Lemma 6.3.4 follows. \qed

For each continuous function $v(z)$ defined on $C(z', r_j) \setminus \{z_1\}$ which has a jump at $z_1$ we write $L^+(v)$ and $L^-(v)$ for the two limits of $v(z)$ as $z \to z_1$ along $C(z', r_j)$. Then, by (6.3.13),

$$|L^+(m) - L^+(h_i)| \leq (1/3)c_k|Z_{k-1}(z_1)|\beta_k(z_1)|$$

and

$$|L^-(m) - L^-(h_i)| \leq (1/3)c_k|Z_{k-1}(z_1)|\beta_k(z_1)|,$$

so

$$|(L^+(m) - L^+(h_i)) - (L^-(m) - L^-(h_i))| \leq (2/3)c_k|Z_{k-1}(z_1)|\beta_k(z_1)|.$$ 

Since $m(z)$ is continuous on $C(z', r_j)$ the jump of $h_i(z)$ at $z_1$ is in modulus less than or equal to $(2/3)c_k|Z_{k-1}(z_1)|\beta_k(z_1)| \neq 0$. But $h_i(z)$ is in $\mathbb{R}$, so its jump at $z_1$ has modulus at least $2c_k|Z_{k-1}(z_1)|\beta_k(z_1)|$. This is a contradiction.
6.3.3 The Sets $E$ and $E_{\mathbb{C}^2}^*$

Denote by $E$ the pluripolar set $E = (S \times \mathbb{C}) \cap X$ where $S$ is a non polar subset of $\mathbb{C}$. Since $X$ is complete pluripolar in $\mathbb{C}^2$ it follows that $E_{\mathbb{C}^2}^* \subset X$. To prove that $X \subset E_{\mathbb{C}^2}^*$ we argue as follows. First we claim that the set $X$ is pseudoconcave. Indeed, by the construction of the set $X$,\[
\mathbb{C}^2 \setminus X = \bigcup_{n=1}^{\infty} \{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1}.
\] (6.3.16)

By the choice of the polynomials $p_n(z, w)$ it follows that\[
\{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1} \subset \{|p_{n+1}(z, w)| > \epsilon_{n+1}\} \cap \mathcal{C}_{n+2}.
\]
Moreover, for each natural number $n$ the set $\{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1}$ is a domain of holomorphy. Hence $\mathbb{C}^2 \setminus X$ is a countable union of increasing domains of holomorphy. By the Behnke-Stein Theorem $\mathbb{C}^2 \setminus X$ is pseudoconvex and the claim follows.

Denote by $u(z, w)$ a globally defined plurisubharmonic function which equals $-\infty$ on $E$. It is shown in [127] that the function $z \mapsto \max\{u(z, w) : (z, w) \in X\}$ is subharmonic in $\mathbb{C}$. Since the projection $S$ of $E$ onto the first coordinate plane is non polar the function $z \mapsto \max\{u(z, w) : (z, w) \in X\}$ will be identically equal to $-\infty$ on $\mathbb{C}$ hence $u(z, w) = -\infty$ on the whole of $X$ and consequently $E_{\mathbb{C}^2}^* = X$.

This ends the proof of Theorem 6.1.2.

6.4 Final Remarks and Open Problems

It follows immediately from Theorem 6.1.1 and the fact that $X$ contains no fine analytic structure that if $\varphi : U \rightarrow \mathbb{C}^2$ is a finely analytic curve, then the set $\varphi^{-1}(\varphi(U) \cap X)$ is polar in $\mathbb{C}$.

Despite the result of Theorem 6.1.2 it should be mentioned here that in the situation where one considers the pluripolar hull of the graph of a finely holomorphic function defined in a fine domain $D$, the following problem still remains open.

**Problem 1.** Let $z \in \Gamma_f(D)_{\mathbb{C}^2}^*$. Does this imply that there is a finely analytic curve contained in $\Gamma_f(D)_{\mathbb{C}^2}^*$ which passes through the point $z$?

It is proved in [40] that the pluripolar hull relative to $\mathbb{C}^n$ of a connected pluripolar $F_\sigma$ subset is a connected set. It is a fairly easy exercise to show that the set $X = E_{\mathbb{C}^2}^*$ in Theorem 6.1.2 is path connected, but in general the pluripolar hull of a connected $(F_\sigma)$ pluripolar set is not path connected. Indeed, denote by $f(z)$ an entire function of order $1/3$. $f(1/z)$ has an essential singularity at $0$ and in [134] Wiegerinck proved that the graph $\Gamma_{f(1/z)}$ of $f(1/z)$ over $\mathbb{C} \setminus \{0\}$ is complete pluripolar in $\mathbb{C}^2$. Consequently, if we put $E = \Gamma_{f(1/z)} \cup (\{0\} \times \mathbb{C})$ then $E$ is complete pluripolar in $\mathbb{C}^2$ and hence $E_{\mathbb{C}^2}^* = E$. Moreover $E$ is a connected $F_\sigma$ subset of $\mathbb{C}^2$. By the famous Denjoy-Carleman-Ahlfors theorem (see e.g. [1]), entire functions of order $1/3$ do not have finite asymptotic values; i.e., there are no curves $\gamma$ ending at infinity such that $f(z)$ approaches a finite value as $z \rightarrow \infty$ along $\gamma$. Hence it is not possible to find a path in $E_{\mathbb{C}^2}^*$ connecting a point on $\Gamma_{f(1/z)}$ with a point in
the set \( \{0\} \times \mathbb{C} \). In view of this remark it would be interesting to know the answer to the following question.

**Problem 2.** Is \( \Gamma_f(D)_{\mathbb{C}^2} \) path connected?