Fine aspects of pluripotential theory
El Marzguioui, S.

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Chapter 6

Fine Analytic Structure

This chapter is joint work with Tomas Edlund and presents, with no changes, the contents of the paper [46] which will appear in Indagationes Mathematicae.

We discuss the relation between pluripolar hulls and fine analytic structure. Our main result is the following. For each non polar subset $S$ of the complex plane $\mathbb{C}$ we prove that there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that the pluripolar hull of $E$ relative to $\mathbb{C}^2$ contains no fine analytic structure and its projection onto the first coordinate plane equals $\mathbb{C}$.

6.1 Introduction

Denote by $\Omega$ an open subset of $\mathbb{C}^n$ and let $E \subset \Omega$ be a pluripolar subset. It might be the case that any plurisubharmonic function $u(z)$ defined in $\Omega$ that is equal to $-\infty$ on the set $E$ is necessarily equal to $-\infty$ on a strictly larger set. For instance, if $E$ contains a non polar proper subset of a connected Riemann surface embedded into $\mathbb{C}^n$, then any plurisubharmonic function defined in a neighborhood of the Riemann surface which is equal to $-\infty$ on $E$ is automatically equal to $-\infty$ on the whole Riemann surface. In order to try to understand some aspect of the underlying mechanism of the described "propagation" property of pluripolar sets, the pluripolar hull of graphs $\Gamma_f(D)$ of analytic functions $f$ in a domain $D \subset \mathbb{C}$ has been studied in a number of papers. (See for instance [38], [44], [93] and [134].)

The pluripolar hull $E^*_\Omega$ relative to $\Omega$ of a pluripolar set $E$ is defined as follows.

$$ E^*_\Omega = \bigcap \{ z \in \Omega : u(z) = -\infty \}, $$

where the intersection is taken over all plurisubharmonic functions defined in $\Omega$ which are equal to $-\infty$ on $E$. The set $E$ is called complete pluripolar in $\Omega$ if there exists a plurisubharmonic function on $\Omega$ which equals $-\infty$ precisely on $E$.

As remarked above a necessary condition for a pluripolar set $E$ to satisfy $E^*_\Omega = E$ is that $E \cap A$ is polar in $A$ (or $E \cap A = A$) for all one-dimensional complex analytic varieties $A \subset \Omega$. The fact that this is not a sufficient condition
was proved by Levenberg in [91]. By using a refinement of Wermer’s example of a polynomially convex compact set with no analytic structure (cf. [132]) Levenberg proved that there exists a compact set \( K \subset \mathbb{C}^2 \) satisfying \( K \neq K_{pl}^2 \), and the intersection of \( K \) with any one dimensional analytic variety \( A \) is polar in \( A \). In this example it is not clear what the pluripolar hull \( K_{pl}^2 \) equals.

We will say that a set \( S \subset \mathbb{C}^n \) contains fine analytic structure if there exists a non constant map \( \varphi : U \to S \) from a fine domain \( U \subset \mathbb{C} \) whose coordinate functions are finely holomorphic in \( U \) (see Definition 2.3 below). Such a map \( \varphi \) will be called a finely analytic curve.

Motivated by recent results of Jörècke and the first author (cf. [44]), the following result was proved in [41].

**Theorem 6.1.1.** Let \( \varphi : U \to \mathbb{C}^n \) be a finely holomorphic map on a fine domain \( U \subset \mathbb{C} \) and let \( E \subset \mathbb{C}^n \) be a pluripolar set. Then the following hold

1. \( \varphi(U) \) is a pluripolar subset of \( \mathbb{C}^n \)
2. If \( \varphi^{-1}(\varphi(U) \cap E) \) is a non polar subset of \( \mathbb{C} \) then \( \varphi(U) \subset E_{pl}^n \).

In view of this result one may expect to get more information on the pluripolar hull \( E_{pl}^n \) by examining the intersection of the pluripolar set \( E \) with finely analytic curves. Since many curves in \( \mathbb{C}^n \) are complete pluripolar (see [45]) one cannot expect that \( E_{pl}^n \) always contains fine analytic structure. However if we consider the non trivial part \( E_{pl}^n \setminus E \) the situation is up to now slightly different. In fact, all examples we have seen so far have the property that if \( E_{pl}^n \setminus E \) is nonempty then for each \( w \in E_{pl}^n \setminus E \) there exists a finely analytic curve \( \varphi \) contained in \( E_{pl}^n \) which passes through the point \( w \). (i.e. \( \varphi : U \to E_{pl}^n \) is a finely analytic curve and \( \varphi(z) = w \) for some \( z \in U \)). In this paper we prove that no such conclusion holds in general. We have the following main result.

**Theorem 6.1.2.** For each proper non polar subset \( S \subset \mathbb{C} \) there exists a pluripolar set \( E \subset (S \times \mathbb{C}) \) with the property that \( E_{pl}^2 \) contains no fine analytic structure and the projection of \( E_{pl}^2 \) onto the first coordinate plane equals \( \mathbb{C} \).

The set \( E \) will be a subset of a complete pluripolar set \( X \) which is constructed in the same spirit as Wermer’s polynomially convex compact set without analytic structure.

Let us describe more precisely the content of the paper. In Section 2 we briefly recall the construction of Wermer’s set and prove that it contains no fine analytic structure. This leads to Theorem 6.2.4 which slightly generalizes a result in [91]. The main result is proved in Section 3. Subsection 3.1 is devoted to construct the above mentioned set \( X \) and in Subsection 3.2 we show that \( X \) contains no fine analytic structure. In Subsection 3.3 we define the set \( E \) and describe \( E_{pl}^2 \). Finally, in Section 4 we make some remarks and pose two open questions.

Readers who are not familiar with basic results on finely holomorphic functions and fine potential theory are referred to [56] and [61].

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6.2. Wermer’s Example

In this Section we sketch the details of Wermer’s construction given in [132]. Denote by $D_r$ the open disk with center zero and radius $r$ and by $C_r$ the open cylinder $D_r \times \mathbb{C}$. Let $a_1, a_2, \ldots$ denote the points in the disk $D_r$ with rational real and imaginary part. For each $j$ we denote by $B_j(z)$ the algebraic $(2\text{-valued})$ function

$$B_j(z) = (z - a_1)(z - a_2)\cdots(z - a_{j-1})\sqrt[2]{z - a_j}.$$ 

To each $n$-tuple of positive constants $c_1, c_2, \ldots, c_n$ we associate the algebraic ($2^n$-valued) function $g_n(z) = \sum_{j=1}^n c_j B_j(z)$. Let $\mathbb{P}(\mathbb{C}_1)$ be the subset of the Riemann surface of $g_n(z)$ which lies in $\mathbb{C}_r$.

**Lemma 6.2.1.** ([132], lemma 1) There exist positive constants $\varepsilon_n$ and $c_n$, $n = 1, 2, \ldots$ such that:

1. $\{p_n = 0\} \cap \{|z| \leq \frac{1}{2}\} = \sum(c_1, \ldots, c_n), n = 1, 2, \ldots$
2. $\{p_{n+1} \leq \varepsilon_{n+1}\} \cap \{|z| \leq \frac{1}{2}\} \subset \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}, n = 1, 2, \ldots$
3. If $|a| \leq \frac{1}{2}$ and $|p_n(a, w)| < \varepsilon_n$, then there is a $w_n$ with $p_n(a, w_n) = 0$ and $|w - w_n| \leq \frac{1}{2^n}$, $n = 1, 2, \ldots$.

With $p_n$, $\varepsilon_n$, $n = 1, 2, \ldots$ chosen as in Lemma 6.2.1, we put

$$Y = \bigcap_{n=1}^{\infty} \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}.$$ 

Clearly, $Y$ is a compact polynomially convex subset of $\mathbb{C}^2$. It was shown by Wermer that $Y$ has no analytic structure i.e. $Y$ contains no non-constant analytic disk. In fact he proves something stronger. The set $Y$ defined above contains no graph of a continuous function defined on a circle in $D_{\frac{1}{2}}$ which avoids all the branch points $\{a_i\}$. Using this observation the following lemma follows.

**Lemma 6.2.2.** There is no finely analytic curve contained in $Y$.

Before we prove Lemma 6.2.2 we recall the following definition (cf. [61], page 75):

**Definition 6.2.3.** Let $U$ be a finely open set in $\mathbb{C}$. A function $f : U \rightarrow \mathbb{C}$ is said to be finely holomorphic if every point of $U$ has a compact (in the usual topology) fine neighborhood $K \subset U$ such that the restriction $f|_K$ belongs to $R(K)$.

Here $R(K)$ denotes the uniform closure of the algebra of all restrictions to $K$ of rational functions on $\mathbb{C}$ with poles off $K$. 
Proof of Lemma 2.2. Let \( \varphi : U \to Y \), \( z \mapsto (\varphi_1(z), \varphi_2(z)) \) be a finely analytic curve contained in \( Y \). If \( \varphi_1(z) \) is constant on \( U \) then \( \varphi_2(z) \) must also be a constant since non constant finely holomorphic functions are finely open maps and by the construction of the set \( Y \) the fibre \( Y \cap \{ \{z \} \times \mathbb{C} \} \) is a Cantor set or a finite set for any point \( z \in \overline{D}_{1/2} \). Assume therefore that \( \varphi_1(z) \) is non-constant. In particular, there is a point \( z_0 \in U \) where the fine derivative of \( \varphi_1(z) \) does not vanish. Hence \( \varphi_1(z) \) is one-to-one on some finely open neighborhood \( V \subset U \) of the point \( z_0 \). By considering the map \( z \mapsto (\varphi_1 \circ \varphi_1^{-1}(z), \varphi_2 \circ \varphi_1^{-1}(z)) \), defined on the finely open set \( \varphi_1(V) \) we may assume that \( \varphi \) is of the form \( z \mapsto (z, g(z)) \) where \( g(z) = \varphi_2 \circ \varphi_1^{-1}(z) \) is finely holomorphic in the finely open set \( V' = \varphi_1(V) \subset D_{1/2} \). By Definition 6.2.3 there exists a compact subset \( K \subset V' \) with non-empty fine interior such that \( g(z) \) is a continuous function on \( K \) (with respect to the Euclidean topology). Shrinking \( K \) if necessary we may assume that \( K \cap \{a_1, a_2, \ldots \} = \emptyset \). Let \( p \) be a point in the fine interior of \( K \). It is well known that there exists a sequence of circles \( \{C(p, r_j)\} \) contained in \( K \) with centers \( p \) and radii \( r_j \to 0 \) as \( j \to \infty \). Clearly, the circle \( C(p, r_j) \) avoids the branch points \( \{a_1, a_2, \ldots \} \) and its image under the continuous map \( z \mapsto (z, g(z)) \) is contained in \( Y \). By the above observation this is not possible. Hence \( Y \) contains no fine analytic structure.

Denote by \( d_n \) the degree of the one variable polynomial \( w \mapsto p_n(z, w) \) where \( p_n(z, w) \) is the polynomial given in Lemma 6.2.1. Assume that the set \( Y \) is constructed using the parameters \( \epsilon_n \) satisfying the following condition

\[
\lim_{n \to \infty} (\epsilon_n)^{1/d_n} = 0.
\]

(6.2.1)

It is shown in [92] that with this choice the set \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \). Using this result and Lemma 6.2.2 we are able to generalize a result in [91].

Theorem 6.2.4. Fix \( \delta \in (0, 1/2) \) and let \( Y_\delta = \bigcap_{n=1}^{\infty} \{ |p_n| \leq \epsilon_n \} \cap \{|z| \leq \delta\} \) be constructed using the parameters \( \epsilon_n \) satisfying (6.2.1). Then

(a) \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a polar subset of \( U \) for all finely analytic curves \( \varphi : U \to \mathbb{C}^2 \).

(b) \( Y_\delta \cap Y_{\delta'} \neq \emptyset \).

Proof of Theorem 6.2.4. In order to prove (a) we argue by contradiction. Assume therefore that \( \varphi : U \to \mathbb{C}^2 \) is a finely analytic curve and \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a non polar subset of \( U \). Then there is a fine domain \( U_{k_0} \subset U \) such that \( \varphi(U_{k_0}) \subset C_{1/2} \) and \( \varphi^{-1}(\varphi(U_{k_0}) \cap Y_\delta) \) is non polar. Indeed, the set \( \varphi^{-1}(\varphi(U) \cap C_{1/2}) \) is a finely open subset of \( U \) and hence has at most countably many finely connected components \( \{U_k\}_{k=1}^{\infty} \). Moreover, \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0} \) is non polar for some natural number \( k_0 \), since otherwise \( \bigcup_{k=1}^{\infty} \{ \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k} \} = \varphi^{-1}(\varphi(U) \cap Y_\delta) \) would be polar contrary to our assumption. Since \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \) there exists a plurisubharmonic function \( u \) defined in \( C_{1/2} \) which is equal to \( -\infty \) exactly on \( Y \cap C_{1/2} \). By Lemma 5.3.6, the function \( u \circ \varphi \) is either finely subharmonic on \( U_{k_0} \) or identically equal to \( -\infty \). Since \( u \) equals \( -\infty \) on the non polar subset \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0} \), it must be identically equal to \( -\infty \) on \( U_{k_0} \). Therefore \( \varphi(U_{k_0}) \subseteq \{ u = -\infty \} = Y \cap C_{1/2} \) contradicting Lemma 6.2.2 and (a) follows.
The proof of assertion (b) follows immediately from the proof of Proposition 3.1 in [91]. Indeed, if \( u \) is a plurisubharmonic function defined in \( \mathbb{C}^2 \) which equals \(-\infty\) on \( Y \) then the function \( z \mapsto \max\{u(z, w) : (z, w) \in Y\} \) is subharmonic in \( \mathcal{D}_{1/2} \) and since it equals \(-\infty\) on \( \mathcal{D}_6 \) it equals \(-\infty\) on \( \mathcal{D}_{1/2} \). Consequently \( Y \cap \mathcal{C}_{1/2} \subset (Y_6)_{\mathbb{C}^2}. \) and hence \( Y_6 \neq (Y_6)_{\mathbb{C}^2}. \)

**Remark.** It follows from the argument used in the proof of assertion (b) in Theorem 6.2.4 that \( Y \cap \mathcal{C}_{1/2} \subset (Y_6)_{\mathbb{C}^2}. \) Since the first set is complete pluripolar in \( \mathcal{C}_{1/2} \) it follows that \( (Y_6)_{\mathbb{C}^2} = Y \cap \mathcal{C}_{1/2}. \) Consequently, \( (Y_6)_{\mathbb{C}^2} \) contains no fine analytic structure. It would be nice to determine what the set \((Y_6)_{\mathbb{C}^2}\) equals and to figure out whether this set contains fine analytic structure. We are unable to do this. But by modifying Wermer’s construction, we will in the next Section construct a complete plurisubharmonic Wermer-like set \( X \subset \mathbb{C}^2 \) with the property that \((X \cap (S \times \mathbb{C}))_{\mathbb{C}^2}\) contains no fine analytic structure for all non polar subset \( S \subset \mathbb{C}. \)

### 6.3 Proof of Theorem 6.1.2

#### 6.3.1 Construction of the Set \( X \)

In this Subsection we construct the set \( X. \) Denote by \( \{a_k\}_{k=1}^{\infty} \) the points in the complex plane both of whose coordinates are rational numbers. Without loss of generality we may assume that \( a_k \in \mathcal{D}_k. \) For any sequence of points \( \{a_i\}_{i=1}^{l} \) we denote by \( B_j(z) \) the algebraic function

\[
B_j(z) = (z - a_1) \ldots (z - a_{j-1}) \sqrt[2]{(z - a_j)}.
\]

Denote by \( \gamma_j \) a simple smooth curve with endpoints \( a_j \) and \( \infty. \) For each \( j, \) \( B_j(z) \) has two single-valued analytic branches on \( \mathbb{C} \setminus \gamma_j. \) Following the notation in [132] we choose one of the branches \( B_j(z) \) arbitrarily and denote it by \( \beta_j(z). \) Then \( |\beta_j(z)| = |B_j(z)| \) is continuous on \( \mathbb{C}. \)

For each \( n + 1 \)-tuple of positive constants \( (c_1, c_2, \ldots, c_{n+1}) \) we denote by \( g_n(z) \) the algebraic function defined recursively in the following way. Put \( g_1(z) = c_1B_1(z) \) and \( g_2(z) = c_1B_1(z) + c_2B_2(z) \) and if \( g_n(z) \) has been chosen we will choose \( g_{n+1}(z) \) as described below. Put \( Z_1(z) = 1 \) and for \( n = 2, 3, \ldots \) define the function \( Z_n(z) \) as follows. Denote by \( z_1, z_2, \ldots, z_l \) all the zeros of all possible differerent differences \( h_j(z) - h_i(z) \) \( (i \neq j) \) of branches \( h_j(z), h_i(z) \) of the function \( g_n(z). \) Suppose \( z_k \) is a zero of \( h_j(z) - h_i(z) \) of order \( m_k \) and put \( Z_n(z) = \Pi_{i=1}^{l} (z - z_i)^{m_i}. \) Note that the zeros of \( Z_n(z) \) are also zeros of the function \( Z_{n+1}(z) \) of the same or greater multiplicity. Define \( g_{n+1}(z) = g_n(z) + c_{n+1}Z_n(z)B_{n+1}(z). \)

By \( \Sigma(c_1, \ldots, c_n) \) we denote the Riemann surface of \( g_n(z) \) which lies in \( \mathbb{C}^2. \) In other words, \( \Sigma(c_1, \ldots, c_n) = \{(z, w) : z \in \mathbb{C}, w = w_j, j = 1, 2, \ldots, 2^n\}, \) where \( w_j, j = 1, 2, \ldots, 2^n \) are the values of \( g_n(z) \) at \( z. \)
We will choose positive constants $c_n, \epsilon_n$ and polynomials $p_n(z, w)$ recursively so that
\[
\{p_n(z, w) = 0\} \cap C_{n+1} = \Sigma(c_1, c_2, \ldots, c_n) \cap C_{n+1} \quad \text{and} \quad (6.3.1)
\]
\[
\{|p_{n+1}(z, w)| \leq \epsilon_{n+1}\} \cap C_{n+1} \subset \{p_n(z, w) < \epsilon_n\} \cap C_{n+1} \quad \text{(6.3.2)}
\]
hold for $n = 1, 2, \ldots$. The set $X$ will be of the form
\[
X = \bigcup_{n=1}^{\infty} \left( \bigcap_{j=n}^{\infty} \{|p_j(z, w)| \leq \epsilon_j\} \cap C_{n+1} \right). \quad (6.3.3)
\]

Put $c_1 = 1$ and let $p_1(z, w) = w^2 - (z - a_1)$. It is clear that $\Sigma(c_1) \cap C_2 = \{p_1(z, w) = 0\} \cap C_2$. Choose $\epsilon_1 > 0$ so that if $z_0 \in D_2$ and $|p_1(z, w)| \leq \epsilon_1$ then there exists $(z_0, w_1) \in \Sigma(c_1) \cap C_2$ with $|w - w_1| \leq 1$. Let $B_2 = D_2 \times D_{p_1}$ be a bidisk where $p_1$ is chosen so that
\[
\{|p_1(z, w)| \leq \epsilon_1\} \cap C_2 = \{p_1(z, w) \leq \epsilon_1\} \cap B_2.
\]

Assume that $c_n, \epsilon_n$ and $p_n(z, w)$ have been chosen so that (6.3.1) and (6.3.2) hold. We will now choose $c_{n+1}$ and $p_{n+1}(z, w)$. We denote by $w_j(z)$, $j = 1, 2, \ldots, 2^n$ the roots of $p_n(z, \cdot) = 0$ and to each positive constant $c$ we assign a polynomial $p_c(z, w)$ by putting
\[
p_c(z, w) = \prod_{j=1}^{2^n} \left( (w - w_j(z))^2 - c^2(Z_n(z)B_{n+1}(z))^2 \right). \quad (6.3.4)
\]

Then $p_c(z, \cdot) = 0$ has the roots $w_j(z) \pm \sqrt{c}Z_n(z)B_{n+1}(z), j = 1, 2, \ldots, 2^n$ and so
\[
\{p_c(z, w) = 0\} = \Sigma(c_1, c_2, \ldots, c_n, c).
\]

Note that from (6.3.4)
\[
p_c = p_n^2 + c^2 q_1 + \ldots + (c^2)^{2^n} q_{2^n},
\]
where the $q_j$ are polynomials in $z$ and $w$, not depending on $c$. Choose $c > 0$ so that the following hold for all $z \in D_{n+1}$.
\[
\Sigma(c_1, c_2, \ldots, c_n, c) \cap C_{n+1} \subset \{p_n(z, w) < \epsilon_n/2\} \cap C_{n+1}\quad \text{and} \quad (6.3.5)
\]
\[
c \cdot |Z_n(z)B_{n+1}(z)| \leq (1/10)|c_n| |Z_{n-1}(z)B_n(z)|. \quad (6.3.6)
\]

Decreasing $c$ if necessary we may assume that if $h_i(z)$ and $h_j(z)$ are any different branches of the function $g_n(z)$ the estimate
\[
|h_j(z) - h_i(z)| \geq 2c|Z_n(z)B_{n+1}(z)| \quad (6.3.7)
\]
holds in $D_{n+1}$ with equality exactly at the zeros of $Z_n(z)$ which are contained in $D_{n+1}$ and at the points $a_1, \ldots, a_n$. This estimate will be needed later when we prove that $X$ contains no fine analytic structure. Choose $c_{n+1} = c$. 

In this Section we show that (6.3.9) above implies that since the bidisks are a local property. By decreasing some natural number \(m_{n+1}\) so that
\[
\frac{1}{m_{n+1}} \log |p_{n+1}(z, w)| \geq -\frac{1}{2^n} \quad \text{for all } (z, w) \in B_{n+1} \setminus \{|p_n(z, w)| \leq \epsilon_n\}. \tag{6.3.8}
\]
Choose \(\epsilon_{n+1} < \epsilon_n\) so that
\[
\frac{1}{m_{n+1}} \log |p_{n+1}(z, w)| \leq -1 \quad \text{for all } (z, w) \in \{|p_{n+1}(z, w)| \leq \epsilon_{n+1}\} \cap C_{n+2}. \tag{6.3.9}
\]
By decreasing \(\epsilon_{n+1}\) we may assume that (6.3.2) and the following assumption hold.

If \((z_0, w) \in C_{n+2}\) and \(|p_{n+1}(z_0, w)| \leq \epsilon_{n+1}\), then there exists \((z_0, w_n) \in C_{n+2}\) such that \(|p_{n+1}(z_0, w_n)| = 0\) and \(|w - w_n| \leq 1/n\). \tag{6.3.10}

This ends the recursion.

Lemma 6.3.1. The set \(X\) defined by (6.3.3) is complete pluripolar in \(\mathbb{C}^2\).

Proof. Define for \(n \geq 2\) the plurisubharmonic function
\[
u_n(z, w) = \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\}
\]
and put \(u(z, w) = \sum_{n \geq 2} u_n(z, w)\). Then \(u(z, w)\) is plurisubharmonic in \(\mathbb{C}^2\). Indeed, since the bidisks \(B_n\) exhaust \(\mathbb{C}^2\) and \(|p_n(z, w)| < 1\) in \(B_{n+1}\) the series \(\sum_{n \geq 2} u_n(z, w)\) will be decreasing on each fixed bidisk \(B_N\) after a finite number of terms and hence plurisubharmonic there. Since plurisubharmonicity is a local property \(u(z, w)\) is plurisubharmonic in \(\mathbb{C}^2\). If \((z_0, w_0) \in X\), then for some natural number \(N\), \((z_0, w_0) \in \bigcap_{j=0}^{n} \{|p_j(z, w)| \leq \epsilon_j\} \cap C_{n+1}\). Condition (6.3.9) above implies that \(u(z_0, w_0) = Const + \sum_{n \geq N} u_n(z_0, w_0) = -\infty\). Finally if \((z_0, w_0) \notin X\) then there exists a natural number \(N\) such that \((z_0, w_0) \in B_N\) and \((z_0, w_0) \notin \{|p_n(z, w)| \leq \epsilon_n\} \cap B_N\) for all \(n \geq N\). By (6.3.8)
\[
u(z, w) = Const + \sum_{n \geq N} \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\} \geq Const + \sum_{n \geq N} -\frac{1}{2^n} > -\infty.
\]
The Lemma follows.

6.3.2 \(X\) Contains No Fine Analytic Structure

In this Section we show that \(X\) contains no fine analytic structure. Suppose that \(z \mapsto (\varphi_1(z), \varphi_2(z))\) is a finely analytic curve whose image is contained in \(X\). If
\( \varphi_1(z) \) is constant then \( \varphi_2(z) \) must be constant since \( X \cap \{ \{z_0\} \times \mathbb{C} \} \) is a Cantor set or a finite set for any point \( z_0 \in \mathbb{C} \). On the other hand, if \( \varphi_1(z) \) is non-constant, then using the arguments given in the proof of Lemma 6.2.2 we may assume that the finely analytic curve contained in \( X \) is given by \( z \mapsto (z, m(z)) \) where \( m(z) \) is a finely holomorphic function defined in \( U \) where \( U \subset \mathcal{D}_n \) for some natural number \( n \). Fix a point \( z' \in U \setminus \{a_1, \ldots, a_n\} \). By the definition of finely holomorphic functions we can find a compact (in the usual topology) fine neighborhood \( \text{Lemma 6.3.3}. \)

\([\text{or a finite set for any point given in} \ [132].\)

Thus \( \text{Lemma 6.3.2}. \)

This follows directly from (6.3.7) since \( \text{Lemma 6.3.12}. \)

holds for all \( z \in C(z', r_j) \setminus \{z_1\} \).

Proof. This follows directly from (6.3.7) since \( C(z', r_j) \subset \mathcal{D}_n \) and \( C(z', r_j) \) does not intersect any of the branch points \( a_1, \ldots, a_k \) or the zeros of \( Z_{k-1}(z) \).

From now on the proof that \( X \) contains no fine analytic structure follows the arguments given in [132].

**Lemma 6.3.3.** Fix \( z_0 \) in \( C(z', r_j) \setminus \{z_1\} \). There exists a function \( h_i(z) \in \mathbb{R}, \) where \( h_i(z) \) depends on \( z_0 \) such that

\[
|m(z_0) - h_i(z_0)| < (1/4)c_k|Z_{k-1}(z_0)|
\]

Proof. By (6.3.10) there exists \( N \geq k \) and \( w_N \) such that \( (z_0, w_N) \) lies on the set \( \Sigma(e_1, \ldots, e_N) \) and \( m(z_0) = w_N + R(z_0) \) where \( |R(z_0)| \leq (1/10)c_k|Z_{k-1}(z_0)| \).

Thus

\[
m(z_0) = \pm c_1 \beta_1(z_0) + \sum_{\nu=2}^N \pm c_{\nu} Z_{k-\nu}(z_0) \beta_{\nu}(z_0) + R(z_0) = \]

\[
def h_i(z_0) + \sum_{\nu=k+1}^N c_{\nu} Z_{k-\nu}(z_0) \beta_{\nu}(z_0) + R(z_0).
\]
Since \( C(z', r_j) \subset \mathcal{D}_{n+1} \) and the constants \( c_v \) are chosen so that (6.3.6) holds, \( m(z_0) - h_i(z_0) \) holds. By Lemma 6.3.3 there is some \( \mathcal{O} \) such that \( (6.3.10) \) holds and the Lemma is proved. \( \square \)

**Lemma 6.3.4.** Fix \( z_0 \in C(z', r_j) \smallsetminus \{ z_1 \} \) and let \( h_i(z) \in \mathbb{R} \) satisfy (6.3.12). Then for all \( z \) in \( C(z', r_j) \smallsetminus \{ z_1 \} \)

\[
|m(z) - h_i(z)| < \frac{1}{3} c_k |Z_{k-1}(z)|. \tag{6.3.13}
\]

**Proof.** The set \( \mathcal{O} = \{ z \in C(z', r_j) \smallsetminus \{ z_1 \} : (6.3.13) \) holds at \( z \} \) is open in \( C(z_0, r_j) \smallsetminus \{ z_1 \} \) and contains \( z_0 \). If \( \mathcal{O} \neq C(z', r_j) \smallsetminus \{ z_1 \} \) then there is a boundary point \( p \) of \( \mathcal{O} \) on \( C(z', r_j) \smallsetminus \{ z_1 \} \) for which

\[
|m(p) - h_i(p)| = \frac{1}{3} c_k |Z_{k-1}(p)|. \tag{6.3.14}
\]

holds. By Lemma 6.3.3 there is some \( h_j(z) \) in \( \mathbb{R} \) such that

\[
|m(p) - h_j(p)| < \frac{1}{3} c_k |Z_{k-1}(p)|. \tag{6.3.15}
\]

Thus \( |h_i(p) - h_j(p)| \leq \frac{7}{12} c_k |Z_{k-1}(p)| \). Also \( h_i(z) \neq h_j(z) \), in view of (6.3.14) and (6.3.15). This contradicts Lemma 6.3.2. Thus \( \mathcal{O} = C(z', r_j) \smallsetminus \{ z_1 \} \) and Lemma 6.3.4 follows. \( \square \)

For each continuous function \( v(z) \) defined on \( C(z', r_j) \smallsetminus \{ z_1 \} \) which has a jump at \( z_1 \) we write \( L^+(v) \) and \( L^-(v) \) for the two limits of \( v(z) \) as \( z \to z_1 \) along \( C(z', r_j) \). Then, by (6.3.13),

\[
|L^+(m) - L^+(h_i)| \leq \frac{1}{3} c_k |Z_{k-1}(z_1)|, 
\]

and

\[
|L^-(m) - L^-(h_i)| \leq \frac{1}{3} c_k |Z_{k-1}(z_1)|, 
\]

so

\[
|(L^+(m) - L^+(h_i)) - (L^-(m) - L^-(h_i))| \leq \frac{2}{3} c_k |Z_{k-1}(z_1)|. 
\]

Since \( m(z) \) is continuous on \( C(z', r_j) \) the jump of \( h_i(z) \) at \( z_1 \) is in modulus less than or equal to \( (2/3)c_k |Z_{k-1}(z_1)| \). But \( h_i(z) \) is in \( \mathbb{R} \), so its jump at \( z_1 \) has modulus at least \( 2c_k |Z_{k-1}(z_1)| \). This is a contradiction.
6.3.3 The Sets $E$ and $E^*_C$ 

Denote by $E$ the pluripolar set $E = (S \times \mathbb{C}) \cap X$ where $S$ is a non polar subset of $\mathbb{C}$. Since $X$ is complete pluripolar in $\mathbb{C}^2$ it follows that $E^*_C \subset X$. To prove that $X \subset E^*_C$ we argue as follows. First we claim that the set $X$ is pseudoconcave. Indeed, by the construction of the set $X$, 

$$\mathbb{C}^2 \setminus X = \cup_{n=1}^{\infty} \{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1}. \quad (6.3.16)$$

By the choice of the polynomials $p_n(z, w)$ it follows that 

$$\{|p_n(z, w)| > \epsilon_n\} \cap \mathcal{C}_{n+1} \subset \{|p_{n+1}(z, w)| > \epsilon_{n+1}\} \cap \mathcal{C}_{n+2}.$$ 

Moreover, for each natural number $n$ the set $\{ |p_n(z, w)| > \epsilon_n \} \cap \mathcal{C}_{n+1}$ is a domain of holomorphy. Hence $\mathbb{C}^2 \setminus X$ is a countable union of increasing domains of holomorphy. By the Behnke-Stein Theorem $\mathbb{C}^2 \setminus X$ is pseudoconvex and the claim follows.

Denote by $u(z, w)$ a globally defined plurisubharmonic function which equals $-\infty$ on $E$. It is shown in [127] that the function $z \mapsto \max \{u(z, w) : (z, w) \in X\}$ is subharmonic in $\mathbb{C}$. Since the projection $S$ of $E$ onto the first coordinate plane is non polar the function $z \mapsto \max \{u(z, w) : (z, w) \in X\}$ will be identically equal to $-\infty$ on $\mathbb{C}$ hence $u(z, w) = -\infty$ on the whole of $X$ and consequently $E^*_C = X$. This ends the proof of Theorem 6.1.2.

6.4 Final Remarks and Open Problems 

It follows immediately from Theorem 6.1.1 and the fact that $X$ contains no fine analytic structure that if $\varphi : U \rightarrow \mathbb{C}^2$ is a finely analytic curve, then the set $\varphi^{-1}(\varphi(U) \cap X)$ is polar in $\mathbb{C}$.

Despite the result of Theorem 6.1.2 it should be mentioned here that in the situation where one considers the pluripolar hull of the graph of a finely holomorphic function defined in a fine domain $D$, the following problem still remains open.

**Problem 1.** Let $z \in \Gamma_f(D)^c_{\mathbb{C}^2}$. Does this imply that there is a finely analytic curve contained in $\Gamma_f(D)^c_{\mathbb{C}^2}$ which passes through the point $z$?

It is proved in [40] that the pluripolar hull relative to $\mathbb{C}^n$ of a connected pluripolar $F_\sigma$ subset is a connected set. It is a fairly easy exercise to show that the set $X = E^*_C$ in Theorem 6.1.2 is path connected, but in general the pluripolar hull of a connected $(F_\sigma)$ pluripolar set is not path connected. Indeed, denote by $f(z)$ an entire function of order $1/3$. $f(1/z)$ has an essential singularity at $0$ and in [134] Wiegerinck proved that the graph $\Gamma_f(1/z)$ of $f(1/z)$ over $\mathbb{C} \setminus \{0\}$ is complete pluripolar in $\mathbb{C}^2$. Consequently, if we put $E = \Gamma_f(1/z) \cup \{(0) \times \mathbb{C}\}$ then $E$ is complete pluripolar in $\mathbb{C}^2$ and hence $E_{\mathbb{C}^2} = E$. Moreover $E$ is a connected $F_\sigma$ subset of $\mathbb{C}^2$. By the famous Denjoy-Carleman-Ahlfors theorem (see e.g. [1]), entire functions of order $1/3$ do not have finite asymptotic values; i.e., there are no curves $\gamma$ ending at infinity such that $f(z)$ approaches a finite value as $z \rightarrow \infty$ along $\gamma$. Hence it is not possible to find a path in $E_{\mathbb{C}^2}$ connecting a point on $\Gamma_f(1/z)$ with a point in...
the set \( \{0\} \times \mathbb{C} \). In view of this remark it would be interesting to know the answer to the following question.

**Problem 2.** Is \( \Gamma_f(D)_{\mathbb{C}^2} \) path connected?