Chapter 6

Fine Analytic Structure

This chapter is joint work with Tomas Edlund and presents, with no changes, the contents of the paper [46] which will appear in Indagationes Mathematicae.

We discuss the relation between pluripolar hulls and fine analytic structure. Our main result is the following. For each non polar subset $S$ of the complex plane $\mathbb{C}$ we prove that there exists a pluripolar set $E \subseteq (S \times \mathbb{C})$ with the property that the pluripolar hull of $E$ relative to $\mathbb{C}^2$ contains no fine analytic structure and its projection onto the first coordinate plane equals $\mathbb{C}$.

6.1 Introduction

Denote by $\Omega$ an open subset of $\mathbb{C}^n$ and let $E \subseteq \Omega$ be a pluripolar subset. It might be the case that any plurisubharmonic function $u(z)$ defined in $\Omega$ that is equal to $-\infty$ on the set $E$ is necessarily equal to $-\infty$ on a strictly larger set. For instance, if $E$ contains a non polar proper subset of a connected Riemann surface embedded into $\mathbb{C}^n$, then any plurisubharmonic function defined in a neighborhood of the Riemann surface which is equal to $-\infty$ on $E$ is automatically equal to $-\infty$ on the whole Riemann surface. In order to try to understand some aspect of the underlying mechanism of the described "propagation" property of pluripolar sets, the pluripolar hull of graphs $\Gamma_f(D)$ of analytic functions $f$ in a domain $D \subseteq \mathbb{C}$ has been studied in a number of papers. (See for instance [38], [44], [93] and [134].)

The pluripolar hull $E^*_\Omega$ relative to $\Omega$ of a pluripolar set $E$ is defined as follows.

$$E^*_\Omega = \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over all plurisubharmonic functions defined in $\Omega$ which are equal to $-\infty$ on $E$. The set $E$ is called complete pluripolar in $\Omega$ if there exists a plurisubharmonic function on $\Omega$ which equals $-\infty$ precisely on $E$.

As remarked above a necessary condition for a pluripolar set $E$ to satisfy $E^*_\Omega = E$ is that $E \cap A$ is polar in $A$ (or $E \cap A = A$) for all one-dimensional complex analytic varieties $A \subset \Omega$. The fact that this is not a sufficient condition
was proved by Levenberg in [91]. By using a refinement of Wermer’s example of a polynomially convex compact set with no analytic structure (cf. [132]) Levenberg proved that there exists a compact set $K \subset \mathbb{C}^2$ satisfying $K \neq K_{\mathbb{C}^2}$, and the intersection of $K$ with any one dimensional analytic variety $A$ is polar in $A$. In this example it is not clear what the pluripolar hull $K_{\mathbb{C}^2}$ equals.

We will say that a set $S \subset \mathbb{C}^n$ contains fine analytic structure if there exists a non constant map $\varphi : U \rightarrow S$ from a fine domain $U \subset \mathbb{C}$ whose coordinate functions are finely holomorphic in $U$ (see Definition 2.3 below). Such a map $\varphi$ will be called a finely analytic curve. Motivated by recent results of Jörjicke and the first author (cf. [44]), the following result was proved in [41].

**Theorem 6.1.1.** Let $\varphi : U \rightarrow \mathbb{C}^n$ be a finely holomorphic map on a fine domain $U \subset \mathbb{C}$ and let $E \subset \mathbb{C}^n$ be a pluripolar set. Then the following hold

1. $\varphi(U)$ is a pluripolar subset of $\mathbb{C}^n$
2. If $\varphi^{-1}(\varphi(U) \cap E)$ is a non polar subset of $\mathbb{C}$ then $\varphi(U) \subset E_{\mathbb{C}^n}$.

In view of this result one may expect to get more information on the pluripolar hull $E_{\mathbb{C}^n}$ by examining the intersection of the pluripolar set $E$ with finely analytic curves. Since many curves in $\mathbb{C}^n$ are complete pluripolar (see [45]) one cannot expect that $E_{\mathbb{C}^n}$ always contains fine analytic structure. However if we consider the non trivial part $E_{\mathbb{C}^n} \setminus E$ the situation is up to now slightly different. In fact, all examples we have seen so far have the property that if $E_{\mathbb{C}^n} \setminus E$ is nonempty then for each $w \in E_{\mathbb{C}^n} \setminus E$ there exists a finely analytic curve $\varphi$ contained in $E_{\mathbb{C}^n}$ which passes through the point $w$. (i.e. $\varphi : U \rightarrow E_{\mathbb{C}^n}$ is a finely analytic curve and $\varphi(z) = w$ for some $z \in U$). In this paper we prove that no such conclusion holds in general. We have the following main result.

**Theorem 6.1.2.** For each proper non polar subset $S \subset \mathbb{C}$ there exists a pluripolar set $E \subset (S \times \mathbb{C})$ with the property that $E_{\mathbb{C}^2}$ contains no fine analytic structure and the projection of $E_{\mathbb{C}^2}$ onto the first coordinate plane equals $\mathbb{C}$.

The set $E$ will be a subset of a complete pluripolar set $X$ which is constructed in the same spirit as Wermer’s polynomially convex compact set without analytic structure.

Let us describe more precisely the content of the paper. In Section 2 we briefly recall the construction of Wermer’s set and prove that it contains no fine analytic structure. This leads to Theorem 6.2.4 which slightly generalizes a result in [91]. The main result is proved in Section 3. Subsection 3.1 is devoted to construct the above mentioned set $X$ and in Subsection 3.2 we show that $X$ contains no fine analytic structure. In Subsection 3.3 we define the set $E$ and describe $E_{\mathbb{C}^2}$. Finally, in Section 4 we make some remarks and pose two open questions.

Readers who are not familiar with basic results on finely holomorphic functions and fine potential theory are referred to [56] and [61].

**Acknowledgments.** Part of this work was completed while the first author was visiting Korteweg-de Vries Institute for Mathematics, University of Amsterdam.
6.2. Wermer’s Example

In this Section we sketch the details of Wermer’s construction given in [132]. Denote by $\mathcal{D}_r$ the open disk with center zero and radius $r$ and by $\mathcal{C}_r$ the open cylinder $\mathcal{D}_r \times \mathbb{C}$. Let $a_1, a_2, \ldots$ denote the points in the disk $\mathcal{D}_1$ with rational real and imaginary part. For each $j$ we denote by $B_j(z)$ the algebraic (2-valued) function

$$B_j(z) = (z - a_1)(z - a_2)\ldots(z - a_{j-1})\sqrt[2]{z - a_j}.$$  

To each $n$-tuple of positive constants $c_1, c_2, \ldots, c_n$ we associate the algebraic ($2^n$-valued) function $g_n(z) = \sum_{j=1}^n c_j B_j(z)$. Let $P_n(c_1, \ldots, c_n)$, $n = 1, 2, \ldots$ be the subset of the Riemann surface of $g_n(z)$ which lies in $\mathcal{C}_n$.

**Lemma 6.2.1.** ([132], lemma 1) There exist positive constants $\varepsilon_n$ and $c_n$, $n = 1, 2, \ldots$, with $c_1 = \frac{1}{10}$ and $c_{n+1} \leq (\frac{1}{10}) c_n$, $n = 1, 2, \ldots$, and a sequence of polynomials $p_n(z, w)$ such that:

1. $\{p_n = 0\} \cap \{|z| \leq \frac{1}{2}\} = \sum (c_1, \ldots, c_n), n = 1, 2, \ldots$
2. $\{|p_n| \leq \varepsilon_{n+1}\} \cap \{|z| \leq \frac{1}{2}\} \subset \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}, n = 1, 2, \ldots$
3. If $|a| \leq \frac{1}{2}$ and $|p_n(a, w)| \leq \varepsilon_n$, then there is a $w_n$ with $p_n(a, w_n) = 0$ and $|w - w_n| \leq \frac{1}{n}$, $n = 1, 2, \ldots$.

With $p_n$, $\varepsilon_n$, $n = 1, 2, \ldots$ chosen as in Lemma 6.2.1, we put

$$Y = \bigcap_{n=1}^{\infty} \{|p_n| \leq \varepsilon_n\} \cap \{|z| \leq \frac{1}{2}\}.$$  

Clearly, $Y$ is a compact polynomially convex subset of $\mathbb{C}^2$. It was shown by Wermer that $Y$ has no analytic structure i.e. $Y$ contains no non-constant analytic disk. In fact he proves something stronger. The set $Y$ defined above contains no graph of a continuous function defined on a circle in $\mathcal{D}_{\frac{1}{2}}$ which avoids all the branch points $\{a_i\}$. Using this observation the following lemma follows.

**Lemma 6.2.2.** There is no finely analytic curve contained in $Y$.

Before we prove Lemma 6.2.2 we recall the following definition (cf. [61], page 75):

**Definition 6.2.3.** Let $U$ be a finely open set in $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is said to be finely holomorphic if every point of $U$ has a compact (in the usual topology) fine neighborhood $K \subset U$ such that the restriction $f|_K$ belongs to $R(K)$.

Here $R(K)$ denotes the uniform closure of the algebra of all restrictions to $K$ of rational functions on $\mathbb{C}$ with poles off $K$.
Proof of Lemma 2.2. Let \( \varphi : U \to Y, z \mapsto (\varphi_1(z), \varphi_2(z)) \) be a finitely analytic curve contained in \( Y \). If \( \varphi_1(z) \) is constant on \( U \) then \( \varphi_2(z) \) must also be a constant since non constant finitely holomorphic functions are finitely open maps and by the construction of the set \( Y \) the fibre \( Y \cap \{z \times \mathbb{C}\} \) is a Cantor set or a finite set for any point \( z \in \overline{D}_{1/2} \). Assume therefore that \( \varphi_1(z) \) is non-constant. In particular, there is a point \( z_0 \in U \) where the fine derivative of \( \varphi_1(z) \) does not vanish. Hence \( \varphi_1(z) \) is one-to-one on some finitely open neighborhood \( V \subset U \) of the point \( z_0 \). By considering the map \( z \mapsto (\varphi_1 \circ \varphi_1^{-1}(z), \varphi_2 \circ \varphi_1^{-1}(z)) \), defined on the finitely open set \( \varphi_1(V) \) we may assume that \( \varphi \) is of the form \( z \mapsto (z, g(z)) \) where \( g(z) = \varphi_2 \circ \varphi_1^{-1}(z) \) is finitely holomorphic in the finitely open set \( V' = \varphi_1(V) \subset D_{1/2} \). By Definition 6.2.3 there exists a compact subset \( K \subset V' \) with non-empty fine interior such that \( g(z) \) is a continuous function on \( K \) (with respect to the Euclidean topology). Shrinking \( K \) if necessary we may assume that \( K \cap \{a_1, a_2, \ldots\} \) = \( \emptyset \). Let \( p \) be a point in the fine interior of \( K \). It is well known that there exists a sequence of circles \( \{C(p, r_j)\} \) contained in \( K \) with centers \( p \) and radii \( r_j \to 0 \) as \( j \to \infty \). Clearly, the circle \( C(p, r_j) \) avoids the branch points \( \{a_1, a_2, \ldots\} \) and its image under the continuous map \( z \mapsto (z, g(z)) \) is contained in \( Y' \). By the above observation this is not possible. Hence \( Y \) contains no fine analytic structure.

Denote by \( d_n \) the degree of the one variable polynomial \( w \mapsto p_n(z, w) \) where \( p_n(z, w) \) is the polynomial given in Lemma 6.2.1. Assume that the set \( Y \) is constructed using the parameters \( \epsilon_n \) satisfying the following condition

\[
\lim_{n \to \infty} (\epsilon_n)^{1/d_n} = 0. \tag{6.2.1}
\]

It is shown in [92] that with this choice the set \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \). Using this result and Lemma 6.2.2 we are able to generalize a result in [91].

Theorem 6.2.4. Fix \( \delta \in (0, 1/2) \) and let \( Y_\delta = \bigcap_{n=1}^\infty \{\{p_n| \leq \epsilon_n\} \cap \{|z| \leq \delta\}\} \) be constructed using the parameters \( \epsilon_n \) satisfying (6.2.1). Then

(a) \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a polar subset of \( U \) for all finitely analytic curves \( \varphi : U \to \mathbb{C}^2 \).

(b) \( Y_\delta \neq (Y_\delta)_{C^2} \).

Proof of Theorem 6.2.4. In order to prove (a) we argue by contradiction. Assume therefore that \( \varphi : U \to \mathbb{C}^2 \) is a finitely analytic curve and \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) is a non polar subset of \( U \). Then there is a fine domain \( U_{k_0} \subseteq U \) such that \( \varphi(U_{k_0}) \subset C_{1/2} \) and \( \varphi^{-1}(\varphi(U_{k_0}) \cap Y_\delta) \) is a finitely open subset of \( U \) and hence has at most countably many finitely connected components \( \{U_k\}_{k=1}^\infty \). Moreover, \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_{k_0} \) is non polar for some natural number \( k_0 \), since otherwise \( \bigcup_{k=1}^\infty \{\varphi^{-1}(\varphi(U) \cap Y_\delta) \cap U_k\} = \varphi^{-1}(\varphi(U) \cap Y_\delta) \) would be polar contrary to our assumption. Since \( Y \cap C_{1/2} \) is complete pluripolar in \( C_{1/2} \) there exists a plurisubharmonic function \( u \) defined in \( C_{1/2} \) which is equal to \( -\infty \) exactly on \( Y \cap C_{1/2} \). By Lemma 5.3.6, the function \( u \circ \varphi \) is either finely subharmonic on \( U_{k_0} \), or identically equal to \( -\infty \). Since \( u \) equals \( -\infty \) on the non polar subset \( \varphi^{-1}(\varphi(U) \cap Y_\delta) \) \( \cap U_{k_0} \), it must be identically equal to \( -\infty \) on \( U_{k_0} \). Therefore \( \varphi(U_{k_0}) \subset \{u = -\infty\} = Y \cap C_{1/2} \) contradicting Lemma 6.2.2 and (a) follows.
6.3. Proof of Theorem 6.1.2

The proof of assertion (b) follows immediately from the proof of Proposition 3.1 in [91]. Indeed, if $u$ is a plurisubharmonic function defined in $\mathbb{C}^2$ which equals $-\infty$ on $Y_5$ then the function $z \mapsto \max \{u(z, w) : (z, w) \in \mathcal{Y} \}$ is subharmonic in $D_{1/2}$ and since it equals $-\infty$ on $D_5$ it equals $-\infty$ on $D_{1/2}$. Consequently $Y \cap \mathcal{C}_{1/2} \subset (Y_5)^c_{\mathbb{C}^2}$ and hence $Y_5 \neq (Y_5)^c_{\mathbb{C}^2}$.

**Remark.** It follows from the argument used in the proof of assertion (b) in Theorem 6.2.4 that $Y \cap \mathcal{C}_{1/2} \subset (Y_5)^c_{\mathbb{C}^2}$. Since the first set is complete pluripolar in $\mathcal{C}_{1/2}$ it follows that $(Y_5)^c_{\mathbb{C}^2} = Y \cap \mathcal{C}_{1/2}$. Consequently, $(Y_5)^c_{\mathbb{C}^2}$ contains no fine analytic structure. It would be nice to determine what the set $(Y_5)^c_{\mathbb{C}^2}$ equals and to figure out whether this set contains fine analytic structure. We are unable to do this. But by modifying Wermer's construction, we will in the next Section construct a complete pluripolar Wermer-like set $X \subset \mathbb{C}^2$ with the property that $(X \cap (S \times \mathbb{C}))^c_{\mathbb{C}^2}$ contains no fine analytic structure for all non polar subset $S \subset \mathbb{C}$.

### 6.3 Proof of Theorem 6.1.2

#### 6.3.1 Construction of the Set $X$

In this Subsection we construct the set $X$. Denote by $\{a_k\}_{k=1}^\infty$ the points in the complex plane both of whose coordinates are rational numbers. Without loss of generality we may assume that $a_k \in D_6$. For any sequence of points $\{a_i\}_{i=1}^l$ we denote by $B_j(z)$ the algebraic function

$$B_j(z) = (z - a_1) \ldots (z - a_j) \sqrt{z - a_j}.$$  

Denote by $\gamma_j$ a simple smooth curve with endpoints $a_j$ and $\infty$. For each $j$, $B_j(z)$ has two single-valued analytic branches on $\mathbb{C} \setminus \gamma_j$. Following the notation in [132] we choose one of the branches $B_j(z)$ arbitrarily and denote it by $\beta_j(z)$. Then $|\beta_j(z)| = |B_j(z)|$ is continuous on $\mathbb{C}$.

For each $n + 1$-tuple of positive constants $(c_1, c_2, \ldots, c_{n+1})$ we denote by $g_n(z)$ the algebraic function defined recursively in the following way. Put $g_1(z) = c_1 B_1(z)$ and $g_2(z) = c_1 B_1(z) + c_2 B_2(z)$ and if $g_n(z)$ has been chosen we will choose $g_{n+1}(z)$ as described below. Put $Z_1(z) = 1$ and for $n = 2, 3, \ldots$ define the function $Z_n(z)$ as follows. Denote by $z_1, z_2, \ldots, z_l$ all the zeros of all possible different differences $h_j(z) - h_i(z)$ ($i \neq j$) of branches $h_j(z), h_i(z)$ of the function $g_n(z)$. Suppose $z_k$ is a zero of $h_j(z) - h_i(z)$ of order $m_k$ and put $Z_n(z) = \Pi_{i=1}^{l} (z - z_i)^{m_i}$. Note that the zeros of $Z_n(z)$ are also zeros of the function $Z_{n+1}(z)$ of the same or greater multiplicity. Define $g_{n+1}(z) = g_n(z) + c_{n+1} Z_n(z) B_{n+1}(z)$.

By $\Sigma(c_1, \ldots, c_n)$ we mean the Riemann surface of $g_n(z)$ which lies in $\mathbb{C}^2$. In other words, $\Sigma(c_1, \ldots, c_n) = \{(z, w) : z \in \mathbb{C}, w = w_j, j = 1, 2, \ldots, 2^n\}$, where $w_j, j = 1, 2, \ldots, 2^n$ are the values of $g_n(z)$ at $z$. 


We will choose positive constants \( c_n, \epsilon_n \) and polynomials \( p_n(z, w) \) recursively so that

\[
\begin{align*}
\{ p_n(z, w) = 0 \} \cap C_{n+1} &= \Sigma(c_1, c_2, \ldots, c_n) \cap C_{n+1} \quad (6.3.1) \\
\{|p_{n+1}(z, w)| \leq \epsilon_n+1 \} \cap C_{n+1} &\subset \{|p_n(z, w)| < \epsilon_n \} \cap C_{n+1} \quad (6.3.2)
\end{align*}
\]

hold for \( n = 1, 2, \ldots \). The set \( X \) will be of the form

\[
X = \bigcup_{n=1}^{\infty} \left( \bigcap_{j=n}^{\infty} \{ |p_j(z, w)| \leq \epsilon_j \} \cap C_{n+1} \right). \quad (6.3.3)
\]

Put \( c_1 = 1 \) and let \( p_1(z, w) = w^2 - (z - a_1) \). It is clear that \( \Sigma(c_1) \cap C_2 = \{ p_1(z, w) = 0 \} \cap C_2 \). Choose \( \epsilon_1 > 0 \) so that if \( z_0 \in D_2 \) and \( |p_1(z_0, w)| \leq \epsilon_1 \) then there exists \( (z_0, w_1) \in \Sigma(c_1) \cap C_2 \) with \( |w - w_1| \leq 1 \). Let \( B_2 = D_2 \times D_{p_1} \) be a bidisk where \( p_1 \) is chosen so that

\[
\{|p_1(z, w)| \leq \epsilon_1 \} \cap C_2 = \{|p_1(z, w)| \leq \epsilon_1 \} \cap B_2.
\]

Assume that \( c_n, \epsilon_n \) and \( p_n(z, w) \) have been chosen so that (6.3.1) and (6.3.2) hold. We will now choose \( c_{n+1} \) and \( p_{n+1}(z, w) \). We denote by \( w_j(z), \ j = 1, 2, \ldots, 2^n \) the roots of \( p_n(z, \cdot) = 0 \) and to each positive constant \( c \) we assign a polynomial \( p_c(z, w) \) by putting

\[
p_c(z, w) = \prod_{j=1}^{2^n} \left( (w - w_j(z))^2 - c^2 (Z_n(z)B_{n+1}(z))^2 \right). \quad (6.3.4)
\]

Then \( p_c(z, \cdot) = 0 \) has the roots \( w_j(z) \pm c Z_n(z) B_{n+1}(z) \), \( j = 1, 2, \ldots, 2^n \) and so

\[
\{ p_c(z, w) = 0 \} = \Sigma(c_1, c_2, \ldots, c_n, c).
\]

Note that from (6.3.4)

\[
p_c = p_n^2 + c^2 q_1 + \cdots + (c^2)^{2^n} q_{2^n},
\]

where the \( q_i \) are polynomials in \( z \) and \( w \), not depending on \( c \). Choose \( c > 0 \) so that the following hold for all \( z \in D_{n+1} \).

\[
\Sigma(c_1, c_2, \ldots, c_n, c) \cap C_{n+1} \subset \{|p_n(z, w)| < \epsilon_n/2 \} \cap C_{n+1} \quad (6.3.5)
\]

\[
c \cdot |Z_n(z)B_{n+1}(z)| \leq (1/10|c_n|Z_{n-1}(z)B_n(z)|. \quad (6.3.6)
\]

Decreasing \( c \) if necessary we may assume that if \( h_1(z) \) and \( h_j(z) \) are any different branches of the function \( g_n(z) \) the estimate

\[
|h_j(z) - h_i(z)| \geq 2c |Z_n(z)B_{n+1}(z)| \quad (6.3.7)
\]

holds in \( D_{n+1} \) with equality exactly at the zeros of \( Z_n(z) \) which are contained in \( D_{n+1} \) and at the points \( a_1, \ldots, a_n \). This estimate will be needed later when we prove that \( X \) contains no fine analytic structure. Choose \( c_{n+1} = c \).
Let \( B_{n+2} = D_{n+2} \times D_{\rho_{n+2}} \) be a bidisk where \( \rho_{n+2} \) is chosen so that \( \{ |p_n(z, w)| \leq \epsilon_n \} \cap C_{n+2} = \{ |p_n(z, w)| \leq \epsilon_n \} \cap B_{n+2} \) and \( \rho_{n+2} > \rho_{n+1} + 1 \). Let \( \delta > 0 \) be a constant such that \( |\delta \cdot p_n(z, w)| < 1 \) in \( B_{n+2} \) and choose \( p_{n+1}(z, w) = \delta \cdot p_n(z, w) \).

We now turn to the choice of \( \epsilon_{n+1} \). Since the part of the zero set of \( p_{n+1}(z, w) \) which is contained in \( B_{n+1} \) is a subset of \( \{ |p_n(z, w)| < \epsilon_n/2 \} \cap B_{n+1} \) it is possible to find a natural number \( m_{n+1} \) so that

\[
\frac{1}{m_{n+1}} \log |p_{n+1}(z, w)| \geq -\frac{1}{2^n} \text{ for all } (z, w) \in B_{n+1} \cap \{ |p_n(z, w)| \leq \epsilon_n \}. \tag{6.3.8}
\]

Choose \( \epsilon_{n+1} < \epsilon_n \) so that

\[
\frac{1}{m_{n+1}} \log |p_{n+1}(z, w)| \leq -1 \text{ for all } (z, w) \in \{ |p_{n+1}(z, w)| \leq \epsilon_{n+1} \} \cap C_{n+2}. \tag{6.3.9}
\]

By decreasing \( \epsilon_{n+1} \) we may assume that (6.3.2) and the following assumption hold.

If \((z_0, w) \in C_{n+2} \) and \(|p_{n+1}(z_0, w)| \leq \epsilon_{n+1}, \) then there exists \((z_0, w_n) \in C_{n+2} \) such that \(|p_{n+1}(z_0, w_n)| = 0 \) and \(|w - w_n| \leq 1/n. \tag{6.3.10}\]

This ends the recursion.

**Lemma 6.3.1.** The set \( X \) defined by (6.3.3) is complete pluripolar in \( \mathbb{C}^2. \)

**Proof.** Define for \( n \geq 2 \) the plurisubharmonic function

\[
u_n(z, w) = \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\}
\]

and put \( u(z, w) = \sum_{n \geq 2} u_n(z, w). \) Then \( u(z, w) \) is plurisubharmonic in \( \mathbb{C}^2. \) Indeed, since the bidisks \( B_n \) exhaust \( \mathbb{C}^2 \) and \( |p_n(z, w)| < 1 \) in \( B_{n+1} \) the series \( \sum_{n \geq 2} u_n(z, w) \) will be decreasing on each fixed bidisk \( B_N \) after a finite number of terms and hence plurisubharmonic here. Since plurisubharmonicity is a local property \( u(z, w) \) is plurisubharmonic in \( \mathbb{C}^2. \) If \((z_0, w_0) \in X, \) then for some natural number \( N, \) \((z_0, w_0) \in \bigcap_{n \geq N} \{ |p_n(z, w)| \leq \epsilon_n \} \cap C_{n+1}. \) Condition (6.3.9) above implies that \( u(z_0, w_0) = \text{Const} + \sum_{n \geq N} u_n(z_0, w_0) = -\infty. \) Finally if \((z_0, w_0) \notin X \) then there exists a natural number \( N \) such that \((z_0, w_0) \in B_N \) and \((z_0, w_0) \notin \{ |p_n(z, w)| \leq \epsilon_n \} \cap B_{n+1} \) for all \( n \geq N. \) By (6.3.8)

\[
u(z, w) = \text{Const} + \sum_{n \geq N} \max \left\{ \frac{1}{m_n} \log |p_n(z, w)|, -1 \right\} \geq \text{Const} + \sum_{n \geq N} -\frac{1}{2^n} > -\infty.
\]

The Lemma follows. \( \square \)

### 6.3.2 \( X \) Contains No Fine Analytic Structure

In this Section we show that \( X \) contains no fine analytic structure. Suppose that \( z \mapsto (\varphi_1(z), \varphi_2(z)) \) is a finely analytic curve whose image is contained in \( X. \) If
\( \varphi_1(z) \) is constant then \( \varphi_2(z) \) must be constant since \( X \cap (\{z_0\} \times \mathbb{C}) \) is a Cantor set or a finite set for any point \( z_0 \in \mathbb{C} \). On the other hand, if \( \varphi_1(z) \) is non-constant, then using the arguments given in the proof of Lemma 6.2.2 we may assume that the finely analytic curve contained in \( X \) is given by \( z \mapsto (z, m(z)) \) where \( m(z) \) is a finely holomorphic function defined in \( U \) where \( U \subset \mathcal{D}_n \) for some natural number \( n \). Fix a point \( z' \in U \setminus \{a_1, \ldots, a_n\} \). By the definition of finely holomorphic functions we can find a compact (in the usual topology) fine neighborhood \( K \subset U \) of \( z' \) where \( m(z) \) is continuous. Shrinking \( K \) if necessary we may assume that \( (K \setminus \{z'\}) \cap (\{a_j\}_{j=1}^\infty \cup \{Z_{k-1}(z) = 0\}_{k=2}^\infty) = \emptyset \). Since the complement of \( K \) is thin at \( z' \), one can find a sequence of circles \( \{C(z', r_i)\} \subset K \) with \( r_i \to 0 \) as \( i \to \infty \). Choose one of the circles \( C(z', r_j) \) so that none of the points \( a_1, \ldots, a_n \) are contained in \( \{|z-z'| \leq r_j\} \). Let \( a_k \) be the first point in the sequence \( \{a_j\}_{j=n+1}^\infty \) which is contained in \( \{|z-z'| < r_j\} \). Note that \( a_k \in \{|z-z'| < r_j\} \), \( m(z) \) is continuous on \( C(z', r_j) \) and the function \( Z_{k-1}(z) \beta_k(z) \neq 0 \) when \( z \in C(z', r_j) \). The fact that the image of \( C(z', r_j) \) under the map \( z \mapsto (z, m(z)) \) is a subset of \( X \) will lead us to a contradiction and hence \( X \) contains no fine analytic structure. In order to prove this fix a point \( z_1 \in C(z', r_j) \) and denote by \( \mathcal{R} \) the \( 2^k \) branches of the algebraic function \( g_k(z) \) defined on \( C(z', r_j) \setminus \{z_1\} \).

**Lemma 6.3.2.** If \( h_i(z) \) and \( h_j(z) \) are any different functions from \( \mathcal{R} \), then

\[
|h_i(z) - h_j(z)| > (3/2)c_k |Z_{k-1}(z) \beta_k(z)|
\]

holds for all \( z \in C(z', r_j) \setminus \{z_1\} \).

**Proof.** This follows directly from (6.3.7) since \( C(z', r_j) \subset \mathcal{D}_n \) and \( C(z', r_j) \) does not intersect any of the branch points \( a_1, \ldots, a_k \) or the zeros of \( Z_{k-1}(z) \).

From now on the proof that \( X \) contains no fine analytic structure follows the arguments given in [132].

**Lemma 6.3.3.** Fix \( z_0 \) in \( C(z', r_j) \setminus \{z_1\} \). There exists a function \( h_i(z) \in \mathcal{R} \), where \( h_i(z) \) depends on \( z_0 \) such that

\[
|m(z_0) - h_i(z_0)| < (1/4)c_k |Z_{k-1}(z_0) \beta_k(z_0)|
\]

**Proof.** By (6.3.10) there exists \( N \geq k \) and \( w_N \) such that \( (z_0, w_N) \) lies on the set \( \Sigma(c_1, \ldots, c_N) \) and \( m(z_0) = w_N + R(z_0) \) where \( |R(z_0)| \leq (1/10)c_k |Z_{k-1}(z_0) \beta_k(z_0)| \). Thus

\[
m(z_0) = \pm c_1 \beta_1(z_0) + \sum_{\nu=2}^N \pm c_\nu Z_{\nu-1}(z_0) \beta_\nu(z_0) + R(z_0) = \]

\[
def \quad h_i(z_0) + \sum_{\nu=k+1}^N c_\nu Z_{\nu-1}(z_0) \beta_\nu(z_0) + R(z_0).
\]
Since $C(z', r_j) \subset D_{n+1}$ and the constants $c_\nu$ are chosen so that (6.3.6) holds,

$$|m(z_0) - h_i(z_0)| \leq \sum_{\nu=k+1}^{N} c_\nu |Z_{\nu-1}(z_0)\beta_\nu(z_0)| + |R(z_0)| \leq$$

$$\leq c_k |Z_{k-1}(z_0)\beta_k(z_0)| \left( \frac{1}{10} + \frac{1}{10^2} + \ldots \right) + |R(z_0)| =$$

$$= \frac{1}{9} c_k |Z_{k-1}(z_0)\beta_k(z_0)| + \frac{1}{10} c_k |Z_{k-1}(z_0)\beta_k(z_0)| <$$

$$(1/4) c_k |Z_{k-1}(z_0)\beta_k(z_0)|.$$

Hence (6.3.12) holds and the Lemma is proved. \(\square\)

**Lemma 6.3.4.** Fix $z_0 \in C(z', r_j) \setminus \{z_1\}$ and let $h_i(z) \in \mathbb{R}$ satisfy (6.3.12). Then for all $z$ in $C(z', r_j) \setminus \{z_1\}$

$$|m(z) - h_i(z)| < (1/3) c_k |Z_{k-1}(z)\beta_k(z)|.$$ \hspace{1cm} (6.3.13)

**Proof.** The set $\mathcal{O} = \{ z \in C(z', r_j) \setminus \{z_1\} : (6.3.13)$ holds at $z \}$ is open in $C(z_0, r_j) \setminus \{z_1\}$ and contains $z_0$. If $\mathcal{O} \neq C(z', r_j) \setminus \{z_1\}$ then there is a boundary point $p$ of $\mathcal{O}$ on $C(z', r_j) \setminus \{z_1\}$ for which

$$|m(p) - h_i(p)| = (1/3) c_k |Z_{k-1}(p)\beta_k(p)|.$$ \hspace{1cm} (6.3.14)

holds. By Lemma 6.3.3 there is some $h_j(z)$ in $\mathbb{R}$ such that

$$|m(p) - h_j(p)| < (1/4) c_k |Z_{k-1}(p)\beta_k(p)|.$$ \hspace{1cm} (6.3.15)

Thus $|h_i(p) - h_j(p)| \leq (7/12) c_k |Z_{k-1}(p)\beta_k(p)|$. Also $h_i(z) \neq h_j(z)$, in view of (6.3.14) and (6.3.15). This contradicts Lemma 6.3.2. Thus $\mathcal{O} = C(z', r_j) \setminus \{z_1\}$ and Lemma 6.3.4 follows. \(\square\)

For each continuous function $v(z)$ defined on $C(z', r_j) \setminus \{z_1\}$ which has a jump at $z_1$ we write $L^+(v)$ and $L^-(v)$ for the two limits of $v(z)$ as $z \to z_1$ along $C(z', r_j)$. Then, by (6.3.13),

$$|L^+(m) - L^+(h_i)| \leq (1/3) c_k |Z_{k-1}(z_1)\beta_k(z_1)|$$

and

$$|L^-(m) - L^-(h_i)| \leq (1/3) c_k |Z_{k-1}(z_1)\beta_k(z_1)|,$$

so

$$|(L^+(m) - L^+(h_i)) - (L^-(m) - L^-(h_i))| \leq (2/3) c_k |Z_{k-1}(z_1)\beta_k(z_1)|.$$  

Since $m(z)$ is continuous on $C(z', r_j)$ the jump of $h_i(z)$ at $z_1$ is in modulus less than or equal to $(2/3) c_k |Z_{k-1}(z_1)\beta_k(z_1)| \neq 0$. But $h_i(z)$ is in $\mathbb{R}$, so its jump at $z_1$ has modulus at least $2c_k |Z_{k-1}(z_1)\beta_k(z_1)|$. This is a contradiction.
6.3.3 The Sets $E$ and $E^*_C$

Denote by $E$ the pluripolar set $E = (S \times \mathbb{C}) \cap X$ where $S$ is a non polar subset of $\mathbb{C}$. Since $X$ is complete pluripolar in $\mathbb{C}^2$ it follows that $E^*_C \subset X$. To prove that $X \subset E^*_C$ we argue as follows. First we claim that the set $X$ is pseudoconcave. Indeed, by the construction of the set $X$, $X = \cup_{n=1}^{\infty} \{|p_n(z,w)| > \epsilon_n\} \cap C_{n+1}$. By the choice of the polynomials $p_n(z,w)$ it follows that $\{|p_n(z,w)| > \epsilon_n\} \cap C_{n+1} \subset \{|p_{n+1}(z,w)| > \epsilon_{n+1}\} \cap C_{n+2}$.

Moreover, for each natural number $n$ the set $\{|p_n(z,w)| > \epsilon_n\} \cap C_{n+1}$ is a domain of holomorphy. Hence $C^2 \setminus X$ is a countable union of increasing domains of holomorphy. By the Behnke-Stein Theorem $C^2 \setminus X$ is pseudoconvex and the claim follows.

Denote by $u(z,w)$ a globally defined plurisubharmonic function which equals $-\infty$ on $E$. It is shown in [127] that the function $z \mapsto \max\{u(z,w) : (z,w) \in X\}$ is subharmonic in $\mathbb{C}$. Since the projection $S$ of $E$ onto the first coordinate plane is non polar the function $z \mapsto \max\{u(z,w) : (z,w) \in X\}$ will be identically equal to $-\infty$ on $\mathbb{C}$ hence $u(z,w) = -\infty$ on the whole of $X$ and consequently $E^*_C = X$. This ends the proof of Theorem 6.1.2.

6.4 Final Remarks and Open Problems

It follows immediately from Theorem 6.1.1 and the fact that $X$ contains no fine analytic structure that if $\varphi : U \rightarrow \mathbb{C}^2$ is a finely analytic curve, then the set $\varphi^{-1}(\varphi(U) \cap X)$ is polar in $\mathbb{C}$.

Despite the result of Theorem 6.1.2 it should be mentioned here that in the situation where one considers the pluripolar hull of the graph of a finely holomorphic function defined in a fine domain $D$, the following problem still remains open.

**Problem 1.** Let $z \in \Gamma_f(D)_{C^2}$. Does this imply that there is a finely analytic curve contained in $\Gamma_f(D)_{C^2}$ which passes through the point $z$?

It is proved in [40] that the pluripolar hull relative to $C^n$ of a connected pluripolar $F_\sigma$ subset is a connected set. It is a fairly easy exercise to show that the set $X = E^*_C$ in Theorem 6.1.2 is path connected, but in general the pluripolar hull of a connected $(F_\sigma)$ pluripolar set is not path connected. Indeed, denote by $f(z)$ an entire function of order $1/3$, $f(1/z)$ has an essential singularity at 0 and in [134] Wiegnerick proved that the graph $\Gamma_f(1/z)$ of $f(1/z)$ over $\mathbb{C} \setminus \{0\}$ is complete pluripolar in $\mathbb{C}^2$. Consequently, if we put $E = \Gamma_f(1/z) \cup \{(0) \times \mathbb{C}\}$ then $E$ is complete pluripolar in $\mathbb{C}^2$ and hence $E^*_C = E$. Moreover $E$ is a connected $F_\sigma$ subset of $\mathbb{C}^2$. By the famous Denjoy-Carleman-Ahlfors theorem (see e.g. [1]), entire functions of order $1/3$ do not have finite asymptotic values; i.e., there are no curves $\gamma$ ending at infinity such that $f(z)$ approaches a finite value as $z \rightarrow \infty$ along $\gamma$. Hence it is not possible to find a path in $E^*_C$ connecting a point on $\Gamma_f(1/z)$ with a point in
the set \( \{0\} \times \mathbb{C} \). In view of this remark it would be interesting to know the answer to the following question.

**Problem 2.** Is \( \Gamma_f(D)_{\mathbb{C}^2} \) path connected?