Chapter 7

Examples and Open Questions

In this last chapter we study the pluripolar hull of graphs of a Borel series. The first example shows that some of the techniques used in [38] can be carried over to the case of a graph of a continuous functions defined on the complement \( \mathbb{C} \setminus E \) of a dense polar set \( E \subset \mathbb{C} \). In order to illustrate results of Chapter 5 a second example, which appeared in [41], is elaborated in subsection 7.2.2. Moreover, a collection of open problems is presented in Section 7.3.

7.1 \( h \)-Hausdorff Measure

Let \( h \) be a real valued, increasing function on the interval \([0, 1)\) with \( \lim_{r \to 0} h(r) = h(0) = 0 \). Such a function is sometimes called a dimension function, see [53]. The \( h \)-Hausdorff measure of a set \( E \subset \mathbb{C} \) is defined as follows:

\[
m_h(E) = \lim_{\delta \to 0} \left( \inf_k \sum h(r_k) \right),
\]

(7.1.1)

where the infimum is taken over all coverings of \( E \) by balls \( B_k \) with radii \( r_k \) not exceeding \( \delta \). Note that if \( h(r) = r^s \), the \( m_{rs} \) is the usual \( s \)-dimensional Hausdorff measure.

We will use the following known result. It is due to Erdős and Gillis [52]. A simpler proof of it was given by Carleson [20].

**Theorem 7.1.1.** Let \( E \subset \mathbb{C} \). If \( E \) has finite \( h \)-Hausdorff measure with respect to the function \( h(r) = (\log(1/r))^{-1} \), then it is polar.

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7.2.1 Example 1

Let \( \{a_j\}_{j=1}^{\infty} \) be a dense sequence in \( \mathbb{C} \) with the property that \( |a_j| < j \), and let \( 0 < c_j < 1 \), \( j = 1, 2, \ldots \). Suppose that

\[
\frac{1}{e^j} \leq c_j < 1, \quad (7.2.1)
\]

and

\[
\sum_{j=n}^{\infty} \frac{-1}{\log c_j} < +\infty. \quad (7.2.2)
\]

The inequality (7.2.1) guarantees the existence of a natural number \( m_0 \geq 4 \) such that

\[
\frac{1}{n^{m_0} \log((\Pi_{j=1}^{m_0} c_j) \to 0, \text{ as } n \to \infty. \quad (7.2.3)}
\]

Let \( E = \bigcap_{m} \bigcup_{j \geq m} B(a_j, c_j) \). By Baire’s theorem, \( E \) is a dense subset of \( \mathbb{C} \). Moreover, \( E \) is polar in view of (7.2.2) and Theorem 7.1.1.

Denote by \( D \) the complement of \( E \):

\[
D = \bigcup_{m} (\mathbb{C} \setminus \bigcup_{j \geq m} B(a_j, c_j)) \overset{\text{def}}{=} \bigcup_{m} F_m.
\]

Observe that \( (F_m) \) is an increasing sequence of closed sets.

Let

\[
f(z) = \sum_{j=1}^{\infty} \frac{c_j^{k(j)+1}}{(z-a_j)}, \quad (7.2.4)
\]

where \( (k(j)) \) is a sequence that will be determined later on.

Let

\[
\Gamma_f(\mathbb{C} \setminus (E \cup \{a_j\}_{j=1}^{\infty})) = \{(z, f(z)) : z \in \mathbb{C} \setminus (E \cup \{a_j\}_{j=1}^{\infty})\}
\]

Proposition 7.2.1. There exist a polar set \( \tilde{E} \supseteq E \cup \{a_j\}_{j=1}^{\infty} \) and a plurisubharmonic function \( \psi \in \text{PSH}(\mathbb{C}^2) \) such that

\[
\{\psi = -\infty\} = \Gamma_f(\mathbb{C} \setminus (E \cup \{a_j\}_{j=1}^{\infty})) \cup (E \times \mathbb{C}). \quad (7.2.5)
\]

Proof. Let \( r_n(z) = \sum_{j=1}^{n} \frac{c_j^{k(j)+1}}{(z-a_j)} \), and let \( q_n(z) = (z-a_1) \ldots (z-a_n) \). Put

\[
h_n(z, w) := \frac{1}{n^{m_0}} \log \left| (w - r_n(z))q_n(z) \right| \quad (7.2.6)
\]

Then \( h_n \) is a continuous plurisubharmonic function in \( \mathbb{C}^2 \).

For any \( \nu \geq 2 \) denote by \( D_\nu \) the disk with center 0 and radius \( \nu \), and put \( D_\nu = D_\nu \cap F_\nu \).
For \( z \in F_{\nu} \setminus \{ a_j : j = 1, \ldots, \nu - 1 \} \) we have, for \( n \geq \nu \),

\[
|f(z) - r_n(z)| = \left| \sum_{j=n+1}^{\infty} c_j^{k(j)}\frac{z^{k(j)+1}}{(z-a_j)} \right| \leq \sum_{j=n+1}^{\infty} c_j^{k(j)}. \tag{7.2.7}
\]

Now we choose \( k(j) \) such that

\[
\sum_{j=n+1}^{\infty} n^{m_0} c_j^{k(j)} \leq n, \text{ for all } n \in \mathbb{N}. \tag{7.2.8}
\]

For \( z \in D_{\nu} \setminus \{ a_j : j = 1, \ldots, \nu - 1 \} \), we have, using (7.2.7) and (7.2.8)

\[
|f(z) - r_n(z)|^{1/n^{m_0}} \leq \left( \frac{1}{n^{m_0}} \sum_{j=n+1}^{\infty} n^{m_0} c_j^{k(j)} \right)^{1/n^{m_0}} \\
\leq \left( \frac{1}{n} \right)^{1/n^{m_0}} \\
= \left( \frac{1}{n} \right)^{1/n^{m_0}}. \tag{7.2.9}
\]

Therefore

\[
||f(z) - r_n(z)||^{1/n^{m_0}}_{D_{\nu}} \to 0. \tag{7.2.10}
\]

For \( z \in D_{\nu} \setminus \{ a_j : j = 1, \ldots, \nu - 1 \} \) and \( n \in \mathbb{N} \) we have

\[
h_n(z, f(z)) := \frac{1}{n^{m_0}} \log \left( |(f(z) - r_n(z)) q_n(z)| \right) \\
= \frac{1}{n^{m_0}} \log \left( |(f(z) - r_n(z))| + \frac{1}{n^{m_0}} \log |q_n(z)| \right) \\
\leq \log |f - r_n|_{D_{\nu}}^{1/n^{m_0}} + \frac{1}{n^{m_0}} \log |z-a_1 \ldots (z-a_n)| \\
\leq \log |f - r_n|_{D_{\nu}}^{1/n^{m_0}} + \frac{1}{n^{m_0}} \log (\nu + n)^n. \tag{7.2.11}
\]

Inequality (7.2.10) holds because \( |a_j| < j \), and \( |z| < \nu \). Hence, for any \( \nu \in \mathbb{N} \) there exists \( n_1(\nu) \) such that, for all \( n \geq n_1(\nu) \), we have

\[
h_n(z, f(z)) \leq -\nu, \quad z \in D_{\nu} \setminus \{ a_j : j = 1, \ldots, \nu - 1 \}. \tag{7.2.11}
\]

Fix \( \nu \in \mathbb{N} \). For \( z \in D_{\nu} \setminus \{ a_j : j = 1, \ldots, \nu - 1 \} \) and \( w \in D_{\nu} \), with \( |w - f(z)| > 1/\nu \) and \( n \geq n_1(\nu) \) we have, using (7.2.11)

\[
h_n(z, w) := \frac{1}{n^{m_0}} \log \left( |(w - f(z)) q_n(z) + (f(z) - r_n(z)) q_n(z)| \right) \\
\geq \frac{1}{n^{m_0}} \log \left( \frac{1}{n^{m_0}} |q_n(z)| - \exp(-\nu n^{m_0}) \right) \\
\geq \frac{1}{n^{m_0}} \log (\Pi_{j=1}^{n} c_j) - \frac{1}{n^{m_0}} \log (\nu) - 1. \tag{7.2.12}
\]
In view of (7.2.3), there exists \( n_2(\nu) \geq n_1(\nu) \) such that for \( n \geq n_2(\nu) \) we have
\[
h_n(z, w) \geq -\log(\nu) - 2. \tag{7.2.13}
\]

Let us estimate \( h_n \) on \( \mathbb{D}_\nu \times \mathbb{D}_\nu \). Let \( \nu + 1 > R > \nu \) such that \( \{ |z| = R \} \subset F_\nu \).

Since \( ||f - r_n||_{1/n\mathbb{C}} \to 0 \), for sufficiently big \( n \in \mathbb{N} \) we have
\[
\max_{|z| \leq R} |p_n(z)| = \max_{|z| = R} |p_n(z)| \leq C_R \max_{|z| = R} |q_n(z)|,
\]
where \( C_R = \max_{|z| = R} |f(z)| + 1 \). Hence there exists \( n_3(\nu) \geq n_2(\nu) \) such that, for \( n \geq n_3(\nu) \), \( (z, w) \in \mathbb{D}_\nu \times \mathbb{D}_\nu \), we have
\[
h_n(z, w) := \frac{1}{n\mathbb{N}_0} \log |\varphi_{n}(z) - p_{n}(z)|
\leq \frac{1}{n\mathbb{N}_0} \log(\nu(\nu + n)^n + C_R(\nu + n)^n)
\leq \log(\nu + 2) \tag{7.2.14}
\]
For any \( \nu \), fix an \( n(\nu) \geq n_3(\nu) \). Now, we consider the plurisubharmonic functions
\[
v_\nu(z, w) = \max\{h_{\nu(n)}(z, w) - \log(\nu + 2), -\nu - \log(\nu + 2)\}
\]
As \( v_\nu(z, w) \) is negative on \( \mathbb{D}_\nu \times \mathbb{D}_\nu \), the series
\[
v(z, w) := \sum_{\nu = 2}^{\infty} \frac{1}{\nu^2} v_\nu(z, w) \tag{7.2.15}
\]
represents a plurisubharmonic function in \( \mathbb{C}^2 \). It is not identically \(-\infty\), because of (7.2.13), while (7.2.11) shows that it is \(-\infty\) on the graph of \( f \) over \( D \setminus \{a_j : j = 1, \ldots\} \). Let \( \varphi \in \text{SH}(\mathbb{C}) \) such that the set \( \{ \varphi = -\infty \} \), which we denote by \( E \), contains \( E \cup \{a_j\}_{j=1}^{\infty} \). Observe now that the function \( \psi(z, w) = v(z, w) + \varphi(z) \) satisfies all our conditions. This ends the proof.

It is possible to obtain more precise information on the pluripolar hull of the graph of \( f \), by making the following extra assumption on the sequence \( \{a_j\}_{j=1}^{\infty} \):

Suppose that for every \( j \), \( a_j \) belongs to infinitely many balls \( B(a_k, c_k) \), \( k \geq 1 \).

Under the above assumption it is clear that \( \{a_j\}_{j=1}^{\infty} \) is a subset of \( E \). Since \( E \) is a polar and a \( G_\delta \), there exists a function \( \varphi \in \text{SH}(\mathbb{C}) \), such that \( E = \{ \varphi = -\infty \} \).

Hence the above propositions can be reformulated as follows.

Corollary 7.2.2. There exists a plurisubharmonic function \( \psi \in \text{PSH}(\mathbb{C}^2) \) such that
\[
\{(z, w) \in \mathbb{C}^2 : \psi(z, w) = -\infty\} = \Gamma_f(\mathbb{C} \setminus E) \cup (E \times \mathbb{C}). \tag{7.2.16}
\]
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7.2.2 Example 2

Here we give an other example in the spirit of Borel to which the results from Chapter 5 apply (cf. [41]). It consists of a finely holomorphic function on an \( \mathcal{F} \)-domain, which is a dense subset of \( \mathbb{C} \) with empty Euclidean interior. Our point is to show that the study of quite natural series in connection with pluripolarity is fruitfully done in the framework of finely holomorphy.

Let \( \{a_j\}_{j=1}^\infty \) be a dense sequence in \( \mathbb{C} \) with the property that \( |a_j| < j \). Let \( r_j = 2^{-j} \). Then \( \cup_{j=1}^\infty B(a_j, r_j) \) has finite area, and its circular projection \( z \mapsto |z| \) has finite length. Next, define subharmonic functions \( g_j(z) = \log |z - a_j| - 3j \) and \( u_n \) by

\[
u_n(z) = \sum_{j=n}^{\infty} j^{-3} g_j(z).
\] (7.2.17)

The terms in the sum of (7.2.17) are subharmonic and they are negative for \( |z| < k \) as soon as \( j > k \). Hence \( u_n \) represents a subharmonic function. Let \( D = (\cup_n \{ u_n > -10 \}) \setminus \{ a_1, a_2, \ldots \} \). We claim that \( D = \{ u_1 > -\infty \} \). Indeed, let \( z_0 \in \cup_n \{ u_n > -\infty \} \setminus \{ a_1, a_2, \ldots \} \). Then there exists a natural number \( k \) such that \( |z_0| < k \) and \( u_k > -\infty \). Since all the terms of the series \( u_k(z_0) \) are negative, a suitable tail, say \( u_N(z_0) \), will be very close to 0. In other words, \( z_0 \in \{ u_N > -10 \} \). Hence \( z_0 \in D \) and consequently \( D = \cup_n \{ u_n > -\infty \} \setminus \{ a_1, a_2, \ldots \} \). Therefore,

\[
\mathbb{C} \setminus D = \cap_{n=1}^\infty \{ u_k = -\infty \} \cup \{ a_1, a_2, \ldots \}.
\]

Since \( \{ u_{k_1} = -\infty \} \setminus \{ a_1, a_2, \ldots \} = \{ u_{k_2} = -\infty \} \setminus \{ a_1, a_2, \ldots \} \) for any natural numbers \( k_1 \) and \( k_2 \), we conclude that

\[
\mathbb{C} \setminus D = \{ u_1 = -\infty \} \cup \{ a_1, a_2, \ldots \} = \{ u_1 = -\infty \}.
\]

This proves the claim. In particular, \( D \) is, by Theorem 2.2.29, an \( \mathcal{F} \)-domain.

For every \( j \) there exists \( 0 < c_j < 1 \) such that if \( |z - a_j| < c_j \), then for \( n \leq j \), \( u_n(z) < -11 \). Indeed, \( \sum_{k>j} k^{-3} g_k(z) < 0 \), while

\[
\sum_{k=n}^{j-1} k^{-3} g_k(z) < \log j \sum_{k=n}^{j-1} k^{-3} < 10 \log j.
\]

So it suffices to take \( c_j = j^{-11} j^3 \).

Next we define a function on \( D \) by

\[
f(z) = \sum_{j=1}^{\infty} \frac{c_j}{2^{j}(z - a_j)},
\] (7.2.18)

We claim that the function \( f \) is finely holomorphic on \( D \). Indeed, let \( z_0 \in D \). For every \( m \) a suitable tail of the series of \( f \) in (7.2.18) is uniformly convergent on the compact set \( K = \{ |z| \leq 2|z_0| \} \setminus \cup_{j \geq m} B(a_j, c_j) \). Now if \( z_0 \in D \), then \( z_0 \) belongs
to the finely open set \( \{ u_m > -10 \} \) for some \( m \). Hence, for all \( j \geq m \) we have \( |z_0 - a_j| > c_j \), and \( K \) is a \( \mathcal{F} \)-neighborhood of \( z_0 \).

Application of Corollary 5.3.2 gives us that the graph of \( \Gamma_f(D) \) of \( f \) over \( D \) is a pluripolar set. The theorem also shows that for a set of positive capacity \( E \subset D \), e.g., a circle in \( \{ u_1 > -10 \} \),

\[
\Gamma_f(D) \subset (\Gamma_f(E))^*_\mathbb{C}^2.
\]

Even for this example there are many questions left. We have no description of the maximal domain \( D_0 \) to which \( f \) extends as a finely holomorphic function, and we don’t know if \( \Gamma_f(D_0) = (\Gamma_f(E))^*_\mathbb{C}^2 \), as one may expect in view of [40].

### 7.3 Unsolved Problems

In this section we collect some problems that we could not solve.

In view of Example 7.2.2 it seems quite natural to ask the following

**Problem 1.** Let \( E \) be a polar dense subset of \( \mathbb{C} \), and let \( f \) be a finely holomorphic functions in the fine domain \( \mathbb{C} \setminus E \). Let \( \Gamma_f(\mathbb{C} \setminus E) \) be the graph of \( f \) over \( \mathbb{C} \setminus E \), and let \( \Gamma_f^*(\mathbb{C} \setminus E) \) be its pluripolar hull. Is it true that \( \Gamma_f^*(\mathbb{C} \setminus E) \) is contained in \( \Gamma_f(\mathbb{C} \setminus E) \cup (E \times \mathbb{C}) \)?

**Problem 2.** Let \( f : U \rightarrow \mathbb{C} \) be a finely continuous function on a finely open subset \( U \) of \( \mathbb{C} \). Suppose that the graph of \( f \) over \( U \) is a pluripolar subset of \( \mathbb{C}^2 \). Is it true that \( f \) is finely holomorphic in \( U \)?

The answer to this problem in the case of a usual open set \( U \) was given by Shcherbina [124]. In the present situation, with \( f \) is a \( C^2 \) function, a positive answer was given by Edlund in his thesis. Very recently, Edigarian and Wiegerinck [37] showed that the answer is still affirmative for \( C^1 \) functions.

We have seen in Proposition 5.5.1 that if a pluripolar set \( E \) hits a finely analytic curve in a “big” set, then the pluripolar hull of \( E \) must contain the whole curve. Thus, in order to understand the phenomenon of propagation of pluripolar sets, it seems necessary to study in more details the concept of fine analytic continuation. In particular we are led to the following two problems.

**Problem 3.** Let \( f : U \rightarrow \mathbb{C} \) be a finely holomorphic functions on a finely open subset \( U \) of \( \mathbb{C} \). The classical process of analytic continuation and the Weierstrass’s concept of the maximal analytic function can be carried over to the present situation, see Definition 5.2.3. The classical Poincaré-Voltera theorem, see [110], states that the maximal analytic function of any given holomorphic function can not have more that countably many values. See also [130] for the history of this result.

The question is: is there an analogue to the classical Poincaré-Voltera theorem for \( f \)? In other words, if \( \mathcal{F} \) is the “maximal” fine analytic functions of \( f \), is it true that \( \mathcal{F} \) has at most countably many values at each point, or at least at each point outside some polar set?.

Of course, Poincaré’s proof relies decisively on the fact that the usual topology has a countable base. Since the fine topology does not have this property, the
Poincaré’s argument can not be directly carried over. However, it seems that the quasi-Lindelöf property, cf. Theorem 2.2.20, might serve as a replacement for the countable base argument.

**Problem 4.** In [45] Edlund proved that for any non-empty closed subset \( K \subseteq \mathbb{C} \) there exists a continuous function \( f(z) \) on \( K \) such that its graph \( \Gamma_f(K) \) over \( K \) is complete pluripolar in \( \mathbb{C}^2 \). The function \( f \) in Edlund’s theorem is certainly finely holomorphic in the \( \mathcal{F} \)-interior \( K' \) of \( K \) (if \( K' \neq \emptyset \)). According to Theorem 5.4.1, \( f \) has no fine analytic continuation outside \( K' \). This means that \( K' \) is an \( \mathcal{F} \)-domain of fine holomorphy. One can therefore ask the following question:

*Let \( U \subseteq \mathbb{C} \) be an \( \mathcal{F} \)-domain. Is it true that \( U \) is an \( \mathcal{F} \)-domain of fine holomorphy?*

In analogy with the theory of finely subharmonic functions one can pose the following problem, see Theorem 2.3.19.

**Problem 5.** Let \( f : U \rightarrow [-\infty, +\infty] \) be \( \mathcal{F} \)-plurisubharmonic in an \( \mathcal{F} \)-domain \( U \subseteq \mathbb{C}^n \). Suppose that every point \( z \in U \) has a compact \( \mathcal{F} \)-neighborhood \( K_z \subseteq U \) such that \( f|_{K_z} \) is continuous in the usual sense.

Is it true that \( f|_{K_z} \) is the uniform limit of a sequence of usual plurisubharmonic functions defined in Euclidean neighborhoods of \( K \).

**Problem 6.** In connections to Problem 5 it seems also interesting to ask whether Theorem 4.3.3 holds without the exceptional pluripolar set \( E \).