Fine aspects of pluripotential theory
El Marzguioui, S.

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Chapter 6

Fine Analytic Structure

This chapter is joint work with Tomas Edlund and presents, with no changes, the contents of the paper [46] which will appear in Indagationes Mathematicae.

We discuss the relation between pluripolar hulls and fine analytic structure. Our main result is the following. For each non polar subset $S$ of the complex plane $\mathbb{C}$ we prove that there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that the pluripolar hull of $E$ relative to $\mathbb{C}^2$ contains no fine analytic structure and its projection onto the first coordinate plane equals $\mathbb{C}$.

6.1 Introduction

Denote by $\Omega$ an open subset of $\mathbb{C}^n$ and let $E \subset \Omega$ be a pluripolar subset. It might be the case that any plurisubharmonic function $u(z)$ defined in $\Omega$ that is equal to $-\infty$ on the set $E$ is necessarily equal to $-\infty$ on a strictly larger set. For instance, if $E$ contains a non polar proper subset of a connected Riemann surface embedded into $\mathbb{C}^n$, then any plurisubharmonic function defined in a neighborhood of the Riemann surface which is equal to $-\infty$ on $E$ is automatically equal to $-\infty$ on the whole Riemann surface. In order to try to understand some aspect of the underlying mechanism of the described "propagation" property of pluripolar sets, the pluripolar hull of graphs $\Gamma_f(D)$ of analytic functions $f$ in a domain $D \subset \mathbb{C}$ has been studied in a number of papers. (See for instance [38], [44], [93] and [134].)

The pluripolar hull $E^*_\Omega$ relative to $\Omega$ of a pluripolar set $E$ is defined as follows.

$$E^*_\Omega = \bigcap \{ z \in \Omega : u(z) = -\infty \},$$

where the intersection is taken over all plurisubharmonic functions defined in $\Omega$ which are equal to $-\infty$ on $E$. The set $E$ is called complete pluripolar in $\Omega$ if there exists a plurisubharmonic function on $\Omega$ which equals $-\infty$ precisely on $E$.

As remarked above a necessary condition for a pluripolar set $E$ to satisfy $E^*_\Omega = E$ is that $E \cap A$ is polar in $A$ (or $E \cap A = A$) for all one-dimensional complex analytic varieties $A \subset \Omega$. The fact that this is not a sufficient condition
was proved by Levenberg in [91]. By using a refinement of Wermer’s example of a polynomially convex compact set with no analytic structure (cf. [132]) Levenberg proved that there exists a compact set $K \subset \mathbb{C}^2$ satisfying $K \neq K^*_2$, and the intersection of $K$ with any one dimensional analytic variety $A$ is polar in $A$. In this example it is not clear what the pluripolar hull $K^*_2$ equals.

We will say that a set $S \subset \mathbb{C}^n$ contains fine analytic structure if there exists a non constant map $\varphi : U \rightarrow S$ from a fine domain $U \subset \mathbb{C}$ whose coordinate functions are finely holomorphic in $U$ (see Definition 2.3 below). Such a map $\varphi$ will be called a finely analytic curve.

Motivated by recent results of Jörıccke and the first author (cf. [44]), the following result was proved in [41].

**Theorem 6.1.1.** Let $\varphi : U \rightarrow \mathbb{C}^n$ be a finely holomorphic map on a fine domain $U \subset \mathbb{C}$ and let $E \subset \mathbb{C}^n$ be a pluripolar set. Then the following hold
(1) $\varphi(U)$ is a pluripolar subset of $\mathbb{C}^n$
(2) If $\varphi^{-1}(\varphi(U) \cap E)$ is a non polar subset of $\mathbb{C}$ then $\varphi(U) \subset E^*_\mathbb{C}^n$.

In view of this result one may expect to get more information on the pluripolar hull $E^*_\mathbb{C}^n$ by examining the intersection of the pluripolar set $E$ with finely analytic curves. Since many curves in $\mathbb{C}^n$ are complete pluripolar (see [45]) one cannot expect that $E^*_\mathbb{C}^n$ always contains fine analytic structure. However if we consider the non trivial part $E^*_\mathbb{C}^n \setminus E$ the situation is up to now slightly different. In fact, all examples we have seen so far have the property that if $E^*_\mathbb{C}^n \setminus E$ is nonempty then for each $w \in E^*_\mathbb{C}^n \setminus E$ there exists a finely analytic curve $\varphi$ contained in $E^*_\mathbb{C}^n$ which passes through the point $w$. (i.e. $\varphi : U \rightarrow E^*_\mathbb{C}^n$ is a finely analytic curve and $\varphi(z) = w$ for some $z \in U$). In this paper we prove that no such conclusion holds in general. We have the following main result.

**Theorem 6.1.2.** For each proper non polar subset $S \subset \mathbb{C}$ there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that $E^*_\mathbb{C}^2$ contains no fine analytic structure and the projection of $E^*_\mathbb{C}^2$ onto the first coordinate plane equals $\mathbb{C}$.

The set $E$ will be a subset of a complete pluripolar set $X$ which is constructed in the same spirit as Wermer’s polynomially convex compact set without analytic structure.

Let us describe more precisely the content of the paper. In Section 2 we briefly recall the construction of Wermer’s set and prove that it contains no fine analytic structure. This leads to Theorem 6.2.4 which slightly generalizes a result in [91]. The main result is proved in Section 3. Subsection 3.1 is devoted to construct the above mentioned set $X$ and in Subsection 3.2 we show that $X$ contains no fine analytic structure. In Subsection 3.3 we define the set $E$ and describe $E^*_\mathbb{C}^2$. Finally, in Section 4 we make some remarks and pose two open questions.

Readers who are not familiar with basic results on finely holomorphic functions and fine potential theory are referred to [56] and [61].

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### 6.2 Wermer’s Example

In this Section we sketch the details of Wermer’s construction given in [132]. Denote by $D_r$ the open disk with center zero and radius $r$ and by $C_r$ the open cylinder $D_r \times \mathbb{C}$. Let $a_1, a_2, \ldots$ denote the points in the disk $D_r$ with rational real and imaginary part. For each $j$ we denote by $B_j(z)$ the algebraic (2-valued) function

$$B_j(z) = (z - a_1)(z - a_2) \cdots (z - a_j).$$

To each $n$-tuple of positive constants $c_1, c_2, \ldots, c_n$ we associate the algebraic ($2^n$-valued) function $g_n(z) = \sum_{j=1}^n c_j B_j(z)$. Let $\mathcal{P}(c_1, \ldots, c_n)$, $n = 1, 2, \ldots$ be the subset of the Riemann surface of $g_n(z)$ which lies in $C_r$.

#### Lemma 6.2.1. ([132], lemma 1)]

There exist positive constants $\epsilon_n$ and $c_n$, $n = 1, 2, \ldots$, with $c_1 = \frac{1}{10}$ and $c_{n+1} \leq (\frac{1}{10}) c_n$, $n = 1, 2, \ldots$ and a sequence of polynomials $\{p_n(z, w)\}$ such that:

1. $\{p_n = 0\} \cap \{|z| \leq \frac{1}{2}\} = \sum (c_1, \ldots, c_n)$, $n = 1, 2, \ldots$
2. $\{\{p_n+1\} \leq \epsilon_{n+1}\} \cap \{|z| \leq \frac{1}{2}\} \subset \{|p_n| \leq \epsilon_n\} \cap \{|z| \leq \frac{1}{2}\}$, $n = 1, 2, \ldots$
3. If $|a| \leq \frac{1}{2}$ and $|p_n(a, w)| \leq \epsilon_n$, then there is a $w_n$ with $p_n(a, w_n) = 0$ and $|w - w_n| \leq \frac{1}{n}$, $n = 1, 2, \ldots$.

With $p_n, \epsilon_n, n = 1, 2, \ldots$ chosen as in Lemma 6.2.1, we put

$$Y = \bigcap_{n=1}^{\infty} \{|p_n| \leq \epsilon_n\} \cap \{|z| \leq \frac{1}{2}\}.$$

Clearly, $Y$ is a compact polynomially convex subset of $\mathbb{C}^2$. It was shown by Wermer that $Y$ has no analytic structure i.e. $Y$ contains no non-constant analytic disk. In fact he proves something stronger. The set $Y$ defined above contains no graph of a continuous function defined on a circle in $D_r$ which avoids all the branch points $\{a_i\}$. Using this observation the following lemma follows.

#### Lemma 6.2.2. There is no finely analytic curve contained in $Y$.

Before we prove Lemma 6.2.2 we recall the following definition (cf. [61], page 75):

**Definition 6.2.3.** Let $U$ be a finely open set in $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is said to be finely holomorphic if every point of $U$ has a compact (in the usual topology) fine neighborhood $K \subset U$ such that the restriction $f|_K$ belongs to $R(K)$.

Here $R(K)$ denotes the uniform closure of the algebra of all restrictions to $K$ of rational functions on $\mathbb{C}$ with poles off $K$. 
Proof of Lemma 2.2. Let \( \varphi : U \to Y \), \( z \mapsto (\varphi_1(z), \varphi_2(z)) \) be a finely analytic curve contained in \( Y \). If \( \varphi_1(z) \) is constant on \( U \) then \( \varphi_2(z) \) must also be a constant since non constant finely holomorphic functions are finely open maps and by the construction of the set \( Y \) the fibre \( Y \cap \{ \{z\} \times \mathbb{C} \} \) is a Cantor set or a finite set for any point \( z \in \overline{D}_{1/2} \). Assume therefore that \( \varphi_1(z) \) is non-constant. In particular, there is a point \( z_0 \in U \) where the fine derivative of \( \varphi_1(z) \) does not vanish. Hence \( \varphi_1(z) \) is one-to-one on some finely open neighborhood \( V \subset U \) of the point \( z_0 \).

By considering the map \( z \mapsto (\varphi_1 \circ \varphi_1^{-1}(z), \varphi_2 \circ \varphi_1^{-1}(z)) \), defined on the finely open set \( \varphi_1(V) \) we may assume that \( \varphi \) is of the form \( z \mapsto (z, g(z)) \) where \( g(z) = \varphi_2 \circ \varphi_1^{-1}(z) \) is finely holomorphic in the finely open set \( V' = \varphi_1(V) \subset D_{1/2} \). By Definition 6.2.3 there exists a compact subset \( K \subset V' \) with non-empty fine interior such that \( g(z) \) is a continuous function on \( K \) (with respect to the Euclidean topology). Shrinking \( K \) if necessary we may assume that \( K \cap \{a_1, a_2, \ldots\} = \emptyset \). Let \( p \) be a point in the fine interior of \( K \). It is well known that there exists a sequence of circles \( \{ C(p, r_j) \} \) contained in \( K \) with centers \( p \) and radii \( r_j \to 0 \) as \( j \to \infty \). Clearly, the circle \( C(p, r_j) \) avoids the branch points \( \{a_1, a_2, \ldots\} \) and its image under the continuous map \( z \mapsto (z, g(z)) \) is contained in \( Y \). By the above observation this is not possible. Hence \( Y \) contains no fine analytic structure.

Denote by \( d_n \) the degree of the one variable polynomial \( w \mapsto p_n(z, w) \) where \( p_n(z, w) \) is the polynomial given in Lemma 6.2.1. Assume that the set \( Y \) is constructed using the parameters \( \epsilon_n \) satisfying the following condition

\[
\lim_{n \to \infty} (\epsilon_n)^{1/d_n} = 0. \tag{6.2.1}
\]

It is shown in [92] that with this choice the set \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \). Using this result and Lemma 6.2.2 we are able to generalize a result in [91].

Theorem 6.2.4. Fix \( \delta \in (0, 1/2) \) and let \( Y_\delta = \bigcap_{n=1}^{\infty} \{ |p_n| \leq \epsilon_n \} \cap \{|z| \leq \delta \} \) be constructed using the parameters \( \epsilon_n \) satisfying (6.2.1). Then

(a) \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a polar subset of \( U \) for all finely analytic curves \( \varphi : U \to \mathbb{C}^2 \).

(b) \( Y_\delta \neq (Y_\delta)_{C^2} \).

Proof of Theorem 6.2.4. In order to prove (a) we argue by contradiction. Assume therefore that \( \varphi : U \to \mathbb{C}^2 \) is a finely analytic curve and \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a non polar subset of \( U \). Then there is a finely open set \( U_{k_0} \subset U \) such that \( \varphi(U_{k_0}) \subset C_{1/2} \) and \( \varphi^{-1}(\varphi(U_{k_0}) \cap Y_\delta) \) is non polar. Indeed, the set \( \varphi^{-1}(\varphi(U) \cap C_{1/2}) \) is a finely open subset of \( U \) and hence has at most countably many finely connected components \( \{U_k\}_{k=1}^{\infty} \). Moreover, \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0} \) is non polar for some natural number \( k_0 \), since otherwise \( \bigcup_{k=1}^{\infty} \{\varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_k\} = \varphi^{-1}(\varphi(U) \cap Y_\delta) \) would be polar contrary to our assumption. Since \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \) there exists a plurisubharmonic function \( u \) defined in \( C_{1/2} \) which is equal to \( -\infty \) exactly on \( Y \cap C_{1/2} \). By Lemma 5.3.6, the function \( u \circ \varphi \) is either finely subharmonic on \( U_{k_0} \) or identically equal to \( -\infty \). Since \( u \) equals \( -\infty \) on the non polar subset \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0} \), it must be identically equal to \( -\infty \) on \( U_{k_0} \). Therefore \( \varphi(U_{k_0}) \subset \{u = -\infty\} = Y \cap C_{1/2} \) contradicting Lemma 6.2.2 and (a) follows.
The proof of assertion (b) follows immediately from the proof of Proposition 3.1 in [91]. Indeed, if \( u \) is a plurisubharmonic function defined in \( \mathbb{C}^2 \) which equals \(-\infty\) on \( Y_3 \) then the function \( z \mapsto \max\{u(z, w) : (z, w) \in Y\} \) is subharmonic in \( D_{1/2} \) and since it equals \(-\infty\) on \( D_6 \) it equals \(-\infty\) on \( D_{1/2} \). Consequently \( Y \cap C_{1/2} \subset (Y_6)_{C^2} \) and hence \( Y_3 \neq (Y_6)_{C^2} \). \( \square \)

Remark. It follows from the argument used in the proof of assertion (b) in Theorem 6.2.4 that \( Y \cap C_{1/2} \subset (Y_6)_{C_{1/2}} \). Since the first set is complete pluripolar in \( C_{1/2} \) it follows that \( (Y_6)_{C_{1/2}} = Y \cap C_{1/2} \). Consequently, \( (Y_6)_{C_{1/2}} \) contains no fine analytic structure. It would be nice to determine what the set \((Y_6)_{C^2}\) equals and to figure out whether this set contains fine analytic structure. We are unable to do this. But by modifying Wermer's construction, we will in the next Section construct a complete pluripolar Wermer-like set \( X \subset \mathbb{C}^2 \) with the property that \((X \cap (S \times \mathbb{C}))_{C^2}\) contains no fine analytic structure for all non polar subset \( S \subset \mathbb{C} \).

## 6.3 Proof of Theorem 6.1.2

### 6.3.1 Construction of the Set \( X \)

In this Subsection we construct the set \( X \). Denote by \( \{a_k\}_{k=1}^\infty \) the points in the complex plane both of whose coordinates are rational numbers. Without loss of generality we may assume that \( a_k \in D_b \). For any sequence of points \( \{a_i\}_{i=1}^n \) we denote by \( B_j(z) \) the algebraic function

\[
B_j(z) = (z - a_1) \ldots (z - a_{j-1}) \sqrt[2]{z - a_j}.
\]

Denote by \( \gamma_j \) a simple smooth curve with endpoints \( a_j \) and \( \infty \). For each \( j \), \( B_j(z) \) has two single-valued analytic branches on \( \mathbb{C} \setminus \gamma_j \). Following the notation in [132] we choose one of the branches \( B_j(z) \) arbitrarily and denote it by \( \beta_j(z) \). Then \( |\beta_j(z)| = |B_j(z)| \) is continuous on \( \mathbb{C} \).

For each \( n + 1 \)-tuple of positive constants \( (c_1, c_2, \ldots, c_{n+1}) \) we denote by \( g_n(z) \) the algebraic function defined recursively in the following way. Put

\[
g_1(z) = c_1 B_1(z) \text{ and } g_2(z) = c_1 B_1(z) + c_2 B_2(z) \text{ and if } g_n(z) \text{ has been chosen we will choose } g_{n+1}(z) \text{ as described below. Put } Z_1(z) = 1 \text{ and for } n = 2, 3, \ldots \text{ define the function } Z_n(z) \text{ as follows. Denote by } z_1, z_2, \ldots, z_1 \text{ all the zeros of all possible different differences } h_j(z) - h_i(z) (i \neq j) \text{ of branches } h_j(z), h_i(z) \text{ of the function } g_n(z). \text{ Suppose } z_k \text{ is a zero of } h_j(z) - h_i(z) \text{ of order } m_k \text{ and put } Z_n(z) = \Pi_{k=1}^{m_i} (z - z_k)^{m_k}. \text{ Note that the zeros of } Z_n(z) \text{ are also zeros of the function } Z_{n+1}(z) \text{ of the same or greater multiplicity. Define } g_{n+1}(z) = g_n(z) + c_{n+1} Z_n(z) B_{n+1}(z). \]

By \( \Sigma(c_1, \ldots, c_n) \) we mean the Riemann surface of \( g_n(z) \) which lies in \( \mathbb{C}^2 \). In other words, \( \Sigma(c_1, \ldots, c_n) = \{(z, w) : z \in \mathbb{C}, w = w_j, j = 1, 2, \ldots, 2^n\} \), where \( w_j, j = 1, 2, \ldots, 2^n \) are the values of \( g_n(z) \) at \( z \).
We will choose positive constants \( c_n, \epsilon_n \) and polynomials \( p_n(z, w) \) recursively so that

\[
\begin{align*}
\{p_n(z, w) = 0\} \cap C_{n+1} &= \Sigma(c_1, c_2, \ldots, c_n) \cap C_{n+1} \quad (6.3.1) \\
\{|p_{n+1}(z, w)| \leq \epsilon_{n+1}\} \cap C_{n+1} &\subset \{|p_n(z, w)| < \epsilon_n\} \cap C_{n+1} \quad (6.3.2)
\end{align*}
\]

hold for \( n = 1, 2, \ldots \). The set \( X \) will be of the form

\[
X = \bigcup_{n=1}^{\infty} \left( \bigcap_{j=n}^{\infty} \{|p_j(z, w)| \leq \epsilon_j\} \cap C_{n+1} \right).
\]

Put \( c_1 = 1 \) and let \( p_1(z, w) = w^2 - (z - a_1) \). It is clear that \( \Sigma(c_1) \cap C_2 = \{p_1(z, w) = 0\} \cap C_2 \). Choose \( \epsilon_1 > 0 \) so that if \( z_0 \in D_2 \) and \( |p_1(z_0, w)| \leq \epsilon_1 \) then there exists \( (z_0, w_1) \in \Sigma(c_1) \cap C_2 \) with \( |w - w_1| \leq 1 \). Let \( B_2 = D_2 \times D_{p_1} \) be a bidisk where \( p_1 \) is chosen so that

\[\{|p_1(z, w)| \leq \epsilon_1\} \cap C_2 = \{|p_1(z, w)| \leq \epsilon_1\} \cap B_2.\]

Assume that \( c_n, \epsilon_n \) and \( p_n(z, w) \) have been chosen so that (6.3.1) and (6.3.2) hold. We will now choose \( c_{n+1} \) and \( p_{n+1}(z, w) \). We denote by \( w_j(z), j = 1, 2, \ldots, 2^n \) the roots of \( p_n(z, \cdot) = 0 \) and to each positive constant \( c \) we assign a polynomial \( p_c(z, w) \) by putting

\[
p_c(z, w) = \prod_{j=1}^{2^n} \left( (w - w_j(z))^2 - c^2(Z_n(z)B_{n+1}(z))^2 \right).
\]

Then \( p_c(z, \cdot) = 0 \) has the roots \( w_j(z) \pm cZ_n(z)B_{n+1}(z), j = 1, 2, \ldots, 2^n \) and so

\[
\{p_c(z, w) = 0\} = \Sigma(c_1, c_2, \ldots, c_n, c).
\]

Note that from (6.3.4)

\[
p_c = p_n^2 + c^2q_1 + \ldots + (c^2)^{2^n}q_{2^n},
\]

where the \( q_i \) are polynomials in \( z \) and \( w \), not depending on \( c \). Choose \( c > 0 \) so that the following hold for all \( z \in D_{n+1} \).

\[
\begin{align*}
\Sigma(c_1, c_2, \ldots, c_n, c) \cap C_{n+1} &\subset \{|p_n(z, w)| < \epsilon_n/2\} \cap C_{n+1} \quad (6.3.5) \\
c \cdot |Z_n(z)B_{n+1}(z)| &\leq (1/10)c_n|Z_{n-1}(z)B_n(z)| \quad (6.3.6)
\end{align*}
\]

Decreasing \( c \) if necessary we may assume that if \( h_i(z) \) and \( h_j(z) \) are any different branches of the function \( g_n(z) \) the estimate

\[
|h_j(z) - h_i(z)| \geq 2c|Z_n(z)B_{n+1}(z)| \quad (6.3.7)
\]

holds in \( D_{n+1} \) with equality exactly at the zeros of \( Z_n(z) \) which are contained in \( D_{n+1} \) and at the points \( a_1, \ldots, a_n \). This estimate will be needed later when we prove that \( X \) contains no fine analytic structure. Choose \( c_{n+1} = c. \)
In this Section we show that \( \text{deed, since the bidisks be of terms and hence plurisubharmonic there. Since plurisubharmonicity is a local property} \)

\[
\text{Choose a local property}
\]

\[
\text{By decreasing}
\]

\[
\text{We now turn to the choice of } \epsilon_{n+1}. \text{ Since the part of the zero set of } p_{n+1}(z, w) \text{ which is contained in } B_{n+1} \text{ is a subset of } \{ |p_n(z, w)| < \epsilon_n/2 \} \cap B_{n+1} \text{ it is possible to find a natural number } m_{n+1} \text{ so that}
\]

\[
\frac{1}{m_{n+1}} \log |p_{n+1}(z, w)| \geq -\frac{1}{2^n} \text{ for all } (z, w) \in B_{n+1} \setminus \{ |p_n(z, w)| \leq \epsilon_n \}. \tag{6.3.8}
\]

Choose \( \epsilon_{n+1} < \epsilon_n \) so that

\[
\frac{1}{m_{n+1}} \log |p_{n+1}(z, w)| \leq -1 \text{ for all } (z, w) \in \{ |p_{n+1}(z, w)| \leq \epsilon_{n+1} \} \cap \mathcal{C}_{n+2}. \tag{6.3.9}
\]

By decreasing \( \epsilon_{n+1} \) we may assume that (6.3.2) and the following assumption hold.

\[
\text{If } (z_0, w) \in \mathcal{C}_{n+2} \text{ and } |p_{n+1}(z_0, w)| \leq \epsilon_{n+1}, \text{ then there exists}
\]

\[
(z_0, w_n) \in \mathcal{C}_{n+2} \text{ such that } |p_{n+1}(z_0, w_n)| = 0 \text{ and } |w - w_n| \leq 1/n. \tag{6.3.10}
\]

This ends the recursion.

**Lemma 6.3.1.** The set \( X \) defined by (6.3.3) is complete pluripolar in \( \mathbb{C}^2 \).

**Proof.** Define for \( n \geq 2 \) the plurisubharmonic function

\[
u_n(z, w) = \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\}
\]

and put \( u(z, w) = \sum_{n \geq 2} u_n(z, w) \). Then \( u(z, w) \) is plurisubharmonic in \( \mathbb{C}^2 \). Indeed, since the bidisks \( B_n \) exhaust \( \mathbb{C}^2 \) and \( |p_n(z, w)| < 1 \) in \( B_{n+1} \) the series \( \sum_{n \geq 2} u_n(z, w) \) will be decreasing on each fixed bidisk \( B_N \) after a finite number of terms and hence plurisubharmonic there. Since plurisubharmonicity is a local property \( u(z, w) \) is plurisubharmonic in \( \mathbb{C}^2 \). If \( (z_0, w_0) \in X \), then for some natural number \( N \), \( (z_0, w_0) \in \bigcap_{N} \{ |p_j(z, w)| \leq \epsilon_j \} \cap \mathcal{C}_{N+1} \). Condition (6.3.9) above implies that \( u(z_0, w_0) = Const + \sum_{n \geq N} u_n(z_0, w_0) = -\infty \). Finally if \( (z_0, w_0) \notin X \) then there exists a natural number \( N \) such that \( (z_0, w_0) \in \mathcal{B}_N \) and \( (z_0, w_0) \notin \{ |p_n(z, w)| \leq \epsilon_n \} \cap \mathcal{B}_N \) for all \( n \geq N \). By (6.3.8)

\[
u(z, w) = Const + \sum_{n > N} \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\} \geq Const + \sum_{n > N} -\frac{1}{2^n} > -\infty.
\]

The Lemma follows. \( \square \)

### 6.3.2 \( X \) Contains No Fine Analytic Structure

In this Section we show that \( X \) contains no fine analytic structure. Suppose that \( z \mapsto (\varphi_1(z), \varphi_2(z)) \) is a finely analytic curve whose image is contained in \( X \). If
\( \varphi_1(z) \) is constant then \( \varphi_2(z) \) must be constant since \( X \cap (\{ z_0 \} \times \mathbb{C}) \) is a Cantor set or a finite set for any point \( z_0 \in \mathbb{C} \). On the other hand, if \( \varphi_1(z) \) is non-constant, then using the arguments given in the proof of Lemma 6.2.2 we may assume that the finely analytic curve contained in \( X \) is given by \( z \mapsto (z, m(z)) \) where \( m(z) \) is a finely holomorphic function defined in \( U \) where \( U \subset \mathcal{D}_n \) for some natural number \( n \). Fix a point \( z' \in U \setminus \{ a_1, \ldots, a_n \} \). By the definition of finely holomorphic functions we can find a compact (in the usual topology) fine neighborhood \( K \subset U \) of \( z' \) where \( m(z) \) is continuous. Shrinking \( K \) if necessary we may assume that \( (K \setminus \{ z' \}) \cap \bigcup_{i=1}^{\infty} \{ Z_{k-1}(z) = 0 \} = \emptyset \). Since the complement of \( K \) is thin at \( z' \), one can find a sequence of circles \( \{ C(z', r_i) \} \subset K \) with \( r_i \to 0 \) as \( i \to \infty \). Choose one of the circles \( C(z', r_j) \) so that none of the points \( a_1, \ldots, a_n \) are contained in \( \{|z - z'| \leq r_j\} \). Let \( a_k \) be the first point in the sequence \( \{ a_j \}_{j=n+1}^\infty \) which is contained in \( \{|z - z'| \leq r_j\} \). Note that \( a_k \in \{|z - z'| < r_j\} \), \( m(z) \) is continuous on \( C(z', r_j) \) and the function \( Z_{k-1}(z) \beta_k(z) \neq 0 \) when \( z \in C(z', r_j) \). The fact that the image of \( C(z', r_j) \) under the map \( z \mapsto (z, m(z)) \) is a subset of \( X \) will lead us to a contradiction and hence \( X \) contains no fine analytic structure.

In order to prove this fix a point \( z_1 \in C(z', r_j) \) and denote by \( \mathcal{R} \) the \( 2^k \) branches of the algebraic function \( g_k(z) \) defined on \( C(z', r_j) \setminus \{ z_1 \} \).

**Lemma 6.3.2.** If \( h_i(z) \) and \( h_j(z) \) are any different functions from \( \mathcal{R} \), then
\[
|h_i(z) - h_j(z)| > (3/2)c_k|Z_{k-1}(z)\beta_k(z)|
\]
holds for all \( z \in C(z', r_j) \setminus \{ z_1 \} \).

**Proof.** This follows directly from (6.3.7) since \( C(z', r_j) \subset \mathcal{D}_n \) and \( C(z', r_j) \) does not intersect any of the branch points \( a_1, \ldots, a_k \) or the zeros of \( Z_{k-1}(z) \).

From now on the proof that \( X \) contains no fine analytic structure follows the arguments given in [132].

**Lemma 6.3.3.** Fix \( z_0 \in C(z', r_j) \setminus \{ z_1 \} \). There exists a function \( h_i(z) \in \mathcal{R} \), where \( h_i(z) \) depends on \( z_0 \) such that
\[
|m(z_0) - h_i(z_0)| < (1/4)c_k|Z_{k-1}(z_0)\beta_k(z_0)|
\]

**Proof.** By (6.3.10) there exists \( N \geq k \) and \( w_N \) such that \((z_0, w_N)\) lies on the set \( \Sigma(c_1, \ldots, c_N) \) and \( m(z_0) = w_N + R(z_0) \) where \( |R(z_0)| \leq (1/10)c_k|Z_{k-1}(z_0)\beta_k(z_0)| \). Thus
\[
m(z_0) = \pm c_1\beta_1(z_0) + \sum_{\nu=2}^{N} \pm c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0) = \]
\[
\def h_i(z_0) + \sum_{\nu=k+1}^{N} c_\nu Z_{\nu-1}(z_0)\beta_\nu(z_0) + R(z_0).
\]
Since \( C(z', r_j) \subset D_{n+1} \) and the constants \( c_\nu \) are chosen so that (6.3.6) holds,

\[
|m(z_0) - h_i(z_0)| \leq \sum_{\nu=k+1}^{N} c_\nu |Z_{\nu-1}(z_0)\beta_\nu(z_0)| + |R(z_0)| \leq \\
\leq c_k |Z_{k-1}(z_0)\beta_k(z_0)|(\frac{1}{10} + \frac{1}{10^2} + \ldots) + |R(z_0)| = \\
= \frac{1}{9} c_k |Z_{k-1}(z_0)\beta_k(z_0)| + \frac{1}{10} c_k |Z_{k-1}(z_0)\beta_k(z_0)| < \\
< (1/4)c_k |Z_{k-1}(z_0)\beta_k(z_0)|.
\]

Hence (6.3.12) holds and the Lemma is proved.

**Lemma 6.3.4.** Fix \( z_0 \in C(z', r_j) \setminus \{z_1\} \) and let \( h_i(z) \in \mathbb{R} \) satisfy (6.3.12). Then for all \( z \) in \( C(z', r_j) \setminus \{z_1\} \)

\[
|m(z) - h_i(z)| < (1/3)c_k |Z_{k-1}(z)\beta_k(z)|.
\]

**Proof.** The set \( \mathcal{O} = \{z \in C(z', r_j) \setminus \{z_1\} : (6.3.13) \) holds at \( z \} \) is open in \( C(z_0, r_j) \setminus \{z_1\} \) and contains \( z_0 \). If \( \mathcal{O} \neq C(z', r_j) \setminus \{z_1\} \) then there is a boundary point \( p \) of \( \mathcal{O} \) on \( C(z', r_j) \setminus \{z_1\} \) for which

\[
|m(p) - h_i(p)| = (1/3)c_k |Z_{k-1}(p)\beta_k(p)|
\] (6.3.14)

holds. By Lemma 6.3.3 there is some \( h_j(z) \) in \( \mathbb{R} \) such that

\[
|m(p) - h_j(p)| < (1/4)c_k |Z_{k-1}(p)\beta_k(p)|. \tag{6.3.15}
\]

Thus \( |h_i(p) - h_j(p)| \leq (7/12)c_k |Z_{k-1}(p)\beta_k(p)| \). Also \( h_i(z) \neq h_j(z) \), in view of (6.3.14) and (6.3.15). This contradicts Lemma 6.3.2. Thus \( \mathcal{O} = C(z', r_j) \setminus \{z_1\} \) and Lemma 6.3.4 follows.

For each continuous function \( v(z) \) defined on \( C(z', r_j) \setminus \{z_1\} \) which has a jump at \( z_1 \) we write \( L^+(v) \) and \( L^-(v) \) for the two limits of \( v(z) \) as \( z \to z_1 \) along \( C(z', r_j) \). Then, by (6.3.13),

\[
|L^+(m) - L^+(h_i)| \leq (1/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)|
\]

and

\[
|L^-(m) - L^-(h_i)| \leq (1/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)|,
\]

so

\[
|(L^+(m) - L^+(h_i)) - (L^-(m) - L^-(h_i))| \leq (2/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)|.
\]

Since \( m(z) \) is continuous on \( C(z', r_j) \) the jump of \( h_i(z) \) at \( z_1 \) is in modulus less than or equal to \( (2/3)c_k |Z_{k-1}(z_1)\beta_k(z_1)| \). But \( h_i(z) \) is in \( \mathbb{R} \), so its jump at \( z_1 \) has modulus at least \( 2c_k |Z_{k-1}(z_1)\beta_k(z_1)| \). This is a contradiction.
6.3.3 The Sets $E$ and $E_{C_2}^*$

Denote by $E$ the pluripolar set $E = (S \times \mathbb{C}) \cap X$ where $S$ is a non polar subset of $\mathbb{C}$. Since $X$ is complete pluripolar in $\mathbb{C}^2$ it follows that $E_{C_2}^* \subset X$. To prove that $X \subset E_{C_2}^*$ we argue as follows. First we claim that the set $X$ is pseudoconcave. Indeed, by the construction of the set $X$,

$$\mathbb{C}^2 \setminus X = \bigcup_{n=1}^{\infty} \{|p_n(z, w)| > \epsilon_n\} \cap C_{n+1}. \quad (6.3.16)$$

By the choice of the polynomials $p_n(z, w)$ it follows that

$$\{|p_n(z, w)| > \epsilon_n\} \cap C_{n+1} \subset \{|p_{n+1}(z, w)| > \epsilon_{n+1}\} \cap C_{n+2}. $$

Moreover, for each natural number $n$ the set $\{|p_n(z, w)| > \epsilon_n\} \cap C_{n+1}$ is a domain of holomorphy. Hence $\mathbb{C}^2 \setminus X$ is a countable union of increasing domains of holomorphy. By the Behnke-Stein Theorem $\mathbb{C}^2 \setminus X$ is pseudoconvex and the claim follows.

Denote by $u(z, w)$ a globally defined plurisubharmonic function which equals $-\infty$ on $E$. It is shown in [127] that the function $z \mapsto \max\{u(z, w) : (z, w) \in X\}$ is subharmonic in $\mathbb{C}$. Since the projection $S$ of $E$ onto the first coordinate plane is non polar the function $z \mapsto \max\{u(z, w) : (z, w) \in X\}$ will be identically equal to $-\infty$ on $\mathbb{C}$ hence $u(z, w) = -\infty$ on the whole of $X$ and consequently $E_{C_2}^* = X$. This ends the proof of Theorem 6.1.2.

6.4 Final Remarks and Open Problems

It follows immediately from Theorem 6.1.1 and the fact that $X$ contains no fine analytic structure that if $\varphi : U \rightarrow \mathbb{C}^2$ is a finely analytic curve, then the set $\varphi^{-1}(\varphi(U) \cap X)$ is polar in $\mathbb{C}$.

Despite the result of Theorem 6.1.2 it should be mentioned here that in the situation where one considers the pluripolar hull of the graph of a finely holomorphic function defined in a finite domain $D$, the following problem still remains open.

**Problem 1.** Let $z \in \Gamma_f(D)^*_{C_2}$. Does this imply that there is a finely analytic curve contained in $\Gamma_f(D)^*_{C_2}$ which passes through the point $z$?

It is proved in [40] that the pluripolar hull relative to $\mathbb{C}^n$ of a connected pluripolar $F_\sigma$ subset is a connected set. It is a fairly easy exercise to show that the set $X = E_{C_2}^*$ in Theorem 6.1.2 is path connected, but in general the pluripolar hull of a connected $(F_\sigma)$ pluripolar set is not path connected. Indeed, denote by $f(z)$ an entire function of order $1/3$, $f(1/z)$ has an essential singularity at $0$ and in [134] Wiegnerick proved that the graph $\Gamma_{f(1/z)}$ of $f(1/z)$ over $\mathbb{C} \setminus \{0\}$ is complete pluripolar in $\mathbb{C}^2$. Consequently, if we put $E = \Gamma_{f(1/z)} \cup \{(0, 0) \times \mathbb{C}\}$ then $E$ is complete pluripolar in $\mathbb{C}^2$ and hence $E_{C_2}^* = E$. Moreover $E$ is a connected $F_\sigma$ subset of $\mathbb{C}^2$.

By the famous Denjoy-Carleman-Ahlfors theorem (see e.g. [1]), entire functions of order $1/3$ do not have finite asymptotic values; i.e., there are no curves $\gamma$ ending at infinity such that $f(z)$ approaches a finite value as $z \rightarrow \infty$ along $\gamma$. Hence it is not possible to find a path in $E_{C_2}^*$ connecting a point on $\Gamma_{f(1/z)}$ with a point in
the set \{0\} \times \mathbb{C}. In view of this remark it would be interesting to know the answer to the following question.

**Problem 2.** Is \( \Gamma_f(D) \mathbb{C} \) path connected?