Fiscal policy under rules and restrictions

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Chapter 2

A comparison of debt versus primary-deficit constraints*

*This chapter is based on Ribeiro, Beetsma, and Schabert (2007).

2.1 Introduction

The recent reform of the European Stability and Growth Pact (SGP) has increased its emphasis on debt sustainability.[1] This way, the reform has broadened the Pact's previous narrow focus on public deficits. The reform ties in closely with the ongoing discussion whether it is preferable to impose constraints on the public debt or on the public deficit (see e.g. Fatás et al., 2003, and Wyplosz 2005).

The literature provides no clear answer to this question. Intuitively, relative advantages of debt-based constraints would be that (1) they reward fiscal discipline in good times, because there is a future benefit to reducing the debt when the economy is doing well; (2) they provide a more accurate picture of the sustainability of the government's finances; (3) as far as the incentive for surprise inflation is concerned, the stock of (nominal) debt is a better indicator than the deficit and (4) they avoid inefficient budget cuts and tax increases that would be sub-optimal from a longer-term perspective when the economy falls victim to sudden (unexpected) adverse developments.[2] However, relative disadvantages of constraints on the public debt would be that (1) the current government, who faces the constraint, is at best partly responsible for the current debt level; and (2) if the reference level of the debt substantially exceeds the actual debt level, there is sufficient room left before the constraint becomes binding that the government may be tempted to spend resources on wasteful projects (by contrast a deficit rule is more immediately binding).

This paper compares constraints on the public debt with constraints on the primary deficit.[3] Although the SGP imposes a constraint on the total deficit in European countries, we concentrate on the primary deficit, since this constraint is not affected by the level of the public debt in an economy, while the total deficit depends positively on the public debt via the interest payments on the debt.[4] In effect, a constraint on the total deficit would become tighter as the public debt increases. The focus of the analysis are the effects of the constraints on public borrowing, government spending and social welfare. The main novelty of this paper is that we take into account how an optimizing government reacts to different constraints when it decides on a time consistent spending/borrowing plan. We set up a simple open economy framework with income shocks and potentially myopic governments as a source of excessive spending. This provides a rationale for imposing fiscal constraints either on the debt or on the primary deficit.

A crucial aspect of our set-up is that the government internalizes the penalties that follow the violation of particular debt or primary-deficit limits, which leads to precautionary behavior. Thus, the constraints will not only affect fiscal policy in cases they are

[1] For a review of the reform of the SGP, see Buti et al. (2005) or Morris et al. (2006).

[2] The Netherlands and Germany provide examples in this respect. Not so long ago, the SGP forced these countries to make budget cuts, thereby reinforcing the adverse economic circumstances they were already facing.

[3] The paper limits itself to these two alternatives and does not aim at finding the generally optimal fiscal rule. It also does not analyze the optimal interaction between monetary policy and fiscal policy under constraints, such as in Lambertini (2006) and Pappa and Vassilatos (2007).

[4] Assuming a primary-deficit constraint is also more convenient analytically.

binding, but will also have an effect on average government expenditures and public debt when they are not binding. We find that the economy behaves in a very similar way under both types of constraint, although welfare is higher under the debt-based constraint. Debt-based constraints imply better smoothing of public spending after an income shock. Further, the appropriate debt constraint is more robust against changes in the interest rate than is the appropriate deficit constraint. This suggests that the former type of constraint might be easier to implement in practice, in view of the fact that it would be politically difficult to make frequent and large adjustments to the constraints. These results support the greater emphasis that the SGP puts on debt after its reform.

Variations in the model parameters yield interesting insights. In particular, an increase in the variance of the endowment shocks or an increase in their persistence produces lower debt on average under a debt constraint (in order to maintain a larger safety margin relative to the reference debt level), but higher average debt under a primary-deficit constraint. Higher average debt corresponds to a higher average primary surplus, which implies a larger safety margin to the reference deficit level, as desired by the government. For similar reasons, an increase in risk aversion produces lower average debt under a debt constraint, but higher average debt under a primary-deficit constraint. Finally, a more myopic government implies a rise in average debt under a primary-deficit constraint, but a fall in average debt under a debt constraint. This is the result of precautionary behavior: by being close to the reference debt level, the government would run the risk of front-loading severe spending cuts when income is hit by a bad shock. This consequence is worse for a more myopic government in view of its preferred time profile of public spending. Hence, in the presence of a debt constraint such a government accumulates *less* debt on average.

The rest of the paper is organized as follows. Section 3.2 describes the model and characterizes the equilibrium for the non-myopic government. In Section 2.3 we solve the model for the myopic government under each type of fiscal constraint. Section 2.4 provides the welfare derivation. Next, Section 2.5 performs a numerical evaluation of the model calibrated to the EU situation and checks the robustness of the results for changes in the parameters. Finally, Section 4.5 concludes the paper.

2.2 The model

In this section we present a model of a small open economy. The model is specified in real terms, such that neither money nor monetary policy is modelled here. Nevertheless, we interpret this country as a member of a monetary union, where its fiscal policy is constrained by unionwide constraints (as is the case for the countries that adopted the Euro). Further, we assume that the constraints are credibly enforced.

2.2.1 The private sector

There exists a continuum of infinitely lived and identical households of total mass one. Their utility increases in consumption C_t and government spending G_t. The objective of a representative household is given by

$$\mathrm{E}_t \sum_{s=t}^{\infty} \beta_w^{s-t} \left[v(C_s) + u(G_s) \right], \tag{2.1}$$

where $\beta_w \in (0,1)$ denotes the discount factor, and E_t the expectations operator conditional on information at time t. The functions u and v are further assumed to be increasing in C and G, strictly concave, twice continuously differentiable, and to satisfy the usual Inada conditions.

In each period households are endowed with a stochastic amount of goods Y_t and with wealth in form of one period risk-free government bonds, B_t, and foreign bonds, F_t. Households have to pay taxes on their endowment Y_t. As a result, the household flow budget constraint reads

$$C_t + B_t + F_t \leq (1 + r_{t-1}^H) B_{t-1} + (1 + r_{t-1}) F_{t-1} + (1 - \tau_t^y)\, Y_t. \tag{2.2}$$

where r_t denotes the exogenous real interest rate on foreign bonds, r_t^H the real interest rate on domestic bonds and τ_t^y the (income) tax rate. Maximizing (2.1) subject to no-Ponzi game conditions for domestic and foreign borrowing, $\lim_{t\to\infty} B_t \prod_{i=1}^{t} (1 + r_{i-1}^H)^{-1} \geq 0$ and $\lim_{t\to\infty} F_t \prod_{i=1}^{t} (1 + r_{i-1})^{-1} \geq 0$, and (2.2) leads to the first-order conditions

$$\begin{aligned} \frac{v'(C_t)}{\mathrm{E}_t v'(C_{t+1})} &= \beta_w \left(1 + r_t\right), & (2.3) \\ r_t^H &= r_t, & (2.4) \end{aligned}$$

where (2.3) presupposes that the state(s) follow a Markov-process, and the transversality conditions $\lim_{s\to\infty} \mathrm{E}_t F_s \prod_{i=1}^{s} (1 + r_{i-1})^{-1} = 0$ and $\lim_{s\to\infty} \mathrm{E}_t B_s \prod_{i=1}^{s} (1 + r_{i-1}^H)^{-1} = 0$. We assume that the endowment follows an exogenous stochastic process

$$Y_t - \overline{Y} = \rho \left(Y_{t-1} - \overline{Y} \right) + \sigma_\varepsilon \varepsilon_t, \qquad \varepsilon_t \sim N(0,1), \tag{2.5}$$

where ε_t is independently, identically and normally distributed with mean zero and unitary variance ($\mathrm{E}_t \varepsilon_s = 0$ and $\mathrm{E}_t \varepsilon_s^2 = 1$ for $s > t$). In addition, ρ satisfies $0 \leq \rho \leq 1$ and $\sigma_\varepsilon \geq 0$ is a known parameter.

2.2.2 The government

The government issues bonds, raises tax revenues T_t and purchases goods G_t from the households. Without any further constraints, its period-by-period budget constraint reads $B_{t+1} = (1 + r_t)\, B_t + G_t - T_t$, where we have used that $r_t^H = r_t$. It chooses sequences for expenditures and debt until infinity. This assumption does not necessarily imply that the government stays in charge from today until infinity. We interpret the infinite planning horizon as an implication of the government's uncertainty about its term in office.

Thus the government chooses the sequences $\{G_s\}_{s=t}^{\infty}$ and $\{B_s\}_{s=t}^{\infty}$. Further, the government features a discount factor β_g that might deviate from the households' discount factor, $\beta_g \leq \beta_w$. The case of $\beta_g < \beta_w$ could be interpreted as corresponding to a situation in which there is a non-zero probability that the government will be removed from office in any future period. The chance of losing office drives the "effective" discount factor of the government below the social discount factor. The government's objective function then takes the form

$$\mathrm{E}_t \sum_{s=t}^{\infty} \beta_g^{s-t} \left[v(C_s) + u\left(G_s\right) \right]. \tag{2.6}$$

When $\beta_g < \beta_w$, the government will tend to frontload government expenditures. This implies excessive borrowing and, hence, a potential role for borrowing constraints. First, a stable equilibrium might not exist in this case, when government debt grows without bounds. Secondly, the equilibrium allocation might be inefficient compared to the case of an unbiased government (or a social planner) that shares the private sector's rate of time preference.

We consider two types of fiscal constraints that are intended to alleviate the incentive for excessive borrowing, namely constraints on public debt and constraints on primary deficits. While the SGP only penalizes excessive deficits, many have argued that a Pact that punishes excessive debt would be preferable. The stock of debt is considered a better (though imperfect) measure of fiscal sustainability. Moreover, one would expect that by putting a constraint on debt, governments are induced to follow prudent fiscal policies also during economic upturns. Below we shall see whether this is indeed the case.

However, such fiscal constraints do not come as a free lunch. Even if the government is not myopic, the constraints may sometimes lead to losses due to the possibility that unexpected, adverse macroeconomic shocks cause the constraints to be violated. In that case, it is simply the macroeconomic uncertainty rather than the opportunistic behavior of the government that gives rise to potential sanctions.

We assume that under a *debt-based* constraint, the government pays a fine if the stock of debt, B_t, exceeds some reference value, B^c, while no fine has to be paid otherwise. The government's period-t budget constraint is then modified into:

$$B_{t+1} = \left(1 + r_t\right) B_t + G_t - T_t + k^B \left(B_t - B^c\right) I_B \left[B_t; B^c\right], \tag{2.7}$$

where the parameter $k^B > 0$ captures the tightness of the constraint and $I_B\left[B_t; B^c\right]$ is an indicator function, such that $I_B\left[B_t; B^c\right] = 1$, if $B_t > B^c$, and $I_B\left[B_t; B^c\right] = 0$, otherwise.

The other possible constraint is expressed in terms of the primary deficit, which is defined as

$$D_{t+1} = B_{t+1} - \left(1 + r_t\right) B_t. \tag{2.8}$$

Under a *primary deficit-based* constraint, the government pays a fine if the primary deficit D_t exceeds some reference value D^c, while it pays no fine otherwise. The government's period-t budget constraint with this type of constraint becomes

$$B_{t+1} = \begin{cases} (1+r_t) B_t + G_t - T_t & if \quad D_t \leq D^c \\ (1+r_t) B_t + G_t - T_t + k^D (D_t - D^c) & if \quad D_t > D^c \end{cases},$$

Using the definition of a primary deficit (2.8) and defining the indicator function $I_D [D_t; D^c]$, such that $I_D [D_t; D^c] = 1$, if $D_t > D^c$, and $I_D [D_t; D^c] = 0$, otherwise, we can rewrite the budget constraint as

$$D_{t+1} = G_t - T_t + k^D (D_t - D^c) I_D [D_t; D^c], \tag{2.9}$$

where the parameter $k^D > 0$ captures the tightness of the constraint when D_t exceeds D^c.

Finally, the government raises its tax revenues by taxing income at the constant and given rate τ^y, such that

$$T_t = \tau^y Y_t, \tag{2.10}$$

Hence, given that taxes are determined in this way, there is one degree of freedom left for the government, which is its choice of $\{G_s\}_{s=t}^{\infty}$. The debt path follows residually.

2.2.3 Equilibrium

We assume that the real interest rate on (foreign and domestic) bonds is exogenously determined and constant at a level r. Given that tax revenues are exogenous, the optimal decisions of the private sector and the government are determined independently of each other. Domestic demand consists of private sector consumption, C_t, and government consumption, G_t. Income can further be invested in internationally traded assets, F_t, such that the resource constraint reads $Y_t = C_t + G_t + F_t - (1+r)F_{t-1} + k_t$, where $k_t = k^B (B_t - B^c) I_B [B_t; B^c]$ or $k_t = k^D (D_t - D^c) I_D [D_t; D^c]$.

Definition 2.1 *For a given endowment sequence $\{Y_s\}_{s=t}^{\infty}$, constant tax rate τ^y, constant interest rate r, and initial values F_t and B_t, a rational expectations equilibrium consists of a set of sequences $\{C_s, G_s, B_s, T_s\}_{s=t}^{\infty}$ satisfying the utility-maximizing plan of the private sector, i.e. (2.3) and the transversality condition $\lim_{s\to\infty} E_t(1+r)^{s-t} B_s = 0$; the plan of the government that maximizes (2.6), subject to (2.7) or (2.9); and the intertemporal resource constraint $F_t = E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (Y_s - C_s - G_s - k_s)$.*

To allow for the existence of a non-explosive equilibrium, we assume that the exogenous interest rate r satisfies $(1+r)\beta_w = 1$. We restrict our attention to rational expectations equilibria that are Markov perfect, such that equilibrium decision rules and, hence, outcomes depend on the current state of the economy only.

2.2.4 Equilibrium under a non-myopic government

To provide a benchmark for the subsequent analysis, we briefly examine the case, in which the government discounts future events at the same rate as households ($\beta_g = \beta_w$) and

$\beta_w\,(1+r) = 1$. The government's problem is then to maximize $\mathrm{E}_t \sum_{s=t}^{\infty} \beta_w^{s-t}\left[v(C_s) + u\,(G_s)\right]$ subject to the budget constraint and given tax revenues, T_s. We note that trades occur sequentially, in contrast to a framework where all trades are contracted in the initial period. The state of the economy is governed by a Markov process. Thus, the government's behavior in any period $s \geq t$ can be described by

$$\mathrm{E}_t u'(G_s) = \beta_w(1+r)\mathrm{E}_t u'(G_{s+1}).$$

Since $\beta_w = 1/(1+r)$, expenditures will therefore satisfy $u'(G_t) = \mathrm{E}_t u'(G_{t+j})$, $\forall\ j \geq 0$, implying a random walk behavior of government expenditures, G.[5] In equilibrium, the household first-order condition (2.3) must further be satisfied, which implies with $\beta_w = 1/(1+r)$ that $v'(C_t) = \mathrm{E}_t v'(C_{t+j})$, $\forall j \geq 0$. Thus, private consumption C also exhibits a unit root. Further, in equilibrium the intertemporal resource constraint, the intertemporal government budget constraint and the household's transversality condition must hold, implying

$$(1+r)B_t = \mathrm{E}_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (T_s - G_s) \tag{2.11}$$

Given these conditions, one can easily solve for the set of equilibrium sequences $\{C_s, G_s, B_s, T_s\}_{s=t}^{\infty}$ given $\{Y_s\}_{s=t}^{\infty}$ and the initial values B_t and F_t (see Appendix 2.A). The state space solution form is linear and given by

$$B_{s+1} = B_s - \tau^y \frac{1-\rho}{1+r-\rho}\left(Y_s - \overline{Y}\right), \tag{2.12}$$

$$G_s = \tau^y Y_s \frac{r}{1+r-\rho} + \tau^y \overline{Y}\frac{1-\rho}{1+r-\rho} - rB_s, \tag{2.13}$$

$$F_{s+1} = F_s + \frac{1-\rho}{1-\rho+r}\left(Y_s - \overline{Y}\right), \tag{2.14}$$

$$C_s = (1-\tau^y)\frac{r}{r-\rho+1}Y_s + (1-\tau^y)\frac{1-\rho}{1+r-\rho}\overline{Y} + r\,(B_s + F_s)\,. \tag{2.15}$$

Due to the unit root(s), the unconditional means are undetermined.

2.3 A myopic government

We maintain the assumption that $\beta_w = 1/\,(1+r)$, but now we allow for $\beta_g < \beta_w$. In this case, the government tends to pre-draw government expenditure and will start borrowing from the private sector. Below we solve the model under our two types of fiscal constraints, which are intended to limit public borrowing.

[5] If $G_t \neq \mathrm{E}_t G_{t+j}$, for $j > 0$, unconditional higher order moments of G_t and G_{t+j} must also differ to satisfy $u'(G_t) = \mathrm{E}_t u'(G_{t+j})$. This would introduce a non-recursive element in equilibrium, such that the state would not follow a Markov-process.

2.3.1 Debt constraints

We start with the case in which the government faces a debt constraint. Under this fiscal constraint, the government maximizes (2.6) subject to (2.7). To simplify the original problem, which exhibits a discontinuity, we approximate the indicator function $I\left[B_s; B^c\right]$ with a transition function, which allows us to apply standard local approximation (perturbation) methods. In particular, we apply the logistic function

$$L_s^B \equiv L\left(B_s; \gamma, B^c\right) = \frac{1}{1+\exp\left(-\gamma\left(B_s - B^c\right)\right)}, \quad \gamma > 0. \tag{2.16}$$

When $\gamma \longrightarrow \infty$, $L\left(B_s; \gamma, B^c\right) \longrightarrow I\left[B_s; B^c\right]$. Hence, for high values of γ, the logistic function will be a good approximation to the indicator function. This alters the intertemporal budget constraint of the government, since there are always fines to be paid (which become negative, but are close to zero when $B_s < B^c$).

$$B_{s+1} = (1+r)\, B_s + G_s - T_s + k^B\left(B_s - B^c\right) L_s^B. \tag{2.17}$$

The problem of the myopic government then reads

$$\max_{\{G_s, B_{s+1}\}} U_g = E_t\left\{\sum_{s=t}^{\infty} \beta_g^{s-t} u\left(G_s\right)\right\} \quad \text{s.t.} \quad (2.17)$$

Given that the approximated problem is now continuous and recursive, the first-order condition for B_{t+1} is given by

$$u'\left(G_t\right) = \beta_g E_t\left\{\left[1 + r + k^B L_{t+1}^B + \gamma k^B\left(B_{t+1} - B^c\right)\left(L_{t+1}^B - \left(L_{t+1}^B\right)^2\right)\right] u'\left(G_{t+1}\right)\right\}. \tag{2.18}$$

We see that the debt constraint affects the "effective" cost of issuing debt by the additional term $k^B L_{t+1}^B + \gamma k^B\left(B_{t+1} - B^c\right)\left(L_{t+1}^B - \left(L_{t+1}^B\right)^2\right)$. The Euler equation contains two unusual elements due to the fiscal constraint. The first element, $k^B L_{t+1}^B$, measures the additional marginal costs of each unit of debt that exceeds the reference value B^c. The second element, $\gamma k^B\left(B_{t+1} - B^c\right)\left(L_{t+1}^B - (L_{t+1}^B)^2\right)$, is the marginal effect of an increase in B_{t+1} on this marginal cost, multiplied by the factor $\left(B_{t+1} - B^c\right)$. In a sense, this term measures how an increase in debt raises the "probability" of hitting the constraint, reflecting the government's internalization of the fiscal constraint.

The set of conditions that characterize the equilibrium sequences $\{G_t, B_t\}_{t=0}^{\infty}$ under the debt constraint are given by (2.17), (2.18), and the terminal condition (see Appendix 2.B)

$$\left(1 + r + k^B L_t^B\right) B_t = \mathrm{E}_t \sum_{s=t}^{\infty} \frac{\tau^y Y_s - G_s + k^B B^c L_s^B}{\prod\limits_{v=t+1}^{s}\left(1 + r + k^B L_v^B\right)}, \tag{2.19}$$

given an initial value B_t.

2.3.2 Deficit constraints

For the case of a primary deficit-based constraint, we apply an analogous approximation to the original indicator function using

$$L_s^D \equiv L\left(D_s;\gamma,D^c\right)=\frac{1}{1+\exp\left(-\gamma\left(D_s-D^c\right)\right)},\quad \gamma>0. \tag{2.20}$$

Rewriting the budget constraint in terms of the deficit, we can summarize the government's problem by

$$\max_{\{G_s D_{s+1}\}} U_g = E_t\left\{\sum_{s=t}^{\infty}\beta_g^{s-t}u\left(G_s\right)\right\} \text{ s.t. } D_{s+1}=G_s-T_s+k^D\left(D_s-D^c\right)L_s^D, \tag{2.21}$$

Using that the problem is continuous and recursive, the government's choice for D_{s+1} is characterized by the following first-order condition

$$u'\left(G_t\right)=\beta_g k^D E_t\left\{\left[L_{t+1}^D+\left(D_{t+1}-D^c\right)\gamma\left(L_{t+1}^D-\left(L_{t+1}^D\right)^2\right)\right]u'\left(G_{t+1}\right)\right\}. \tag{2.22}$$

The set of conditions that characterize the equilibrium sequences $\{G_t,B_t\}_{t=0}^{\infty}$ under the deficit constraint are given by (2.21), (2.22), and the terminal condition (see Appendix 2.C)

$$\left(1+r\right)B_t=\mathrm{E}_t\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\frac{\tau^y Y_s-G_s+k^D D^c L_s^D}{1-k^D L_s^D}, \tag{2.23}$$

given an initial value B_t.

2.3.3 The solution

In order to solve the model when fiscal constraints are imposed and to evaluate welfare departures from the solution under a non-myopic government, we use a perturbation method following Judd (1998), Collard and Juillard (2001), and Schmitt-Grohé and Uribe (2004). For a generic variable X_t, we define (1) its stochastic steady state value as its unconditional expectation: $\widehat{X} \equiv \mathrm{E}[X_t]$, and (2) deviations from a deterministic steady state fixed point, $\overline{X}$, as $\widetilde{X}_t \equiv X_t-\overline{X}$.

The values of government expenditures $\overline{G}$ and debt $\overline{B}$ (deficit $\overline{D}$) in a deterministic steady state under the two fiscal constraints are found by solving the systems formed by the government budget constraint and the Euler equation under the respective constraints (see Appendices 2.D and 2.E). The values $\overline{G}$ and $\overline{B}$ ($\overline{D}$) will generally differ, given the differences in the two systems. Nevertheless, as we shall discuss in Section 2.5, we can always calibrate the parameters k^B and k^D and the reference values under the fiscal constraints, B^c and D^c, so as to obtain the same deterministic steady states under the two constraints.

The solution under a debt-based constraint

For the case where the myopic government faces a debt-based constraint, the solution has to satisfy the set of conditions (2.17), (2.18), and (2.19). In order to solve that system, we apply a second order Taylor expansion at the deterministic steady state. To this end, we postulate the existence of two auxiliary functions that describe the system's evolution as a function of observable state variables $(B_t, Y_t, \sigma_\varepsilon)$ at period t. The general solution form for G_t and B_{t+1} is given by

$$\begin{aligned} G_t &= f\left(B_t, Y_t, \sigma_\varepsilon\right), \qquad (2.24)\\ B_{t+1} &= h\left(B_t, Y_t, \sigma_\varepsilon\right). \qquad (2.25)\end{aligned}$$

Eliminating G_t with (2.17), (2.18) can be written as $0 = E_t[g\left(B_{t+2}, B_{t+1}, B_t, Y_{t+1}, Y_t, \sigma_\varepsilon\right)]$. The latter condition can, by using (2.25) and $B_{t+2} = h\left(h\left(B_t, Y_t, \sigma_\varepsilon\right), \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}, \sigma_\varepsilon\right)$, by rewritten as a function of the state variables $B_t, Y_t, \sigma_\varepsilon, \varepsilon_{t+1}$ and the function $h(\cdot)$:

$$\mathrm{E}_t\left\{g\left(\begin{array}{c} h\left(h\left(B_t, Y_t, \sigma_\varepsilon\right), \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}, \sigma_\varepsilon\right), \\ h\left(B_t, Y_t, \sigma_\varepsilon\right), B_t, \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}, Y_t, \sigma_\varepsilon \end{array}\right)\right\} = 0. \qquad (2.26)$$

We then identify the approximated solution for debt (2.25) by taking first- and second-order Taylor expansions of (2.26). Appendix 2.F describes in detail the derivation of the second-order approximation. The mean, i.e. the unconditional expectation, of debt under the debt constraint will be denoted by $\widehat{B^B}$ and the unconditional variance of B by $\mathrm{Var}(B_t) = \mathrm{E}\left(B_t - \widehat{B^B}\right)^2$.[6] The approximated solution of G_t is then derived using (2.17) and the solution for B_{t+1}. The unconditional expectation of government expenditure under the debt constraint will be denoted by $\widehat{G^B}$ (see Appendix 2.F).

The solution under primary deficit-based constraint

The same procedure is used under a primary deficit-based constraint to solve for sequences for D_t and G_t satisfying the conditions (2.21), (2.22), and (2.23). The general solution form for G_t and D_t is given by

$$\begin{aligned} G_t &= j\left(D_t, Y_t, \sigma_\varepsilon\right), \qquad (2.27)\\ D_{t+1} &= l\left(D_t, Y_t, \sigma_\varepsilon\right), \qquad (2.28)\end{aligned}$$

Eliminating G_t in (2.22) with (2.21), we can write $0 = E_t[i\left(D_{t+2}, D_{t+1}, D_t, Y_{t+1}, Y_t, \sigma_\varepsilon\right)]$. Using (2.28) and $D_{t+2} = l\left(l\left(D_t, Y_t, \sigma_\varepsilon\right), \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}, \sigma_\varepsilon\right)$, we get

$$\mathrm{E}_t\left[i\left(\begin{array}{c} l\left(l\left(D_t, Y_t, \sigma_\varepsilon\right), \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}, \sigma_\varepsilon\right), \\ l\left(D_t, Y_t, \sigma_\varepsilon\right), D_t, \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}, Y_t, \sigma_\varepsilon \end{array}\right)\right] = 0. \qquad (2.29)$$

The approximated solution for the deficit (2.28) is found by taking first- and second-order Taylor expansions of (2.29) (derived analytically in Appendix 2.G). The means of D_t and G_t under the debt constraint will be denoted by $\widehat{D^D}$ and $\widehat{G^D}$.

[6] The variance is obtained by taking unconditional expectation of the square of the first-order approximation of B_{t+1}. See Appendix 2.F for the complete derivation of $\widehat{B^B}$.

2.4 Welfare

To assess the efficiency of different fiscal constraints, we refer to household welfare (2.1). Given that the consumption decision is independent of fiscal policy, we restrict our attention to the welfare effects of government expenditures. Specifically, we apply a second-order Taylor expansion of (2.1) at the deterministic steady state and use the solutions for the mean and the unconditional variance of government expenditures, $\widehat{G}$ and $\mathrm{Var}(G)$, respectively, to compute household welfare under both types of constraints. Our welfare measure is given by[7]:

$$\begin{aligned} U_w &\approx \sum_{t=0}^{\infty} \beta_w^t \left[u\left(\overline{G}\right) + u'\left(\overline{G}\right) * \left(\widehat{G}^z - \overline{G}\right) + \frac{1}{2} u''\left(\overline{G}\right) * \mathrm{Var}\left(G^z\right) \right] \\ &\approx \frac{u\left(\overline{G}\right) + u'\left(\overline{G}\right) * \left(\widehat{G}^z - \overline{G}\right) + \frac{1}{2} u''\left(\overline{G}\right) * \mathrm{Var}\left(G^z\right)}{1-\beta_w}, \end{aligned} \tag{2.30}$$

where z denotes the type of fiscal constraint that is imposed, B (debt-based) or D (primary-deficit based). Based on our welfare measure (2.30) we compare welfare under both constraints and analyze which constraint is preferred from the households' perspective.

2.5 Numerical evaluation

In this section we perform a numerical evaluation of the model and study public debt, government expenditures and welfare under the two different types of fiscal constraints. We also analyze the consequences of changes in both the "deep" parameters and the policy parameters.

2.5.1 Calibration

We calibrate the model using average values for eleven members of the Eurozone, which we denote by "Euro-11".[8] The calibration is based on annual data from the OECD Economic Outlook database (OECD, 2006, n^o. 79) for 1970-2006.

Table 2.1 summarizes the calibration of our benchmark specification. It describes the parameters, lists the values chosen for them and provides the motivation for these choices. For simplicity, we normalize average income, $\overline{Y}$, to 100 and set the income tax rate, τ^y, to 1. This choice is immaterial given that the private sector behavior is not affected by public policy in any case. Initial public debt, B_0, is set at 63.14, which is the average

[7] Notice that, given our hypothesis about partisan government, the intertemporal discount rate of society can differ from that of the government, $\beta_g \leq \beta_w$. Thus, while the government maximizes its expenditure level using its intertemporal discount factor β_g, society's welfare is evaluated using the discount factor β_w (see Appendix 2.H for further details).

[8] These countries are Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, The Netherlands, Portugal and Spain.

debt/GDP ratio over all observations in our panel (the Euro-11 over 1970 to 2006). The initial amount of foreign debt, F_0, is set to 0. The real interest rate, r, is fixed at 0.0262, which is the average ex-post long-term real interest rate over all observations in our panel. The coefficient of income autocorrelation, ρ, is set at 0.264. It is obtained by taking log-deviations of real GDP from its trend for each country,[9] and estimating an AR-1 process on these constructed series for the Euro-11 area over the period 1970-2006. We then take the non-weighted average of the estimated AR-1 coefficients, which provides us with our choice of ρ. The variance of the income shocks, $\sigma_\varepsilon^2 = 38.821$, is the non-weighted average for the Euro-11 of the sum of the squared residuals of the estimated AR-1 income process.

Following Grossman and van Huyck (1988) and Kumhof and Yakadina (2007), the myopic government discount factor, β_g, is expressed as a fraction of the social discount factor β_w:

$$\beta_g = \alpha\beta_w, \qquad 0 < \alpha < 1.$$

As the benchmark value for α we choose 0.933.[10] Further, utility from government expenditure is specified as

$$u\left(G\right) \equiv \frac{G^{1-\frac{1}{\mu}}}{1-\frac{1}{\mu}}, \tag{2.31}$$

where μ is the constant elasticity of intertemporal substitution. For the benchmark parametrization we set μ equal to 0.7.[11] Finally, the smoothness parameter γ of the logistic function is set at 300. This value enables a good approximation of the indicator function in (2.7) and (2.9), as shown in Franses and Van Dijk (2004).

As regards to the policy parameters, we set k^B and k^D, which govern the tightness of the constraints, at very high values ($k^B = k^D = 100$) so that the constraints are almost strictly binding. Then, we search for those reference values (B^c and D^c) for the debt, respectively primary deficit, constraints that yield as the deterministic steady state the level of the public debt that satisfies the intertemporal government budget constraints (2.19) and (2.23), i.e. $\overline{B^B} = \overline{B^D} = B_0 = 63.140$. Specifically, this results in a value of $B^c = 63.144$ under our benchmark parameter setting. For the primary deficit-based constraint, under the baseline parameter setting, the intertemporal government budget constraint (2.23) holds only when the reference value D^c is not higher than -1.649, implying that the reference value corresponds to a surplus.

[9] We use the Hodrick-Prescott filter with smoothing parameter $\lambda = 100$.

[10] The term $1/(1-\alpha)$ can also be interpreted as a measure of the planning horizon of the politician (see Grossman and van Huyck, 1998, and Kumhof and Yakadina, 2007). Our benchmark value of α corresponds to a planning horizon of the government of 15 periods (the same value as in the calibration by Kumhof and Yakadina, 2007).

[11] There is no consensus on the most appropriate value of μ. The value chosen here provides a coefficient of constant relative risk aversion close to 1.43, which falls within the range of values used in the literature. In particular, our benchmark value for μ is close to the value of 1.391 estimated for the Euro Area by Smets and Wouters (2003) and to the value of 1.5 assumed for the US by Ayagari and McGrattan (1998). In our sensitivity analyses we vary μ to check the robustness of our results for different degrees of relative risk aversion $1/\mu$.

2.5.2 Results

We are now ready to discuss our numerical results. First, we present our findings for the benchmark case. Next, we perform a sensitivity analysis in which we investigate how changes in the parameters affect welfare and the dynamics of the economy.

The benchmark case

Stochastic steady state Table 2.2 provides the results for our benchmark calibration, under which the respective intertemporal government budget constraints are strictly satisfied. Conform our earlier definition, bars above variables denote deterministic steady state values, while hats denote the mean of a variable. Further, making use of (2.8), Table 2.2 also reports the value $D^{Bc} = -1.653$ of the primary deficit that corresponds to the reference value B^c, and the level of the debt $B^{Dc} = 62.98$ that corresponds to the reference deficit level D^c. We separate the outcomes in four blocks. The first block presents the outcomes for a non-myopic government (i.e. $\beta_g = \beta_w$). Even though we consider a tax rate of 1 and initial net foreign debt of zero, because the government starts with positive initial public debt, $\overline{G}$ is smaller than $\overline{Y}$ and the steady state level of private consumption is positive ($\overline{C} = 1.65$). The steady state primary surplus $\overline{D}$ is then equal to the value of consumption in absolute terms.

The second and third blocks of Table 2.2 report the outcomes under a myopic government (i.e., $\beta_g < \beta_w$) subject to a debt, respectively primary deficit, constraint.[12] By construction, the deterministic steady states of debt under both types of fiscal constraints ($\overline{B^B}$ and $\overline{B^D}$) are equal to the initial value of the public debt B_0. However, the deterministic steady state government expenditure level is marginally higher (approximately 0.09% of GDP) under the myopic than under the non-myopic government.

This is the result of the approximation of the indicator function by the logistic functions $\overline{L^B}$ and $\overline{L^D}$ in (2.83) and (2.143),[13] which for the benchmark specification are equal to 0.218 and 0.226 respectively. These equilibrium values of the logistic function L are positive and non-negligible (even for a very large γ) because of the marginal effect of the debt (primary deficit) on the logistic approximation $\overline{L^B}$ ($\overline{L^D}$) in the government's Euler equation under the debt constraint (2.18) (under the primary deficit constraint (2.22)). The government takes into account that the tightness of the constraint varies when the debt or the primary deficit are close to their respective reference values. On the one hand, these are not fully binding when they are marginally exceeded. On the other hand, they will already be felt even before they are hit. The government faces a continuous trade-off between higher fines and being able to spend more in the short run. The trade-off is reflected by the additional non-zero terms $-k^B\left(\overline{B} - B^c\right)\overline{L^B}$ and $-k^D\left(\overline{D} - D^c\right)\overline{L^D}$ in the government's budget constraint when determining the solutions for $\overline{G^B}$ and $\overline{G^D}$, respectively. These

[12] With the benchmark value of α and the value of β_w given in Table 2.2, the myopic government discount factor becomes $\beta_g = 0.91$.

[13] The respective expressions are given in (2.83) and (2.143) in the Appendix.

additional terms are positive, implying steady state levels of government spending under the myopic government that slightly exceed those under the non-myopic government.[14]

Besides comparing steady state outcomes, we also investigate the responses of our variables to an income shock (i.e., we study the transition path to the steady state after an income shock). For the non-myopic government we do this using the benchmark specification and the system of linear dynamic equations (2.12) to (2.15). For the myopic government, we employ our second-order approximations.

We calculate the impulse responses to a transitory negative income shock of one percent of GDP in period t ($\sigma_\varepsilon \varepsilon_t = -1$), assuming that the economy is initially in its deterministic steady state and assuming no further shocks. Hence, $Y_t = 99$. The impulse responses over 10 periods for the non-myopic government, the myopic government under a debt constraint, and the myopic government under a primary deficit constraint are displayed in Figures 2.1, 2.2 and 2.3, respectively. In those figures, we plot the dynamics of income, public debt, government expenditure and the primary deficit. The dashed lines represent the deterministic steady states of the variables, whereas the solid lines show the impulse responses to the income shock. For the case of a non-myopic government we plot in addition the dynamics of the net foreign debt and of private consumption.

Under the non-myopic government (Figure 2.1), the negative income shock causes an immediate fall in the level of government expenditure in period t, which remains constant from then onwards. The fall in government spending is smaller than the initial drop in tax revenues and additional debt is accumulated until it reaches a new steady state level. This debt build up is consistent with the temporary decline in the primary surplus. However, the primary surplus converges to a new steady state level that is higher than before, consistent with government solvency (2.11). Private consumption remains constant throughout at its original steady state level. That is because the government and households can freely borrow or lend on the international capital market at a given interest rate, so that C_t is chosen independently of G_t to satisfy the household's first-order conditions. Therefore, the additional issuance of government bonds due to the shock in tax revenues combined with the constant level of private consumption implies an accumulation of foreign debt (negative impulse response of F_t in Figure 2.1).[15]

For the myopic government facing a debt constraint (Figure 2.2), the negative income shock leads to a sharper decrease in government expenditure in period t than for the non-myopic government. Under the debt constraint, the government abstains from excessive borrowing and will rather reduce its expenditures to meet the budget constraint. Over time, tax revenue and government expenditure converge back to the stochastic steady state level, which deviates only marginally from the initial, deterministic steady state. Hence, in the long run public debt and the primary deficit are almost unaffected by the

[14] Under the benchmark calibration the two terms are virtually equal (0.0928) and, therefore, neutral in terms of the difference between $\overline{G^B}$ and $\overline{G^D}$.

[15] Since all private income is taxed away and taxes fall below government spending in the short run, foreign debt must rise.

shock.

For the myopic government under the primary deficit constraint (Figure 2.3), the negative income shock leads to a pattern of government expenditure that is similar to that under the debt constraint but with a marginally stronger contraction in period t. The primary deficit remains virtually constant and close to D^c.

The results described so far provide the benchmark for the sensitivity analysis that we perform next. The benchmark setting is virtually neutral in terms of welfare for the two types of fiscal constraint as we can observe from the fourth block of Table 2.2. There, we also report social welfare U_w^B and U_w^D associated with public spending under the debt, respectively primary-deficit constraint, as well as the difference $W = U_w^B - U_w^D$. Because the difference in utility levels is uninformative about the true size of the welfare impact of different fiscal constraints, we also express it in terms of the permanent difference in government spending, G_{dif}, that generates this utility difference. To this end, we use the inverse function of (2.31) to compute the permanent constant government spending streams corresponding to U_w^B and U_w^D, respectively, and then take the difference between these two streams.[16] This results in a very small permanent spending difference $G_{dif} = 8.864E-05$. The final line of Table 2.2 reports the difference $Err_{dif} = k^D\left(\overline{D} - D^c\right)\overline{L^D} - k^B\left(\overline{B} - B^c\right)\overline{L^B}$ in the approximation error, which is virtually zero.

Sensitivity analysis

First, we investigate how changes in the deep parameters affect welfare and the impulse responses to income shocks under the two types of fiscal constraints. We adjust B^c, D^c, and k^D to keep the deterministic steady state of debt, $\overline{B}$, and the approximation error, Err_{dif}, always equal to their benchmark values.[17]

The government's discount factor (β_g) Because the introduction of our fiscal constraints is motivated by government myopia and the consequent lack of fiscal discipline, we start the sensitivity analysis by investigating the consequences of an increase in the degree of government myopia (i.e., β_g falls), while keeping all the other parameters at their benchmark values.

Table 2.3 reports the outcomes under the two fiscal constraints for different values of β_g. Under a debt constraint, if the government becomes more myopic, government spending $\widehat{G^B}$ rises, while the public debt $\widehat{B^B}$ falls. Also under the primary deficit constraint, government spending $\widehat{G^D}$ rises as β_g falls. However, public debt $\widehat{B^D}$ now increases. This

[16] More precisely, we compute the permanent government spending streams G_w^B and G_w^D from $\left(G_w^B\right)^{1-\frac{1}{\mu}} / \left[\left(1-\frac{1}{\mu}\right)(1-\beta_w)\right] = U_w^B$, respectively $\left(G_w^D\right)^{1-\frac{1}{\mu}} / \left[\left(1-\frac{1}{\mu}\right)(1-\beta_w)\right] = U_w^D$. Then, $G_{dif} = G_w^B - G_w^D$. For similar transformations of utility differences into permanent consumption equivalents, see e.g. Jensen (2002) and Beetsma and Jensen (2005).

[17] By adjusting B^c, D^c, and k^D, we keep Err_{dif} constant so that the difference in welfare under the two fiscal constraints is not affected by changes in the approximation errors when the deep parameters are varied.

occurs in spite of the tightening of the penalty parameter k^D needed to keep Err_{dif} constant. Otherwise, $\widehat{B^D}$ would increase even more as β_g falls, implying lower government spending. The intuition behind these outcomes is the following. Consider first the primary deficit constraint. More myopia provides the government with a stronger incentive to shift spending from the future to the present, implying a higher primary deficit for given reference value for the primary deficit and, hence, more borrowing. This implies that *in the steady state* debt and the primary surplus will be higher. Under a debt constraint a similar mechanism implies an increase in the steady state level of the debt. However, there is a mechanism pushing steady state debt in the opposite direction. With a direct constraint on the debt, a more myopic government has a stronger incentive to limit indebtedness to protect itself against bad income shocks that would force it to cut current spending in the very short run. Hence, a more myopic government engages in more precautionary saving. Notice that welfare in terms of permanent government consumption rises more under the debt- than under the primary deficit constraint (G_{dif} rises for a constant Err_{dif}) as we make the government more short-sighted.

The outcomes for this case can be summarized as:

Result 2.1 *Ceteris paribus, an increase in government myopia (a fall in β_g) leads to a fall (rise) in the stochastic steady state level of the public debt under a debt (primary deficit) constraint. Welfare in terms of permanent government consumption under a debt constraint rises relative to welfare under a primary-deficit constraint.*

The interest rate (r) Next, we investigate the welfare consequences of a change in the interest rate r. Changes in the interest rate affect the steady state outcomes under both the non-myopic and myopic government.

Table 2.4 reports the outcomes. In all cases, the lower is the interest rate, the lower is the steady state level of consumption $\overline{C}$ and higher is the steady state of government spending $\overline{G}$. A lower interest rate reduces interest income earned by the private sector on its assets, which thus reduces its private consumption. Lower interest spending also allows the government to have a higher amount of debt outstanding without violating the public budget constraint. Hence, the mean of debt under both constraints, $\widehat{B^B}$ and $\widehat{B^D}$, increases when r is smaller.

In order to keep the deterministic steady state value of the public debt at B_0 when the interest rate falls, the reference value D^c under the primary deficit constraint has to rise substantially (i.e., amount to a lower primary surplus), while the reference value B^c under the debt constraint needs to change only slightly. To see the intuition, notice that the deterministic steady state debt and primary deficit are linked as $\overline{B} = -(1/r)\,\overline{D}$ ($-\overline{D}$ is the constant primary surplus that pays off an initial debt $\overline{B}$). A given fall in r thus requires a relatively large increase in $\overline{D}$ to maintain $\overline{B} = B_0$. In turn, this rather large shift in $\overline{D}$ translates into a large shift in the reference value D^c.

Result 2.2 *In contrast to the reference value B^c under the debt-based constraint, the reference value D^c under the primary deficit-based constraint is relatively sensitive to changes in the interest rate. Further, a lower interest rate reduces the social welfare difference in terms of permanent government consumption under the debt based constraint relative to that under the primary-deficit based constraint.*

Because making frequent and large changes in reference values of fiscal constraints may be politically problematic (every change in the fiscal constraint is likely to re-open a political debate on what should be the appropriate design of the constraint).[18] Result 2.2 suggests that with frequent and large changes in the interest rate, a debt-based fiscal constraint becomes more attractive relative to a deficit-based constraint. In addition, Table 2.4 shows that as the interest rate falls, the relative disadvantage of the deficit-based constraint decreases. A lower interest rate reduces the cost of debt service, which, in particular, reduces the penalty parameter k^D associated with the primary deficit constraint that is needed to keep the steady state debt level and Err_{dif} constant.

The variance (σ_ε^2) and persistence (ρ) of the income shock To see how the relative desirability of the two fiscal constraints depends on the volatility of the business cycle, we vary σ_ε^2, keeping all other parameters at their benchmark values. Table 2.5 shows the outcomes of this numerical variation. Of course, changes in the variance of the income shock do not affect the deterministic steady state outcomes under both fiscal constraints. However, they do affect the stochastic steady state.

Under the debt constraint, a rise in uncertainty induces the government to reduce debt on average, thereby being able to absorb shocks that have become larger on average without hitting the reference debt level. By contrast, under the primary deficit constraint, a higher σ_ε^2 implies an increase in steady state public debt. The reason is that the latter corresponds to a higher stochastic steady state surplus, which is the result of a desire to maintain a larger safety margin relative to its reference level. Table 2.5 further shows that welfare falls as a result of the higher ex ante uncertainty about the available resources. The rise in the average surplus for higher σ_ε^2 reduces steady state government expenditure under the primary deficit constraint more than under the debt constraint.

Table 2.6 shows that welfare falls with the persistence of the income process. Higher income persistence implies a larger unconditional variance of income, for given σ_ε^2. The intuitions for the consequences of a rise in ρ are, therefore, very similar to the intuitions behind the effects of an increase in σ_ε^2.

Result 2.3 *Ceteris paribus, both an increase in income uncertainty (a higher σ_ε^2) and an increase in income persistence (a higher ρ), lower (raise) the expected debt level under the debt (primary-deficit) constraint. Under both constraints, social welfare falls in terms of*

[18]For a more formal discussion of the political issues involved in the implementation of fiscal constraints, see Krogstrup and Wyplosz (2006), Debrun and Kumar (2007), and Ribeiro and Beetsma (forthcoming).

permanent government consumption equivalents, but this fall is relatively larger under a primary deficit-based constraint than under a debt-based constraint.

The elasticity of intertemporal substitution (μ): Table 2.7 shows that the average debt level $\widehat{B^B}$ under the debt constraint increases when the relative risk aversion $1/\mu$ falls. The myopic government then prefers to keep a smaller margin relative to the reference debt level, because it is less bothered by the higher likelihood that the debt constraint becomes binding and that it has to implement an unexpected spending cut. Obviously, the higher average debt level is accompanied by lower average public spending. By contrast, the average debt level $\widehat{B^D}$ under the primary-deficit constraint falls as risk aversion falls, the reason being that the government now prefers to maintain a smaller safety margin to the reference deficit level. Hence, the average primary surplus falls, thereby also supporting a lower average debt level. The results and intuitions behind the increase in μ are very similar to those behind a fall in σ_ε^2. They are summarized as:

Result 2.4 *Ceteris paribus, a fall in relative risk aversion (a rise in μ), raises (lowers) the average public debt level under the debt (primary-deficit) constraint. As a result the difference between average public spending under the debt constraint and that under the primary-deficit constraint shrinks.*

Notice that changes in μ change the utility function itself, which makes it impossible to make a comparison of the change in the welfare difference as we increase μ.

2.6 Conclusion

This paper has compared primary-deficit constraints with debt constraints in a simple macroeconomic model with myopic governments. In our framework and based on our calibration, debt constraints yield higher welfare than primary deficit-based constraints. This finding is reinforced when (i) government myopia is stronger, (ii) the interest rate is higher and (iii) the income shocks have higher variances and are more persistent. Moreover, we also find that the debt constraint is more likely to be politically feasible.

Our analysis is of particular relevance for the design of the Stability and Growth Pact, which was reformed in July 2005. Our results support the shift from the narrow focus on the public deficit to the enhanced emphasis on the public debt in the Pact.

Our analysis also opens several avenues for further research. Throughout the paper we have assumed an exogenous and constant interest rate. Hence, neither the possibility of a myopic government, nor the presence of fiscal constraints, affect the risk premium on the interest rate as we would expect to observe it in reality. An interesting extension would be to allow for an endogenous interest rate in the presence of debt default risk. We have also not included public capital in our model. This would allow us to study golden rules as a third alternative fiscal constraint.

2.7 Tables and figures

Table 2.1: Calibration of the model

Par.	Value	Description and computation
Deep parameters		
$\overline{Y}$	100	*Average Income*: normalized value.
τ^y	1	*Income tax*: equal to 1 to focus on the utility of the public good.
B_0	63.14	*Initial domestic debt*: average $B_0/\overline{Y}$ for Euro-11 countries between 1970-2006.
F_0	0	*Initial foreign debt*: calibrated to zero.
r	0.02617585	*Long-term real interest rate*: average real interest rate for Euro-11 panel.
ρ	0.264	*Income autocorrelation*: average coefficient of an AR-1 estimation of log-deviations of the income trend for the Euro-11 countries between 1970-2006.
σ_ϵ^2	38.821	*Variance of the income shock*: average (over all observations) of the squared residuals of the above income regression.
α	0.933	Parameter that determines the ratio between the myopic government discount factor β_g and the social discount factor β_w, so that $\beta_g = \alpha\beta_w$.
μ	0.7	*Constant elasticity of intertemporal substitution*: provides a coefficient of relative risk aversion close to Ayagari and McGrattan (1998) and Smets and Wouters (2003).
γ	300	*Smoothness of the logistic function*: value that sufficiently approximates the indicator function.
Policy parameters		
k^B	100	*Severity of the debt-based sanction*: implies a very tight restriction.
k^D	100	*Severity of the primary deficit-based sanction*: implies a very tight restriction.
B^c	63.14425	*Reference value in the debt constraint*: provides a deterministic steady state of debt equal to B_0 and satisfies the transversality condition (tvc).
D^c	-1.64864435	*Reference value in the primary deficit constraint*: provides a deterministic steady state of debt equal to B_0 and satisfies the tvc.

Source: OECD (2006) and own calculations.

Table 2.2: Results of the model with benchmark calibration

Variables	Result	Description
Non-Myopic Government		
β_w	0.974492	Social discount factor.
$\overline{G}$	98.347257	Deterministic steady state of government spending.
$\overline{C}$	1.652743	Deterministic steady state of private consumption.
$\overline{D}$	-1.652743	Deterministic steady state of primary deficit.
Myopic Government with Debt Constraint		
B^c	63.144250	Reference value of debt constraint.
D^{Bc}	-1.652854	Primary deficit corresponding to reference value of debt under debt constraint.
$\overline{G^B}$	98.440078	Deterministic steady state of government spending.
$\overline{D^B}$	-1.652743	Deterministic steady state of primary deficit.
$\overline{B^B}$	63.140000	Deterministic steady state of public debt.
$\widehat{G^B}$	98.440082	Stochastic steady state of government spending.
$\widehat{D^B}$	-1.652743	Stochastic steady state of primary deficit.
$\widehat{B^B}$	63.139955	Stochastic steady state of public debt.
Myopic Government with Primary Deficit Constraint		
B^{Dc}	62.983415	Debt level corresponding to reference value of primary deficit.
D^c	-1.648644	Reference value of primary deficit constraint.
$\overline{G^D}$	98.439990	Deterministic steady state of government spending.
$\overline{D^D}$	-1.652743	Deterministic steady state of primary deficit.
$\overline{B^D}$	63.140000	Deterministic steady state of public debt.
$\widehat{G^D}$	98.439994	Stochastic steady state of government spending.
$\widehat{D^D}$	-1.652785	Stochastic steady state of primary deficit.
$\widehat{B^D}$	63.141595	Stochastic steady state of public debt.
Welfare		
U_w^B	-12.8130749	Welfare (utility of government spending) with debt constraint.
U_w^D	-12.81307986	Welfare (utility of government spending) with primary deficit constraint.
W	4.960E-06	Welfare difference under the two types of constraints ($U_w^B - U_w^D$).
G_{dif}	8.864E-05	Difference between the level of public consumption that provides U_w^B and U_w^D.
Err_{dif}	8.808E-05	Difference in the "approximation error" under the debt and primary deficit rule.

Table 2.3: Results of the model with different β_g

Variables[a,b]	$\alpha = 0.667$	$\alpha = 0.833$	$\alpha = 0.889$	$\alpha = 0.916$	Benchmark
β_g *	0.65	0.81	0.86	0.89	0.91
B^c	63.144185	63.144230	63.144241	63.144247	63.144250
D^{Bc}	-1.652853	-1.652854	-1.652854	-1.652854	-1.652854
k^B	100.000000	100.000000	100.000000	100.000000	100.000000
$\overline{G^B}$	98.440059	98.440075	98.440077	98.440078	98.440078
$\overline{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^B}$	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^B}$	98.440090	98.440086	98.440084	98.440083	98.440082
$\widehat{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\widehat{B^B}$	63.139940	63.139951	63.139954	63.139955	63.139955
B^{Dc}	62.985790	62.984154	62.983764	62.983550	62.983415
D^c	-1.648707	-1.648664	-1.648653	-1.648648	-1.648644
k^D	100.067622	100.021245	100.010050	100.003897	100.000000
$\overline{G^D}$	98.439971	98.439987	98.439989	98.439990	98.439990
$\overline{D^D}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^D}$	63.140000	63.140000	63.140000	63.140001	63.140000
$\widehat{G^D}$	98.440001	98.439997	98.439996	98.439994	98.439994
$\widehat{D^D}$	-1.652800	-1.652790	-1.652787	-1.652786	-1.652785
$\widehat{B^D}$	63.142179	63.141778	63.141682	63.141629	63.141595
U_w^B	-12.813074	-12.813075	-12.813075	-12.813075	-12.813075
U_w^D	-12.813079	-12.813080	-12.813080	-12.813080	-12.813080
W	5.046E-06	4.982E-06	4.969E-06	4.963E-06	4.960E-06
G_{dif}	9.018E-05	8.903E-05	8.880E-05	8.870E-05	8.864E-05
Err_{dif}	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05

Notes: [a] The variables of the first column are defined as in Table 2.2. [b] In the other columns, in addition, we vary the value of β_g while keeping the other parameters at their benchmark values indicated in Table 2.1. * This line indicates the value of the myopic government discount factor, which is computed via $\beta_g = \alpha\beta_w$.

Table 2.4: Results of the model with different interest rates (r)

Var.[a,b]	$r = 0.002$	$r = 0.014$	Benchmark	$r = 0.041$	$r = 0.056$	$r = 0.071$	$r = 0.09$
β_w	0.998004	0.986193	0.974492	0.960615	0.946970	0.933707	0.917431
$\overline{G}$	99.873720	99.116040	98.347257	97.411260	96.464160	95.517060	94.317400
$\overline{C}$	0.126280	0.883960	1.652743	2.588740	3.535840	4.482940	5.682600
$\overline{D}$	-0.126280	-0.883960	-1.652743	-2.588740	-3.535840	-4.482940	-5.682600
B^c	63.144247	63.144249	63.144250	63.144253	63.144255	63.144257	63.144260
D^{Bc}	-0.126288	-0.884019	-1.652854	-2.588914	-3.536078	-4.483242	-5.682983
k^B	100.000000	100.000000	100.000000	100.000000	100.000000	100.000000	100.000000
$\overline{G^B}$	99.966541	99.208861	98.440078	97.504081	96.556981	95.609881	94.410222
$\overline{D^B}$	-0.126280	-0.883960	-1.652743	-2.588740	-3.535840	-4.482940	-5.682600
$\overline{B^B}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^B}$	99.966545	99.208865	98.440082	97.504085	96.556985	95.609886	94.410226
$\widehat{D^B}$	-0.126280	-0.883960	-1.652743	-2.588740	-3.535840	-4.482940	-5.682600
$\widehat{B^B}$	63.139957	63.139957	63.139955	63.139955	63.139954	63.139953	63.139952
B^{Dc}	61.090621	62.847231	62.983415	63.040030	63.066808	63.082271	63.094458
D^c	-0.122181	-0.879861	-1.648644	-2.584641	-3.531741	-4.478841	-5.678501
k^D	99.999664	99.999843	100.000000	100.000159	100.000283	100.000370	100.000428
$\overline{G^D}$	99.966453	99.208773	98.439990	97.503993	96.556893	95.609793	94.410133
$\overline{D^D}$	-0.126280	-0.883960	-1.652743	-2.588740	-3.535840	-4.482940	-5.682600
$\overline{B^D}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^D}$	99.966456	99.208777	98.439994	97.503997	96.556897	95.609797	94.410137
$\widehat{D^D}$	-0.12632	-0.88400	-1.65278	-2.58878	-3.53588	-4.48298	-5.68265
$\widehat{B^D}$	63.16024	63.14294	63.14159	63.14104	63.14077	63.14062	63.14050
U_w^B	-162.66298	-23.59313	-12.81307	-8.33275	-6.21478	-4.99261	-4.03038
U_w^D	-162.66304	-23.59314	-12.81308	-8.33276	-6.21478	-4.99261	-4.03038
W	6.197E-05	9.058E-06	4.960E-06	3.257E-06	2.454E-06	1.991E-06	1.628E-06
G_{dif}	8.860E-05	8.861E-05	8.864E-05	8.864E-05	8.867E-05	8.868E-05	8.869E-05
Err_{dif}	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05

Notes: [a] The variables in the first column are defined in Table 2.2. [b] In the other lines, in addition, we vary the value of the interest rate (r) while keeping the other parameters at their benchmark values indicated in Table 2.1.

Table 2.5: Results of the model with different σ_ϵ^2

Var.[a,b]	$\sigma_\epsilon^2=1$	$\sigma_\epsilon^2=12$	$\sigma_\epsilon^2=24$	Benchmark	$\sigma_\epsilon^2=50$	$\sigma_\epsilon^2=75$	$\sigma_\epsilon^2=100$
B^c	63.144250	63.144250	63.144250	63.144250	63.144250	63.144250	63.144250
D^{Bc}	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854
k^B	100.000000	100.000000	100.000000	100.000000	100.000000	100.000000	100.000000
$\overline{G^B}$	98.440078	98.440078	98.440078	98.440078	98.440078	98.440078	98.440078
$\overline{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^B}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^B}$	98.440078	98.440078	98.440079	98.440082	98.440085	98.440094	98.440106
$\widehat{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\widehat{B^B}$	63.140000	63.139996	63.139983	63.139955	63.139926	63.139833	63.139702
B^{Dc}	62.983415	62.983415	62.983415	62.983415	62.983415	62.983415	62.983415
D^c	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644
k^D	100.000000	100.000000	100.000000	100.000000	100.000000	100.000000	100.000000
$\overline{G^D}$	98.439990	98.439990	98.439990	98.439990	98.439990	98.439990	98.439990
$\overline{D^D}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^D}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^D}$	98.439990	98.439990	98.439991	98.439994	98.439996	98.440005	98.440017
$\widehat{D^D}$	-1.652743	-1.652747	-1.652759	-1.652785	-1.652813	-1.652900	-1.653023
$\widehat{B^D}$	63.140000	63.140146	63.140602	63.141595	63.142657	63.146008	63.150706
U_w^B	-12.796641	-12.801421	-12.806635	-12.813075	-12.817932	-12.828795	-12.839657
U_w^D	-12.796646	-12.801426	-12.806640	-12.813080	-12.817937	-12.828800	-12.839662
W	4.909E-06	4.921E-06	4.937E-06	4.960E-06	4.980E-06	5.034E-06	5.100E-06
G_{dif}	8.811E-05	8.821E-05	8.837E-05	8.864E-05	8.889E-05	8.960E-05	9.052E-05
Err_{dif}	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05

Notes: [a] The variables of the first column are defined as in Table 2.2. [b] In the other columns, in addition, we vary the value of σ_ϵ^2 while keeping the other parameters at their benchmark values indicated in Table 2.1.

Table 2.6: Results of the model with different ρ

Var.[a,b]	$\rho = 0.01$	Benchmark	ρ= 0.4	$\rho = 0.6$	$\rho = 0.8$	$\rho = 0.99$
B^c	63.144250	63.144250	63.144250	63.144250	63.144250	63.144250
D^{Bc}	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854
k^B	100.00000	100.00000	100.00000	100.00000	100.00000	100.00000
$\overline{G^B}$	98.440078	98.440078	98.440078	98.440078	98.440078	98.440078
$\overline{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^B}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^B}$	98.440082	98.440082	98.440082	98.440082	98.440082	98.440082
$\widehat{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\widehat{B^B}$	63.1399555	63.1399552	63.1399551	63.1399550	63.1399549	63.1399549
B^{Dc}	62.983415	62.983415	62.983415	62.983415	62.983415	62.983415
D^c	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644
k^D	100.00000	100.00000	100.00000	100.00000	100.00000	100.00000
$\overline{G^D}$	98.439990	98.439990	98.439990	98.439990	98.439990	98.439990
$\overline{D^D}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^D}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^D}$	98.439993	98.439994	98.439994	98.439994	98.439994	98.439994
$\widehat{D^D}$	-1.652785	-1.652785	-1.652785	-1.652785	-1.652785	-1.652785
$\widehat{B^D}$	63.141586	63.141595	63.141598	63.141603	63.141606	63.141608
U_w^B	-12.811901	-12.813075	-12.814888	-12.820726	-12.839797	-13.584787
U_w^D	-12.811906	-12.813080	-12.814893	-12.820731	-12.839803	-13.584794
W	4.958E-06	4.960E-06	4.964E-06	4.976E-06	5.017E-06	6.636E-06
G_{dif}	8.864E-05	8.864E-05	8.866E-05	8.875E-05	8.904E-05	9.759E-05
Err_{dif}	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05

Notes: [a] The variables of the first column are defined as in Table 2.2. [b] In the other columns, in addition, we vary the value of ρ while keeping the other parameters at their benchmark values indicated in Table 2.1.

Table 2.7: Results of the model with different μ

Var.[a,b]	$\mu = 0.3$	$\mu = 0.5$	Benchmark	$\mu = 1.01$	$\mu = 2$	$\mu = 3$	$\mu = 4$
B^c	63.144250	63.144250	63.144250	63.144250	63.144250	63.144250	63.144250
D^{Bc}	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854	-1.652854
k^B	100.00000	100.00000	100.00000	100.00000	100.00000	100.00000	100.00000
$\overline{G^B}$	98.440078	98.440078	98.440078	98.440078	98.440078	98.440078	98.440078
$\overline{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^B}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^B}$	98.440094	98.440085	98.440082	98.440080	98.440079	98.440079	98.440078
$\widehat{D^B}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\widehat{B^B}$	63.139815	63.139923	63.139955	63.139974	63.139990	63.139994	63.139996
B^{Dc}	62.983415	62.983415	62.983415	62.983415	62.983415	62.983415	62.983415
D^c	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644	-1.648644
k^D	100.00000	100.00000	100.00000	100.00000	100.00000	100.00000	100.00000
$\overline{G^D}$	98.439990	98.439990	98.439990	98.439990	98.439990	98.439990	98.439990
$\overline{D^D}$	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743	-1.652743
$\overline{B^D}$	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000	63.140000
$\widehat{G^D}$	98.440005	98.439996	98.439994	98.439992	98.439991	98.439990	98.439990
$\widehat{D^D}$	-1.652917	-1.652815	-1.652785	-1.652767	-1.652752	-1.652748	-1.652747
$\widehat{B^D}$	63.146654	63.142761	63.141595	63.140905	63.140344	63.140204	63.140143
U_w^B	-0.000382	-0.399959	-12.813075	4143.502	777.505	1253.100	1632.914
U_w^D	-0.000382	-0.399959	-12.813080	4143.502	777.505	1253.099	1632.913
W	8.189E-10	3.629E-07	4.960E-06	3.694E-05	3.489E-04	7.490E-04	1.098E-03
G_{dif}	8.985E-05	8.895E-05	8.864E-05	8.844E-05	8.825E-05	8.820E-05	8.817E-05
Err_{dif}	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05	8.808E-05

Notes: [a] The variables of the first column are defined as in Table 2.2. [b] In the other columns, in addition, we vary the value of μ while keeping the other parameters at their benchmark values indicated in Table 2.1.

Figure 2.1: Impulse responses for non-myopic government with benchmark parameters

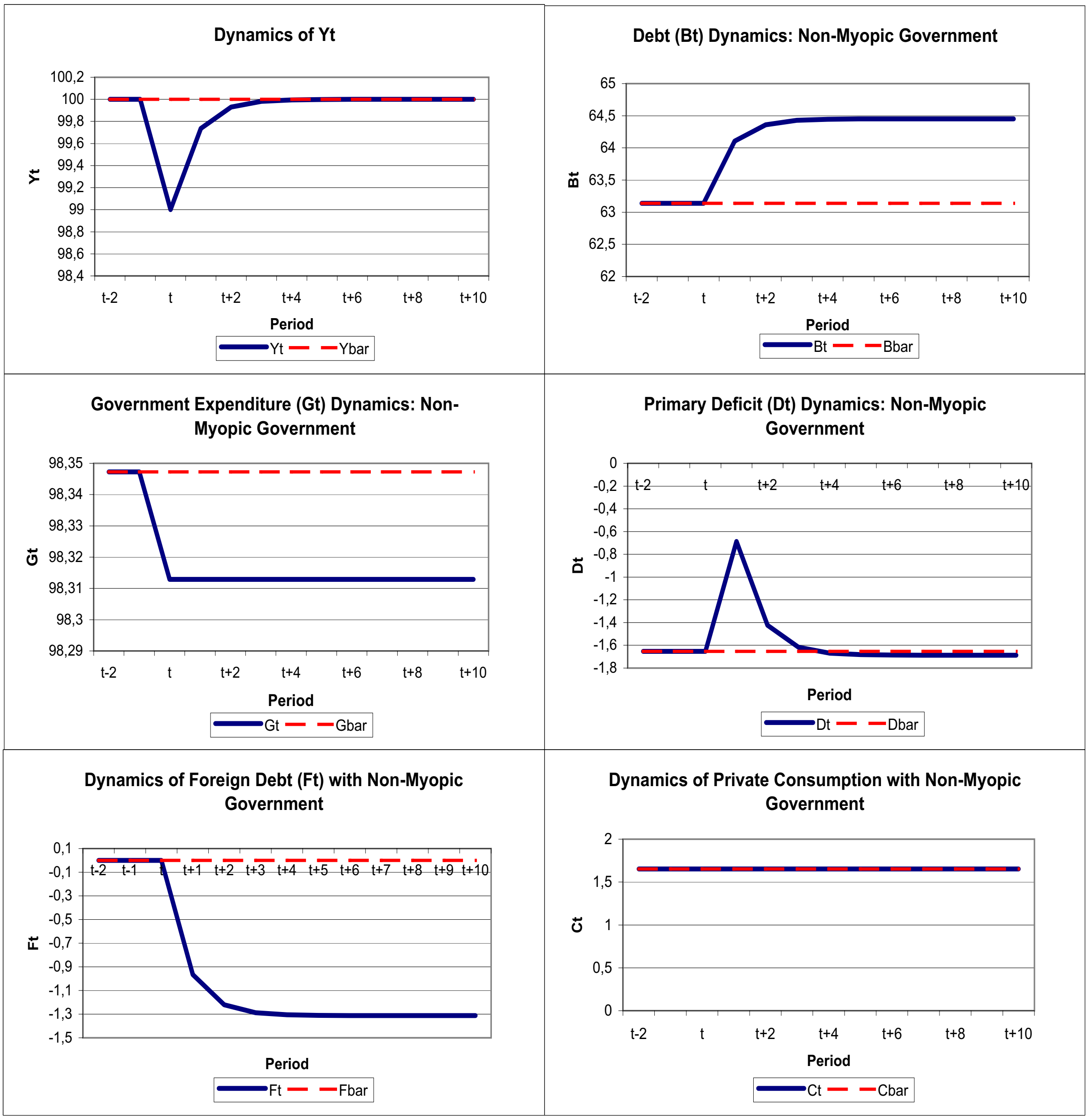


Figure 2.2: Impulse responses for myopic government under debt constraint with benchmark parameters

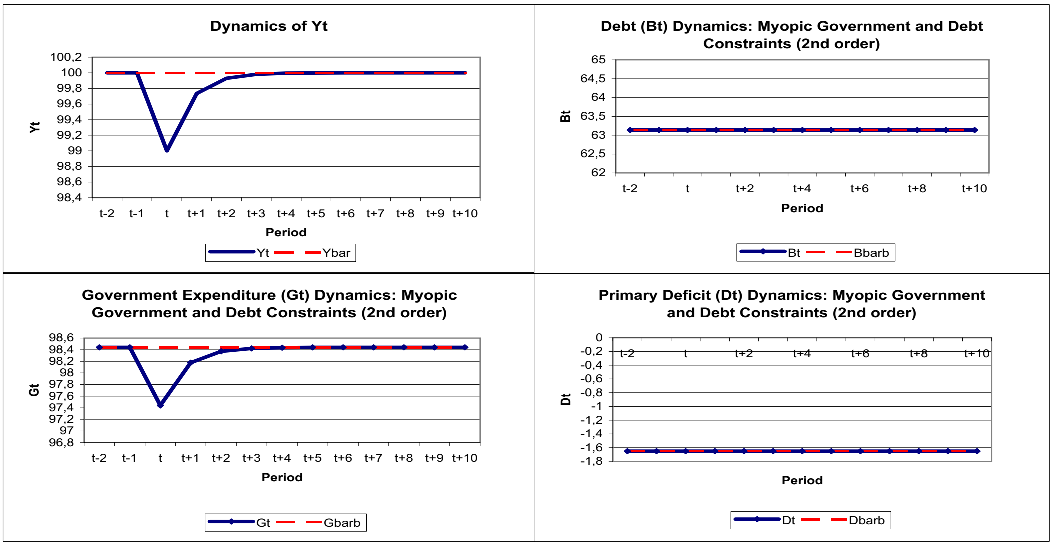
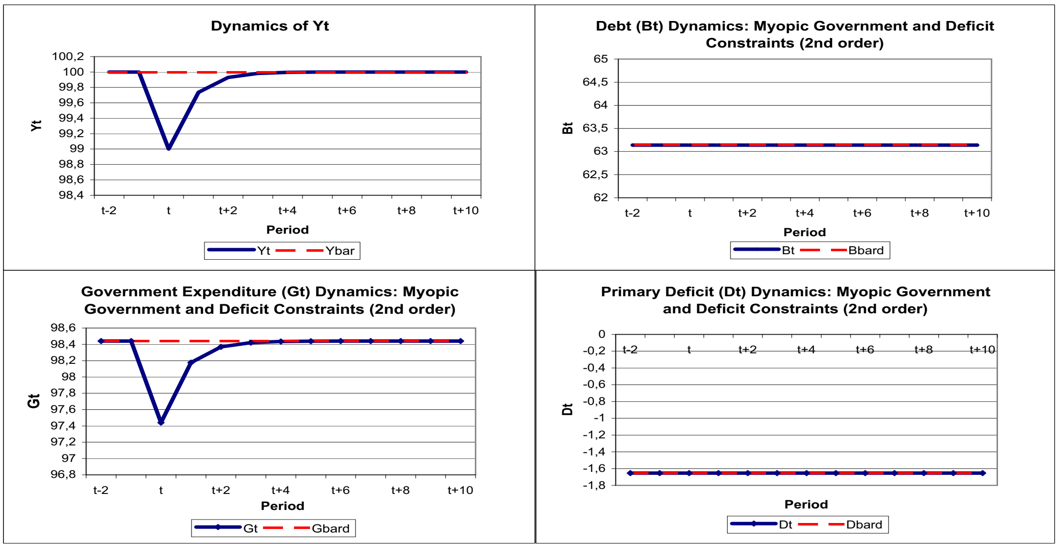


Figure 2.3: Impulse responses for myopic government under primary deficit constraint with benchmark parameters



Appendices to Chapter 2

2.A Solution for the non-myopic government

For a given endowment sequence $\{Y_t\}_{t=0}^{\infty}$, constant tax rate τ^y, and initial values F_0 and B_0, a rational expectations equilibrium under a non-myopic government consists of a set of sequences $\{C_t, G_t, B_{t+1}, F_{t+1}\}_{t=0}^{\infty}$ satisfying

$$\begin{aligned} u'(G_t) &= \mathrm{E}_t u'(G_{t+1}), \\ (1+r)B_t &= \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (\tau^y Y_s - G_s), \\ v'(C_t) &= \mathrm{E}_t v'(C_{t+1}), \\ -(1+r)F_t &= \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (Y_s - C_s - G_s). \end{aligned}$$

The first two conditions feature two unknowns, G_t and B_{t+1}. Hence, one can in principle separately solve for G_t and B_{t+1} and subsequently for C_t and F_{t+1}, using the last two conditions and the solution for G_t.

Since we restrict the equilibrium to be recursive, we know that the condition $u'(G_t) = \mathrm{E}_t u'(G_{t+1})$ implies that the conditional expectations of $E_t G_{t+j}$ are constant for all $j \geq 0$ and thus equal to G_t, $E_t G_{t+j} = G_t, \forall j \geq 0$. Similarly, $v'(C_t) = \mathrm{E}_t v'(C_{t+1})$, implies that $E_t C_{t+j} = C_t$, $\forall j \geq 0$.

We, firstly, aim to derive solutions for G_t and B_{t+1}, i.e., to identify the state space representation of G_t and B_{t+1}. We start with the subset of equilibrium conditions

$$\begin{aligned} u'(G_t) &= \mathrm{E}_t u'(G_{t+1}) \\ (1+r)B_t &= \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (\tau^y Y_s - G_s), \end{aligned}$$

given B_0. Taking expectations conditional on the information in period t, the intertemporal budget constraint implies

$$\mathrm{E}_t \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} G_s = \tau^y Y_t + \mathrm{E}_t \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \tau^y Y_s - (1+r)B_t.$$

Using $\mathrm{E}_t G_{t+j} = G_t, \forall j > 0$, and $\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} = 1/\left(1 - \frac{1}{1+r}\right) = \frac{r+1}{r}$ given that $1+r > 1$, leads to

$$\frac{r+1}{r} G_t = \tau^y Y_t + \mathrm{E}_t \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \tau^y Y_s - (1+r)B_t.$$

To determine the period t expectation of Y_s for $s \geq t$, we use that Y_t follows the stochastic process

$$Y_t - \overline{Y} = \rho \left(Y_{t-1} - \overline{Y}\right) + \sigma_\varepsilon \varepsilon_t.$$

Expectations conditional on the information in period t are thus

$$\mathrm{E}_t Y_s = (1-\rho)\overline{Y} + \rho E_t Y_{s-1}.$$

To write $\mathrm{E}_t Y_s$ as a function of Y_t, we iterate backwards to get $E_t Y_s = (1-\rho)\overline{Y} + \rho(1-\rho)\overline{Y} + \ldots + \rho^{(s-t)-1}(1-\rho)\overline{Y} + \rho^{(s-t)} E_t Y_t$, or, concisely

$$\mathrm{E}_t Y_s = (1-\rho)\overline{Y} \sum_{k=0}^{(s-t)-1} \rho^k + \rho^{(s-t)} Y_t.$$

Eliminating $E_t Y_s$ in $\frac{r+1}{r} G_t = \tau^y Y_t + \tau^y \mathrm{E}_t \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s - (1+r) B_t$ then gives

$$\begin{aligned}
\frac{r+1}{r} G_t &= \tau^y Y_t + \tau^y \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \left((1-\rho)\overline{Y} \sum_{k=0}^{(s-t)-1} \rho^k + \rho^{(s-t)} Y_t \right) - (1+r) B_t \\
&= \left\{ \begin{array}{c} \tau^y Y_t + \tau^y \sum\limits_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (1-\rho)\overline{Y} \sum_{k=0}^{(s-t)-1} \rho^k + \tau^y \sum\limits_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \rho^{(s-t)} Y_t \\ -(1+r) B_t \end{array} \right\} \\
&= \tau^y Y_t \left(1 + \frac{\rho}{r-\rho+1}\right) + \left[(1-\rho)\overline{Y} \tau^y \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \sum_{k=0}^{(s-t)-1} \rho^k \right] - (1+r) B_t.
\end{aligned}$$

Using $\sum_{k=0}^{(s-t)-1} \rho^k = \left(\frac{1}{1-\rho} - \frac{\rho^{s-t}}{1-\rho}\right)$ for any $\rho \neq 1$, the term in the square bracket can further be simplified to

$$\begin{aligned}
& \tau^y (1-\rho)\overline{Y} \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \sum_{k=0}^{(s-t)-1} \rho^k \\
&= \tau^y (1-\rho)\overline{Y} \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \left(\frac{1}{1-\rho} - \frac{\rho^{s-t}}{1-\rho}\right) \\
&= \tau^y \overline{Y} \sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \left(1 - \rho^{s-t}\right).
\end{aligned}$$

By using $\sum_{k=0}^{\infty} a^{k+1} = a/(1-a)$ for any $|a| < 1$, such that $\sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} = 1/r$ and $\sum_{s=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} \rho^{s-t} = \frac{\rho}{r-\rho+1}$, the last expression becomes

$$\begin{aligned}
& \tau^y \overline{Y} \left(\frac{1}{r} - \frac{\rho}{r-\rho+1}\right) \\
&= \tau^y \overline{Y} \frac{(1-\rho)(1+r)}{r(1+r-\rho)}.
\end{aligned}$$

Hence, we end up with the following state space solution for G_t :

$$\begin{aligned}
\frac{r+1}{r} G_t &= \tau^y Y_t \left(1 + \frac{\rho}{r-\rho+1}\right) + \tau^y \overline{Y} \frac{(1-\rho)(1+r)}{r(1+r-\rho)} - (1+r) B_t \\
&= \tau^y Y_t \frac{r+1}{r-\rho+1} + \tau^y \overline{Y} \frac{(1-\rho)(1+r)}{r(1+r-\rho)} - (1+r) B_t \\
&\Rightarrow G_t = \tau^y Y_t \frac{r}{1+r-\rho} + \tau^y \overline{Y} \frac{1-\rho}{1+r-\rho} - r B_t.
\end{aligned}$$

The solution for B_{t+1} can then simply be found by eliminating G_t in the period-by-period budget constraint

$$\begin{aligned} B_{t+1} &= (1+r)\,B_t + G_t - \tau^y Y_t \\ &= (1+r)\,B_t + \left(\tau^y Y_t \frac{r}{1+r-\rho} + \tau^y \overline{Y}\frac{1-\rho}{1+r-\rho} - rB_t\right) - \tau^y Y_t \\ &= B_t - \tau^y \frac{1-\rho}{1+r-\rho}\left(Y_t - \overline{Y}\right), \end{aligned}$$

which discloses the unit root in B_t and thus in G_t. Hence, for a given initial value B_0 and a given endowment sequence $\{Y_t\}_{t=0}^{\infty}$ we obtain a unique set of sequences $\{G_t, B_{t+1}\}_{t=0}^{\infty}$. Due to the unit root, there are infinitely many solutions for the unconditional expectations for G and B, which satisfy

$$\overline{G} = \tau^y \overline{Y} - r\overline{B}.$$

We can now derive the solutions for C_t and F_{t+1}, for which we use the remaining conditions (taking the solution for G_t as given)

$$\begin{aligned} v'(C_t) &= \mathrm{E}_t v'(C_{t+1}) \\ -(1+r)F_t &= \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - C_s - G_s)\,, \end{aligned}$$

given $F_0 = 0$. Taking expectations conditional on the information in period t, the intertemporal resource constraint leads to

$$\begin{aligned} \mathrm{E}_t \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} C_s &= \mathrm{E}_t \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(Y_s - G_s) + (1+r)F_t \\ \Rightarrow \frac{1+r}{r} C_t &= \mathrm{E}_t \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s - \frac{1+r}{r} G_t + (1+r)F_t \end{aligned}$$

where we used $\mathrm{E}_t C_{t+j} = C_t$, $\forall j > 0$, and $\mathrm{E}_t G_{t+j} = G_t$, $\forall j > 0$. Using that

$$\mathrm{E}_t \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s = Y_t \frac{r+1}{r-\rho+1} + \overline{Y}\frac{(1-\rho)\,(1+r)}{r\,(1+r-\rho)},$$

holds (see above), consumption C_t can be written as a function of Y_t, G_t, and F_t :

$$\begin{aligned} \frac{1+r}{r} C_t &= Y_t \frac{r+1}{r-\rho+1} + \overline{Y}\frac{(1-\rho)\,(1+r)}{r\,(1+r-\rho)} - \frac{1+r}{r} G_t + (1+r)F_t \\ &\Rightarrow \; C_t = Y_t \frac{r}{r-\rho+1} + \overline{Y}\frac{1-\rho}{1+r-\rho} - G_t + rF_t. \end{aligned}$$

Plugging in the solution for G_t, $G_t = \tau^y Y_t \frac{r}{1+r-\rho} + \tau^y \overline{Y}\frac{1-\rho}{1+r-\rho} - rB_t$, we get the following state space solution for consumption

$$\begin{aligned} C_t &= Y_t \frac{r}{r-\rho+1} + \overline{Y}\frac{(1-\rho)}{(1+r-\rho)} - \left(\tau^y Y_t \frac{r}{1+r-\rho} + \tau^y \overline{Y}\frac{1-\rho}{1+r-\rho} - rB_t\right) + rF_t \\ &= (1-\tau^y)\,\frac{r}{r-\rho+1} Y_t + (1-\tau^y)\,\frac{1-\rho}{1+r-\rho}\overline{Y} + rB_t + rF_t. \end{aligned}$$

Using the period-by-period resource constraint, $F_{t+1} = (1+r)F_t + Y_t - C_t - G_t$, and the solutions for C_t and G_t, we can compute the solution for F_t with

$$\begin{aligned} F_{t+1} &= (1+r)F_t + Y_t \\ &\quad - \left((1-\tau^y)\frac{r}{r-\rho+1}Y_t + (1-\tau^y)\frac{1-\rho}{1+r-\rho}\overline{Y} + rB_t + rF_t\right) \\ &\quad - \left(\tau^y Y_t \frac{r}{1+r-\rho} + \tau^y\overline{Y}\frac{1-\rho}{1+r-\rho} - rB_t\right) \\ &= F_t + \frac{1-\rho}{1-\rho+r}\left(Y_t - \overline{Y}\right), \end{aligned}$$

and the initial value $F_0 = 0$. The unconditional means of F and C, which satisfy $-r\overline{F} = \overline{Y} - \overline{C} - \overline{G}$ are not determined.

To summarize, the solution $X_t = \Gamma_X(B_t, F_t, Y_t)$ for $X_t \in (G_t, C_t, B_{t+1}, F_{t+1})$ is represented by (2.12) to (2.15) given B_0 and F_0, with unconditional means satisfying

$$\overline{G} = \tau^y\overline{Y} - r\overline{B} \quad \text{and} \quad \overline{C} = (1-\tau^y)\overline{Y} + r\overline{F} + r\overline{B}$$

2.B Intertemporal budget constraint under debt sanction

Substitute (2.10) and $I\left[B_s;\overline{B}\right]$ by L_s^B in (2.7) and rewrite it for $s = t$:

$$\begin{aligned} B_{t+1} &= (1+r)B_t + G_t - \tau^y Y_t + k^B(B_t - B^c)L_t^B \Leftrightarrow \\ \left(1+r+k^B L_t^B\right)B_t &= \tau^y Y_t - G_t + k^B B^c L_t^B + B_{t+1}. \end{aligned}$$

Iterating this process forward one period, we can jot down this last expression like:

$$B_{t+1} = \frac{\tau^y Y_{t+1} - G_{t+1} + k^B B^c L_{t+1}^B}{\left(1+r+k^B L_{t+1}^B\right)} + \frac{B_{t+2}}{\left(1+r+k^B L_{t+1}^B\right)}.$$

So, the intertemporal budget constraint with debt sanction is equal to:

$$\left(1+r+k^B L_t^B\right)B_t = \left\{\begin{array}{c} \tau^y Y_t - G_t + k^B B^c L_t^B + \frac{\tau^y Y_{t+1} - G_{t+1} + k^B B^c L_{t+1}^B}{\left(1+r+k^B L_{t+1}^B\right)} \\ + \frac{\tau^y Y_{t+2} - G_{t+2} + k^B B^c L_{t+2}^B}{\left(1+r+k^B L_{t+1}^B\right)\left(1+r+k^B L_{t+2}^B\right)} + \ldots \end{array}\right\},$$

that can be represented by:

$$\begin{aligned} \left(1+r+k^B L_t^B\right)B_t &= \left[\begin{array}{c} \tau^y Y_t - G_t + k^B B^c L_t^B + \frac{\tau^y Y_{t+1} - G_{t+1} + k^B B^c L_{t+1}^B}{\left(1+r+k^B L_{t+1}^B\right)} \\ + \frac{\tau^y Y_{t+2} - G_{t+2} + k^B B^c L_{t+2}^B}{\prod_{v=t+1}^{t+2}\left(1+r+k^B L_v^B\right)} + \ldots + \frac{\tau^y Y_{t+T} - G_{t+T} + k^B B^c L_{t+T}^B}{\prod_{v=t+1}^{t+T}\left(1+r+k^B L_v^B\right)} \\ + \lim_{T\to\infty} \frac{B_{t+T+1}}{\prod_{v=t+1}^{t+T}\left(1+r+k^B L_v^B\right)} \end{array}\right] \Leftrightarrow \\ \left(1+r+k^B L_t^B\right)B_t &= \sum_{s=t}^{\infty} \frac{\tau^y Y_s - G_s + k^B B^c L_s^B}{\prod_{v=t+1}^{s}\left(1+r+k^B L_v^B\right)} + \lim_{T\to\infty} \frac{B_{t+T+1}}{\prod_{v=t+1}^{t+T}\left(1+r+k^B L_v^B\right)}, \end{aligned}$$

where we define $\prod_{v=t+1}^{t}\left(1+r+k^B L_v^B\right)\equiv 1$. In equilibrium that constraint is then equal to (2.19). Moreover, for the deterministic steady state, the transversality condition imposes:

$$\left(1+r+k^B L_0^B\right)B_0-\frac{1+r+k^B\overline{L^B}}{r+k^B\overline{L^B}}\left(\tau^y\overline{Y}-\overline{G}+k^B B^c\overline{L^B}\right)=0.$$

Note that if $B_0=\overline{B}$, as we assume in our numerical analysis, then the last equation simplifies as follows:

$$\begin{aligned}
\left(1+r+k^B\overline{L^B}\right)\overline{B}-\frac{1+r+k^B\overline{L^B}}{r+k^B\overline{L^B}}\left(\tau^y\overline{Y}-\overline{G}+k^B B^c\overline{L^B}\right) &= 0\Leftrightarrow\\
\left(r+k^B\overline{L^B}\right)\overline{B} &= \tau^y\overline{Y}-\overline{G}+k^B B^c\overline{L^B}\Leftrightarrow\\
r\overline{B}+k^B\left(\overline{B}-B^c\right)\overline{L^B} &= \tau^y\overline{Y}-\overline{G}.
\end{aligned}$$

For large values of γ and thus good approximations of the indicator function, this simplifies to:

$$r\overline{B}=\tau^y\overline{Y}-\overline{G},$$

which is the usual constraint that should hold in steady state.

2.C Intertemporal budget constraint under deficit sanction

Substituting (2.8), $I\left[D_t>D^c\right]$ by L_t^D, and (2.10) in (2.21):

$$\begin{aligned}
B_{t+1} &= (1+r)B_t+G_t-\tau^y Y_t+k^D\left(D_t-D^c\right)L_t^D\Leftrightarrow\\
B_{t+1} &= (1+r)B_t+G_t-\tau^y Y_t+k^D\left(B_{t+1}-(1+r)B_t-D^c\right)L_t^D\Leftrightarrow\\
(1+r)\left(1-k^D L_t^D\right)B_t &= \tau^y Y_t-G_t+k^D D^c L_t^D+\left(1-k^D L_t^D\right)B_{t+1}\Leftrightarrow\\
(1+r)B_t &= \frac{\tau^y Y_t-G_t+k^D D^c L_t^D}{1-k^D L_t^D}+B_{t+1}.
\end{aligned}$$

Iterating this process forward one period, we can rewrite this last expression like:

$$B_{t+1}=\frac{\tau^y Y_{t+1}-G_{t+1}+k^D D^c L_{t+1}^D}{(1+r)\left(1-k^D L_{t+1}^D\right)}+\frac{B_{t+2}}{1+r}.$$

So, the intertemporal budget constraint with debt sanction is equal to:

$$(1+r)B_t=\left\{\begin{array}{c}
\frac{\tau^y Y_t-G_t+k^D D^c L_t^D}{1-k^D L_t^D}+\frac{\tau^y Y_{t+1}-G_{t+1}+k^D D^c L_{t+1}^D}{(1+r)\left(1-k^D L_{t+1}^D\right)}+\\
\frac{\tau^y Y_{t+2}-G_{t+2}+k^D D^c L_{t+2}^D}{(1+r)^2\left(1-k^D L_{t+2}^D\right)}+\frac{\tau^y Y_{t+3}-G_{t+3}+k^D D^c L_{t+3}^D}{(1+r)^3\left(1-k^D L_{t+3}^D\right)}+\ldots+\\
\frac{\tau^y Y_{t+T}-G_{t+T}+k^D D^c L_{t+T}^D}{(1+r)^T\left(1-k^D L_{t+T}^D\right)}+\lim_{T\to\infty}\frac{B_{t+T+1}}{(1+r)^{T+1}}
\end{array}\right\},$$

which in equilibrium becomes (2.23). Finally, for the deterministic steady state, the transversality condition imposes:

$$(1+r)\,B_0 - \frac{1+r}{r}\left(\frac{\tau^y\overline{Y} - \overline{G} + k^D D^c \overline{L^D}}{1 - k^D\overline{L^D}}\right) = 0.$$

Note that if $B_0 = \overline{B}$, as we assume in our numerical analysis, then the last equation simplifies as follows:

$$\begin{aligned}
(1+r)\,\overline{B} - \frac{1+r}{r}\left(\frac{\tau^y\overline{Y} - \overline{G} + k^D D^c \overline{L^D}}{1 - k^D\overline{L^D}}\right) &= 0 \Leftrightarrow \\
r\left(1 - k^D\overline{L^D}\right)\overline{B} &= \tau^y\overline{Y} - \overline{G} + k^D D^c \overline{L^D}.
\end{aligned}$$

For large values of γ and thus good approximations of the indicator function, this simplifies to:

$$r\overline{B} = \tau^y\overline{Y} - \overline{G},$$

which is the usual constraint that should hold in steady state.

2.D Deterministic steady state under debt-based sanction

Under the debt constraint the equilibrium conditions are given by (2.17) and (2.18). Hence, a deterministic steady state has to satisfy the combination

$$\begin{aligned}
1 &= \beta_g\left[1 + r + k^B\overline{L^B} + \gamma k^B\left(\overline{B} - B^c\right)\overline{L^B}(1 - \overline{L^B})\right], \\
0 &= r\overline{B} + \overline{G} - \tau^y\overline{Y} + k^B\left(\overline{B} - B^c\right)\overline{L^B}
\end{aligned}$$

and thus

$$\begin{aligned}
\beta_g^{-1} - (1+r) &= k^B\overline{L^B}\left(1 + \gamma\left(\overline{B} - B^c\right)(1 - \overline{L^B})\right), \\
\overline{L^B} &= \frac{\tau^y\overline{Y} - r\overline{B} - \overline{G}}{k^B\left(\overline{B} - B^c\right)}.
\end{aligned} \tag{2.32}$$

Eliminating $\overline{L^B}$ gives

$$\begin{aligned}
\beta_g^{-1} - (1+r) &= k^B\frac{\tau^y\overline{Y} - r\overline{B} - \overline{G}}{k^B\left(\overline{B} - B^c\right)}\left(1 + \gamma\left(\overline{B} - B^c\right)\left(1 - \frac{\tau^y\overline{Y} - r\overline{B} - \overline{G}}{k^B\left(\overline{B} - B^c\right)}\right)\right) \\
&= \frac{\tau^y\overline{Y} - r\overline{B} - \overline{G}}{\overline{B} - B^c}\left(1 + \gamma\left(\overline{B} - B^c\right) - \gamma\frac{\tau^y\overline{Y} - r\overline{B} - \overline{G}}{k^B}\right).
\end{aligned} \tag{2.33}$$

Or, instead, using (2.16) to eliminate $\overline{G}$, in the previous system of equations, we obtain:

$$\beta_g^{-1} - (1+r) = \frac{k^B \left\{\left[1+\exp\left(-\gamma\left(\overline{B}-B^c\right)\right)\right] * \left[1+\gamma\left(\overline{B}-B^c\right)\right] - \gamma\left(\overline{B}-B^c\right)\right\}}{\left[1+\exp\left(-\gamma\left(\overline{B}-B^c\right)\right)\right]^2}.$$

This equation determines $\overline{B}$. We can substitute it into (2.32) to find then $\overline{G}$. Moreover, if we pass the denominator multiplying the left-hand side of the previous equation we get:

$$\left[\beta_g^{-1} - (1+r)\right] * \left[1+\exp\left(-\gamma\left(\overline{B}-B^c\right)\right)\right]^2 = k^B \left\{1+\exp\left(-\gamma\left(\overline{B}-B^c\right)\right) * \left[1+\gamma\left(\overline{B}-B^c\right)\right]\right\}.$$

The derivative of the left-hand side of this expression is:

$$\left[\beta_g^{-1} - (1+r)\right] * 2\left[1+\exp\left(-\gamma\left(\overline{B}-B^c\right)\right)\right] * \left[-\gamma\exp\left(-\gamma\left(\overline{B}-B^c\right)\right)\right] < 0,$$

if $\beta_g < \beta_w$, $\gamma >> 0$ and $\overline{B} < B^c$. Further, the derivative of the right-hand side is:

$$k^B \left\{-\gamma\exp\left(-\gamma\left(\overline{B}-B^c\right)\right) * \left[1+\gamma\left(\overline{B}-B^c\right)\right] + \gamma\exp\left(-\gamma\left(\overline{B}-B^c\right)\right)\right\} > 0,$$

using the same assumptions as before. Therefore, this steady state is unique.

2.E Deterministic steady state under deficit-based sanction

Under the primary deficit-based constraint the equilibrium conditions are given by (2.21) and (2.22). Hence, a deterministic steady state has to satisfy

$$\begin{aligned} 1 &= \beta_g k^D \left[\overline{L^D} + \gamma\left(\overline{D}-D^c\right)\overline{L^D}(1-\overline{L^D})\right], \\ \overline{D} &= \overline{G} - \tau^y\overline{Y} + k^D\left(\overline{D}-D^c\right)\overline{L^D} \end{aligned}$$

and thus

$$\begin{aligned} \beta_g^{-1} &= k^D\overline{L^D}\left(1+\gamma\left(\overline{D}-D^c\right)(1-\overline{L^D})\right), && (2.34) \\ \overline{L^D} &= \frac{\tau^y\overline{Y}+\overline{D}-\overline{G}}{k^D\left(\overline{D}-D^c\right)}. && (2.35) \end{aligned}$$

Eliminating $\overline{L^D}$ gives

$$\begin{aligned} \beta_g^{-1} &= k^D\frac{\tau^y\overline{Y}+\overline{D}-\overline{G}}{k^D\left(\overline{D}-D^c\right)}\left(1+\gamma\left(\overline{D}-D^c\right)\left(1-\frac{\tau^y\overline{Y}+\overline{D}-\overline{G}}{k^D\left(\overline{D}-D^c\right)}\right)\right) \Leftrightarrow && (2.36) \\ \beta_g^{-1} &= \frac{\tau^y\overline{Y}+\overline{D}-\overline{G}}{\overline{D}-D^c}\left(1+\gamma\left(\overline{D}-D^c\right)-\gamma\frac{\tau^y\overline{Y}+\overline{D}-\overline{G}}{k^D}\right). && (2.37) \end{aligned}$$

Instead, if we use (2.20) in (2.35) and eliminate $\overline{G}$ in that system of equations, we arrive at:

$$\beta_g^{-1} = \frac{k^D \left\{\left[1+\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)\right] * \left[1+\gamma\left(\overline{D}-D^c\right)\right] - \gamma\left(\overline{D}-D^c\right)\right\}}{\left[1+\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)\right]^2}.$$

This determines $\overline{D}$. We can substitute the solution into (2.35) to find $\overline{G}$. Moreover, If we pass the denominator multiplying the left-hand side of the previous equation we get:

$$\beta_g^{-1}\left[1+\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)\right]^2 = k^D\left\{1+\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)*\left[1+\gamma\left(\overline{D}-D^c\right)\right]\right\}.$$

The derivative of the left-hand side of this expression is:

$$\beta_g^{-1}*2\left[1+\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)\right]*\left[-\gamma\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)\right]<0,$$

if $\gamma >> 0$ and $\overline{D} < D^c$. Further, the derivative of the right-hand side is:

$$k^D\left\{-\gamma\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)*\left[1+\gamma\left(\overline{D}-D^c\right)\right]+\gamma\exp\left(-\gamma\left(\overline{D}-D^c\right)\right)\right\}>0,$$

using the same assumptions as before. Therefore, this steady state is unique.

2.F Stochastic steady state with debt-based sanction

For the partisan government that faces debt-based sanction, the solution has to fulfill the system of equations given by the Euler equation (2.18) when $\gamma < \infty$, and the budget constraint (2.7) approximated by (2.16):

$$G_t = B_{t+1}-(1+r)B_t+\tau^y Y_t - k^B\left(B_t-B^c\right)L_t^B. \tag{2.38}$$

Hence, substituting (2.5) and (2.38) into the Euler equation (2.18) for $s=t$ implies

$$\mathrm{E}_t\left\{\begin{array}{c} u'\left(B_{t+1}-(1+r)B_t+\tau^y Y_t-k^B\left(B_t-B^c\right)L_t^B\right)- \\ \beta_g\left[\begin{array}{c}1+r+k^B L_{t+1}^B+\\ \gamma k^B\left(B_{t+1}-B^c\right)\left(L_{t+1}^B-\left(L_{t+1}^B\right)^2\right)\end{array}\right]u'\left[\begin{array}{c}B_{t+2}-(1+r)B_{t+1}+\\ \tau^y Y_{t+1}-k^B\left(B_{t+1}-B^c\right)L_{t+1}^B\end{array}\right]\end{array}\right\}=0, \tag{2.39}$$

where we have used that

$$\partial L_{t+1}^B/\partial B_{t+1} = \gamma\left(L_{t+1}^B-\left(L_{t+1}^B\right)^2\right). \tag{2.40}$$

Thus (2.39) is of the format:

$$\mathrm{E}_t\left[g\left(B_t,B_{t+1},B_{t+2},Y_t,Y_{t+1}\right)\right]=0,$$

where $g(.)$ is a function implicitly defined by (2.39). Hence, by using (2.25) the unknown function $h(\cdot)$ satisfies

$$\begin{aligned}
\mathrm{E}_t\left[q\left(B_t,Y_t,\sigma_\varepsilon\right)\right] &= \mathrm{E}_t\left\{g\left(B_t,h\left(B_t,Y_t,\sigma_\varepsilon\right),h\left(h\left(B_t,Y_t,\sigma_\varepsilon\right),Y_{t+1},\sigma_\varepsilon\right),Y_t,Y_{t+1}\right)\right\}=0\Rightarrow \\
\mathrm{E}_t\left\{q\left(B_t,Y_t,\sigma_\varepsilon\right)\right] &= \mathrm{E}_t\left\{g\left(\begin{array}{c}B_t,h\left(B_t,Y_t,\sigma_\varepsilon\right),\\ h\left(h\left(B_t,Y_t,\sigma_\varepsilon\right),\rho Y_t+(1-\rho)\overline{Y}+\sigma_\varepsilon\varepsilon_{t+1},\sigma_\varepsilon\right),\\ Y_t,\rho Y_t+(1-\rho)\overline{Y}+\sigma_\varepsilon\varepsilon_{t+1}\end{array}\right)\right\}=0,
\end{aligned}$$

where we have defined the new function $q\left(B_t,Y_t,\sigma_\varepsilon\right)$.

For $\sigma_\varepsilon = 0$, $B_{t+2} = B_{t+1} = B_t = \overline{B}$ and $Y_{t+1} = Y_t = \overline{Y}$, (2.26) becomes:

$$\mathrm{E}_t\left[q\left(\overline{B}, \overline{Y}, 0\right)\right] = \mathrm{E}_t\left[g\left(\overline{B}, \overline{B}, \overline{B}, \overline{Y}, \overline{Y}\right)\right] = 0, \tag{2.41}$$

where we have used that $h\left(\overline{B}, \overline{Y}, 0\right) = \overline{B}$ and $h\left(h\left(\overline{B}, \overline{Y}, 0\right), \overline{Y}, 0\right) = h\left(\overline{B}, \overline{Y}, 0\right) = \overline{B}$. Because $u' > 0$, $\mathrm{E}_t\left[q\left(\overline{B}, \overline{Y}, 0\right)\right] = 0$ is equivalent to

$$\beta_g\left[1 + r + k^B\overline{L^B} + \gamma k^B\left(\overline{B} - B^c\right)\left(\overline{L^B} - \overline{L^B}^2\right)\right] = 1, \tag{2.42}$$

where $\overline{L^B}$ and u' are evaluated at the point $\left(\overline{B}, \overline{Y}, 0\right)$. This expression yields a solution for $\overline{B}$. $\overline{B}$ and $\overline{Y}$ correspond to the average value of B_t and Y_t, since they are computed when $\sigma_\varepsilon = 0$. Thus, using (2.5) we know that $E_0\left\{Y_t\right\} = \widehat{Y} = \overline{Y}$ and $E_0\left\{\left(Y_t - \overline{Y}\right)^2\right\}$ =Var$(Y_t) = \frac{\sigma_\varepsilon^2}{1-\rho^2}$.

We find an approximate solution of (2.25) by taking first- and second-order Taylor expansions of (2.26)

2.F.1 First-order approximation of (2.26)

A first-order Taylor expansion of (2.26) yields:

$$\mathrm{E}_t\left[q\left(B_t, Y_t, \sigma_\varepsilon\right)\right] \simeq \mathrm{E}_t\left[q\left(\overline{B}, \overline{Y}, 0\right) + q_B\left(\overline{B}, \overline{Y}, 0\right)\widetilde{B}_t + q_Y\left(\overline{B}, \overline{Y}, 0\right)\widetilde{Y}_t + q_{\sigma_\varepsilon}\left(\overline{B}, \overline{Y}, 0\right)\sigma_\varepsilon\right] = 0, \tag{2.43}$$

where $q_B \equiv \frac{\partial q}{\partial B_t}$, $q_Y \equiv \frac{\partial q}{\partial Y_t}$, and $q_{\sigma_\varepsilon} \equiv \frac{\partial q}{\partial \sigma_\varepsilon}$.

From (2.41), we already know that $\mathrm{E}_t\left[q\left(\overline{B}, \overline{Y}, 0\right)\right] = 0$. Thus, since we consider approximations of (2.43) around the point $\left(\overline{B}, \overline{Y}, 0\right)$ the remaining unknown coefficients of that first-order approximation are found by solving

$$\mathrm{E}_t\left[q_B\left(\overline{B}, \overline{Y}, 0\right)\widetilde{B}_t + q_{Y_t}\left(\overline{B}, \overline{Y}, 0\right)\widetilde{Y}_t + q_{\sigma_\varepsilon}\left(\overline{B}, \overline{Y}, 0\right)\sigma_\varepsilon\right] = 0.$$

This equation implies a system of three equations with three unknown coefficients related to the effects of B_t, Y_t and σ_ε in (2.25).

Derivation of the first order partial derivatives q_B, q_{Y_t}, and q_{σ_ε}

In this section, we compute q_B, q_{Y_t}, and q_{σ_ε}. We start by computing the partial derivatives of (2.25) with respect to $B_t, Y_t, \sigma_\varepsilon$:

First-order partial derivatives of (2.25) Differentiate B_{t+1} with respect to B_t:

$$\frac{\partial B_{t+1}}{\partial B_t} = \frac{\partial h\left(B_t, Y_t, \sigma_\varepsilon\right)}{\partial B_t} = \frac{\partial h_t}{\partial B_t} \equiv h_1^B, \tag{2.44}$$

where $h_t \equiv h\left(B_t, Y_t, \sigma_\varepsilon\right)$. Further, the derivative of B_{t+1} with respect to Y_t is

$$\frac{\partial B_{t+1}}{\partial Y_t} = \frac{\partial h\left(B_t, Y_t, \sigma_\varepsilon\right)}{\partial Y_t} = \frac{\partial h_t}{\partial Y_t} \equiv h_1^Y. \tag{2.45}$$

Finally, the derivative of B_{t+1} with respect to σ_ε is

$$\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} = \frac{\partial h\left(B_t, Y_t, \sigma_\varepsilon\right)}{\partial \sigma_\varepsilon} = \frac{\partial h_t}{\partial \sigma_\varepsilon} \equiv h_1^\sigma. \tag{2.46}$$

Derivation of the first-order partial derivative of B_{t+2} with respect to B_t, Y_t and σ_ε From 2.25, we can write B_{t+2} as

$$B_{t+2} = h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right) = h\left(h\left(B_t, Y_t, \sigma_\varepsilon\right), \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon \varepsilon_{t+1}, \sigma_\varepsilon\right). \tag{2.47}$$

Hence, differentiating (2.47) with respect to B_t, we obtain:

$$\begin{aligned}\frac{\partial B_{t+2}}{\partial B_t} &= \frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial B_t} = \frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial B_t} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial B_t} \\ &= \frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial h_t}{\partial B_t} = \left(\frac{\partial h_t}{\partial B_t}\right)^2 = \left(h_1^B\right)^2,\end{aligned}$$

where we have used (2.44), and the facts that $(i)\frac{\partial Y_{t+1}}{\partial B_t} = \frac{\partial \sigma_\varepsilon}{\partial B_t} = 0$, and (ii) the partials $\frac{\partial h_{t+1}}{\partial B_{t+1}}$ and $\frac{\partial h_t}{\partial B_t}$ are evaluated at the same point $\left(\overline{B}, \overline{Y}, 0\right)$.

Differentiate of B_{t+2} with respect to Y_t:

$$\begin{aligned}\frac{\partial B_{t+2}}{\partial Y_t} &= \frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial Y_t} = \frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t} \\ &= \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial h_t}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\rho = h_1^B h_1^Y + h_1^Y \rho,\end{aligned}$$

where we have used (2.5), (2.44) and (2.45).

Finally, differentiate B_{t+2} with respect to σ_ε:

$$\begin{aligned}\frac{\partial B_{t+2}}{\partial \sigma_\varepsilon} &= \frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial \sigma_\varepsilon} = \frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon} \\ &= h_1^B \frac{\partial h_t}{\partial \sigma_\varepsilon} + h_1^Y \varepsilon_{t+1} + h_1^\sigma = h_1^B h_1^\sigma + h_1^Y \varepsilon_{t+1} + h_1^\sigma.\end{aligned}$$

Derivation of q_B Taking the partial derivative of (2.39) with respect to B_t at the point $\left(\overline{B}, \overline{Y}, 0\right)$, and multiplying it by $\widetilde{B}_t$, yields:

$$\mathrm{E}_t\left[q_B \widetilde{B}_t\right] \simeq E_t\left\{\left[\begin{array}{c} u''\left(G_t\right) * \left(\frac{\partial B_{t+1}}{\partial B_t} - (1+r) - k^B \overline{L^B} - k^B\left(\overline{B} - B^c\right)\frac{\partial L_t^B}{\partial B_t}\right) - \\ \beta_g k^B\left[\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t} + \frac{\partial B_{t+1}}{\partial B_t}\frac{\partial L_{t+1}^B}{\partial B_{t+1}} + \left(\overline{B} - B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right] u'\left(G_{t+1}\right) \\ -\beta_g\left[\begin{array}{c} 1 + r + k^B \overline{L^B} + \\ \gamma k^B\left(\overline{B} - B^c\right)\overline{L^B}(1 - \overline{L^B}) \end{array}\right] u''\left(G_{t+1}\right) * \\ \left(\begin{array}{c} \frac{\partial B_{t+2}}{\partial B_t} - (1+r)\frac{\partial B_{t+1}}{\partial B_t} - k^B \overline{L^B}\frac{\partial B_{t+1}}{\partial B_t} \\ -k^B\left(\overline{B} - B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t} \end{array}\right) \end{array}\right] \widetilde{B}_t\right\},$$

where u' and $u''$are the first- and second-order derivatives of the utility function evaluated at the non-stochastic steady state. Now, we need to compute the terms $\frac{\partial L_t^B}{\partial B_t}$, $\frac{\partial L_{t+1}^B}{\partial B_{t+1}}$ and $\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}$.

Then, differentiating (2.40) once more with respect to B_{t+1} we have that

$$\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}} = \gamma\left[\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\left(1 - L_{t+1}^B\right) - L_{t+1}^B \frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right].$$

Substituting (2.40) in the last equation:

$$\begin{aligned}\frac{\partial^2 L_{t+1}^B}{\partial B_{t+1}^2} &= \gamma\left[\gamma L_{t+1}^B\left(1-L_{t+1}^B\right)\left(1-L_{t+1}^B\right)-L_{t+1}^B\gamma L_{t+1}^B\left(1-L_{t+1}^B\right)\right] \\ &= \gamma\left[\gamma L_{t+1}^B\left(1-L_{t+1}^B\right)\left(1-L_{t+1}^B-L_{t+1}^B\right)\right]=\gamma^2\left(L_{t+1}^B-3\left(L_{t+1}^B\right)^2+2\left(L_{t+1}^B\right)^3\right)\end{aligned} \tag{2.48}$$

Using the functional form of L^B, we can also represent (2.48) by

$$\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}=\frac{\gamma^2\exp\left(-\gamma\left(B_{t+1}-B^c\right)\right)*\left[-1+\exp\left(-\gamma\left(B_{t+1}-B^c\right)\right)\right]}{\left[1+\exp\left(-\gamma\left(B_{t+1}-B^c\right)\right)\right]^3}.$$

Therefore, using (2.40) and (2.48), combined with the expressions for $\frac{\partial B_{t+1}}{\partial B_t}$ and $\frac{\partial B_{t+2}}{\partial B_t}$, we write $\mathrm{E}_t\left[q_B\widetilde{B}_t\right]$ as

$$\mathrm{E}_t\left[q_B\widetilde{B}_t\right]\simeq\left[\begin{array}{c} u''*\left(h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}(1-\overline{L^B})\right)\right)- \\ \beta_g k^B\left[\begin{array}{c} 2\gamma\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)h_1^B+ \\ \left(\overline{B}-B^c\right)\gamma^2\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)h_1^B \end{array}\right]*u' \\ -\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}(1-\overline{L^B})\right]* \\ u''\left(\left(h_1^B\right)^2-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}(1-\overline{L^B})\right)h_1^B\right) \end{array}\right]\widetilde{B}_t,$$

where we exclude the expectations operator from the right-hand side of the previous equation, since all its elements are known in period t. Using (2.42), it follows that

$$\mathrm{E}_t\left[q_B\widetilde{B}_t\right]\simeq\left[\begin{array}{c} u''*\left(h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right) \\ -\beta_g k^B\gamma\left[\begin{array}{c} 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right) \end{array}\right]*u'h_1^B- \\ u''*\left(\left(h_1^B\right)^2-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^B\right) \end{array}\right]\widetilde{B}_t. \tag{2.49}$$

Derivation of q_{Y_t} Differentiating the partial derivative of (2.39) with respect to Y_t at the point $\left(\overline{B},\overline{Y},0\right)$ and multiplying it by $\widetilde{Y}_t$, yields:

$$\mathrm{E}_t\left[q_{Y_t}\widetilde{Y}_t\right]\simeq\mathrm{E}_t\left\{\left[\begin{array}{c} \left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right)u''\left(G_t\right)- \\ \beta_g k^B\left[\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}+\frac{\partial B_{t+1}}{\partial Y_t}\frac{\partial L_{t+1}^B}{\partial B_{t+1}}+\left(\overline{B}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right]u'\left(G_{t+1}\right) \\ -\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right]* \\ \left(\begin{array}{c} \frac{\partial B_{t+2}}{\partial Y_t}-\left(1+r+k^B\overline{L^B}\right)\frac{\partial B_{t+1}}{\partial Y_t} \\ +\tau^y\rho-k^B\left(\overline{B}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} \end{array}\right)u''\left(G_{t+1}\right) \end{array}\right]\widetilde{Y}_t\right\},$$

since $\frac{\partial B_t}{\partial Y_t}=0$ (see the government's budget constraint) and from (2.5), $\frac{\partial Y_{t+1}}{\partial Y_t}=\rho$. Using (2.40), (2.42), (2.48) and the expressions for $\frac{\partial B_{t+1}}{\partial Y_t}$ and $\frac{\partial B_{t+2}}{\partial Y_t}$, we write

$$\mathrm{E}_t\left[q_{Y_t}\widetilde{Y}_t\right]\simeq\left\{\begin{array}{c}\left(h_1^Y+\tau^y\right)u''-\\ \beta_g k^B\gamma\left[\begin{array}{c}2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\end{array}\right]h_1^Y u'-\\ \left[\begin{array}{c}h_1^B h_1^Y+h_1^Y\rho+\tau^y\rho-\\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Y\end{array}\right]u''\end{array}\right\}\widetilde{Y}_t. \tag{2.50}$$

Derivation of q_{σ_ε} Taking the derivative of (2.39) with respect to σ_ε at the point $\left(\overline{B},\overline{Y},0\right)$ and multiplying it by σ_ε yields:

$$\mathrm{E}_t\left[q_{\sigma_\varepsilon}\sigma_\varepsilon\right]\simeq\mathrm{E}_t\left\{\left[\begin{array}{c}\left(\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)u''\left(G_t\right)-\\ \beta_g k^B\left[\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\frac{\partial L_{t+1}^B}{\partial B_{t+1}}+\left(\overline{B}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right]u'\left(G_{t+1}\right)\\ -\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right]*\\ \left(\begin{array}{c}\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}-\left(1+r+k^B\overline{L^B}\right)\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}-\\ k^B\left(\overline{B}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}\end{array}\right)u''\left(G_{t+1}\right)\end{array}\right]\sigma_\varepsilon\right\},$$

since $\frac{\partial B_t}{\partial\sigma_\varepsilon}=\frac{\partial Y_t}{\partial\sigma_\varepsilon}=0$. Using (2.42), (2.46) and the expression for $\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}$, we can rewrite the last expression as

$$\mathrm{E}_t\left[q_{\sigma_\varepsilon}\sigma_\varepsilon\right]\simeq\mathrm{E}_t\left\{\left[\begin{array}{c}h_1^\sigma u''-\beta_g k^B\gamma\left[\begin{array}{c}2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\end{array}\right]h_1^\sigma u'\\ -\left(\begin{array}{c}h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\tau^y\varepsilon_{t+1}-\\ \left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right]h_1^\sigma\end{array}\right)u''\end{array}\right]\sigma_\varepsilon\right\}. \tag{2.51}$$

First-order Taylor approximation of (2.43)

With $q\left(\overline{B},\overline{Y},0\right)=0$ and (2.49), (2.50), and (2.51), we can write (2.43) as

$$
A_1 \equiv
\left\{
\begin{array}{c}
\left[
\begin{array}{c}
\left(h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right)u'' \\
-\beta_g k^B\gamma\left[
\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}
\right]h_1^B u'- \\
\left(\left(h_1^B\right)^2-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^B\right)u''
\end{array}
\right]\widetilde{B}_t \\
+\left[
\begin{array}{c}
\left(h_1^Y+\tau^y\right)u''- \\
\beta_g k^B\gamma\left[
\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}
\right]h_1^Y u'- \\
\left(
\begin{array}{c}
h_1^B h_1^Y+h_1^Y\rho+\tau^y\rho- \\
\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Y u''
\end{array}
\right)
\end{array}
\right]\widetilde{Y}_t+ \\
h_1^\sigma\left[
\begin{array}{c}
u''-\beta_g k^B\gamma\left[
\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}
\right]u' \\
-\left[
\begin{array}{c}
h_1^B+1- \\
\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)
\end{array}
\right]u''
\end{array}
\right]\sigma_\varepsilon
\end{array}
\right\}
\tag{2.52}
$$

where we have lost the terms in ε_{t+1} by taking expectations.

Then, because (2.26) must be equal to zero for any potential combination $\{B_t, Y_t, \sigma_\varepsilon\}$, it must be the case that the derivatives of any order of that expression must also be equal to zero. Hence the coefficients of B_t, Y_t and σ_ε in (2.52) should be zero. This allows us to compute the three unknown variables of that equation, namely h_1^B, h_1^Y, and h_1^σ.

Computation of h_1^B The coefficient of $\widetilde{B}_t$ in (2.52) is zero, hence

$$
\left[
\begin{array}{c}
-u''*\left(h_1^B\right)^2-u''*\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)+ \\
\left[
\begin{array}{c}
u''+u''*\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)- \\
\beta_g k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]u'
\end{array}
\right]\left(h_1^B\right)
\end{array}
\right]=0.
$$

Multiplying the last equation by $-\frac{\beta_g}{u''}$ and using (2.42) twice, we simplify it to

$$
\left[
\begin{array}{c}
\beta_g\left(h_1^B\right)^2+1+ \\
\left[
\begin{array}{c}
-\beta_g-1+ \\
\frac{u'}{u''}\beta_g^2 k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]
\end{array}
\right]\left(h_1^B\right)
\end{array}
\right]=0,
\tag{2.53}
$$

since u'' is always different from zero. Thus, the equation above is quadratic in h_1^B and by solving it, we obtain

$$h_1^B = \frac{\left\{\begin{array}{c} \left(\beta_g + 1 - \frac{u'}{u''}\beta_g^2 k^B \gamma \left[\begin{array}{c} 2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \\ \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right) \end{array}\right]\right) \pm \\ \sqrt{\left(\beta_g + 1 - \frac{u'}{u''}\beta_g^2 k^B \gamma \left[\begin{array}{c} 2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \\ \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right) \end{array}\right]\right)^2 - 4\beta_g} \end{array}\right\}}{2\beta_g} \tag{2.54}$$

Equation (2.54) features two solutions (roots). To ensure a non-explosive path for the public debt, we need to pick the solution that is smaller than 1 in absolute value.

Computation of h_1^Y The coefficient of $\widetilde{Y}_t$ in (2.52) equals zero, hence

$$\left[\begin{array}{c} h_1^Y \left[\begin{array}{c} \left(1 - \rho - h_1^B + 1 + r + k^B\overline{L^B} + \gamma k^B\left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) u'' - \\ \beta_g k^B \gamma \left[2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\right] u' \end{array}\right] \\ + \tau^y (1-\rho) u'' \end{array}\right] = 0.$$

Hence,

$$h_1^Y = \frac{\tau^y(\rho - 1)u''}{\left[\begin{array}{c} \left(2 + r + k^B\overline{L^B} + \gamma k^B\left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) - \rho - h_1^B\right) u'' - \\ \beta_g k^B \gamma \left[2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\right] u' \end{array}\right]}. \tag{2.55}$$

By plugging in the stable root of h_1^B (2.54), we thus find the solution for h_1^Y.

Computation of h_1^σ Because the coefficient of σ_ε in (2.52) is also zero, we have

$$h_1^\sigma = 0. \tag{2.56}$$

First-order approximation of B_{t+1}

Using (2.54), (2.55), and (2.56), the first order approximation of the true non-linear solution of B_{t+1} around the point $\left(\overline{B}, \overline{Y}, 0\right)$ becomes

$$B_{t+1} = h\left(B_t, Y_t, \sigma_\varepsilon\right) \approx h\left(\overline{B}, \overline{Y}, 0\right) + h_1^B \widetilde{B}_t + h_1^Y \widetilde{Y}_t. \tag{2.57}$$

2.F.2 Second-order approximation of (2.26)

The Second-Order Taylor Expansion of (2.26) around the point $\left(\overline{B}, \overline{Y}, 0\right)$ is

$$\mathrm{E}_t\left\{q\left(B_t, Y_t, \sigma_\varepsilon\right)\right\} \simeq \tfrac{1}{2}\mathrm{E}_t\begin{bmatrix} q_{BB}\widetilde{B}_t^2 + q_{YY}\widetilde{Y}_t^2 + q_{\sigma_\varepsilon\sigma_\varepsilon}\sigma_\varepsilon^2 + \\ 2q_{BY}\widetilde{B}_t\widetilde{Y}_t + 2q_{B\sigma_\varepsilon}\widetilde{B}_t\sigma_\varepsilon + 2q_{Y\sigma_\varepsilon}\widetilde{Y}_t\sigma_{\varepsilon\varepsilon}^2 \end{bmatrix} = 0, \tag{2.58}$$

where all partial derivatives are evaluated at $\left(\overline{B}, \overline{Y}, 0\right)$ and where we have used that $\mathrm{E}_t\left[q + q_B\widetilde{B}_t + q_Y\widetilde{Y}_t + q_{\sigma_\varepsilon}\sigma_\varepsilon\right] = 0$, where $q \equiv q\left(\overline{B}, \overline{Y}, 0\right)$.

Derivation of the second-order partial derivatives q_{BB}, q_{BY}, $q_{B\sigma_\varepsilon}$, q_{YY}, $q_{Y\sigma_\varepsilon}$, and $q_{\sigma_\varepsilon\sigma_\varepsilon}$

For future use, we first calculate the second-order partial derivatives of (2.25) and (2.47) with respect to B_t, Y_t and σ_ε:

Derivation of the second-order partial derivative of (2.25) with respect to $B_t, Y_t, \sigma_\varepsilon$ We have:

$$\frac{\partial^2 B_{t+1}}{\partial^2 B_t} = \frac{\partial\left(\partial h_t/\partial B_t\right)}{\partial B_t} = \frac{\partial^2 h_t}{\partial^2 B_t} \equiv h_{11}^B. \tag{2.59}$$

- Further,
$$\frac{\partial^2 B_{t+1}}{\partial B_t \partial Y_t} = \frac{\partial^2 h_t}{\partial B_t \partial Y_t} \equiv h_{11}^{BY}. \tag{2.60}$$
- Further,
$$\frac{\partial^2 B_{t+1}}{\partial Y_t^2} = \frac{\partial^2 h_t}{\partial^2 Y_t} \equiv h_{11}^{Y}. \tag{2.61}$$
- Further,
$$\frac{\partial^2 B_{t+1}}{\partial B_t \partial \sigma_\varepsilon} = \frac{\partial^2 h_t}{\partial B_t \partial \sigma_\varepsilon} \equiv h_{11}^{B\sigma}. \tag{2.62}$$
- Further,
$$\frac{\partial^2 B_{t+1}}{\partial Y_t \partial \sigma_\varepsilon} = \frac{\partial^2 h_t}{\partial Y_t \partial \sigma_\varepsilon} \equiv h_{11}^{Y\sigma}. \tag{2.63}$$
- Finally,
$$\frac{\partial^2 B_{t+1}}{\partial^2 \sigma_\varepsilon} = \frac{\partial^2 h_t}{\partial^2 \sigma_\varepsilon} \equiv h_{11}^{\sigma}. \tag{2.64}$$

Derivation of the second-order partial derivatives of (2.47) with respect to $B_t, Y_t, \sigma_\varepsilon$

- Computation of $\frac{\partial^2 B_{t+2}}{\partial^2 B_t}$:

$$\begin{aligned}
\frac{\partial^2 B_{t+2}}{\partial^2 B_t} &= \frac{\partial\left(\frac{\partial B_{t+2}}{\partial B_t}\right)}{\partial B_t} = \frac{\partial\left(\frac{\partial B_{t+2}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right)}{\partial B_t} = \frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial B_t} * \frac{\partial B_{t+1}}{\partial B_t} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial\left(\frac{\partial B_{t+1}}{\partial B_t}\right)}{\partial B_t} \\
&= \left[\frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial B_{t+1}} * \frac{\partial B_{t+1}}{\partial B_t}\right] * \frac{\partial B_{t+1}}{\partial B_t} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial^2 B_{t+1}}{\partial^2 B_t},
\end{aligned}$$

since the first order derivative of Y_{t+1} and σ_ε with respect to B_t is equal to zero. The second equality maked use of the chain rule. Using (2.44) and (2.59), we can rewrite it as

$$\frac{\partial^2 B_{t+2}}{\partial^2 B_t} = \left[\frac{\partial^2 h_{t+1}}{\partial^2 B_{t+1}} * \frac{\partial B_{t+1}}{\partial B_t}\right] * \frac{\partial B_{t+1}}{\partial B_t} + h_1^B * h_{11}^B = h_{11}^B \left(h_1^B\right)^2 + h_1^B h_{11}^B.$$

- Computation of $\frac{\partial^2 B_{t+2}}{\partial B_t \partial Y_t}$:

$$\begin{aligned}
\frac{\partial^2 B_{t+2}}{\partial B_t \partial Y_t} &= \frac{\partial}{\partial B_t}\left(\frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial Y_t}\right) \Rightarrow \\
\frac{\partial^2 B_{t+2}}{\partial B_t \partial Y_t} &= \frac{\partial}{\partial B_t}\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right) \Rightarrow \\
&= \frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial B_t}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial\left(\frac{\partial B_{t+1}}{\partial Y_t}\right)}{\partial B_t} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial Y_{t+1}}\right)}{\partial B_t}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial\left(\frac{\partial Y_{t+1}}{\partial Y_t}\right)}{\partial B_t} \\
&= \frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial B_t}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial\left(\frac{\partial B_{t+1}}{\partial Y_t}\right)}{\partial B_t} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial Y_{t+1}}\right)}{\partial B_t}\rho \\
&= \left[\frac{\partial^2 h_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial B_t} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial B_t}\right]\frac{\partial B_{t+1}}{\partial Y_t} + \\
& h_1^B * \left[\frac{\partial^2 B_{t+1}}{\partial Y_t \partial B_t}\frac{\partial B_t}{\partial B_t} + \frac{\partial^2 B_{t+1}}{\partial^2 Y_t}\frac{\partial Y_t}{\partial B_t} + \frac{\partial^2 B_{t+1}}{\partial Y_t \partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial B_t}\right] + \\
& \rho * \left[\frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t} + \frac{\partial^2 h_{t+1}}{\partial^2 Y_{t+1}}\frac{\partial Y_{t+1}}{\partial B_t} + \frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial B_t}\right] \\
&= h_{11}^B h_1^B h_1^Y + (1+\rho)\, h_1^B h_{11}^{BY},
\end{aligned}$$

using $\frac{\partial \sigma_\varepsilon}{\partial Y_t} = 0$, $\frac{\partial Y_{t+1}}{\partial Y_t} = \rho$, (2.44), and (2.60).

- Computation of $\frac{\partial^2 B_{t+2}}{\partial B_t \partial \sigma_\varepsilon}$: knowing (2.44) and (2.62), we compute this derivative as

$$\begin{aligned}
\frac{\partial^2 B_{t+2}}{\partial B_t \partial \sigma_\varepsilon} &= \frac{\partial}{\partial B_t}\left(\frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial \sigma_\varepsilon}\right) = \frac{\partial}{\partial B_t}\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right) \\
&= \left[\begin{array}{c} \frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial B_t} * \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial\left(\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial B_t} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial Y_{t+1}}\right)}{\partial B_t} * \frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} \\ + \frac{\partial h_{t+1}}{\partial Y_{t+1}} * \frac{\partial\left(\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial B_t} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial B_t} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon} * \frac{\partial(1)}{\partial B_t} \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\partial^2 h_{t+1}}{\partial^2 B_{t+1}} * \frac{\partial B_{t+1}}{\partial B_t} * \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial^2 B_{t+1}}{\partial B_t \partial \sigma_\varepsilon} + \frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial B_{t+1}} * \frac{\partial B_{t+1}}{\partial B_t} * \frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} \\ + \frac{\partial h_{t+1}}{\partial Y_{t+1}} * \frac{\partial \varepsilon_{t+1}}{\partial B_t} + \frac{\partial^2 h_{t+1}}{\partial \sigma_\varepsilon \partial B_{t+1}} * \frac{\partial B_{t+1}}{\partial B_t} \end{array}\right] \\
&= h_{11}^B h_1^B h_1^\sigma + h_1^B h_{11}^{B\sigma} + h_1^B h_{11}^{BY}\varepsilon_{t+1} + h_1^B h_{11}^{B\sigma} = h_{11}^B h_1^B h_1^\sigma + 2h_1^B h_{11}^{B\sigma} + h_1^B h_{11}^{BY}\varepsilon_{t+1}.
\end{aligned}$$

- Computation of $\frac{\partial^2 B_{t+2}}{\partial^2 Y_t}$:

$$
\begin{aligned}
\frac{\partial^2 B_{t+2}}{\partial^2 Y_t} &= \frac{\partial}{\partial Y_t}\left(\frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial Y_t}\right) = \frac{\partial}{\partial Y_t}\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right) \\
&= \frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial Y_t}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial\left(\frac{\partial B_{t+1}}{\partial Y_t}\right)}{\partial Y_t} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial Y_{t+1}}\right)}{\partial Y_t}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial\left(\frac{\partial Y_{t+1}}{\partial Y_t}\right)}{\partial Y_t} \\
&= \left[\begin{array}{c}
\left[\frac{\partial^2 h_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right] * \frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial^2 B_{t+1}}{\partial^2 Y_t} \\
+\left[\frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial^2 Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right] * \frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial Y_{t+1}} * \frac{\partial \rho}{\partial Y_t}
\end{array}\right],
\end{aligned}
$$

since $\frac{\partial \sigma_\varepsilon}{\partial Y_t} = 0$ and $\frac{\partial Y_{t+1}}{\partial Y_t} = \rho$. In addition, given (2.45), (2.60) and (2.61); we rewrite the last equation as

$$
\begin{aligned}
\frac{\partial^2 B_{t+2}}{\partial^2 Y_t} &= h_{11}^B\left(h_1^Y\right)^2 + h_{11}^{BY}\rho h_1^Y + h_1^B h_{11}^Y + h_{11}^{BY} h_1^Y \rho + h_{11}^Y \rho^2 \\
&= h_{11}^B\left(h_1^Y\right)^2 + 2h_{11}^{BY}\rho h_1^Y + h_{11}^Y\left(h_1^B + \rho^2\right).
\end{aligned}
$$

- Computation of $\frac{\partial^2 B_{t+2}}{\partial Y_t \partial \sigma_\varepsilon}$:

$$
\begin{aligned}
\frac{\partial^2 B_{t+2}}{\partial Y_t \partial \sigma_\varepsilon} &= \frac{\partial}{\partial Y_t}\left(\frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial \sigma_\varepsilon}\right) \\
&= \frac{\partial}{\partial Y_t}\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right) \\
&= \left[\begin{array}{c}
\frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial Y_t} * \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial\left(\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial Y_t} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial Y_{t+1}}\right)}{\partial Y_t} * \varepsilon_{t+1} \\
+\frac{\partial h_{t+1}}{\partial Y_{t+1}} * \frac{\partial(\varepsilon_{t+1})}{\partial Y_t} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial Y_t} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon} * \frac{\partial(1)}{\partial Y_t}
\end{array}\right] \\
&= \left[\begin{array}{c}
\left[\frac{\partial^2 h_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right] * \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial^2 B_{t+1}}{\partial \sigma_\varepsilon \partial Y_t} \\
+\left[\frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial^2 Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right] * \varepsilon_{t+1} + \\
\frac{\partial^2 h_{t+1}}{\partial \sigma_\varepsilon \partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial \sigma_\varepsilon \partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial^2 h_{t+1}}{\partial^2 \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}
\end{array}\right] \\
&= \left(h_{11}^B h_1^Y + h_{11}^{BY}\rho\right) h_1^\sigma + h_1^B h_{11}^{Y\sigma} + \left(h_{11}^{BY} h_1^Y + h_{11}^Y \rho\right)\varepsilon_{t+1} + h_{11}^{B\sigma} h_1^Y + h_{11}^{Y\sigma}\rho.
\end{aligned}
$$

- Computation of $\frac{\partial^2 B_{t+2}}{\partial^2 \sigma_\varepsilon}$:

$$\begin{aligned}
\frac{\partial^2 B_{t+2}}{\partial^2 \sigma_\varepsilon} &= \frac{\partial}{\partial \sigma_\varepsilon}\left(\frac{\partial h\left(B_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial \sigma_\varepsilon}\right) \\
&= \frac{\partial}{\partial \sigma_\varepsilon}\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right) \\
&= \left[\begin{array}{c}
\frac{\partial\left(\frac{\partial h_{t+1}}{\partial B_{t+1}}\right)}{\partial \sigma_\varepsilon} * \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial\left(\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial \sigma_\varepsilon} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial Y_{t+1}}\right)}{\partial \sigma_\varepsilon} * \varepsilon_{t+1} \\
+\frac{\partial h_{t+1}}{\partial Y_{t+1}} * \frac{\partial(\varepsilon_{t+1})}{\partial \sigma_\varepsilon} + \frac{\partial\left(\frac{\partial h_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial \sigma_\varepsilon} * \frac{\partial(1)}{\partial \sigma_\varepsilon}
\end{array}\right] \\
&= \left[\begin{array}{c}
\left[\frac{\partial^2 h_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\sigma_\varepsilon} + \frac{\partial^2 h_{t+1}}{\partial B_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right] * \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial h_{t+1}}{\partial B_{t+1}} * \frac{\partial^2 B_{t+1}}{\partial^2 \sigma_\varepsilon} \\
+\left[\frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial^2 h_{t+1}}{\partial^2 Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial^2 h_{t+1}}{\partial Y_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right] * \varepsilon_{t+1} + \\
\frac{\partial^2 h_{t+1}}{\partial \sigma_\varepsilon \partial B_{t+1}}\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial^2 h_{t+1}}{\partial \sigma_\varepsilon \partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial^2 h_{t+1}}{\partial^2 \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}
\end{array}\right] \\
&= \left[\begin{array}{c}
\left(h_{11}^B h_1^\sigma + h_{11}^{BY}\varepsilon_{t+1} + h_{11}^{B\sigma}\right) h_1^\sigma + \left(h_{11}^{BY} h_1^\sigma + h_{11}^{Y\sigma}\right)\varepsilon_{t+1} \\
+h_{11}^{Y}\varepsilon_{t+1}^2 + h_{11}^{B\sigma} h_1^\sigma + h_{11}^{Y\sigma}\varepsilon_{t+1} + \left(1 + h_1^B\right) h_{11}^\sigma
\end{array}\right].
\end{aligned}$$

Derivation of q_{BB} That is easier found by taking the partial derivative of q_B with respect to B_t and approximating it around the point $\left(\overline{B}, \overline{Y}, 0\right)$:

$$q_{BB} = \left[\begin{array}{c}
\frac{\partial}{\partial B_t}\left(u''\left(G_t\right)\right) * \left(\frac{\partial B_{t+1}}{\partial B_t} - \left(1 + r + k^B L_t^B\right) - k^B\left(B_t - B^c\right)\frac{\partial L_t^B}{\partial B_t}\right) \\
+u''\left(G_t\right) * \frac{\partial}{\partial B_t}\left(\frac{\partial B_{t+1}}{\partial B_t} - \left(1 + r + k^B L_t^B\right) - k^B\left(B_t - B^c\right)\frac{\partial L_t^B}{\partial B_t}\right) - \\
\beta_g k^B \left[\begin{array}{c}
\frac{\partial}{\partial B_t}\left(\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right) + \frac{\partial}{\partial B_t}\left(\frac{\partial B_{t+1}}{\partial B_t}\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right) + \\
\frac{\partial}{\partial B_t}\left(\left(B_{t+1} - B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right)
\end{array}\right] u' \\
-\beta_g k^B \left[\begin{array}{c}
\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t} + \frac{\partial B_{t+1}}{\partial B_t}\frac{\partial L_{t+1}^B}{\partial B_{t+1}} + \\
\left(B_{t+1} - B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial B_{t+1}^2}\frac{\partial B_{t+1}}{\partial B_t}
\end{array}\right] * \frac{\partial}{\partial B_t}\left(u'\left(G_{t+1}\right)\right) - \\
\beta_g * \frac{\partial}{\partial B_t}\left(\begin{array}{c}
1 + r + k^B L_{t+1}^B + \\
k^B\left(B_{t+1} - B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}
\end{array}\right) * u'' * \left(\begin{array}{c}
\frac{\partial B_{t+2}}{\partial B_t} - \left(1 + r + k^B L_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial B_t} \\
-k^B\left(B_{t+1} - B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}
\end{array}\right) - \\
\beta_g \left[\begin{array}{c}
1 + r + k^B L_{t+1}^B + \\
k^B\left(B_{t+1} - B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}
\end{array}\right] * \frac{\partial}{\partial B_t}\left(u''\left(G_{t+1}\right)\right) * \left(\begin{array}{c}
\frac{\partial B_{t+2}}{\partial B_t} - \\
\left(1 + r + k^B L_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial B_t} \\
-k^B\left(B_{t+1} - B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}
\end{array}\right) \\
-\beta_g \left[\begin{array}{c}
1 + r + k^B L_{t+1}^B + \\
k^B\left(B_{t+1} - B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}
\end{array}\right] * u'' * \frac{\partial}{\partial B_t}\left(\begin{array}{c}
\frac{\partial B_{t+2}}{\partial B_t} - \left(1 + r + k^B L_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial B_t} \\
-k^B\left(B_{t+1} - B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}
\end{array}\right).
\end{array}\right].$$

Then, using (2.40) and (2.48) we already simplify the last expression around the point

$\left(\overline{B},\overline{Y},0\right)$ to:

$$q_{BB}=\left[\begin{array}{c}
\left[\frac{\partial B_{t+1}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right]^2 u'''+\\
\left[\frac{\partial^2 B_{t+1}}{\partial^2 B_t}-k^B\left(2\frac{\partial L_t^B}{\partial B_t}+\left(\overline{B}-B^c\right)\frac{\partial^2 L_t^B}{\partial^2 B_t}\right)\right]u''-\\
\beta_g k^B\left[\begin{array}{c}
\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial B_t}\right)^2+\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial^2 B_t}+\frac{\partial^2 B_{t+1}}{\partial^2 B_t}\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\\
+\left(\frac{\partial B_{t+1}}{\partial B_t}\right)^2\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}+\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial B_t}\right)^2+\\
\left(\overline{B}-B^c\right)\frac{\partial^3 L^B_{t+1}}{\partial^3 B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial B_t}\right)^2+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial^2 B_t}
\end{array}\right]u'-\\
\beta_g k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]\frac{\partial B_{t+1}}{\partial B_t}*\\
\left[\frac{\partial B_{t+2}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t}\right]u''-\\
\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}+\left(B_{t+1}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right]*\\
\left[\frac{\partial B_{t+2}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t}\right]u''-\\
\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}(1-\overline{L^B})\right]*\\
\left[\frac{\partial B_{t+2}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t}\right]^2 u'''-\\
\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}(1-\overline{L^B})\right]*\\
\left[\begin{array}{c}
\frac{\partial^2 B_{t+2}}{\partial^2 B_t}-2k^B\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial B_t}\right)^2-\\
\left(1+r+k^B\overline{L^B}+k^B\left(\overline{B}-B^c\right)\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\right)\frac{\partial^2 B_{t+1}}{\partial^2 B_t}\\
-k^B\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial B_t}\right)^2
\end{array}\right]u''
\end{array}\right],$$

where all functions are evaluated at $\left(\overline{B},\overline{Y},0\right)$.

Using the derivatives of B_{t+1} and B_{t+2} computed above and again (2.40) and (2.48), we simplify the last equation once more to

$$
q_{BB} = \left[\begin{array}{c}
\left[h_1^B - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right) \left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right)\right]^2 u''' + \\
\left[h_{11}^B - k^B \gamma \left(2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\right)\right] u'' - \\
\beta_g k^B \left[\begin{array}{c}
\left[3\gamma^2 \left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right) + \left(\overline{B} - B^c\right) \frac{\partial^3 L^B_{t+1}}{\partial B^3_{t+1}}\right] \left(h_1^B\right)^2 + \\
\left[2\gamma\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \gamma^2\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\right] h_{11}^B
\end{array} \right] u' \\
-2\beta_g k^B \gamma \left[2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\right] u'' * \\
h_1^B \left[\left(h_1^B\right)^2 - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) h_1^B\right] - \\
\beta_g \left[1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right] * \\
\left[\left(h_1^B\right)^2 - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) h_1^B\right]^2 u''' \\
-\beta_g \left[1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \overline{L^B}^2\right)\right] * \\
\left[\begin{array}{c}
h_{11}^B \left(h_1^B\right)^2 + h_1^B h_{11}^B - \\
\left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) h_{11}^B - \\
k^B \left(\begin{array}{c}
2\gamma\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \\
\gamma^2\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)
\end{array} \right) \left(h_1^B\right)^2
\end{array} \right] u''
\end{array} \right].
$$

The only additional term that we have to calculate in the equation above is $\frac{\partial^3 L^B_{t+1}}{\partial^3 B_{t+1}}$, but from (2.48) we get

$$
\begin{aligned}
\frac{\partial^3 L^B_{t+1}}{\partial B^3_{t+1}} &= \frac{\partial}{\partial B_{t+1}} \left[\gamma^2 \left(L^B_{t+1} - 3\left(L^B_{t+1}\right)^2 + 2\left(L^B_{t+1}\right)^3\right)\right] \\
&= \gamma^2 \left(\begin{array}{c}
\gamma(L^B_{t+1} - \left(L^B_{t+1}\right)^2) - 6L^B_{t+1}\gamma(L^B_{t+1} - \left(L^B_{t+1}\right)^2) \\
+6\left(L^B_{t+1}\right)^2 \gamma(L^B_{t+1} - \left(L^B_{t+1}\right)^2)
\end{array} \right) \\
&= \gamma^3 \left(L^B_{t+1} - 7\left(L^B_{t+1}\right)^2 + 12\left(L^B_{t+1}\right)^3 - 6\left(L^B_{t+1}\right)^4\right). \qquad (2.65)
\end{aligned}
$$

Thus, using (2.65) in our equation of q_{BB} together with (2.42), we have that:

$$
q_{BB} = \left[\begin{array}{c}
\left[h_1^B - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c \right) \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) \right) \right]^2 u''' + \\
\left[h_{11}^B - k^B \gamma \left(2 \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) + \gamma \left(\overline{B} - B^c \right) \left(\overline{L^B} - 3 \left(\overline{L^B} \right)^2 + 2 \left(\overline{L^B} \right)^3 \right) \right) \right] u'' - \\
\beta_g k^B \left[\begin{array}{c}
\left(\begin{array}{c}
3\gamma^2 \left(\overline{L^B} - 3 \left(\overline{L^B} \right)^2 + 2 \left(\overline{L^B} \right)^3 \right) + \\
\left(\overline{B} - B^c \right) \gamma^3 \left(\overline{L^B} - 7 \left(\overline{L^B} \right)^2 + 12 \left(\overline{L^B} \right)^3 - 6 \left(\overline{L^B} \right)^4 \right)
\end{array} \right) \left(h_1^B \right)^2 \\
+ \left(\begin{array}{c}
2\gamma \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) + \\
\gamma^2 \left(\overline{B} - B^c \right) \left(\overline{L^B} - 3 \left(\overline{L^B} \right)^2 + 2 \left(\overline{L^B} \right)^3 \right)
\end{array} \right) h_{11}^B
\end{array} \right] u' \\
-2\beta_g k^B \gamma \left[2 \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) + \gamma \left(\overline{B} - B^c \right) \left(\overline{L^B} - 3 \left(\overline{L^B} \right)^2 + 2 \left(\overline{L^B} \right)^3 \right) \right] * \\
\left[\left(h_1^B \right)^3 - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c \right) \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) \right) \left(h_1^B \right)^2 \right] u'' \\
- \left[\left(h_1^B \right)^2 - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c \right) \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) \right) h_1^B \right]^2 u''' - \\
\left[\begin{array}{c}
\left(\left(h_1^B \right)^2 + h_1^B \right) h_{11}^B - \\
\left[1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c \right) \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) \right] h_{11}^B - \\
k^B \left[\begin{array}{c}
2\gamma \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) + \\
\gamma^2 \left(\overline{B} - B^c \right) \left(\overline{L^B} - 3 \left(\overline{L^B} \right)^2 + 2 \left(\overline{L^B} \right)^3 \right)
\end{array} \right] \left(h_1^B \right)^2
\end{array} \right] u''
\end{array} \right].
$$

Finally, isolating the only unknown term h_{11}^B of the equation above and multiplying it to $\widetilde{B}_t^2$:

$$q_{BB}\widetilde{B}_t^2$$

$$= \left[\begin{array}{c} h_{11}^B \left[\begin{array}{c} \left(2+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)-\left(h_1^B\right)^2-h_1^B\right)u'' \\ -\beta_g k^B\gamma \left(\begin{array}{c} 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right) \end{array}\right)u' \end{array}\right] \\ +\left[h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right]^2\left[1-\left(h_1^B\right)^2\right]u''' \\ -\beta_g k^B\gamma^2\left[\begin{array}{c} 3\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)+ \\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-7\overline{L^B}^2+12\overline{L^B}^3-6\overline{L^B}^4\right) \end{array}\right]\left(h_1^B\right)^2u'+ \\ k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]* \\ \left(-1+3\left(h_1^B\right)^2-2\beta_g\left(h_1^B\right)^3\right)u'' \end{array}\right]\widetilde{B}_t^2$$

$$= \left[\begin{array}{c} h_{11}^B\left[\begin{array}{c}\left(1+\frac{1}{\beta_g}-\left(h_1^B\right)^2-h_1^B\right)u''- \\ \beta_g k^B\gamma z_1 u'\end{array}\right]+ \\ k^B\gamma z_1\left(-1+3\left(h_1^B\right)^2-2\beta_g\left(h_1^B\right)^3\right)u''+\left(h_1^B-\frac{1}{\beta_g}\right)^2\left[1-\left(h_1^B\right)^2\right]u'''- \\ \beta_g k^B\gamma^2\left[\begin{array}{c} 3\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)+ \\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-7\overline{L^B}^2+12\overline{L^B}^3-6\overline{L^B}^4\right)\end{array}\right]\left(h_1^B\right)^2u' \end{array}\right]\widetilde{B}_t^2$$

$$\equiv \left(Mh_{11}^B+N\right)\widetilde{B}_t^2. \tag{2.66}$$

where

$$z_1 \equiv 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right).$$

Derivation of the q_{BY_t} Differentiating q_{Y_t} with respect to B_t at the point $\left(\overline{B},\overline{Y},0\right)$ yields:

$$q_{BY_t} = \left[\begin{array}{c} \frac{\partial}{\partial B_t}\left(u''\left(G_t\right)\right)*\left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right)+u''*\frac{\partial}{\partial B_t}\left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right) \\ -\beta_g k^B\left[2\frac{\partial}{\partial B_t}\left(\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right)+\frac{\partial}{\partial B_t}\left(\left(B_{t+1}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right)\right]u'- \\ \beta_g k^B*\left[2\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}+\left(B_{t+1}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right]*\frac{\partial}{\partial B_t}\left(u'\left(G_{t+1}\right)\right) \\ -\beta_g*\frac{\partial}{\partial B_t}\left(1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)* \\ \left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)\frac{\partial B_{t+1}}{\partial Y_t}\right]u'' \\ -\beta_g*\left[1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*\frac{\partial}{\partial B_t}\left(u''\left(G_{t+1}\right)\right)* \\ \left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)\frac{\partial B_{t+1}}{\partial Y_t}\right] \\ -\beta_g*\left[1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*u''* \\ \frac{\partial}{\partial B_t}\left(\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)\frac{\partial B_{t+1}}{\partial Y_t}\right) \end{array}\right].$$

Then, using (2.40), (2.48) and the derivation of (2.66), we already simplify the last expression around the point $\left(\overline{B},\overline{Y},0\right)$ to:

$$
q_{BY_t}=\left[\begin{array}{c}
\left[\frac{\partial B_{t+1}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right]\left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right)u'''\\
+u''\frac{\partial^2 B_{t+1}}{\partial B_t\partial Y_t}-\\
\beta_g k^B\left[\begin{array}{c}2\left(\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\frac{\partial B_{t+1}}{\partial Y_t}+\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial B_t\partial Y_t}\right)+\left(\frac{\partial B_{t+1}}{\partial B_t}\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right)\\
+\left(\overline{B}-B^c\right)\left(\frac{\partial^3 L^B_{t+1}}{\partial^3 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\frac{\partial B_{t+1}}{\partial B_t}+\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial B_t\partial Y_t}\right)\end{array}\right]u'\\
-\beta_g k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]*\\
h_1^Y\left[\frac{\partial B_{t+2}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t}\right]u''\\
-\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}+\left(B_{t+1}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right]*\\
\left[h_1^B h_1^Y+h_1^Y\rho+\tau^y\rho-\left(\begin{array}{c}1+r+k^B\overline{L^B}+\\ \gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\end{array}\right)h_1^Y\right]u''\\
-\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right]*\\
\left\{\begin{array}{c}\left[\frac{\partial B_{t+2}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t}\right]\\
*\left[\begin{array}{c}h_1^B h_1^Y+h_1^Y\rho+\tau^y\rho-\\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Y\end{array}\right]\end{array}\right\}u'''\\
-\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right]*\\
\left[\begin{array}{c}\frac{\partial^2 B_{t+2}}{\partial B_t\partial Y_t}-\\
k^B\left(\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}+\frac{\partial B_{t+1}}{\partial B_t}\frac{\partial L^B_{t+1}}{\partial B_{t+1}}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right)\frac{\partial B_{t+1}}{\partial Y_t}\\
-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial^2 B_{t+1}}{\partial B_t\partial Y_t}\end{array}\right]u''.
\end{array}\right].
$$

Using the derivatives of B_{t+1} and B_{t+2}, (2.40), (2.48) and (2.65), and (2.42), we simplify the last equation once more to

$$
q_{BY_t} = \left[\begin{array}{c}
\left[h_1^B - \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right]\left(h_1^Y+\tau^y\right)u''' \\
+h_{11}^{BY}u''- \\
\beta_g k^B \left[\begin{array}{c} 3\gamma^2\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)h_1^Bh_1^Y+2\gamma\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)h_{11}^{BY}+ \\ +\left(\overline{B}-B^c\right)\gamma^3\left(\overline{L^B}-7\left(\overline{L^B}\right)^2+12\left(\overline{L^B}\right)^3-6\left(\overline{L^B}\right)^4\right)h_1^Bh_1^Y+ \\ \left(\overline{B}-B^c\right)\gamma^2\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)h_{11}^{BY} \end{array}\right]u' \\
-\beta_g k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]* \\
\left(\left(h_1^B\right)^2h_1^Y-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Bh_1^Y\right]u''- \\
\beta_g k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]h_1^B* \\
\left[\begin{array}{c} h_1^Bh_1^Y+h_1^Y\rho+\tau^y\rho- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Y \end{array}\right]u'' \\
-\left\{\begin{array}{c} \left(\left(h_1^B\right)^2-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^B\right)* \\ \left(\begin{array}{c} h_1^Bh_1^Y+h_1^Y\rho+\tau^y\rho- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Y \end{array}\right) \end{array}\right\}u''' \\
-\left(\begin{array}{c} h_{11}^Bh_1^Bh_1^Y+h_1^Bh_{11}^{BY}+\rho h_1^Bh_{11}^{BY}- \\ k^B\gamma\left[\begin{array}{c} 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right) \end{array}\right]h_1^Bh_1^Y \\ -\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_{11}^{BY} \end{array}\right)u''
\end{array}\right].
$$

Thus, using (2.42) again and isolating the two unknown coefficients h_{11}^{BY} and h_{11}^{B}, we can rewrite the equation above as:

$$
q_{BY_t} = \left[\begin{array}{c}
h_{11}^{BY} \left[\begin{array}{c}
u''- \\
\beta_g k^B \gamma \left[2(\overline{L^B}-\overline{L^B}^2)+\left(\overline{B}-B^c\right)\gamma\left(\overline{L^B}-3\overline{L^B}^2+2\overline{L^B}^3\right)\right] u' \\
-\left[\begin{array}{c} h_1^B\left(1+\rho\right)- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\overline{L^B}^2\right)\right) \end{array} \right] u''
\end{array} \right] \\
-h_{11}^B h_1^B h_1^Y u''+ \\
\left[h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\overline{L^B}^2\right)\right)\right]\left(h_1^Y+\tau^y\right)u''' \\
-\beta_g k^B \gamma^2 \left[\begin{array}{c} 3\left(\overline{L^B}-3\overline{L^B}^2+2\overline{L^B}^3\right)+ \\ \left(\overline{B}-B^c\right)\gamma\left(\overline{L^B}-7\overline{L^B}^2+12\overline{L^B}^3-6\overline{L^B}^4\right) \end{array} \right] h_1^B h_1^Y u'- \\
2\beta_g k^B \gamma \left[2(\overline{L^B}-\overline{L^B}^2)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\overline{L^B}^2+2\overline{L^B}^3\right)\right] * \\
\left[\left(h_1^B\right)^2 h_1^Y-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\overline{L^B}^2\right)\right)h_1^B h_1^Y\right]u''- \\
\beta_g k^B \gamma \left[2(\overline{L^B}-\overline{L^B}^2)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\overline{L^B}^2+2\overline{L^B}^3\right)\right]\left(h_1^B h_1^Y\rho+h_1^B\tau^y\rho\right)u'' \\
-h_1^B\left[h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\overline{L^B}^2\right)\right)\right] * \\
\left[h_1^B h_1^Y+h_1^Y\rho+\tau^y\rho-\left(\begin{array}{c} 1+r+k^B\overline{L^B}+ \\ \gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\overline{L^B}^2\right) \end{array} \right)h_1^Y\right]u''' \\
+k^B\gamma\left[2(\overline{L^B}-\overline{L^B}^2)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\overline{L^B}^2+2\overline{L^B}^3\right)\right]h_1^B h_1^Y u''
\end{array} \right],
$$

which can be simplified once more to

$$
q_{BY_t} = \left[\begin{array}{c}
h_{11}^{BY} \left\{ \begin{array}{c}
u'' * \left[2+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)-h_1^B\left(1+\rho\right)\right] \\
-u'\beta_g k^B\gamma \left[\begin{array}{c} 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\ \left(\overline{B}-B^c\right)\gamma\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right) \end{array} \right]
\end{array} \right\} \\
-h_{11}^B u'' h_1^B h_1^Y+ \\
u''' * \left[h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right] * \\
\left[\begin{array}{c} h_1^Y+\tau^y-\left(h_1^B\right)^2 h_1^Y-h_1^B h_1^Y\rho-h_1^B\tau^y\rho+ \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^B h_1^Y \end{array} \right]- \\
u'\beta_g k^B\gamma^2 \left[\begin{array}{c} 3\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)+ \\ \left(\overline{B}-B^c\right)\gamma\left(\overline{L^B}-7\left(\overline{L^B}\right)^2+12\left(\overline{L^B}\right)^3-6\left(\overline{L^B}\right)^4\right) \end{array} \right] h_1^B h_1^Y+ \\
u'' k^B\gamma \left\{ \begin{array}{c}
\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right] * \\
\left\{ \begin{array}{c}
h_1^B h_1^Y- \\
2\beta_g\left[\left(h_1^B\right)^2 h_1^Y-\left(\begin{array}{c} 1+r+k^B\overline{L^B}+ \\ \gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\overline{L^B}^2\right) \end{array} \right)h_1^B h_1^Y\right] \\
-\beta_g\left(h_1^B h_1^Y\rho+h_1^B\tau^y\rho\right)
\end{array} \right\}
\end{array} \right\}
\end{array} \right].
$$

Finally, multiplying the last equation by $\widetilde{B}_t\widetilde{Y}_t$ leads to

$$
\begin{aligned}
& q_{BY_t}\widetilde{B}_t\widetilde{Y}_t \\
= & \left[\begin{array}{c}
h_{11}^{BY}\left\{\begin{array}{c}
\left[2+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)-h_1^B\left(1+\rho\right)\right]u'' \\
-\beta_g k^B\gamma\left[\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\left(\overline{B}-B^c\right)\gamma\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}\right]u'
\end{array}\right\} \\
-h_{11}^B h_1^B h_1^Y u''+ \\
\left[h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right]* \\
\left[\begin{array}{c}
h_1^Y+\tau^y-\left(h_1^B\right)^2 h_1^Y-h_1^B\tau^y\rho+ \\
\left(1+r+k^B\overline{L^B}-\rho+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^B h_1^Y
\end{array}\right]u'''- \\
\beta_g k^B\gamma^2\left[\begin{array}{c}
3\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)+ \\
\left(\overline{B}-B^c\right)\gamma\left(\overline{L^B}-7\left(\overline{L^B}\right)^2+12\left(\overline{L^B}\right)^3-6\left(\overline{L^B}\right)^4\right)
\end{array}\right]h_1^B h_1^Y u' \\
+k^B\gamma\left\{\begin{array}{c}
\left[\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}\right]* \\
\left(3h_1^B h_1^Y-2\beta_g\left(h_1^B\right)^2 h_1^Y-\beta_g h_1^B\tau^y\rho-\beta_g h_1^B h_1^Y\rho\right)
\end{array}\right\}u''
\end{array}\right]\widetilde{B}_t\widetilde{Y}_t \\
= & \left(Oh_{11}^{BY}+Ph_{11}^B+Q\right)\widetilde{B}_t\widetilde{Y}_t. \qquad (2.67)
\end{aligned}
$$

Therefore, this equation has two linear unknown coefficients h_{11}^{BY} and h_{11}^B. All other terms are numbers already known.

Derivation of $q_{B\sigma_\varepsilon}$ Next, we compute $q_{B\sigma_\varepsilon}$. For that, we derive q_{σ_ε} with respect to B_t around the point $\left(\overline{B},\overline{Y},0\right)$:

$$
q_{B\sigma_\varepsilon}=\left[\begin{array}{c}
\frac{\partial}{\partial B_t}\left(u''\left(G_t\right)\right)*\left(\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)+u''*\frac{\partial}{\partial B_t}\left(\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)- \\
\beta_g k^B*\left[2\frac{\partial}{\partial B_t}\left(\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)+\frac{\partial}{\partial B_t}\left(\left(B_{t+1}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)\right]*u' \\
-\beta_g k^B\left[2\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\left(\overline{B}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right]*\frac{\partial}{\partial B_t}\left(u'\left(G_{t+1}\right)\right)- \\
\beta_g*\frac{\partial}{\partial B_t}\left(1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)*u''* \\
\left(\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}-\left(1+r+k^B L_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}-k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\varepsilon_{t+1}\right) \\
-\beta_g\left[1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*\frac{\partial}{\partial B_t}\left(u''\left(G_{t+1}\right)\right)* \\
\left(\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}-\left(1+r+k^B L_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}-k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\varepsilon_{t+1}\right) \\
-\beta_g\left[1+r+k^B L_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*u''* \\
\frac{\partial}{\partial B_t}\left(\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}-\left(1+r+k^B L_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}-k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\varepsilon_{t+1}\right)
\end{array}\right].
$$

Using our earlier expressions for the derivatives of B_{t+1} and B_{t+2} with respect to σ_ε, the last expression becomes

$$q_{B\sigma_\varepsilon}=\left[\begin{array}{c}
\left[h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\right]h_1^\sigma u'''+u''*\frac{\partial^2 B_{t+1}}{\partial B_t\partial\sigma_\varepsilon}\\
-\beta_g k^B\left[\begin{array}{c}2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial B_t\partial\sigma_\varepsilon}+3\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}h_1^\sigma+\\ \left(\overline{B}-B^c\right)\frac{\partial^3 L^B_{t+1}}{\partial^3 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}h_1^\sigma+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial B_t\partial\sigma_\varepsilon}\end{array}\right]u'\\
-\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}h_1^\sigma+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}h_1^\sigma\right]*\\
\left[\frac{\partial B_{t+2}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t}\right]u''-\\
\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right]*\\
\left[\begin{array}{c}h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\varepsilon_{t+1}-\\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^\sigma\end{array}\right]u''-\\
\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right]*\\
\left\{\begin{array}{c}\left[\frac{\partial B_{t+2}}{\partial B_t}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t}\right]*\\ \left[h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\varepsilon_{t+1}-\left(\begin{array}{c}1+r+k^B\overline{L^B}+\\ \gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\end{array}\right)h_1^\sigma\right]\end{array}\right\}u'''\\
-\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right]*\\
\left[\begin{array}{c}\frac{\partial^2 B_{t+2}}{\partial B_t\partial\sigma_\varepsilon}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial B_t\partial\sigma_\varepsilon}\\ -\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial B_t}\right]h_1^\sigma\end{array}\right]u''
\end{array}\right].$$

From (2.56), we know that $h_1^\sigma=0$. Using, in addition, (2.42) and the derivatives of B_{t+1} and B_{t+2}, we can simplify this expression as:

$$q_{B\sigma_\varepsilon}=\left[\begin{array}{c}
+h_{11}^{B\sigma}u''-\\
\beta_g k^B\gamma\left[\begin{array}{c}2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\end{array}\right]h_{11}^{B\sigma}u'-\\
\beta_g k^B\gamma\left[\begin{array}{c}2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\end{array}\right]h_1^B\left(h_1^Y+1\right)\varepsilon_{t+1}u''-\\
\left[\begin{array}{c}\left(h_1^B\right)^2-\\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^B\end{array}\right]\left(h_1^Y+1\right)\varepsilon_{t+1}u'''\\
-\left[\begin{array}{c}2h_1^B h_{11}^{B\sigma}+h_1^B h_{11}^{BY}\varepsilon_{t+1}-\\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_{11}^{B\sigma}\end{array}\right]u''
\end{array}\right].$$

Grouping terms and multiplying it by $\widetilde{B}_t\sigma_\varepsilon$, we have

$$
\begin{aligned}
& q_{B\sigma_\varepsilon}\widetilde{B}_t\sigma_\varepsilon \\
= & \left\{\begin{array}{c}
h_{11}^{B\sigma}\left[\begin{array}{c}
\left[2+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)-2h_1^B\right]u''- \\
\beta_g k^B\gamma\left[\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}\right]u'
\end{array}\right] \\
-u''h_{11}^{BY}h_1^B\varepsilon_{t+1}- \\
h_1^B\left(h_1^Y+1\right)\left[\begin{array}{c}
\beta_g k^B\gamma\left[\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}\right]u'' \\
+\left[h_1^B-\left(\begin{array}{c}
1+r+k^B\overline{L^B}+ \\
\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)
\end{array}\right)\right]u'''
\end{array}\right]\varepsilon_{t+1}
\end{array}\right\}\widetilde{B}_t\sigma_\varepsilon \\
\equiv & \left(Rh_{11}^{B\sigma}+S\varepsilon_{t+1}\right)\widetilde{B}_t\sigma_\varepsilon.
\end{aligned}
$$

Finally, applying expectations, it follows that

$$
\mathrm{E}_t\left[q_{B\sigma_\varepsilon}\widetilde{B}_t\sigma_\varepsilon\right]=\mathrm{E}_t\left[\left(Rh_{11}^{B\sigma}+S\varepsilon_{t+1}\right)\widetilde{B}_t\sigma_\varepsilon\right]=Rh_{11}^{B\sigma}\widetilde{B}_t\sigma_\varepsilon. \tag{2.68}
$$

Derivation of $q_{Y_tY_t}$ Differentitae q_{Y_t} at the point $\left(\overline{B},\overline{Y},0\right)$:

$$
q_{Y_tY_t}=\left[\begin{array}{c}
\frac{\partial}{\partial Y_t}\left(u''\left(G_t\right)\right)*\left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right)+u''*\frac{\partial}{\partial Y_t}\left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right)- \\
\beta_g k^B*\left[2\frac{\partial}{\partial Y_t}\left(\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right)+\frac{\partial}{\partial Y_t}\left(\left(B_{t+1}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial B_{t+1}^2}\frac{\partial B_{t+1}}{\partial Y_t}\right)\right]*u' \\
-\beta_g k^B*\left[2\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}+\left(B_{t+1}-B^c\right)\frac{\partial^2 L_{t+1}^B}{\partial B_{t+1}^2}\frac{\partial B_{t+1}}{\partial Y_t}\right]*\frac{\partial}{\partial Y_t}\left(u'\left(G_{t+1}\right)\right) \\
-\beta_g*\frac{\partial}{\partial Y_t}\left(1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)*u''* \\
\left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)\frac{\partial B_{t+1}}{\partial Y_t}\right] \\
-\beta_g\left[1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*\frac{\partial}{\partial Y_t}\left(u''\left(G_{t+1}\right)\right)* \\
\left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)\frac{\partial B_{t+1}}{\partial Y_t}\right] \\
-\beta_g\left[1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*u''* \\
\frac{\partial}{\partial Y_t}\left(\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)\frac{\partial B_{t+1}}{\partial Y_t}\right)
\end{array}\right].
$$

Hence,

$$
q_{Y_tY_t}=\left[\begin{array}{c}
\left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right)^2u'''+\frac{\partial^2 B_{t+1}}{\partial^2 Y_t}u'' \\
-\beta_g k^B\left[\begin{array}{c}
2\left(\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial Y_t}\right)^2+\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial^2 Y_t}\right)+\left(\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial Y_t}\right)^2\right) \\
+\left(\overline{B}-B^c\right)\frac{\partial^3 L^B_{t+1}}{\partial^3 B_{t+1}}\left(\frac{\partial B_{t+1}}{\partial Y_t}\right)^2+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial^2 Y_t}
\end{array}\right]u' \\
-\beta_g k^B\gamma\left[\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}\right]h^Y_1 * \\
\left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial Y_t}\right]u'' \\
-\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}+\left(B_{t+1}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right] * \\
\left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial Y_t}\right]u'' \\
-\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right] * \\
\left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial Y_t}\right]^2u''' \\
-\beta_g\left[1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right] * \\
\left[\begin{array}{c}
\frac{\partial^2 B_{t+2}}{\partial^2 Y_t}-k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}+\left(B_{t+1}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right]\frac{\partial B_{t+1}}{\partial Y_t} \\
-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\overline{L^B}^2\right)\right)\frac{\partial^2 B_{t+1}}{\partial^2 Y_t}
\end{array}\right]u''
\end{array}\right]
$$

Hence, also using (2.42)

$$
q_{Y_tY_t} = \left[\begin{array}{c}
\left(h_1^Y + \tau^y\right)^2 u''' + h_{11}^Y u'' - \\
\beta_g k^B \left[\begin{array}{c}
2\left[\gamma^2 \left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\left(h_1^Y\right)^2 + \gamma\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) h_{11}^Y\right] \\
+\gamma^2 \left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\left(h_1^Y\right)^2 + \\
\left(\overline{B} - B^c\right)\gamma^2 \left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right) h_{11}^Y \\
+\left(\overline{B} - B^c\right)\gamma^3 \left(\overline{L^B} - 7\left(\overline{L^B}\right)^2 + 12\left(\overline{L^B}\right)^3 - 6\left(\overline{L^B}\right)^4\right)\left(h_1^Y\right)^2
\end{array} \right] u' \\
-\beta_g k^B \gamma \left[2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\right] h_1^Y * \\
u'' \left[h_1^B h_1^Y + h_1^Y \rho + \tau^y \rho - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) h_1^Y\right] \\
-\beta_g k^B \gamma \left[2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)\right] h_1^Y * \\
u'' \left[h_1^B h_1^Y + h_1^Y \rho + \tau^y \rho - \left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) h_1^Y\right] \\
-u''' \left[\begin{array}{c}
h_1^B h_1^Y + h_1^Y \rho + \tau^y \rho - \\
\left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) h_1^Y
\end{array} \right]^2 \\
-u'' \left[\begin{array}{c}
h_{11}^B \left(h_1^Y\right)^2 + 2h_{11}^{BY} \rho h_1^Y + h_{11}^Y \left(h_1^B + \rho^2\right) - \\
k^B \gamma \left[\begin{array}{c}
2\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right) + \\
\gamma\left(\overline{B} - B^c\right)\left(\overline{L^B} - 3\left(\overline{L^B}\right)^2 + 2\left(\overline{L^B}\right)^3\right)
\end{array} \right] \left(h_1^Y\right)^2 \\
-\left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c\right)\left(\overline{L^B} - \left(\overline{L^B}\right)^2\right)\right) h_{11}^Y
\end{array} \right]
\end{array} \right].
$$

Thus, using (2.42) again, and isolating the three unknown terms h_{11}^{Y}, h_{11}^{BY} and h_{11}^{B}; we obtain

$$
\begin{aligned}
q_{Y_tY_t} &= \left[\begin{array}{c}
h_{11}^{Y}\left[\begin{array}{c}
\left[2+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)-h_1^B-\rho^2\right]u''- \\
\beta_g k^B\gamma\left[\begin{array}{c}
2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\
\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)
\end{array}\right]u'
\end{array}\right] \\
-2\rho h_1^Y h_{11}^{BY}u''-\left(h_1^Y\right)^2 h_{11}^B u''- \\
\beta_g k^B\gamma^2\left[\begin{array}{c}
3\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)+ \\
\left(\overline{B}-B^c\right)\gamma\left(\overline{L^B}-7\left(\overline{L^B}\right)^2+12\left(\overline{L^B}\right)^3-6\left(\overline{L^B}\right)^4\right)
\end{array}\right]\left(h_1^Y\right)^2u' \\
+k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]* \\
\left[3\left(h_1^Y\right)^2-2\beta_g h_1^B\left(h_1^Y\right)^2-2\beta_g\left(h_1^Y\right)^2\rho-2\beta_g h_1^Y\tau^y\rho\right]u''+ \\
\left\{\begin{array}{c}
\left(h_1^Y+\tau^y\right)^2- \\
\left[\begin{array}{c}
h_1^Bh_1^Y+h_1^Y\rho+\tau^y\rho- \\
\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Y
\end{array}\right]^2
\end{array}\right\}u'''
\end{array}\right] \\
\Rightarrow\quad q_{Y_tY_t}\widetilde{Y}_t^2 &= \left(Vh_{11}^Y+Xh_{11}^{BY}+Th_{11}^B+Z\right)\widetilde{Y}_t^2. \qquad (2.69)
\end{aligned}
$$

This equation contains three variables that need to be solved for (h_{11}^{Y}, h_{11}^{BY} and h_{11}^{B}).

Derivation of $q_{Y_t\sigma_\varepsilon}$ Next, we compute $q_{Y\sigma_\varepsilon}$. For that, we derive q_{σ_ε} with respect to Y_t around the point $\left(\overline{B},\overline{Y},0\right)$:

$$
q_{Y\sigma_\varepsilon} = \left[\begin{array}{c}
\frac{\partial}{\partial Y_t}\left(u''\left(G_t\right)\right)*\left(\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)+u''*\frac{\partial}{\partial Y_t}\left(\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)- \\
\beta_g k^B*\left[2\frac{\partial}{\partial Y_t}\left(\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)+\frac{\partial}{\partial Y_t}\left(\left(B_{t+1}-B^c\right)\frac{\partial^2L_{t+1}^B}{\partial^2B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right)\right]*u' \\
-\beta_g k^B\left[2\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\left(\overline{B}-B^c\right)\frac{\partial^2L_{t+1}^B}{\partial^2B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}\right]*\frac{\partial}{\partial Y_t}\left(u'\left(G_{t+1}\right)\right)- \\
\beta_g*\frac{\partial}{\partial Y_t}\left(1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right)*u''* \\
\left(\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}-\left(1+r+k^BL_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}-k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}\right) \\
-\beta_g\left[1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*\frac{\partial}{\partial Y_t}\left(u''\left(G_{t+1}\right)\right)* \\
\left(\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}-\left(1+r+k^BL_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}-k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}\right) \\
-\beta_g\left[1+r+k^BL_{t+1}^B+k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\right]*u''* \\
\frac{\partial}{\partial Y_t}\left(\begin{array}{c}
\frac{\partial B_{t+2}}{\partial\sigma_\varepsilon}-\left(1+r+k^BL_{t+1}^B\right)\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon} \\
-k^B\left(B_{t+1}-B^c\right)\frac{\partial L_{t+1}^B}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}
\end{array}\right)
\end{array}\right].
$$

Hence, also using (2.42),

$$
q_{Y\sigma_\varepsilon}=\left[\begin{array}{c}
\left(\frac{\partial B_{t+1}}{\partial Y_t}+\tau^y\right)h_1^\sigma u'''+u''\frac{\partial^2 B_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}- \\
\beta_g k^B\begin{bmatrix} 2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}+3\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}h_1^\sigma+ \\ \left(\overline{B}-B^c\right)\frac{\partial^3 L^B_{t+1}}{\partial^3 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}h_1^\sigma+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}\end{bmatrix}u' \\
-\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\right]h_1^\sigma * \\
\left[\frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial Y_t}\right]u'' \\
-\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right]* \\
\begin{bmatrix} h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\tau^y\varepsilon_{t+1}- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^\sigma\end{bmatrix}u'' \\
-\begin{bmatrix} \frac{\partial B_{t+2}}{\partial Y_t}+\tau^y\rho- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial Y_t}\end{bmatrix}* \\
\begin{bmatrix} h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\tau^y\varepsilon_{t+1}- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^\sigma\end{bmatrix}u''' \\
-\begin{bmatrix} \frac{\partial^2 B_{t+2}}{\partial Y_t\partial\sigma_\varepsilon}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial B_{t+1}}{\partial Y_t\partial\sigma_\varepsilon} \\ -\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial^2 B_{t+1}}\frac{\partial B_{t+1}}{\partial Y_t}\right]h_1^\sigma\end{bmatrix}u''
\end{array}\right].
$$

From (2.56), we know that $h_1^\sigma=0$. Using this, and working out, yields:

$$
q_{Y\sigma_\varepsilon}=\left[\begin{array}{c}
+h_{11}^{Y\sigma}u''- \\
\beta_g k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]h_{11}^{Y\sigma}u' \\
-\beta_g k^B\gamma\begin{bmatrix} 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\end{bmatrix}h_1^Y\left(h_1^Y+\tau^y\right)\varepsilon_{t+1}u''- \\
\begin{bmatrix} h_1^B h_1^Y+h_1^Y\rho+\tau^y\rho- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^Y\end{bmatrix}\left(h_1^Y+\tau^y\right)\varepsilon_{t+1}u'''- \\
\begin{bmatrix} \left(h_{11}^B h_1^Y+h_{11}^{BY}\rho\right)h_1^\sigma+h_1^B h_{11}^{Y\sigma}+\left(h_{11}^{BY}h_1^Y+h_{11}^Y\rho\right)\varepsilon_{t+1}+h_{11}^{B\sigma}h_1^Y+h_{11}^{Y\sigma}\rho \\ -\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_{11}^{Y\sigma}\end{bmatrix}u''
\end{array}\right].
$$

Then, grouping terms, again using $h_1^\sigma = 0$ and multiplying by $\widetilde{Y}_t \sigma_\varepsilon$, we obtain:

$$
\begin{aligned}
& q_{Y\sigma_\varepsilon} \widetilde{Y}_t \sigma_\varepsilon \\
= & \left\{ \begin{array}{c}
h_{11}^{Y\sigma} \left[\begin{array}{c}
u'' \left(2 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c \right) \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) - h_1^B - \rho \right) \\
-u' \beta_g k^B \gamma \left[\begin{array}{c}
2 \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) + \\
\gamma \left(\overline{B} - B^c \right) \left(\overline{L^B} - 3 \left(\overline{L^B} \right)^2 + 2 \left(\overline{L^B} \right)^3 \right)
\end{array} \right]
\end{array} \right] \\
-h_{11}^{B\sigma} u'' h_1^Y - u'' \left(h_{11}^Y \rho + h_{11}^{BY} h_1^Y \right) \varepsilon_{t+1} - \\
\left(h_1^Y + \tau^y \right) * \\
\left[\begin{array}{c}
u'' \beta_g k^B \gamma \left[\begin{array}{c}
2 \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) + \\
\gamma \left(\overline{B} - B^c \right) \left(\overline{L^B} - 3 \left(\overline{L^B} \right)^2 + 2 \left(\overline{L^B} \right)^3 \right)
\end{array} \right] h_1^Y + \\
u''' \left[\begin{array}{c}
h_1^B h_1^Y + h_1^Y \rho + \tau^y \rho - \\
\left(1 + r + k^B \overline{L^B} + \gamma k^B \left(\overline{B} - B^c \right) \left(\overline{L^B} - \left(\overline{L^B} \right)^2 \right) \right) h_1^Y
\end{array} \right]
\end{array} \right] \varepsilon_{t+1}
\end{array} \right\} \widetilde{Y}_t \sigma_\varepsilon \\
= & \left(O_1 h_{11}^{Y\sigma} - W h_{11}^{B\sigma} - \Delta \varepsilon_{t+1} \right) \widetilde{Y}_t \sigma_\varepsilon.
\end{aligned}
$$

Applying expectations yields

$$
\mathrm{E}_t \left[q_{Y\sigma_\varepsilon} \widetilde{Y}_t \sigma_\varepsilon \right] = \left(O_1 h_{11}^{Y\sigma} - W h_{11}^{B\sigma} \right) \widetilde{Y}_t \sigma_\varepsilon. \tag{2.70}
$$

Derivation of $q_{\sigma_\varepsilon \sigma_\varepsilon}$ Differentiating q_{σ_ε} with respect to σ_ε at the point $\left(\overline{B}, \overline{Y}, 0 \right)$:

$$
q_{\sigma_\varepsilon \sigma_\varepsilon} = \left[\begin{array}{c}
\frac{\partial}{\partial \sigma_\varepsilon} \left(u'' \left(G_t \right) \right) * \left(\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} \right) + u'' * \frac{\partial}{\partial \sigma_\varepsilon} \left(\frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} \right) - \\
\beta_g k^B * \left[2 \frac{\partial}{\partial \sigma_\varepsilon} \left(\frac{\partial L_{t+1}^B}{\partial B_{t+1}} \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} \right) + \frac{\partial}{\partial \sigma_\varepsilon} \left(\left(B_{t+1} - B^c \right) \frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}} \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} \right) \right] * u' \\
-\beta_g k^B \left[2 \frac{\partial L_{t+1}^B}{\partial B_{t+1}} \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \left(\overline{B} - B^c \right) \frac{\partial^2 L_{t+1}^B}{\partial^2 B_{t+1}} \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} \right] * \frac{\partial}{\partial \sigma_\varepsilon} \left(u' \left(G_{t+1} \right) \right) - \\
\beta_g * \frac{\partial}{\partial \sigma_\varepsilon} \left(1 + r + k^B L_{t+1}^B + k^B \left(B_{t+1} - B^c \right) \frac{\partial L_{t+1}^B}{\partial B_{t+1}} \right) * u'' * \\
\left(\frac{\partial B_{t+2}}{\partial \sigma_\varepsilon} - \left(1 + r + k^B L_{t+1}^B \right) \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} - k^B \left(B_{t+1} - B^c \right) \frac{\partial L_{t+1}^B}{\partial B_{t+1}} \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \tau^y \varepsilon_{t+1} \right) \\
-\beta_g \left[1 + r + k^B L_{t+1}^B + k^B \left(B_{t+1} - B^c \right) \frac{\partial L_{t+1}^B}{\partial B_{t+1}} \right] * \frac{\partial}{\partial \sigma_\varepsilon} \left(u'' \left(G_{t+1} \right) \right) * \\
\left(\frac{\partial B_{t+2}}{\partial \sigma_\varepsilon} - \left(1 + r + k^B L_{t+1}^B \right) \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} - k^B \left(B_{t+1} - B^c \right) \frac{\partial L_{t+1}^B}{\partial B_{t+1}} \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \tau^y \varepsilon_{t+1} \right) \\
-\beta_g \left[1 + r + k^B L_{t+1}^B + k^B \left(B_{t+1} - B^c \right) \frac{\partial L_{t+1}^B}{\partial B_{t+1}} \right] * u'' * \\
\frac{\partial}{\partial \sigma_\varepsilon} \left(\frac{\partial B_{t+2}}{\partial \sigma_\varepsilon} - \left(1 + r + k^B L_{t+1}^B \right) \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} - k^B \left(B_{t+1} - B^c \right) \frac{\partial L_{t+1}^B}{\partial B_{t+1}} \frac{\partial B_{t+1}}{\partial \sigma_\varepsilon} + \tau^y \varepsilon_{t+1} \right)
\end{array} \right].
$$

Working out and using (2.42)

$$q_{\sigma_\varepsilon\sigma_\varepsilon}=\left[\begin{array}{c} u'''\left(h_1^\sigma\right)^2+u''\frac{\partial^2 B_{t+1}}{\partial\sigma_\varepsilon^2}- \\ \beta_g k^B\left[\begin{array}{c} 2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}\frac{\partial^2 B_{t+1}}{\partial\sigma_\varepsilon^2}+3\frac{\partial^2 L^B_{t+1}}{\partial B^2_{t+1}}\left(h_1^\sigma\right)^2+ \\ \left(\overline{B}-B^c\right)\frac{\partial^3 L^B_{t+1}}{\partial B^3_{t+1}}\left(h_1^\sigma\right)^2+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial B^2_{t+1}}\frac{\partial^2 B_{t+1}}{\partial\sigma_\varepsilon^2}\end{array}\right]u' \\ -\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial B^2_{t+1}}\right]h_1^\sigma * \\ \left[\begin{array}{c} h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\tau^y\varepsilon_{t+1}- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^\sigma\end{array}\right]u'' \\ -\beta_g k^B\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial B^2_{t+1}}\right]h_1^\sigma * \\ \left[\begin{array}{c} h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\tau^y\varepsilon_{t+1}- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^\sigma\end{array}\right]u''- \\ \left(\begin{array}{c} h_1^B h_1^\sigma+h_1^Y\varepsilon_{t+1}+h_1^\sigma+\tau^y\varepsilon_{t+1}- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_1^\sigma\end{array}\right)^2 u'''- \\ \left[\begin{array}{c} \frac{\partial^2 B_{t+2}}{\partial\sigma_\varepsilon^2}-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)\frac{\partial^2 B_{t+1}}{\partial\sigma_\varepsilon^2} \\ -\left[2\frac{\partial L^B_{t+1}}{\partial B_{t+1}}+\left(\overline{B}-B^c\right)\frac{\partial^2 L^B_{t+1}}{\partial B^2_{t+1}}\right]\left(h_1^\sigma\right)^2\end{array}\right]u''\end{array}\right].$$

Using $h_1^\sigma=0$ and working out further, we obtain

$$q_{\sigma_\varepsilon\sigma_\varepsilon}=\left[\begin{array}{c} +u''h_{11}^\sigma-u'''\left(\left(h_1^Y+\tau^y\right)\varepsilon_{t+1}\right)^2- \\ u'\beta_g k^B\gamma\left[2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\right]h_{11}^\sigma \\ -u''\left[\begin{array}{c} \left(h_{11}^B h_1^\sigma+h_{11}^{BY}\varepsilon_{t+1}+h_{11}^{B\sigma}\right)h_1^\sigma+\left(h_{11}^{BY}h_1^\sigma+h_{11}^{Y\sigma}\right)\varepsilon_{t+1} \\ +h_{11}^Y\varepsilon_{t+1}^2+h_{11}^{B\sigma}h_1^\sigma+h_{11}^{Y\sigma}\varepsilon_{t+1}+\left(1+h_1^B\right)h_{11}^\sigma- \\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)\right)h_{11}^\sigma\end{array}\right]\end{array}\right].$$

Next, grouping terms, using again $h_1^\sigma=0$, and multiplying by σ_ε^2:

$$\begin{aligned} & q_{\sigma_\varepsilon\sigma_\varepsilon}\sigma_\varepsilon^2 \\ = & \left\{\begin{array}{c} h_{11}^\sigma\left[\begin{array}{c} u''\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)-h_1^B\right) \\ -u'\beta_g k^B\gamma\left[\begin{array}{c} 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+ \\ \gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)\end{array}\right]\end{array}\right] \\ -u''\left(h_{11}^{BY}h_1^\sigma+2h_{11}^{Y\sigma}\right)\varepsilon_{t+1}-\left[u'''\left(h_1^Y+\tau^y\right)^2+u''h_{11}^Y\right]\varepsilon_{t+1}^2\end{array}\right\}\sigma_\varepsilon^2 \\ = & \left\{\Theta h_{11}^\sigma-2u''h_{11}^{Y\sigma}\varepsilon_{t+1}-\left[u'''\left(h_1^Y+\tau^y\right)^2+u''h_{11}^Y\right]\varepsilon_{t+1}^2\right\}\sigma_\varepsilon^2,\end{aligned}$$

since again $h_1^\sigma=0$. Finally, applying expectations:

$$\begin{aligned}\mathrm{E}_t\left[q_{\sigma_\varepsilon\sigma_\varepsilon}\sigma_\varepsilon^2\right] &= \mathrm{E}_t\left\{\left\{\Theta h_{11}^\sigma-2u''h_{11}^{Y\sigma}\varepsilon_{t+1}-\left[u'''\left(h_1^Y+\tau^y\right)^2+u''h_{11}^Y\right]\varepsilon_{t+1}^2\right\}\sigma_\varepsilon^2\right\} \\ &= \left\{\Theta h_{11}^\sigma-\left[u'''\left(h_1^Y+\tau^y\right)^2+u''h_{11}^Y\right]\sigma_\varepsilon^2\right\}\sigma_\varepsilon^2. \qquad (2.71)\end{aligned}$$

Working out the second-order Taylor expansion

Substituting (2.66), (2.67), (2.68), (2.69), (2.70) and (2.71) into (2.58):

$$\mathrm{E}_t\left[q\left(B_t,Y_t,\sigma_\varepsilon\right)\right]\simeq\frac{1}{2}\left\{\left[\begin{array}{c}\left(Mh_{11}^B+N\right)\widetilde{B}_t^2+2\left(Oh_{11}^{BY}+Ph_{11}^B+Q\right)\widetilde{B}_t\widetilde{Y}_t\\+2Rh_{11}^{B\sigma}\widetilde{B}_t\sigma_\varepsilon+\left(Vh_{11}^Y+Xh_{11}^{BY}+Th_{11}^B+Z\right)\widetilde{Y}_t^2\\+2\left(O_1h_{11}^{Y\sigma}-Wh_{11}^{B\sigma}\right)\widetilde{Y}_t\sigma_\varepsilon+\\\left\{\Theta h_{11}^\sigma-\left[u'''\left(h_1^Y+\tau^y\right)^2+u''h_{11}^Y\right]\sigma_\varepsilon^2\right\}\sigma_\varepsilon^2\end{array}\right]\right\}=0. \tag{2.72}$$

This must hold for all possible 6-tuples $\left\{\widetilde{B}_t^2,\widetilde{B}_t\widetilde{Y}_t,\widetilde{B}_t\sigma_\varepsilon,\widetilde{Y}_t^2,\widetilde{Y}_t\sigma_\varepsilon,\sigma_\varepsilon^2\right\}$, implying six equations in six unknowns. First,

$$Mh_{11}^B+N=0\Leftrightarrow h_{11}^B=-N/M,$$

since $M\neq 0$. So, $h_{11}^B\equiv\frac{\partial^2 h}{\partial B_t^2}$ is equal to:

$$h_{11}^B=\frac{\left[\begin{array}{c}\beta_g k^B\gamma^2\left[\begin{array}{c}3\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right)+\\\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-7\overline{L^B}^2+12\overline{L^B}^3-6\overline{L^B}^4\right)\end{array}\right]\left(h_1^B\right)^2u'-\\k^B\gamma z_8\left(-1+3\left(h_1^B\right)^2-2\beta_g\left(h_1^B\right)^3\right)u''-\left(h_1^B-\frac{1}{\beta_g}\right)^2\left[1-\left(h_1^B\right)^2\right]u'''\end{array}\right]}{\left[\left(1+\frac{1}{\beta_g}-\left(h_1^B\right)^2-h_1^B\right)u''-\beta_g k^B\gamma z_8u'\right]}, \tag{2.73}$$

where

$$z_8\equiv 2\left(\overline{L^B}-\left(\overline{L^B}\right)^2\right)+\gamma\left(\overline{B}-B^c\right)\left(\overline{L^B}-3\left(\overline{L^B}\right)^2+2\left(\overline{L^B}\right)^3\right).$$

Using h_{11}^B, we can solve for h_{11}^{BY} as:

$$Oh_{11}^{BY}+Ph_{11}^B+Q=0\Leftrightarrow h_{11}^{BY}=-\frac{Ph_{11}^B+Q}{O},$$

or

$$h_{11}^{BY}=\frac{\left[\begin{array}{c}h_{11}^Bh_1^Bh_1^Yu''-\\\left[h_1^B-\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}\left(1-\overline{L^B}\right)\right)\right]u'''*\\\left[\begin{array}{c}h_1^Y+\tau^y-\left(h_1^B\right)^2h_1^Y-h_1^B\tau^y\rho+\\\left(1+r+k^B\overline{L^B}-\rho+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}\left(1-\overline{L^B}\right)\right)h_1^Bh_1^Y\end{array}\right]+\\\beta_g k^B\gamma^2\overline{L^B}\left(1-\overline{L^B}\right)\left[3\left(1-2\overline{L^B}\right)+\gamma\left(\overline{B}-B^c\right)\left(1-6\overline{L^B}+6\left(\overline{L^B}\right)^2\right)\right]h_1^Bh_1^Yu'\\-k^B\gamma z_8\left(3h_1^Bh_1^Y-2\beta_g\left(h_1^B\right)^2h_1^Y-\beta_g h_1^B\tau^y\rho-\beta_g h_1^Bh_1^Y\rho\right)u''\end{array}\right]}{\left[2+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}\left(1-\overline{L^B}\right)-h_1^B\left(1+\rho\right)\right]u''-\beta_g k^B\gamma z_8u'}. \tag{2.74}$$

Next, we find h_{11}^Y as:

$$Vh_{11}^Y+Xh_{11}^{BY}+Th_{11}^B+Z=0\Leftrightarrow h_{11}^Y=-\frac{Xh_{11}^{BY}+Th_{11}^B+Z}{V},$$

or

$$h_{11}^{Y}=\frac{\left[\begin{array}{c}2\rho h_1^Y h_{11}^{BY}u''+\left(h_1^Y\right)^2 h_{11}^B u''+\\ \beta_g k^B\gamma^2\overline{L^B}\left(1-\overline{L^B}\right)\left[\begin{array}{c}3\left(1-2\overline{L^B}\right)+\\ \gamma\left(\overline{B}-B^c\right)\left(1-6\overline{L^B}+6\left(\overline{L^B}\right)^2\right)\end{array}\right]\left(h_1^Y\right)^2u'-\\ k^B\gamma\overline{L^B}\left(1-\overline{L^B}\right)\left[\begin{array}{c}2+\\ \gamma\left(\overline{B}-B^c\right)\left(1-2\overline{L^B}\right)\end{array}\right]\left[\begin{array}{c}3\left(h_1^Y\right)^2-2\beta_g h_1^B\left(h_1^Y\right)^2-\\ 2\beta_g\left(h_1^Y\right)^2\rho-2\beta_g h_1^Y\tau^y\rho\end{array}\right]u''\\ -\left\{\begin{array}{c}\left(h_1^Y+\tau^y\right)^2-\\ \left[\begin{array}{c}h_1^B h_1^Y+h_1^Y\rho+\tau^y\rho-\\ \left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}\left(1-\overline{L^B}\right)\right)h_1^Y\end{array}\right]^2\end{array}\right\}u'''\end{array}\right]}{\left[\begin{array}{c}\left[2+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}\left(1-\overline{L^B}\right)-h_1^B-\rho^2\right]u''\\ -\beta_g k^B\gamma\overline{L^B}\left(1-\overline{L^B}\right)\left[2+\gamma\left(\overline{B}-B^c\right)\left(1-2\overline{L^B}\right)\right]u'\end{array}\right]}. \tag{2.75}$$

Further,

$$h_{11}^{B\sigma}=0. \tag{2.76}$$

Hence,

$$h_{11}^{Y\sigma}=0. \tag{2.77}$$

Finally, given (2.55) and (2.75), we obtain:

$$\Theta h_{11}^{\sigma}-\left[u'''\left(h_1^Y+\tau^y\right)^2+u''h_{11}^Y\right]\sigma_\varepsilon^2=0\Leftrightarrow h_{11}^{\sigma}=\frac{\left[u'''\left(h_1^Y+\tau^y\right)^2+u''h_{11}^Y\right]\sigma_\varepsilon^2}{\Theta},$$

or

$$h_{11}^{\sigma}=\frac{\left[\left(h_1^Y+\tau^y\right)^2u'''+h_{11}^Y u''\right]\sigma_\varepsilon^2}{\left\{\begin{array}{c}\left(1+r+k^B\overline{L^B}+\gamma k^B\left(\overline{B}-B^c\right)\overline{L^B}\left(1-\overline{L^B}\right)-h_1^B\right)u''\\ -\beta_g k^B\gamma\overline{L^B}\left(1-\overline{L^B}\right)\left[2+\gamma\left(\overline{B}-B^c\right)\left(1-2\overline{L^B}\right)\right]u'\end{array}\right\}}. \tag{2.78}$$

Second-order approximation of B_{t+1}

The second-order approximation of the true non-linear solution of B_{t+1} around the point $\left(\overline{B},\overline{Y},0\right)$ can be written as

$$B_{t+1}\approx\left\{\begin{array}{c}h\left(\overline{B},\overline{Y},0\right)+h_1^B\widetilde{B}_t+h_1^Y\widetilde{Y}_t+h_1^\sigma\sigma_\varepsilon+\\ \frac{1}{2}\left[h_{11}^B\widetilde{B}_t^2+2h_{11}^{BY}\widetilde{B}_t\widetilde{Y}_t+2h_{11}^{B\sigma}\widetilde{B}_t\sigma_\varepsilon+h_{11}^Y\widetilde{Y}_t^2+2h_{11}^{Y\sigma}\widetilde{Y}_t\sigma_\varepsilon+h_{11}^\sigma\sigma_\varepsilon^2\right]\end{array}\right\}.$$

Hence, using $h_1^\sigma=h_{11}^{B\sigma}=h_{11}^{Y\sigma}=0$,

$$B_{t+1}\approx\left\{\begin{array}{c}h\left(\overline{B},\overline{Y},0\right)+h_1^B\widetilde{B}_t+h_1^Y\widetilde{Y}_t+\\ \frac{1}{2}\left[h_{11}^B\widetilde{B}_t^2+2h_{11}^{BY}\widetilde{B}_t\widetilde{Y}_t+h_{11}^Y\widetilde{Y}_t^2+h_{11}^\sigma\sigma_\varepsilon^2\right]\end{array}\right\}. \tag{2.79}$$

2.F.3 Stochastic steady state

We take the unconditional expectation of debt in (2.79) and use the definition of the variables with a tilde:

$$\mathrm{E}\left[B_{t+1}\right] \approx \left\{ \begin{array}{c} \overline{B} + h_1^B \left[\mathrm{E}\left(B_t\right) - \overline{B}\right] + h_1^Y \left[\mathrm{E}\left(Y_t\right) - \overline{Y}\right] + \\ \frac{1}{2} \left[\begin{array}{c} h_{11}^B \mathrm{E}\left[\left(B_t - \overline{B}\right)^2\right] + 2h_{11}^{BY} \mathrm{E}\left[\left(B_t - \overline{B}\right)\left(Y_t - \overline{Y}\right)\right] \\ + h_{11}^Y \mathrm{E}\left[\left(Y_t - \overline{Y}\right)^2\right] + h_{11}^{\sigma} \sigma_\varepsilon^2 \end{array} \right] \end{array} \right\}.$$

We have that $\mathrm{E}(Y_t) = \overline{Y}$ and $\mathrm{Var}(Y) = \frac{\sigma_\varepsilon^2}{1-\rho^2}$. Define

$$\widehat{B} \equiv \mathrm{E}\left(B_t\right) = \mathrm{E}\left[B_{t+1}\right].$$

Let's work out the terms in the expression for $\mathrm{E}[B_{t+1}]$:

$$\mathrm{E}\left(Y_t\right) - \overline{Y} = 0,$$

$$\mathrm{E}\left[\left(Y_t - \overline{Y}\right)^2\right] = \mathrm{Var}\left(Y_t\right) = \frac{\sigma_\varepsilon^2}{1-\rho^2},$$

$$\begin{aligned} \mathrm{E}\left[\left(B_t - \overline{B}\right)^2\right] &= \mathrm{E}\left\{\left[\left(B_t - \widehat{B}\right) + \left(\widehat{B} - \overline{B}\right)\right]^2\right\} \\ &= \mathrm{E}\left[\left(B_t - \widehat{B}\right)^2\right] + 2\mathrm{E}\left[\left(B_t - \widehat{B}\right)\left(\widehat{B} - \overline{B}\right)\right] + \mathrm{E}\left[\left(\widehat{B} - \overline{B}\right)^2\right] \\ &= \mathrm{E}\left[\left(B_t - \widehat{B}\right)^2\right] + \left(\widehat{B} - \overline{B}\right)^2 = \mathrm{Var}\left(B_t\right) + \left(\widehat{B} - \overline{B}\right)^2, \end{aligned}$$

$$\begin{aligned} \mathrm{E}\left[\left(B_t - \overline{B}\right)\left(Y_t - \overline{Y}\right)\right] &= \mathrm{E}\left\{\left[\left(B_t - \widehat{B}\right) + \left(\widehat{B} - \overline{B}\right)\right]\left(Y_t - \overline{Y}\right)\right\} \\ &= \mathrm{E}\left[\left(B_t - \widehat{B}\right)\left(Y_t - \overline{Y}\right)\right] + \mathrm{E}\left[\left(\widehat{B} - \overline{B}\right)\left(Y_t - \overline{Y}\right)\right] \\ &= \mathrm{E}\left[\left(B_t - \widehat{B}\right)\left(Y_t - \overline{Y}\right)\right] = \mathrm{Cov}\left(B_t, Y_t\right). \end{aligned}$$

Substitute these terms into the expression for $\mathrm{E}[B_{t+1}]$, to give:

$$\widehat{B} - \overline{B} \approx \frac{1}{1-h_1^B} \left\{ \left[\begin{array}{c} \frac{1}{2} h_{11}^B \left[\mathrm{E}\left(B_t - \widehat{B}\right)^2 + \left(\widehat{B} - \overline{B}\right)^2\right] + \\ h_{11}^{BY} \mathrm{E}\left[\left(B_t - \widehat{B}\right)\left(Y_t - \overline{Y}\right)\right] + \frac{1}{2} h_{11}^Y \frac{\sigma_\varepsilon^2}{1-\rho^2} + \frac{1}{2} h_{11}^{\sigma} \sigma_\varepsilon^2 \end{array} \right] \right\}.$$

Notice that the difference between $\widehat{B}$ and $\overline{B}$ is of second-order magnitude. Approximating terms of higher than second order by zero, we have:

$$\mathrm{E}\left(\widehat{B} - \overline{B}\right)^2 \simeq 0,$$

and

$$\begin{aligned}
\mathrm{E}\left[\left(B_t-\widehat{B}\right)\widetilde{Y}_t\right] &= \mathrm{E}\left[\left(B_{t+1}-\widehat{B}\right)\widetilde{Y}_{t+1}\right]=\mathrm{E}\left[\left(\overline{B}+h_1^B\widetilde{B}_t+h_1^Y\widetilde{Y}_t-\widehat{B}\right)\left(\rho\widetilde{Y}_t+\sigma_\varepsilon\varepsilon_{t+1}\right)\right], \\
&= \mathrm{E}\left[\left(\rho h_1^B\widetilde{B}_t\widetilde{Y}_t+\rho h_1^Y\widetilde{Y}_t^2\right)\right].
\end{aligned}$$

Hence,

$$\mathrm{E}\left[\widetilde{B}_t\widetilde{Y}_t\right]\simeq\left(\frac{\rho h_1^Y}{1-\rho h_1^B}\right)\frac{\sigma_\varepsilon^2}{1-\rho^2},$$

Further,

$$\begin{aligned}
\mathrm{E}\left(B_t-\widehat{B}\right)^2 &= \mathrm{E}\left(B_{t+1}-\widehat{B}\right)^2 \\
&\simeq \mathrm{E}\left[\left(\overline{B}+h_1^B\widetilde{B}_t+h_1^Y\widetilde{Y}_t-\widehat{B}\right)^2\right]\simeq\mathrm{E}\left[\left(h_1^B\widetilde{B}_t+h_1^Y\widetilde{Y}_t\right)^2\right] \\
&= \mathrm{E}\left[\left(h_1^B\right)^2\left(B_t-\overline{B}\right)^2+2h_1^Bh_1^Y\left(B_t-\overline{B}\right)\left(Y_t-\overline{Y}\right)+\left(h_1^Y\right)^2\left(Y_t-\overline{Y}\right)^2\right] \\
&\simeq \mathrm{E}\left[\left(h_1^B\right)^2\left(B_t-\overline{B}\right)^2\right]+2\left(\frac{\rho h_1^B\left(h_1^Y\right)^2}{1-\rho h_1^B}\right)\frac{\sigma_\varepsilon^2}{1-\rho^2}+\left(h_1^Y\right)^2\mathrm{E}\left(Y_t-\overline{Y}\right)^2 \\
&= \left(h_1^B\right)^2\mathrm{E}\left[\left(B_t-\overline{B}\right)^2\right]+\left(\frac{2\rho h_1^B}{1-\rho h_1^B}+1\right)\left(h_1^Y\right)^2\tfrac{\sigma_\varepsilon^2}{1-\rho^2} \\
&= \left(h_1^B\right)^2\mathrm{E}\left[\left(B_t-\overline{B}\right)^2\right]+\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right)\left(h_1^Y\right)^2\tfrac{\sigma_\varepsilon^2}{1-\rho^2}.
\end{aligned}$$

Hence,

$$\mathrm{Var}\left(B_t\right)=\mathrm{E}\left(B_t-\widehat{B}\right)^2\simeq\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right)\frac{\left(h_1^Y\right)^2}{1-\left(h_1^B\right)^2}\frac{\sigma_\varepsilon^2}{1-\rho^2}. \tag{2.80}$$

Hence, we obtain:

Solution 2.1 *The unconditional expectation or "stochastic steady state" value of the debt in the debt-based sanction case is given by*

$$\widehat{B}\simeq\overline{B}+\frac{1}{2}\frac{\sigma_\varepsilon^2}{1-h_1^B}\left[\left(\frac{h_{11}^B\left(h_1^Y\right)^2}{1-\left(h_1^B\right)^2}\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right)+2\frac{\rho h_{11}^{BY}h_1^Y}{1-\rho h_1^B}+h_{11}^Y\right)\frac{1}{1-\rho^2}+h_{11}^\sigma\right]. \tag{2.81}$$

2.F.4 Stochastic steady state of government expenditure with debt-based sanction

In (2.38) the only unknown variable is B_{t+1}. However, (2.79) provides a second-order approximation of that variable. Hence, plugging this into (2.38), we can approximate G_t as

$$G_t \approx \left[\begin{array}{c} \overline{B} + h_1^B \left(B_t - \overline{B}\right) + h_1^Y \left(Y_t - \overline{Y}\right) + \\ \frac{1}{2} \left[h_{11}^B \left(B_t - \overline{B}\right)^2 + 2h_{11}^{BY} \left(B_t - \overline{B}\right)\left(Y_t - \overline{Y}\right) + h_{11}^Y \left(Y_t - \overline{Y}\right)^2 + h_{11}^\sigma \sigma_\varepsilon^2 \right] \\ -\left(1+r\right) B_t + \tau^y Y_t - k^B \left(B_t - B^c\right) L_t^B \end{array} \right]. \quad (2.82)$$

Next, from (2.38) we know that

$$\overline{G} \equiv f\left(\overline{B}, \overline{Y}, 0\right) = \tau^y \overline{Y} - r\overline{B} - k^B \left(\overline{B} - B^c\right) \overline{L^B}. \quad (2.83)$$

First-order approximation of G_t

Analogous to what we did for B_{t+1}, we take a first-order Taylor expansion of (2.24) around the point $\left(\overline{B}, \overline{Y}, 0\right)$:

$$G_t \approx f\left(\overline{B}, \overline{Y}, 0\right) + f_1^B \widetilde{B}_t + f_1^Y \widetilde{Y}_t + f_1^\sigma \sigma_\varepsilon, \quad (2.84)$$

where $f_1^B \equiv \frac{\partial G_t}{\partial B_t}$, $f_1^Y \equiv \frac{\partial G_t}{\partial Y_t}$ and $f_1^\sigma \equiv \frac{\partial G_t}{\partial \sigma_\varepsilon}$ are evaluated at the point $\left(\overline{B}, \overline{Y}, 0\right)$. Hence, using (2.82) we can calculate these derivatives as follows:

- First order partial derivative of (2.82) with respect to B_t:

  $$\frac{\partial G_t}{\partial B_t} = h_1^B + \frac{1}{2}\left[2h_{11}^B \left(B_t - \overline{B}\right) + 2h_{11}^{BY}\left(Y_t - \overline{Y}\right)\right] - (1+r) - k^B L_t^B - k^B \left(B_t - B^c\right) \frac{\partial L_t^B}{\partial B_t}.$$

  If we evaluate this derivative at the point $(B_t, Y_t, \sigma_\varepsilon) = \left(\overline{B}, \overline{Y}, 0\right)$, use (2.40) and (2.42), we arrive at

  $$f_1^B = h_1^B - 1/\beta_g. \quad (2.85)$$

- First order partial derivative of (2.82) with respect to Y_t:

  $$\frac{\partial G_t}{\partial Y_t} = h_1^Y + \frac{1}{2}\left[2h_{11}^{BY}\left(B_t - \overline{B}\right) + 2h_{11}^Y \left(Y_t - \overline{Y}\right)\right] + \tau^y.$$

  If we evaluate this derivative at the point $(B_t, Y_t, \sigma_\varepsilon) = \left(\overline{B}, \overline{Y}, 0\right)$, we arrive at

  $$f_1^Y = h_1^Y + \tau^y. \quad (2.86)$$

- First order partial derivative of (2.82) with respect to σ_ε:

  $$\frac{\partial G_t}{\partial \sigma_\varepsilon} = \frac{1}{2}\left[2h_{11}^\sigma \sigma_\varepsilon\right].$$

  Since we evaluate at $(B_t, Y_t, \sigma_\varepsilon) = \left(\overline{B}, \overline{Y}, 0\right)$, we obtain

  $$f_1^\sigma = 0. \quad (2.87)$$

- Evaluating the right-hand side of (2.82) at $(B_t, Y_t, \sigma_\varepsilon) = \left(\overline{B}, \overline{Y}, 0\right)$ yields:

  $$\overline{G} \equiv f\left(\overline{B}, \overline{Y}, 0\right) = \tau^y \overline{Y} - r\overline{B} - k^B \left(\overline{B} - B^c\right) \overline{L^B}. \quad (2.88)$$

- Hence

  $$G_t \approx \overline{G} + \left[h_1^B - (1/\beta_g)\right] \widetilde{B}_t + \left(h_1^Y + \tau^y\right) \widetilde{Y}_t. \quad (2.89)$$

Second-order approximation of G_t

The second-order approximation of (2.24) around the point $\left(\overline{B},\overline{Y},0\right)$ is:

$$G_t \approx \left\{ \begin{array}{c} f\left(\overline{B},\overline{Y},0\right) + f_1^B \widetilde{B}_t + f_1^Y \widetilde{Y}_t + f_1^{\sigma}\sigma_{\varepsilon} + \\ \frac{1}{2}\left[f_{11}^B \widetilde{B}_t^2 + 2 f_{11}^{BY} \widetilde{B}_t \widetilde{Y}_t + 2 f_{11}^{B\sigma} \widetilde{B}_t \sigma_{\varepsilon} + f_{11}^Y \widetilde{Y}_t^2 + 2 f_{11}^{Y\sigma} \widetilde{Y}_t \sigma_{\varepsilon} + f_{11}^{\sigma} \sigma_{\varepsilon}^2 \right] \end{array} \right\},$$

where for generic variables X and Z, $f_{11}^X \equiv \frac{\partial^2 G_t}{\partial^2 X_t}$ and $f_{11}^{XZ} \equiv \frac{\partial^2 G_t}{\partial X_t \partial Z_t}$.

- Computation of f_{11}^B: Differentiating $\frac{\partial G_t}{\partial B_t}$ with respect to B_t, we obtain:

$$\frac{\partial^2 G_t}{\partial^2 B_t} = \frac{1}{2}\left[2h_{11}^B\right] - k^B \frac{\partial L_t^B}{\partial B_t} - k^B \frac{\partial L_t^B}{\partial B_t} - k^B \left(B_t - B^c\right) \frac{\partial^2 L_t^B}{\partial B_t^2}.$$

If we evaluate this derivative at the point $\left(\overline{B},\overline{Y},0\right)$ and use (2.48), we arrive to

$$f_{11}^B = h_{11}^B - k^B \gamma \overline{L^B}\left(1 - \overline{L^B}\right)\left[2 + \left(\overline{B} - B^c\right)\gamma\left(1 - 2\overline{L^B}\right)\right]. \tag{2.90}$$

- Computation of f_{11}^{BY}: Differentiating $\frac{\partial G_t}{\partial Y_t}$ with respect to B_t, we obtain:

$$\frac{\partial^2 G_t}{\partial B_t \partial Y_t} = \frac{1}{2}\left[2h_{11}^{BY}\right] \Longrightarrow f_{11}^{BY} = h_{11}^{BY}. \tag{2.91}$$

- Computation of $f_{11}^{B\sigma}$ and $f_{11}^{Y\sigma}$: Differentiating $\frac{\partial G_t}{\partial B_t}$ and $\frac{\partial G_t}{\partial Y_t}$ with respect to σ_{ε}, we obtain:

$$f_{11}^{B\sigma} = f_{11}^{Y\sigma} = \frac{\partial^2 G_t}{\partial \sigma_{\varepsilon} \partial B_t} = \frac{\partial^2 G_t}{\partial \sigma_{\varepsilon} \partial Y_t} = 0. \tag{2.92}$$

- Computation of f_{11}^Y: Differentiating $\frac{\partial G_t}{\partial Y_t}$ with respect to Y_t, we obtain:

$$\frac{\partial^2 G_t}{\partial Y_t^2} = \frac{1}{2}\left[2h_{11}^Y\right] \Longrightarrow f_{11}^Y = h_{11}^Y. \tag{2.93}$$

- Computation of f_{11}^{σ}: Differentiating $\frac{\partial G_t}{\partial \sigma_{\varepsilon}}$ with respect to σ_{ε}, we obtain:

$$\frac{\partial^2 G_t}{\partial \sigma_{\varepsilon}^2} = \frac{1}{2}\left[2h_{11}^{\sigma}\right] \Longrightarrow f_{11}^{\sigma} = h_{11}^{\sigma}. \tag{2.94}$$

With all those derivatives, we write down the second-order approximation for G_t as

$$G_t \approx \left\{ \begin{array}{c} \overline{G} + \left[h_1^B - (1/\beta_g)\right]\widetilde{B}_t + \left(h_1^Y + \tau^y\right)\widetilde{Y}_t + \\ \frac{1}{2}\left\{ \begin{array}{c} \left[h_{11}^B - k^B \gamma \overline{L^B}\left(1 - \overline{L^B}\right)\left(2 + \left(\overline{B} - B^c\right)\gamma\left(1 - 2\overline{L^B}\right)\right)\right]\widetilde{B}_t^2 \\ +2h_{11}^{BY}\widetilde{B}_t\widetilde{Y}_t + h_{11}^Y \widetilde{Y}_t^2 + h_{11}^{\sigma}\sigma_{\varepsilon}^2 \end{array} \right\} \end{array} \right\}. \tag{2.95}$$

Unconditional expectation of government expenditure with a debt-based sanction

Next, using that

$$\mathrm{E}\left[B_t\right] = \widehat{B}, \quad \mathrm{E}\left[\widetilde{B}_t^2\right] \simeq \mathrm{Var}\left(B_t\right), \quad \mathrm{E}\left[\widetilde{B}_t \widetilde{Y}_t\right] \simeq \left(\frac{\rho h_1^Y}{1-\rho h_1^B}\right) \frac{\sigma_\varepsilon^2}{1-\rho^2},$$

the unconditional expectation of (2.95) can be written as:

$$\widehat{G} \approx \left\{ \begin{array}{c} \overline{G} + \left[h_1^B - (1/\beta_g)\right]\left(\widehat{B} - \overline{B}\right) + \left(\frac{\rho h_1^Y h_{11}^{BY}}{1-\rho h_1^B}\right)\frac{\sigma_\varepsilon^2}{1-\rho^2} + \\ \frac{1}{2}\left\{ \begin{array}{c} \left[h_{11}^B - k^B \gamma \overline{L^B}\left(1-\overline{L^B}\right)\left(2 + \left(\overline{B} - B^c\right)\gamma\left(1 - 2\overline{L^B}\right)\right)\right] \mathrm{Var}\left(B_t\right) \\ + \left(\frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma\right)\sigma_\varepsilon^2 \end{array} \right\} \end{array} \right\}.$$

Then, substituting (2.80), (2.81) and (2.88), it follows that

$$\widehat{G} \approx \left\{ \begin{array}{l} \overline{G} + \frac{\left[h_1^B - (1/\beta_g)\right]\sigma_\varepsilon^2}{2\left(1-h_1^B\right)} \left[\frac{h_{11}^B \left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)\left(1-\rho^2\right)} \left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) + 2\frac{\rho h_{11}^{BY} h_1^Y}{1-\rho h_1^B}\frac{1}{1-\rho^2} + \frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma \right] \\ +\frac{1}{2}\left[\begin{array}{c} \frac{\left[h_{11}^B - k^B\gamma\overline{L^B}\left(1-\overline{L^B}\right)\left(2+\left(\overline{B}-B^c\right)\gamma\left(1-2\overline{L^B}\right)\right)\right]\left(h_1^Y\right)^2\sigma_\varepsilon^2}{\left(1-\left(h_1^B\right)^2\right)\left(1-\rho^2\right)} \left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) \\ +2\left(\frac{\rho h_1^Y h_{11}^{BY}}{1-\rho h_1^B}\right)\frac{\sigma_\varepsilon^2}{1-\rho^2} + \left(\frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma\right)\sigma_\varepsilon^2 \end{array} \right] \end{array} \right\},$$

or

Solution 2.2 *The unconditional expectation or "stochastic steady state" value of the government expenditures in the debt-based sanction case is given by*

$$\widehat{G} \approx \left\{ \begin{array}{c} \tau^y \overline{Y} - r\overline{B} - k^B\left(\overline{B} - B^c\right)\overline{L^B} + \\ \frac{\sigma_\varepsilon^2}{2}\left\{ \begin{array}{l} \frac{h_1^B - (1/\beta_g)}{1-h_1^B}\left[\frac{h_{11}^B\left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)\left(1-\rho^2\right)}\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) + 2\frac{\rho h_{11}^{BY} h_1^Y}{1-\rho h_1^B}\frac{1}{1-\rho^2} + \frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma \right] \\ + \left[\begin{array}{c} \frac{\left[h_{11}^B - k^B\gamma\overline{L^B}\left(1-\overline{L^B}\right)\left(2+\left(\overline{B}-B^c\right)\gamma\left(1-2\overline{L^B}\right)\right)\right]\left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)\left(1-\rho^2\right)}\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) \\ +2\left(\frac{\rho h_1^Y h_{11}^{BY}}{1-\rho h_1^B}\right)\frac{1}{1-\rho^2} + \frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma \end{array} \right] \end{array} \right\} \end{array} \right\}. \tag{2.96}$$

2.G Stochastic steady state with primary deficit-based sanction

The same procedure is used to obtain the stochastic steady state of debt with a primary deficit-based sanction. First, we isolate G_s in (2.21) for $s = t$ and $t+1$, and substitute into (2.22), to give

$$\mathrm{E}_t \left\{ \begin{array}{c} u'\left(D_{t+1} + Y_t - k^D\left(D_t - D^c\right)L_t^D\right) \\ -\beta_g k^D\left[L_{t+1}^D + \left(D_{t+1} - D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\right] u'\left(\begin{array}{c} D_{t+2} + \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon \varepsilon_{t+1} \\ -k^D\left(D_{t+1} - D^c\right)L_{t+1}^D \end{array} \right) \end{array} \right\} = 0, \tag{2.97}$$

Using that

$$\frac{\partial L_{t+1}^D}{\partial D_{t+1}} = \frac{\gamma \exp\left(-\gamma\left(D_{t+1} - D^c\right)\right)}{\left(1 + \exp\left(-\gamma\left(D_{t+1} - D^c\right)\right)\right)^2} = \gamma L_{t+1}^D\left(1 - L_{t+1}^D\right), \tag{2.98}$$

this becomes

$$\mathrm{E}_t\left\{\begin{array}{c} u'\left(D_{t+1} + Y_t - k^D\left(D_t - D^c\right)L_t^D\right) - \\ \beta_g k^D\left[\begin{array}{c} L_{t+1}^D + \\ \left(D_{t+1} - D^c\right)\gamma\left(L_{t+1}^D - \left(L_{t+1}^D\right)^2\right)\end{array}\right] u'\left(\begin{array}{c} D_{t+2} + \rho Y_t + (1-\rho)\overline{Y} + \\ \sigma_\varepsilon \varepsilon_{t+1} - k^D\left(D_{t+1} - D^c\right)L_{t+1}^D\end{array}\right)\end{array}\right\} = 0. \tag{2.99}$$

Thus, (2.99) has the format

$$\mathrm{E}_t\left\{i\left(D_t, D_{t+1}, D_{t+2}, Y_t, Y_{t+1}\right)\right\} = 0.$$

Hence, using (2.5) and (2.28) the unknown $l(\cdot)$ satisfies

$$\begin{aligned} \mathrm{E}_t\left[p\left(D_t, Y_t, \sigma_\varepsilon\right)\right] &= \mathrm{E}_t\left[i\left(D_t, l\left(D_t, Y_t, \sigma_\varepsilon\right), l\left(l\left(D_t, Y_t, \sigma_\varepsilon\right), Y_{t+1}, \sigma_\varepsilon\right), Y_t, Y_{t+1}\right)\right] = 0 \Rightarrow \\ \mathrm{E}_t\left[p\left(D_t, Y_t, \sigma_\varepsilon\right)\right] &= \mathrm{E}_t\left[i\left(\begin{array}{c} D_t, l\left(D_t, Y_t, \sigma_\varepsilon\right), \\ l\left(l\left(D_t, Y_t, \sigma_\varepsilon\right), \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}, \sigma_\varepsilon\right), \\ Y_t, \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon\varepsilon_{t+1}\end{array}\right)\right] = 0. \end{aligned}$$

For $\sigma_\varepsilon = 0$, $D_{t+2} = D_{t+1} = D_t = \overline{D}$ and $Y_{t+1} = Y_t = \overline{Y}$, (2.29) becomes:

$$\mathrm{E}_t\left[p\left(\overline{B}, \overline{Y}, 0\right)\right] = \mathrm{E}_t\left[i\left(\overline{D}, \overline{D}, \overline{D}, \overline{Y}, \overline{Y}\right)\right] = 0,$$

where we have used that $l\left(\overline{D}, \overline{Y}, 0\right) = \overline{D}$ and $l\left(l\left(\overline{D}, \overline{Y}, 0\right), \overline{Y}, 0\right) = l\left(\overline{D}, \overline{Y}, 0\right) = \overline{D}$. Because $u' > 0$, $\mathrm{E}_t\left[p\left(\overline{D}, \overline{Y}, 0\right)\right] = 0$ is equivalent to

$$\beta_g k^D \overline{L^D}\left[1 + \left(\overline{D} - D^c\right)\gamma\left(1 - \overline{L^D}\right)\right] = 1, \tag{2.100}$$

where $\overline{L^D}$ and u' are evaluated at the point $\left(\overline{D}, \overline{Y}, 0\right)$. This expression determines $\overline{D}$.

2.G.1 First-order approximation of (2.29)

Derivation of p_D, p_{Y_t} and p_{σ_ε}

In this section, we compute p_D, p_{Y_t} and p_{σ_ε}. We start by computing the partial derivatives of (2.28) with respect to B_t, Y_t and σ_ε.

First-order partial derivatives of (2.28) Derivative of D_{t+1} with respect to D_t:

$$\frac{\partial D_{t+1}}{\partial D_t} = \frac{\partial l\left(D_t, Y_t, \sigma_\varepsilon\right)}{\partial D_t} = \frac{\partial l_t}{\partial D_t} \equiv l_1^D, \tag{2.101}$$

where l_t represents $l\left(D_t, Y_t, \sigma_\varepsilon\right)$, and since $\frac{\partial Y_t}{\partial D_t} = \frac{\partial \sigma_\varepsilon}{\partial D_t} = 0$. Further, the derivative of D_{t+1} with respect to Y_t is

$$\frac{\partial D_{t+1}}{\partial Y_t} = \frac{\partial l\left(D_t, Y_t, \sigma_\varepsilon\right)}{\partial Y_t} = \frac{\partial l_t}{\partial Y_t} \equiv l_1^Y, \tag{2.102}$$

since at the initial period $\frac{\partial D_t}{\partial Y_t} = 0$. Finally, the derivative of D_{t+1} with respect to σ_ε is

$$\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} = \frac{\partial l\left(D_t, Y_t, \sigma_\varepsilon\right)}{\partial \sigma_\varepsilon} = \frac{\partial l_t}{\partial \sigma_\varepsilon} = l_1^\sigma, \tag{2.103}$$

since $\frac{\partial D_t}{\partial \sigma_\varepsilon} = 0$ and in the initial value of Y_t is also not correlated with σ_ε, causing $\frac{\partial Y_t}{\partial \sigma_\varepsilon} = 0$.

Derivation of the first order partial derivative of D_{t+2} with respect to $D_t, Y_t, \sigma_\varepsilon$
Using 2.28, we can write D_{t+2} as

$$D_{t+2} = l\left(l\left(D_t, Y_t, \sigma_\varepsilon\right), \rho Y_t + (1-\rho)\overline{Y} + \sigma_\varepsilon \varepsilon_{t+1}, \sigma_\varepsilon\right). \tag{2.104}$$

Hence, differentiating D_{t+2} with respect to D_t:

$$\frac{\partial D_{t+2}}{\partial D_t} = \frac{\partial l\left(D_{t+1}, Y_{t+1}, \sigma_\varepsilon\right)}{\partial D_t} = \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t} + \frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial D_t} + \frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial D_t}.$$

Because $\frac{\partial Y_{t+1}}{\partial D_t} = 0$ and $\frac{\partial \sigma_\varepsilon}{\partial D_t} = 0$, and using (2.101), we have:

$$\frac{\partial D_{t+2}}{\partial D_t} = \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial l_t}{\partial D_t} = \left(\frac{\partial l}{\partial D_t}\right)^2 = \left(l_1^D\right)^2,$$

where we use that $\frac{\partial l_{t+1}}{\partial D_{t+1}}$ and $\frac{\partial l_t}{\partial D_t}$ are evaluated at the same point. Further,

$$\frac{\partial D_{t+2}}{\partial Y_t} = \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t} + \frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t} = l_1^D l_1^Y + l_1^Y \rho.$$

Finally,

$$\begin{aligned}
\frac{\partial D_{t+2}}{\partial \sigma_\varepsilon} &= \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon} \\
&= l_1^D l_1^\sigma + l_1^Y \varepsilon_{t+1} + l_1^\sigma.
\end{aligned}$$

Derivation of p_D Differentiate (2.97) with respect to D_t around the point $\left(\overline{D}, \overline{Y}, 0\right)$ and multiply by $\widetilde{D}_t$:

$$\mathrm{E}_t\left[p_D \widetilde{D}_t\right] = E_t\left\{\left[\begin{array}{c}
u''\left(G_t\right) * \left(\frac{\partial D_{t+1}}{\partial D_t} - k^D\overline{L^D} - k^D\left(\overline{D} - D^c\right)\frac{\partial L_t^D}{\partial D_t}\right) - \\
\beta_g k^D * \left[2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t} + \left(\overline{D} - D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right] * u'\left(G_{t+1}\right) \\
-\beta_g k^D\left[\overline{L^D} + \gamma\left(\overline{D} - D^c\right)\overline{L^D}(1 - \overline{L^D})\right] * \\
\left(\frac{\partial D_{t+2}}{\partial D_t} - k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t} - k^D\left(\overline{D} - D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)u''\left(G_{t+1}\right)
\end{array}\right]\widetilde{D}_t\right\}.$$

Notice that:

$$\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}} = \gamma^2 L_{t+1}^D\left(1 - L_{t+1}^D\right)\left(1 - 2L_{t+1}^D\right) = \gamma^2\left(L_{t+1}^D - 3\left(L_{t+1}^D\right)^2 + 2\left(L_{t+1}^D\right)^3\right). \tag{2.105}$$

Using (2.105), the expressions for $\frac{\partial D_{t+1}}{\partial D_t}$ and $\frac{\partial D_{t+2}}{\partial D_t}$, and (2.100), we have

$$\mathrm{E}_t\left[p_D\widetilde{D}_t\right]=\left[\begin{array}{c} u''*\left(l_1^D-k^D\overline{L^D}-k^D\left(\overline{D}-D^c\right)\gamma\overline{L^D}(1-\overline{L^D})\right)- \\ \beta_g\gamma k^D l_1^D*\left[2\overline{L^D}(1-\overline{L^D})+\left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})(1-2\overline{L^D})\right]*u' \\ -u''*\left(\left(l_1^D\right)^2-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^D\right) \end{array}\right]\widetilde{D}_t,$$

where we exclude the expectations operator on the right-hand side, because all the values are known in period t. In addition, given (2.100), it follows that

$$\mathrm{E}_t\left[p_D\widetilde{D}_t\right]=\left[\begin{array}{c} -\left(l_1^D\right)^2u''+ \\ l_1^D\left(\begin{array}{c} u''+u''*k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)- \\ u'\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})*\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right] \end{array}\right) \\ -u''*k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right) \end{array}\right]\widetilde{D}_t. \tag{2.106}$$

Derivation of p_Y Differentiate (2.97) with respect to Y_t around the point $\left(\overline{D},\overline{Y},0\right)$:

$$p_Y=\left[\begin{array}{c} \left(\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\right)u''\left(G_t\right)- \\ \beta_g k^D\left[\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial L_{t+1}^D}{\partial D_{t+1}}+\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right]u'\left(G_{t+1}\right) \\ -u''\left(G_{t+1}\right)*\left(\frac{\partial D_{t+2}}{\partial Y_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\rho-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right) \end{array}\right].$$

Hence,

$$\mathrm{E}_t\left[p_Y\widetilde{Y}_t\right]=\left[\begin{array}{c} \left(l_1^Y+\tau^y\right)u''- \\ \beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\gamma\left(\overline{D}-D^c\right)\left(1-2\overline{L^D}\right)\right]l_1^Y u'- \\ \left[l_1^D l_1^Y+l_1^Y\rho+\tau^y\rho-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^Y\right]u'' \end{array}\right]\widetilde{Y}_t. \tag{2.107}$$

Again, we exclude the expectations operator on the right-hand side since all values are known in period t.

Derivation of p_{σ_ε} Differentiate (2.97) with respect to σ_ε around the point $\left(\overline{D},\overline{Y},0\right)$, multiply by σ_ε, take expectations and use (2.100), to yield:

$$\mathrm{E}_t\left[p_{\sigma_\varepsilon}\sigma_\varepsilon\right]=E_t\left\{\left[\begin{array}{c} \left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)u''\left(G_t\right)- \\ \beta_g k^D\left[2\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\frac{\partial L_{t+1}^D}{\partial D_{t+1}}+\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right]u'\left(G_{t+1}\right) \\ -\left[\begin{array}{c} \frac{\partial D_{t+2}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon} \\ -k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon} \end{array}\right]u''\left(G_{t+1}\right) \end{array}\right]\sigma_\varepsilon\right\}.$$

Hence,

$$\mathrm{E}_t\left[p_{\sigma_\varepsilon}\sigma_\varepsilon\right]=E_t\left\{\left[\begin{array}{c} l_1^\sigma u''- \\ \beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\gamma\left(\overline{D}-D^c\right)\left(1-2\overline{L^D}\right)\right]l_1^\sigma u' \\ -\left[\begin{array}{c} l_1^D l_1^\sigma+l_1^Y\varepsilon_{t+1}+l_1^\sigma+\tau^y\varepsilon_{t+1}- \\ k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^\sigma \end{array}\right]u'' \end{array}\right]\sigma_\varepsilon\right\}.$$

Hence,

$$\mathrm{E}_t\left[p_{\sigma_\varepsilon}\sigma_\varepsilon\right] = E_t\left\{\left[\begin{array}{c} u''*l_1^\sigma - \beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\gamma\left(\overline{D}-D^c\right)\left(1-2\overline{L^D}\right)\right]*l_1^\sigma u' \\ -u''*\left[\left(1+l_1^D\right)l_1^\sigma - k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^\sigma\right] \end{array}\right]\sigma_\varepsilon\right\}. \tag{2.108}$$

Hence,

$$l_1^\sigma = 0. \tag{2.109}$$

First-order Taylor expansion

Substituting (2.106), (2.107), $\mathrm{E}_t\left[p\left(\overline{B},\overline{Y},0\right)\right] = 0$ and $\mathrm{E}_t\left[p_{\sigma_\varepsilon}\sigma_\varepsilon\right] = 0$, we can write the first-order Taylor expansion of 2.29 as

$$\left\{\begin{array}{l} \left[\begin{array}{c} l_1^D\left(\begin{array}{c} -\left(l_1^D\right)^2 u''+ \\ u''+k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)u''- \\ \beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]u' \end{array}\right) \\ -k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)u'' \end{array}\right]\widetilde{D}_t \\ +\left[\begin{array}{c} \left(l_1^Y+\tau^y\right)u''- \\ \beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\gamma\left(\overline{D}-D^c\right)\left(1-2\overline{L^D}\right)\right]l_1^Y u'- \\ \left[l_1^D l_1^Y + l_1^Y\rho + \tau^y\rho - k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^Y\right]u'' \end{array}\right]\widetilde{Y}_t \end{array}\right\} = 0. \tag{2.110}$$

This expression must hold for $\left(\widetilde{D}_t,\widetilde{Y}_t\right)$, which allows us to solve for l_1^D and l_1^Y.

Computation of l_1^D Multiply the coefficient of $\widetilde{D}_t$ in (2.110) by $-\frac{\beta_g}{u''}$ and set the result equal to zero:

$$\left[\beta_g\left(l_1^D\right)^2 - l_1^D\left(\begin{array}{c} (\beta_g+1) - \frac{u'}{u''}\beta_g^2\gamma k^D\overline{L^D}(1-\overline{L^D})* \\ \left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right] \end{array}\right)+1\right] = 0.$$

where we have used (2.100). This a quadratic equation in l_1^D, with solutions:

$$l_1^D = \frac{\begin{array}{c}\beta_g+1-\frac{u'}{u''}\beta_g^2\gamma k^D\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\pm \\ \sqrt{\left(\beta_g+1-\frac{u'}{u''}\beta_g^2\gamma k^D\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\right)^2-4\beta_g}\end{array}}{2\beta_g}. \tag{2.111}$$

To ensure a non-explosive path for the primary deficit, we need to pick the solution that is smaller than unity in absolute value.

Computation of l_1^Y The third term of (2.110) also has to be zero for any value of Y_t. This requires that:

$$l_1^Y \left\{ \begin{array}{c} \left[1-\rho-l_1^D+k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right]u'' \\ -\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\gamma\left(\overline{D}-D^c\right)\left(1-2\overline{L^D}\right)\right]u' \end{array} \right\} + \tau^y\left(1-\rho\right)u'' = 0.$$

Thus, plugging the stable root of l_1^D (2.111), we find that

$$l_1^Y = \frac{\tau^y\left(\rho-1\right)u''}{\left\{ \begin{array}{c} \left[1-\rho-l_1^D+k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right]u'' \\ -\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\gamma\left(\overline{D}-D^c\right)\left(1-2\overline{L^D}\right)\right]u' \end{array} \right\}}. \tag{2.112}$$

Finally, $l_1^\sigma = 0$.

2.G.2 Second-order approximation of (2.29)

Derivation of the second-order partial derivatives p_{DD}, p_{DY_t}, $p_{D\sigma_\varepsilon}$, $p_{Y_tY_t}$, $p_{Y_t\sigma_\varepsilon}$, and $p_{\sigma_\varepsilon\sigma_\varepsilon}$

For later use, we compute the second-order partial derivatives of (2.28) and (2.104) with respect to D_t, Y_t and σ_ε.

Derivation of the second-order partial derivative of (2.28) with respect to $D_t, Y_t, \sigma_\varepsilon$ With (2.101), we have:

$$\frac{\partial^2 D_{t+1}}{\partial^2 D_t} = \frac{\partial\left(\partial l_t/\partial D_t\right)}{\partial D_t} = \frac{\partial^2 l_t}{\partial^2 D_t} \equiv l_{11}^D. \tag{2.113}$$

- Further,

$$\frac{\partial^2 D_{t+1}}{\partial D_t \partial Y_t} = \frac{\partial^2 l_t}{\partial D_t \partial Y_t} \equiv l_{11}^{DY}. \tag{2.114}$$

- Further,

$$\frac{\partial^2 D_{t+1}}{\partial^2 Y_t} = \frac{\partial^2 l_t}{\partial^2 Y_t} \equiv l_{11}^{Y}. \tag{2.115}$$

- Further,

$$\frac{\partial^2 D_{t+1}}{\partial D_t \partial \sigma_\varepsilon} = \frac{\partial^2 l_t}{\partial D_t \partial \sigma_\varepsilon} \equiv l_{11}^{D\sigma}. \tag{2.116}$$

- Further,

$$\frac{\partial^2 D_{t+1}}{\partial Y_t \partial \sigma_\varepsilon} = \frac{\partial^2 l_t}{\partial Y_t \partial \sigma_\varepsilon} \equiv l_{11}^{Y\sigma}. \tag{2.117}$$

- Finally,

$$\frac{\partial^2 D_{t+1}}{\partial^2 \sigma_\varepsilon} = \frac{\partial^2 l_t}{\partial^2 \sigma_\varepsilon} \equiv l_{11}^{\sigma}. \tag{2.118}$$

Derivation of the second-order partial derivatives of (2.104) with respect to D_t, Y_t and σ_ε

- Computation of $\frac{\partial^2 D_{t+2}}{\partial^2 D_t}$:

$$
\begin{aligned}
\frac{\partial^2 D_{t+2}}{\partial^2 D_t} &= \frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial D_t}\frac{\partial D_{t+1}}{\partial D_t} + \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial\left(\frac{\partial D_{t+1}}{\partial D_t}\right)}{\partial D_t} \\
&= \left[\frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t} + \frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial D_t} + \frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial D_t}\right]\frac{\partial D_{t+1}}{\partial D_t} + \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial^2 D_t} \\
&= \left[\frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial D_{t+1}} * \frac{\partial D_{t+1}}{\partial D_t}\right]\frac{\partial D_{t+1}}{\partial D_t} + \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial D_t^2}.
\end{aligned}
$$

Hence,

$$
\frac{\partial^2 D_{t+2}}{\partial^2 D_t} = l_{11}^D\left(l_1^D\right)^2 + l_1^D l_{11}^D.
$$

- Computation of $\frac{\partial^2 D_{t+2}}{\partial D_t \partial Y_t}$:

$$
\begin{aligned}
\frac{\partial^2 D_{t+2}}{\partial D_t \partial Y_t} &= \frac{\partial}{\partial D_t}\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t} + \frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t} + \frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right) \Rightarrow \\
\frac{\partial^2 D_{t+2}}{\partial D_t \partial Y_t} &= \left[\frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial D_t}\frac{\partial D_{t+1}}{\partial Y_t} + \frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial\left(\frac{\partial D_{t+1}}{\partial Y_t}\right)}{\partial D_t} + \frac{\partial\left(\frac{\partial l_{t+1}}{\partial Y_{t+1}}\right)}{\partial D_t}\rho + \frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial(\rho)}{\partial D_t}\right] \\
&= \left[\begin{array}{c} \frac{\partial^2 l_{t+1}}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial Y_t} + l_1^D * \left[\frac{\partial^2 D_{t+1}}{\partial Y_t \partial D_t} + \frac{\partial^2 D_{t+1}}{\partial^2 Y_t}\frac{\partial Y_t}{\partial D_t} + \frac{\partial^2 D_{t+1}}{\partial Y_t \partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial D_t}\right] \\ +\rho * \left[\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t} + \frac{\partial^2 l_{t+1}}{\partial^2 Y_{t+1}}\frac{\partial Y_{t+1}}{\partial D_t} + \frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial D_t}\right] \end{array}\right] \\
&= l_{11}^D l_1^D l_1^Y + l_1^D l_{11}^{DY} + \rho l_1^D l_{11}^{DY},
\end{aligned}
$$

using that $\frac{\partial \sigma_\varepsilon}{\partial Y_t} = 0$, $\frac{\partial Y_{t+1}}{\partial Y_t} = \rho$, (2.101), and (2.114).

- Computation of $\frac{\partial^2 D_{t+2}}{\partial D_t \partial \sigma_\varepsilon}$:

$$
\begin{aligned}
\frac{\partial^2 D_{t+2}}{\partial D_t \partial \sigma_\varepsilon} &= \frac{\partial}{\partial D_t}\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right) \\
&= \left[\begin{array}{c} \frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial D_t} * \frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial l_{t+1}}{\partial D_{t+1}} * \frac{\partial\left(\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial D_t} + \frac{\partial\left(\frac{\partial l_{t+1}}{\partial Y_{t+1}}\right)}{\partial D_t} * \varepsilon_{t+1} \\ +\frac{\partial l_{t+1}}{\partial Y_{t+1}} * \frac{\partial(\varepsilon_{t+1})}{\partial D_t} + \frac{\partial\left(\frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\right)}{\partial D_t} + \frac{\partial l_{t+1}}{\partial \sigma_\varepsilon} * \frac{\partial(1)}{\partial B_t} \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\partial^2 l_{t+1}}{\partial D_{t+1}^2} * \frac{\partial D_{t+1}}{\partial D_t} * \frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial l_{t+1}}{\partial D_{t+1}} * \frac{\partial^2 D_{t+1}}{\partial D_t \partial \sigma_\varepsilon} + \\ \frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial D_{t+1}} * \frac{\partial D_{t+1}}{\partial D_t} * \varepsilon_{t+1} + \frac{\partial^2 l_{t+1}}{\partial \sigma_\varepsilon \partial D_{t+1}} * \frac{\partial D_{t+1}}{\partial D_t} \end{array}\right] \\
&= l_{11}^D l_1^D l_1^\sigma + l_1^D l_{11}^{D\sigma} + l_1^D l_{11}^{DY}\varepsilon_{t+1} + l_1^D l_{11}^{D\sigma} = l_{11}^D l_1^D l_1^\sigma + 2 l_1^D l_{11}^{D\sigma} + l_1^D l_{11}^{DY}\varepsilon_{t+1}.
\end{aligned}
$$

- Computation of $\frac{\partial^2 D_{t+2}}{\partial^2 Y_t}$:

$$\begin{aligned}
\frac{\partial^2 D_{t+2}}{\partial^2 Y_t} &= \frac{\partial}{\partial Y_t}\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t}+\frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial Y_t}\right)\\
&= \left[\frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial Y_t}*\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial l_{t+1}}{\partial D_{t+1}}*\frac{\partial\left(\frac{\partial D_{t+1}}{\partial Y_t}\right)}{\partial Y_t}+\frac{\partial\left(\frac{\partial l_{t+1}}{\partial Y_{t+1}}\right)}{\partial Y_t}*\rho+\frac{\partial l_{t+1}}{\partial Y_{t+1}}*\frac{\partial(\rho)}{\partial Y_t}\right]\\
&= \left[\begin{array}{c}\left[\frac{\partial^2 l_{t+1}}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial D_{t+1}\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial D_{t+1}\partial\sigma_\varepsilon}\frac{\partial\sigma_\varepsilon}{\partial Y_t}\right]*\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial l_{t+1}}{\partial D_{t+1}}*\frac{\partial^2 D_{t+1}}{\partial Y_t^2}\\
+\left[\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}^2}\frac{\partial Y_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial\sigma_\varepsilon}\frac{\partial\sigma_\varepsilon}{\partial Y_t}\right]*\rho\end{array}\right],
\end{aligned}$$

since $\frac{\partial\sigma_\varepsilon}{\partial Y_t}=0$ and $\frac{\partial Y_{t+1}}{\partial Y_t}=\rho$. Using (2.102), (2.114) and (2.115), we can rewrite the last equation as

$$\frac{\partial^2 D_{t+2}}{\partial^2 Y_t}=l_{11}^D\left(l_1^Y\right)^2+l_{11}^{DY}l_1^Y\rho+l_1^Dl_{11}^Y+l_{11}^{DY}l_1^Y\rho+l_{11}^Y\rho^2=l_{11}^D\left(l_1^Y\right)^2+2l_{11}^{DY}l_1^Y\rho+l_{11}^Y\left(l_1^D+\rho^2\right).$$

- Computation of $\frac{\partial^2 D_{t+2}}{\partial Y_t\partial\sigma_\varepsilon}$:

$$\begin{aligned}
\frac{\partial^2 D_{t+2}}{\partial Y_t\partial\sigma_\varepsilon} &= \frac{\partial}{\partial Y_t}\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right)\\
&= \left[\begin{array}{c}\frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial Y_t}*\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial D_{t+1}}*\frac{\partial\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)}{\partial Y_t}+\frac{\partial\left(\frac{\partial l_{t+1}}{\partial Y_{t+1}}\right)}{\partial Y_t}*\varepsilon_{t+1}\\
+\frac{\partial l_{t+1}}{\partial Y_{t+1}}*\frac{\partial(\varepsilon_{t+1})}{\partial Y_t}+\frac{\partial\left(\frac{\partial l_{t+1}}{\partial\sigma_\varepsilon}\right)}{\partial Y_t}+\frac{\partial l_{t+1}}{\partial\sigma_\varepsilon}*\frac{\partial(1)}{\partial Y_t}\end{array}\right]\\
&= \left[\begin{array}{c}\left[\frac{\partial^2 l_{t+1}}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial D_{t+1}\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial D_{t+1}\partial\sigma_\varepsilon}\frac{\partial\sigma_\varepsilon}{\partial Y_t}\right]*\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial D_{t+1}}*\frac{\partial^2 D_{t+1}}{\partial\sigma_\varepsilon\partial Y_t}\\
+\left[\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}^2}\frac{\partial Y_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial\sigma_\varepsilon}\frac{\partial\sigma_\varepsilon}{\partial Y_t}\right]*\varepsilon_{t+1}+\\
\frac{\partial^2 l_{t+1}}{\partial\sigma_\varepsilon\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial\sigma_\varepsilon\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial Y_t}+\frac{\partial^2 l_{t+1}}{\partial\sigma_\varepsilon^2}\frac{\partial\sigma_\varepsilon}{\partial Y_t}\end{array}\right]\\
&= \left(l_{11}^Dl_1^Y+l_{11}^{DY}\rho\right)l_1^\sigma+l_1^Dl_{11}^{Y\sigma}+\left(l_{11}^{DY}l_1^Y+l_{11}^Y\rho\right)\varepsilon_{t+1}+l_{11}^{D\sigma}l_1^Y+l_{11}^{Y\sigma}\rho.
\end{aligned}$$

- Computation of $\frac{\partial^2 D_{t+2}}{\partial\sigma_\varepsilon^2}$:

$$\begin{aligned}
\frac{\partial^2 D_{t+2}}{\partial\sigma_\varepsilon^2} &= \frac{\partial}{\partial \sigma_\varepsilon}\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial \sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial \sigma_\varepsilon}\frac{\partial \sigma_\varepsilon}{\partial \sigma_\varepsilon}\right)\\
&= \left[\begin{array}{c}\frac{\partial\left(\frac{\partial l_{t+1}}{\partial D_{t+1}}\right)}{\partial\sigma_\varepsilon}*\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial D_{t+1}}*\frac{\partial\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)}{\partial\sigma_\varepsilon}+\frac{\partial\left(\frac{\partial l_{t+1}}{\partial Y_{t+1}}\right)}{\partial\sigma_\varepsilon}*\varepsilon_{t+1}\\
+\frac{\partial l_{t+1}}{\partial Y_{t+1}}*\frac{\partial(\varepsilon_{t+1})}{\partial\sigma_\varepsilon}+\frac{\partial\left(\frac{\partial l_{t+1}}{\partial\sigma_\varepsilon}\right)}{\partial\sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial\sigma_\varepsilon}*\frac{\partial(1)}{\partial\sigma_\varepsilon}\end{array}\right]\\
&= \left[\begin{array}{c}\left[\frac{\partial^2 l_{t+1}}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial^2 l_{t+1}}{\partial D_{t+1}\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\sigma_\varepsilon}+\frac{\partial^2 l_{t+1}}{\partial D_{t+1}\partial\sigma_\varepsilon}\frac{\partial\sigma_\varepsilon}{\partial\sigma_\varepsilon}\right]*\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial l_{t+1}}{\partial D_{t+1}}*\frac{\partial^2 D_{t+1}}{\partial\sigma_\varepsilon^2}\\
+\left[\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}^2}\frac{\partial Y_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial^2 l_{t+1}}{\partial Y_{t+1}\partial\sigma_\varepsilon}\frac{\partial\sigma_\varepsilon}{\partial\sigma_\varepsilon}\right]*\varepsilon_{t+1}+\\
\frac{\partial^2 l_{t+1}}{\partial\sigma_\varepsilon\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial^2 l_{t+1}}{\partial\sigma_\varepsilon\partial Y_{t+1}}\frac{\partial Y_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial^2 l_{t+1}}{\partial\sigma_\varepsilon^2}\frac{\partial\sigma_\varepsilon}{\partial\sigma_\varepsilon}\end{array}\right]\\
&= \left(l_{11}^Dl_1^\sigma+l_{11}^{DY}\varepsilon_{t+1}+2l_{11}^{D\sigma}\right)l_1^\sigma+\left(l_{11}^{DY}l_1^\sigma+2l_{11}^{Y\sigma}\right)\varepsilon_{t+1}+l_{11}^Y\varepsilon_{t+1}^2+\left(1+l_1^D\right)l_{11}^\sigma.
\end{aligned}$$

Derivation of p_{DD} Differentiate p_D with respect to D_t, and evaluate at $\left(\overline{D},\overline{Y},0\right)$:

$$p_{DD}=\left[\begin{array}{c}
\frac{\partial(u''(G_t))}{\partial D_t}\left(\frac{\partial D_{t+1}}{\partial D_t}-k^D\overline{L^D}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_t^D}{\partial D_t}\right)+\\
\frac{\partial}{\partial D_t}\left(\frac{\partial D_{t+1}}{\partial D_t}-k^D L_t^D-k^D\left(D_t-D^c\right)\frac{\partial L_t^D}{\partial D_t}\right)u''-\\
\left[2\frac{\partial}{\partial D_t}\left(\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)+\frac{\partial}{\partial D_t}\left(\left(D_{t+1}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)\right]\beta_g k^D u'\\
-\left[2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}+\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right]\beta_g k^D\frac{\partial}{\partial D_t}\left(u'\left(G_{t+1}\right)\right)-\\
\left[\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}+\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial L_{t+1}^D}{\partial D_{t+1}}+\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right]*\beta_g k^D u''*\\
\left(\frac{\partial D_{t+2}}{\partial D_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)-\\
\beta_g k^D\left[\overline{L^D}+\gamma\left(\overline{D}-D^c\right)\overline{L^D}(1-\overline{L^D})\right]*\frac{\partial}{\partial D_t}\left(u''\left(G_{t+1}\right)\right)*\\
\left(\frac{\partial D_{t+2}}{\partial D_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)\\
-\beta_g k^D\left[\overline{L^D}+\gamma\left(\overline{D}-D^c\right)\overline{L^D}(1-\overline{L^D})\right]*u''*\\
\frac{\partial}{\partial D_t}\left(\frac{\partial D_{t+2}}{\partial D_t}-k^D L_{t+1}^D\frac{\partial D_{t+1}}{\partial D_t}-k^D\left(D_{t+1}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)
\end{array}\right].$$

Using (2.98) and (2.105), as well as (2.100), this becomes:

$$p_{DD}=\left[\begin{array}{c}
\left[\frac{\partial D_{t+1}}{\partial D_t}-k^D\overline{L^D}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_t^D}{\partial D_t}\right]^2 u'''+\\
\left[\frac{\partial^2 D_{t+1}}{\partial^2 D_t}-k^D\left(2\frac{\partial L_t^D}{\partial D_t}+\left(\overline{D}-D^c\right)\frac{\partial^2 L_t^D}{\partial^2 D_t}\right)\right]u''-\\
\beta_g k^D\left[\begin{array}{c}
2\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial D_t}\right)^2+2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial^2 D_t}+\\
\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial D_t}\right)^2+\left(\overline{D}-D^c\right)\frac{\partial^3 L_{t+1}^D}{\partial^3 D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial D_t}\right)^2\\
+\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial^2 D_t}
\end{array}\right]u'-\\
\beta_g k^D\left[2l_1^D\gamma\overline{L^D}(1-\overline{L^D})+l_1^D\left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right]*u''*\\
\left(\frac{\partial D_{t+2}}{\partial D_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)-\\
\beta_g k^D\left[2l_1^D\gamma\overline{L^D}(1-\overline{L^D})+l_1^D\left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right]*u''*\\
\left(\frac{\partial D_{t+2}}{\partial D_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)\\
-u'''\left(\frac{\partial D_{t+2}}{\partial D_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right)^2-\\
\left[\begin{array}{c}
\frac{\partial^2 D_{t+2}}{\partial^2 D_t}-k^D\left(2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}+\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\right)\left(\frac{\partial D_{t+1}}{\partial D_t}\right)^2\\
-k^D\left(\overline{L^D}+\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\right)\frac{\partial^2 D_{t+1}}{\partial^2 D_t}
\end{array}\right]u''
\end{array}\right].$$

Using the derivatives of D_{t+1} and D_{t+2}, (2.98) and (2.105), we simplify the last equation once more to

$$p_{DD} = \left[\begin{array}{c} \left[l_1^D - k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1-\overline{L^D})\right)\right]^2 u''' + \\ \left[l_{11}^D - k^D\gamma\overline{L^D}(1-\overline{L^D})\left(2+\gamma\left(\overline{D} - D^c\right)\left(1-2\overline{L^D}\right)\right)\right] u'' - \\ \beta_g k^D \left[\begin{array}{c} \left[3\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right) + \left(\overline{D} - D^c\right)\frac{\partial^3 L_{t+1}^D}{\partial^3 D_{t+1}}\right]\left(l_1^D\right)^2 + \\ \left[2\gamma\overline{L^D}(1-\overline{L^D}) + \left(\overline{D} - D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right] l_{11}^D \end{array} \right] u' - \\ 2l_1^D\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D}) * \\ \left[2 + \left(\overline{D} - D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\left[\left(l_1^D\right)^2 - k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1-\overline{L^D})\right) l_1^D\right] u'' \\ -\left[\left(l_1^D\right)^2 - k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1-\overline{L^D})\right) l_1^D\right]^2 u''' \\ -\left[\begin{array}{c} l_{11}^D\left(l_1^D\right)^2 + l_1^D l_{11}^D - k^D\gamma\overline{L^D}(1-\overline{L^D})\left(2+\left(\overline{D} - D^c\right)\gamma\left(1-2\overline{L^D}\right)\right)\left(l_1^D\right)^2 \\ -k^D\overline{L^D}\left[1 + \left(\overline{D} - D^c\right)\gamma(1-\overline{L^D})\right] l_{11}^D \end{array} \right] u'' \end{array} \right].$$

The only additional term that we have to compute in the equation above is $\frac{\partial^3 L_{t+1}^D}{\partial^3 D_{t+1}}$, but from (2.105) we get

$$\begin{aligned} \frac{\partial^3 L_{t+1}^D}{\partial D_{t+1}^3} &= \frac{\partial}{\partial D_{t+1}}\left[\gamma^2 L_{t+1}^D\left(1-L_{t+1}^D\right)\left(1-2L_{t+1}^D\right)\right] \\ &= \gamma^2 \left[\begin{array}{c} \frac{\partial L_{t+1}^D}{\partial D_{t+1}}\left(1-L_{t+1}^D\right)\left(1-2L_{t+1}^D\right) - L_{t+1}^D\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\left(1-2L_{t+1}^D\right) \\ -L_{t+1}^D\left(1-L_{t+1}^D\right)2\frac{\partial L_{t+1}^D}{\partial D_{t+1}} \end{array} \right] \\ &= \gamma^2 \left[\begin{array}{c} \gamma L_{t+1}^D\left(1-L_{t+1}^D\right)^2\left(1-2L_{t+1}^D\right) - \gamma\left(L_{t+1}^D\right)^2\left(1-L_{t+1}^D\right)\left(1-2L_{t+1}^D\right) \\ -2\gamma\left(L_{t+1}^D\left(1-L_{t+1}^D\right)\right)^2 \end{array} \right] \\ &= \gamma^3 L_{t+1}^D\left(1-L_{t+1}^D\right) \left[\begin{array}{c} \left(1-L_{t+1}^D\right)\left(1-2L_{t+1}^D\right) - L_{t+1}^D\left(1-2L_{t+1}^D\right) \\ -2L_{t+1}^D\left(1-L_{t+1}^D\right) \end{array} \right] \qquad (2.119) \\ &= \gamma^3 L_{t+1}^D\left(1-L_{t+1}^D\right)\left[\left(1-2L_{t+1}^D\right)\left(1-2L_{t+1}^D\right) - 2L_{t+1}^D\left(1-L_{t+1}^D\right)\right] \\ &= \gamma^3 L_{t+1}^D\left(1-L_{t+1}^D\right)\left[1 - 6L_{t+1}^D + 6\left(L_{t+1}^D\right)^2\right]. \end{aligned}$$

Hence,

$$
p_{DD}=\left[\begin{array}{c}
\left[l_1^D-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right]^2u'''+\\
\left[l_{11}^D-k^D\gamma\overline{L^D}(1-\overline{L^D})\left(2+\gamma\left(\overline{D}-D^c\right)\left(1-2\overline{L^D}\right)\right)\right]u''-\\
\beta_gk^D\gamma\overline{L^D}(1-\overline{L^D})\left[\begin{array}{c}\left[\begin{array}{c}3\gamma\left(1-2\overline{L^D}\right)+\\ \left(\overline{D}-D^c\right)\gamma^2\left(1-6\overline{L^D}+6\left(\overline{L^D}\right)^2\right)\end{array}\right]\left(l_1^D\right)^2\\ +\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]l_{11}^D\end{array}\right]u'\\
-2l_1^D\beta_gk^D\gamma\overline{L^D}(1-\overline{L^D})*\\
\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\left[\left(l_1^D\right)^2-\frac{1}{\beta_g}l_1^D\right]u''\\
-\left[\left(l_1^D\right)^2-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^D\right]^2u'''\\
-\left[\begin{array}{c}\left(\left(l_1^D\right)^2+l_1^D-k^D\overline{L^D}\left[1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right]\right)l_{11}^D-\\ k^D\gamma\overline{L^D}(1-\overline{L^D})\left(2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right)\left(l_1^D\right)^2\end{array}\right]u''
\end{array}\right].
$$

Finally, we can isolate the only unknown term l_{11}^D of that equation and multiply it to $\widetilde{D}_t^2$:

$$
\begin{aligned}
&p_{DD}\widetilde{D}_t^2\\
=&\left[\begin{array}{c}
l_{11}^D\left[\begin{array}{c}\left(1+k^D\overline{L^D}\left[1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right]-\left(l_1^D\right)^2-l_1^D\right)u''-\\ \beta_gk^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]u'\end{array}\right]\\
+\left[l_1^D-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right]^2\left(1-\left(l_1^D\right)^2\right)u'''-\\
\beta_gk^D\gamma^2\overline{L^D}(1-\overline{L^D})\left[\begin{array}{c}3\left(1-2\overline{L^D}\right)+\\ \left(\overline{D}-D^c\right)\gamma\left(1-6\overline{L^D}+6\left(\overline{L^D}\right)^2\right)\end{array}\right]\left(l_1^D\right)^2u'\\
+k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\left[-1+3\left(l_1^D\right)^2-2\beta_g\left(l_1^D\right)^3\right]u''
\end{array}\right]\widetilde{D}_t^2\\
\equiv&\left(\Lambda l_{11}^D+\Xi\right)\widetilde{D}_t^2. \qquad (2.120)
\end{aligned}
$$

Derivation of the p_{DY} Differentiate p_Y with respect to D_t around the point $\left(\overline{D},\overline{Y},0\right)$ and use (2.100) to give:

$$
p_{DY}=\left[\begin{array}{c}
\frac{\partial}{\partial D_t}\left(u''\left(G_t\right)\right)*\left(\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\right)+u''*\frac{\partial}{\partial D_t}\left(\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\right)-\\
\beta_gk^D*\left[2\frac{\partial}{\partial D_t}\left(\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right)+\frac{\partial}{\partial D_t}\left(\left(D_{t+1}-D^c\right)\frac{\partial^2L_{t+1}^D}{\partial^2D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right)\right]*u'\\
-\beta_gk^D\left[2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}+\left(D_{t+1}-D^c\right)\frac{\partial^2L_{t+1}^D}{\partial^2D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right]*\frac{\partial}{\partial D_t}\left(u'\left(G_{t+1}\right)\right)-\\
\beta_gk^D\left[\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}+\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial L_{t+1}^D}{\partial D_{t+1}}+\left(\overline{D}-D^c\right)\frac{\partial^2L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial D_t}\right]*u''*\\
\left(\frac{\partial D_{t+2}}{\partial Y_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\rho-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right)-\\
\frac{\partial}{\partial D_t}\left(u''\left(G_{t+1}\right)\right)*\left(\frac{\partial D_{t+2}}{\partial Y_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\rho-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right)\\
-u''*\frac{\partial}{\partial D_t}\left(\frac{\partial D_{t+2}}{\partial Y_t}-k^DL_{t+1}^D\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\rho-k^D\left(D_{t+1}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right)
\end{array}\right].
$$

Then, using (2.98), (2.100) and (2.105), we can rewrite this expression as:

$$
p_{DY}=\left[\begin{array}{c}
\left[\frac{\partial D_{t+1}}{\partial D_t}-k^D\overline{L^D}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_t^D}{\partial D_t}\right]\left(\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\right)u'''+u''\frac{\partial^2 D_{t+1}}{\partial D_t\partial Y_t}- \\
\beta_g k^D*\left\{\begin{array}{c}
2\left[\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial D_t\partial Y_t}\right]+\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial D_{t+1}}{\partial D_t} \\
+\left(\overline{D}-D^c\right)\frac{\partial^2 D_{t+1}}{\partial D_t\partial Y_t}\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}+\left(\overline{D}-D^c\right)\frac{\partial^3 L_{t+1}^D}{\partial^3 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial Y_t}
\end{array}\right\}*u' \\
-\beta_g k^D\left[2l_1^Y\gamma\overline{L^D}(1-\overline{L^D})+l_1^Y\left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right]* \\
\left[\frac{\partial D_{t+2}}{\partial D_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right]u''- \\
\beta_g k^D\left[2l_1^D\gamma\overline{L^D}(1-\overline{L^D})+l_1^D\left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right]* \\
\left[\frac{\partial D_{t+2}}{\partial Y_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial Y_t}+\tau^y\rho-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right]u''- \\
u'''*\left[\begin{array}{c}
\frac{\partial D_{t+2}}{\partial D_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t}- \\
k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}
\end{array}\right]*\left[\begin{array}{c}
\frac{\partial D_{t+2}}{\partial Y_t}-k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial Y_t}+ \\
\tau^y\rho-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}
\end{array}\right] \\
-u''*\left[\begin{array}{c}
\frac{\partial^2 D_{t+2}}{\partial D_t\partial Y_t}-k^D\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial Y_t}-k^D\overline{L^D}\frac{\partial^2 D_{t+1}}{\partial D_t\partial Y_t}-k^D\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial Y_t} \\
-k^D\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial Y_t}-k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial D_t\partial Y_t}
\end{array}\right]
\end{array}\right].
$$

Using again the derivatives of D_{t+1} and D_{t+2} computed above, (2.98), (2.105) and (2.119), we simplify the last equation once more to

$$
p_{DY}=\left[\begin{array}{c}
\left[l_1^D-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right]\left(l_1^Y+\tau^y\right)u'''+l_{11}^{DY}u''- \\
\beta_g k^D*\left\{\begin{array}{c}
\left[\begin{array}{c}
3\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)+ \\
\left(\overline{D}-D^c\right)\gamma^3\overline{L^D}(1-\overline{L^D})\left(1-6\overline{L^D}+6\left(\overline{L^D}\right)^2\right)
\end{array}\right]l_1^D l_1^Y+ \\
\left[2\gamma\overline{L^D}(1-\overline{L^D})+\left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right]l_{11}^{DY}
\end{array}\right\}*u' \\
-\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]* \\
\left[\left(l_1^D\right)^2 l_1^Y-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^D l_1^Y\right]u''- \\
\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]* \\
\left[\left(l_1^D\right)^2 l_1^Y+l_1^D l_1^Y\rho+l_1^D\tau^y\rho-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^D l_1^Y\right]u''- \\
u'''*\left[\left(l_1^D\right)^2-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^D\right]* \\
\left[l_1^D l_1^Y+l_1^Y\rho+\tau^y\rho-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^Y\right] \\
-u''*\left[\begin{array}{c}
l_{11}^D l_1^D l_1^Y+(1+\rho)\,l_1^D l_{11}^{DY}- \\
k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]l_1^D l_1^Y \\
-k^D\overline{L^D}\left[1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right]l_{11}^{DY}
\end{array}\right]
\end{array}\right].
$$

Next, we isolate the two unknown variables, namely l_{11}^{DY} and l_{11}^D, and multiply p_{DY} by

$\widetilde{D}_t\widetilde{Y}_t$ to give:

$$
\begin{aligned}
& p_{DY}\widetilde{D}_t\widetilde{Y}_t \\
= & \left[\begin{array}{c}
l_{11}^{DY} \left\{ \begin{array}{c} \left[1 + k^D\overline{L^D}\left[1 + \left(\overline{D} - D^c\right)\gamma(1 - \overline{L^D})\right] - (1+\rho)\, l_1^D\right] u'' \\ -\beta_g k^D \gamma \overline{L^D}(1 - \overline{L^D})\left[2 + \left(\overline{D} - D^c\right)\gamma\left(1 - 2\overline{L^D}\right)\right] u' \end{array} \right\} - \\
l_{11}^D l_1^D l_1^Y u'' + \left[l_1^D - k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1 - \overline{L^D})\right)\right] * \\
\left[\begin{array}{c} l_1^Y + \tau^y - \left(l_1^D\right)^2 l_1^Y - l_1^D l_1^Y \rho - l_1^D \tau^y \rho + \\ k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1 - \overline{L^D})\right) l_1^D l_1^Y \end{array} \right] u''' - \\
\beta_g k^D \gamma^2 \overline{L^D}(1 - \overline{L^D}) \left[\begin{array}{c} 3\left(1 - 2\overline{L^D}\right) + \\ \left(\overline{D} - D^c\right)\gamma\left(1 - 6\overline{L^D} + 6\left(\overline{L^D}\right)^2\right) \end{array} \right] l_1^D l_1^Y u' - \\
k^D\gamma\overline{L^D}(1 - \overline{L^D})\left[2 + \left(\overline{D} - D^c\right)\gamma\left(1 - 2\overline{L^D}\right)\right] \left[\begin{array}{c} 2\beta_g\left(l_1^D\right)^2 l_1^Y + \beta_g l_1^D l_1^Y \rho \\ +\beta_g l_1^D \tau^y \rho - 3 l_1^D l_1^Y \end{array} \right] u''
\end{array} \right] \widetilde{D}_t\widetilde{Y}_t \\
= & \left(\Pi l_{11}^{DY} + \Upsilon l_{11}^D + \Phi\right)\widetilde{D}_t\widetilde{Y}_t. \qquad (2.121)
\end{aligned}
$$

Derivation of $p_{D\sigma_\varepsilon}$ Differentiate p_{σ_ε} with respect to D_t around the point $\left(\overline{D}, \overline{Y}, 0\right)$ and use

$$
p_{D\sigma_\varepsilon} = \left[\begin{array}{c}
\frac{\partial}{\partial D_t}\left(u''\left(G_t\right)\right) * \left(\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right) + u'' * \frac{\partial}{\partial D_t}\left(\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right) - \\
\beta_g k^D * \left[2\frac{\partial}{\partial D_t}\left(\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right) + \frac{\partial}{\partial D_t}\left(\left(D_{t+1} - D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right)\right] * u' \\
-\beta_g k^D\left[2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \left(\overline{D} - D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right] * \frac{\partial}{\partial D_t}\left(u'\left(G_{t+1}\right)\right) - \\
\beta_g k^D * \frac{\partial}{\partial D_t}\left(L_{t+1}^D + \left(D_{t+1} - D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\right) * u'' * \\
\left(\frac{\partial D_{t+2}}{\partial \sigma_\varepsilon} + \varepsilon_{t+1} - k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1 - \overline{L^D})\right)\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right) - \\
\frac{\partial}{\partial D_t}\left(u''\left(G_{t+1}\right)\right) * \left(\frac{\partial D_{t+2}}{\partial \sigma_\varepsilon} + \varepsilon_{t+1} - k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1 - \overline{L^D})\right)\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right) \\
-u'' * \frac{\partial}{\partial D_t}\left(\frac{\partial D_{t+2}}{\partial \sigma_\varepsilon} + \varepsilon_{t+1} - k^D L_{t+1}^D\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} - k^D\left(D_{t+1} - D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon}\right)
\end{array} \right].
$$

Recall from (2.109) that $\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} = l_1^\sigma = 0$. In addition, using the first- and second-order derivatives of D_{t+1} and D_{t+2} with respect to σ_ε, we obtain

$$
p_{D\sigma_\varepsilon} = \left[\begin{array}{c}
\frac{\partial^2 D_{t+1}}{\partial D_t \partial \sigma_\varepsilon} u'' - \\
\beta_g k^D \left[\begin{array}{c} 2\left(\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial D_t \partial \sigma_\varepsilon}\right) + \frac{\partial D_{t+1}}{\partial D_t}\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \\ \left(\overline{D} - D^c\right)\left(\frac{\partial^3 L_{t+1}^D}{\partial D_{t+1}^3}\frac{\partial D_{t+1}}{\partial D_t}\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} + \frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial^2 D_{t+1}}{\partial D_t \partial \sigma_\varepsilon}\right) \end{array} \right] u' - \\
\left[2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t} + \left(\overline{D} - D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial D_t}\right]\left(l_1^D l_1^\sigma + l_1^Y \varepsilon_{t+1} + l_1^\sigma + \varepsilon_{t+1}\right)\beta_g k^D u'' - \\
\left[\frac{\partial D_{t+2}}{\partial D_t} - k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial D_t} - k^D\left(\overline{D} - D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial D_t}\right]\left(l_1^D l_1^\sigma + l_1^Y \varepsilon_{t+1} + l_1^\sigma + \varepsilon_{t+1}\right) u''' \\
-u'' * \left(\frac{\partial^2 D_{t+2}}{\partial D_t \partial \sigma_\varepsilon} - k^D\overline{L^D}\frac{\partial^2 D_{t+1}}{\partial D_t \partial \sigma_\varepsilon} - k^D\left(\overline{D} - D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial D_t \partial \sigma_\varepsilon}\right)
\end{array} \right].
$$

We can rewrite the last equation further as

$$p_{D\sigma_\varepsilon} = \left[\begin{array}{c} u'' l_{11}^{D\sigma} - \\ \beta_g k^D * \left[2\gamma \overline{L^D}(1-\overline{L^D}) + \left(\overline{D}-D^c\right)\gamma^2 \overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right] * l_{11}^{D\sigma} u' \\ -\beta_g k^D * \left[\begin{array}{c} 2\gamma \overline{L^D}(1-\overline{L^D}) + \\ \left(\overline{D}-D^c\right)\gamma^2 \overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right) \end{array}\right] * l_1^D u'' \left(l_1^Y+1\right)\varepsilon_{t+1} \\ -u''' * \left[\left(l_1^D\right)^2 - k^D \overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right) l_1^D \right] * \left(l_1^Y+1\right)\varepsilon_{t+1} \\ -u'' * \left(2 l_1^D l_{11}^{D\sigma} + l_1^D l_{11}^{DY}\varepsilon_{t+1} - k^D \overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right) l_{11}^{D\sigma}\right) \end{array} \right].$$

Then, grouping the terms of the equation above and multiplying it by $\widetilde{D}_t\sigma_\varepsilon$, we have

$$\begin{aligned} & p_{D\sigma_\varepsilon}\widetilde{D}_t\sigma_\varepsilon \\ = & \left\{ \begin{array}{c} l_{11}^{D\sigma} \left[\begin{array}{c} u'' \left[1 + k^D \overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right) - 2 l_1^D\right] - \\ u'\beta_g k^D \gamma \overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right] \end{array}\right] \\ -u'' l_{11}^{DY} l_1^D \varepsilon_{t+1} - \\ l_1^D\left(l_1^Y+1\right) \left[\begin{array}{c} u''\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right] + \\ u'''\left[l_1^D - k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right] \end{array}\right] \varepsilon_{t+1} \end{array} \right\} \widetilde{D}_t\sigma_\varepsilon \\ = & \left(\Psi l_{11}^{D\sigma} + \Omega\varepsilon_{t+1}\right)\widetilde{D}_t\sigma_\varepsilon. \end{aligned}$$

Finally, applying expectations to the equation above, we have that

$$E_t\left[p_{D\sigma_\varepsilon}\widetilde{D}_t\sigma_\varepsilon\right] = E_t\left[\left(\Psi l_{11}^{D\sigma} + \Omega\varepsilon_{t+1}\right)\widetilde{D}_t\sigma_\varepsilon\right] = \Psi l_{11}^{D\sigma}\widetilde{D}_t\sigma_\varepsilon. \tag{2.122}$$

Derivation of p_{YY} Differentiate p_Y once more with respect to Y_t around the point $\left(\overline{D},\overline{Y},0\right)$ and use (2.100):

$$p_{YY} = \left[\begin{array}{c} \frac{\partial}{\partial Y_t}\left(u''\left(G_t\right)\right) * \left(\frac{\partial D_{t+1}}{\partial Y_t} + \tau^y\right) + u'' * \frac{\partial}{\partial Y_t}\left(\frac{\partial D_{t+1}}{\partial Y_t} + \tau^y\right) \\ -\beta_g k^D * \left[2\frac{\partial}{\partial Y_t}\left(\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right) + \frac{\partial}{\partial Y_t}\left(\left(D_{t+1}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial Y_t}\right)\right] * u' \\ -\beta_g k^D\left[2\left(\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right) + \left(D_{t+1}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial Y_t}\right] * \frac{\partial}{\partial Y_t}\left(u'\left(G_{t+1}\right)\right) - \\ \beta_g k^D \left[\begin{array}{c} \frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t} + \frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial L_{t+1}^D}{\partial D_{t+1}} \\ + \left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial Y_t} \end{array}\right] * u'' * \left(\begin{array}{c} \frac{\partial D_{t+2}}{\partial Y_t} - k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial Y_t} + \\ \tau^y\rho - k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t} \end{array}\right) \\ -\frac{\partial}{\partial Y_t}\left(u''\left(G_{t+1}\right)\right) * \left(\frac{\partial D_{t+2}}{\partial Y_t} - k^D\overline{L^D}\frac{\partial D_{t+1}}{\partial Y_t} + \tau^y\rho - k^D\left(\overline{D}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right) \\ -u'' * \frac{\partial}{\partial Y_t}\left(\frac{\partial D_{t+2}}{\partial Y_t} - k^D L_{t+1}^D\frac{\partial D_{t+1}}{\partial Y_t} + \tau^y\rho - k^D\left(D_{t+1}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right) \end{array} \right].$$

Hence, also using (2.100),

$$p_{YY} = \left[\begin{array}{c} \left(l_1^Y + \tau^y\right)^2 u''' + \frac{\partial^2 D_{t+1}}{\partial Y_t^2} u'' - \\ u' \beta_g k^D * \left[\begin{array}{c} 2\left[\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}} \left(\frac{\partial D_{t+1}}{\partial Y_t}\right)^2 + \frac{\partial L_{t+1}^D}{\partial D_{t+1}} \frac{\partial^2 D_{t+1}}{\partial^2 Y_t}\right] + \frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}} \left(\frac{\partial D_{t+1}}{\partial Y_t}\right)^2 \\ + \left(\overline{D} - D^c\right) \frac{\partial^3 L_{t+1}^D}{\partial^3 D_{t+1}} \left(\frac{\partial D_{t+1}}{\partial Y_t}\right)^2 + \left(\overline{D} - D^c\right) \frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}} \frac{\partial^2 D_{t+1}}{\partial^2 Y_t} \end{array} \right] \\ -2\beta_g k^D \left[2\gamma \overline{L^D}(1 - \overline{L^D}) + \left(\overline{D} - D^c\right) \gamma^2 \overline{L^D}(1 - \overline{L^D}) \left(1 - 2\overline{L^D}\right)\right] l_1^Y u'' * \\ \left[l_1^D l_1^Y + l_1^Y \rho + \tau^y \rho - k^D \overline{L^D} \left(1 + \left(\overline{D} - D^c\right) \gamma(1 - \overline{L^D})\right) l_1^Y\right] \\ -u''' * \left[l_1^D l_1^Y + l_1^Y \rho + \tau^y \rho - k^D \overline{L^D} \left(1 + \left(\overline{D} - D^c\right) \gamma(1 - \overline{L^D})\right) l_1^Y\right]^2 - \\ u'' * \left(\begin{array}{c} \frac{\partial^2 D_{t+2}}{\partial Y_t^2} - k^D \frac{\partial L_{t+1}^D}{\partial D_{t+1}} \left(\frac{\partial D_{t+1}}{\partial Y_t}\right)^2 - k^D \overline{L^D} \frac{\partial^2 D_{t+1}}{\partial Y_t^2} - k^D \frac{\partial L_{t+1}^D}{\partial D_{t+1}} \left(\frac{\partial D_{t+1}}{\partial Y_t}\right)^2 \\ -k^D \left(\overline{D} - D^c\right) \frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2} \left(\frac{\partial D_{t+1}}{\partial Y_t}\right)^2 - k^D \left(\overline{D} - D^c\right) \frac{\partial L_{t+1}^D}{\partial D_{t+1}} \frac{\partial^2 D_{t+1}}{\partial Y_t^2} \end{array} \right) \end{array} \right].$$

Again, given (2.98), (2.105), (2.119) and first- and second-order derivatives of D_{t+1} and D_{t+2}, it follows that

$$p_{YY} = \left[\begin{array}{c} \left(l_1^Y + \tau^y\right)^2 u''' + l_{11}^Y u'' - \\ \beta_g k^D \left[\begin{array}{c} \left[2\gamma \overline{L^D}(1 - \overline{L^D}) + \left(\overline{D} - D^c\right) \gamma^2 \overline{L^D}(1 - \overline{L^D}) \left(1 - 2\overline{L^D}\right)\right] l_{11}^Y \\ + \left[\begin{array}{c} 3\gamma^2 \overline{L^D}(1 - \overline{L^D}) \left(1 - 2\overline{L^D}\right) + \\ \left(\overline{D} - D^c\right) \gamma^3 \overline{L^D}(1 - \overline{L^D}) \left(1 - 6\overline{L^D} + 6\left(\overline{L^D}\right)^2\right) \end{array} \right] \left(l_1^Y\right)^2 \end{array} \right] u' \\ -2k^D \gamma \overline{L^D}(1 - \overline{L^D}) \left[2 + \left(\overline{D} - D^c\right) \gamma \left(1 - 2\overline{L^D}\right)\right] * \\ \left[\begin{array}{c} \beta_g l_1^D \left(l_1^Y\right)^2 + \beta_g \left(l_1^Y\right)^2 \rho + \beta_g l_1^Y \tau^y \rho - \\ \beta_g k^D \overline{L^D} \left(1 + \left(\overline{D} - D^c\right) \gamma(1 - \overline{L^D})\right) \left(l_1^Y\right)^2 \end{array} \right] u'' \\ - \left[l_1^D l_1^Y + l_1^Y \rho + \tau^y \rho - k^D \overline{L^D} \left(1 + \left(\overline{D} - D^c\right) \gamma(1 - \overline{L^D})\right) l_1^Y\right]^2 u''' - \\ \left(\begin{array}{c} l_{11}^D \left(l_1^Y\right)^2 + 2 l_{11}^{DY} l_1^Y \rho + l_{11}^Y \left(l_1^D + \rho^2\right) - \\ k^D \left[\begin{array}{c} \left[2\gamma \overline{L^D}(1 - \overline{L^D}) + \left(\overline{D} - D^c\right) \gamma^2 \overline{L^D}(1 - \overline{L^D}) \left(1 - 2\overline{L^D}\right)\right] \left(l_1^Y\right)^2 \\ + \left(\overline{L^D} + \left(\overline{D} - D^c\right) \gamma \overline{L^D}(1 - \overline{L^D})\right) l_{11}^Y \end{array} \right] \end{array} \right) u'' \end{array} \right].$$

Using (2.100) again and isolating the three unknown coefficients l_{11}^{Y}, l_{11}^{DY} and l_{11}^{D}, we obtain

$$
p_{YY} = \left[\begin{array}{c}
l_{11}^{Y}\left[\begin{array}{c}\left[1+k^{D}\overline{L^{D}}\left(1+\left(\overline{D}-D^{c}\right)\gamma(1-\overline{L^{D}})\right)-\left(l_{1}^{D}+\rho^{2}\right)\right]u''- \\ \beta_{g}k^{D}\gamma\overline{L^{D}}(1-\overline{L^{D}})\left[2+\left(\overline{D}-D^{c}\right)\gamma\left(1-2\overline{L^{D}}\right)\right]u'\end{array}\right] \\
-2\rho l_{1}^{Y}l_{11}^{DY}u''-\left(l_{1}^{Y}\right)^{2}l_{11}^{D}u''- \\
\beta_{g}k^{D}\gamma^{2}\overline{L^{D}}(1-\overline{L^{D}})\left[\begin{array}{c}3\left(1-2\overline{L^{D}}\right)+ \\ \left(\overline{D}-D^{c}\right)\gamma\left(1-6\overline{L^{D}}+6\left(\overline{L^{D}}\right)^{2}\right)\end{array}\right]\left(l_{1}^{Y}\right)^{2}u'+ \\
k^{D}\gamma\overline{L^{D}}(1-\overline{L^{D}})\left[2+\left(\overline{D}-D^{c}\right)\gamma\left(1-2\overline{L^{D}}\right)\right]\left[\begin{array}{c}3\left(l_{1}^{Y}\right)^{2}-2\beta_{g}l_{1}^{D}\left(l_{1}^{Y}\right)^{2}- \\ 2\beta_{g}\left(l_{1}^{Y}\right)^{2}\rho-2\beta_{g}l_{1}^{Y}\tau^{y}\rho\end{array}\right]u'' \\
+\left\{\begin{array}{c}\left(l_{1}^{Y}+\tau^{y}\right)^{2}- \\ \left[l_{1}^{D}l_{1}^{Y}+l_{1}^{Y}\rho+\tau^{y}\rho-k^{D}\overline{L^{D}}\left(1+\left(\overline{D}-D^{c}\right)\gamma(1-\overline{L^{D}})\right)l_{1}^{Y}\right]^{2}\end{array}\right\}u'''
\end{array}\right]
$$

$$
\Rightarrow\quad p_{YY}\widetilde{Y}_{t}^{2}=\left(\digamma l_{11}^{Y}-2\rho l_{1}^{Y}l_{11}^{DY}u''-\left(l_{1}^{Y}\right)^{2}l_{11}^{D}u''+\yen\right)\widetilde{Y}_{t}^{2}. \tag{2.123}
$$

This equation contains three unknown variables (l_{11}^{Y}, l_{11}^{DY} and l_{11}^{D}).

Derivation of $p_{Y\sigma_\varepsilon}$ Differentiate p_{σ_ε} with respect to Y_t around the point $\left(\overline{D},\overline{Y},0\right)$ and use (2.100)

$$
p_{Y\sigma_\varepsilon} = \left[\begin{array}{c}
\frac{\partial(u''(G_t))}{\partial Y_t}*\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)+u''*\frac{\partial}{\partial Y_t}\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)- \\
\beta_{g}k^{D}*\left[2\frac{\partial}{\partial Y_t}\left(\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)+\frac{\partial}{\partial Y_t}\left(\left(D_{t+1}-D^{c}\right)\frac{\partial^{2}L_{t+1}^{D}}{\partial^{2}D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)\right]*u' \\
-\beta_{g}k^{D}\left[2\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\left(\overline{D}-D^{c}\right)\frac{\partial^{2}L_{t+1}^{D}}{\partial^{2}D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right]*\frac{\partial(u'(G_{t+1}))}{\partial Y_t}- \\
\beta_{g}k^{D}*\frac{\partial}{\partial Y_t}\left(L_{t+1}^{D}+\left(D_{t+1}-D^{c}\right)\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\right)*u''* \\
\left(\frac{\partial D_{t+2}}{\partial\sigma_\varepsilon}+\tau^{y}\varepsilon_{t+1}-k^{D}\overline{L^{D}}\left(1+\left(\overline{D}-D^{c}\right)\gamma(1-\overline{L^{D}})\right)\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)- \\
\frac{\partial}{\partial Y_t}\left(u''\left(G_{t+1}\right)\right)*\left(\frac{\partial D_{t+2}}{\partial\sigma_\varepsilon}+\tau^{y}\varepsilon_{t+1}-k^{D}\overline{L^{D}}\left(1+\left(\overline{D}-D^{c}\right)\gamma(1-\overline{L^{D}})\right)\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right) \\
-u''*\frac{\partial}{\partial Y_t}\left(\frac{\partial D_{t+2}}{\partial\sigma_\varepsilon}+\tau^{y}\varepsilon_{t+1}-k^{D}L_{t+1}^{D}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}-k^{D}\left(D_{t+1}-D^{c}\right)\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)
\end{array}\right].
$$

Using that $\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}=l_{1}^{\sigma}=0$ and using the first- and second-order derivatives of D_{t+1} and D_{t+2} with respect to σ_ε, this expression becomes

$$
p_{Y\sigma_\varepsilon} = \left[\begin{array}{c}
u''\frac{\partial^{2}D_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}- \\
\beta_{g}k^{D}*\left[\begin{array}{c}2\left(\frac{\partial^{2}L_{t+1}^{D}}{\partial^{2}D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial^{2}D_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}\right)+\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial^{2}L_{t+1}^{D}}{\partial^{2}D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon} \\ +\left(\overline{D}-D^{c}\right)\left(\frac{\partial^{3}L_{t+1}^{D}}{\partial^{3}D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\frac{\partial^{2}L_{t+1}^{D}}{\partial^{2}D_{t+1}}\frac{\partial^{2}D_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}\right)\end{array}\right]*u'- \\
\beta_{g}k^{D}*\left[\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}+\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}+\left(\overline{D}-D^{c}\right)\frac{\partial^{2}L_{t+1}^{D}}{\partial^{2}D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right]*u''* \\
\left(l_{1}^{D}l_{1}^{\sigma}+l_{1}^{Y}\varepsilon_{t+1}+l_{1}^{\sigma}+\tau^{y}\varepsilon_{t+1}\right) \\
-u'''*\left[\frac{\partial D_{t+2}}{\partial Y_t}-k^{D}\overline{L^{D}}\frac{\partial D_{t+1}}{\partial Y_t}+\tau^{y}\rho-k^{D}\left(\overline{D}-D^{c}\right)\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\right]* \\
\left(l_{1}^{D}l_{1}^{\sigma}+l_{1}^{Y}\varepsilon_{t+1}+l_{1}^{\sigma}+\tau^{y}\varepsilon_{t+1}\right)- \\
u''*\left(\begin{array}{c}\frac{\partial^{2}D_{t+2}}{\partial Y_t\partial\sigma_\varepsilon}-k^{D}\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}-k^{D}\overline{L^{D}}\frac{\partial^{2}D_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}-k^{D}\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon} \\ -k^{D}\left(\overline{D}-D^{c}\right)\frac{\partial^{2}L_{t+1}^{D}}{\partial^{2}D_{t+1}}\frac{\partial D_{t+1}}{\partial Y_t}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}-k^{D}\left(\overline{D}-D^{c}\right)\frac{\partial L_{t+1}^{D}}{\partial D_{t+1}}\frac{\partial^{2}D_{t+1}}{\partial Y_t\partial\sigma_\varepsilon}\end{array}\right)
\end{array}\right].
$$

We can simplify the last equation once more to

$$p_{Y\sigma_\varepsilon}=\left[\begin{array}{c} l_{11}^{Y\sigma}u''-\beta_g k^D*\left[2\gamma\overline{L^D}(1-\overline{L^D})+\left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\right]*l_{11}^{Y\sigma}u' \\ -\beta_g k^D*\left[\begin{array}{c} 2\gamma\overline{L^D}(1-\overline{L^D})+ \\ \left(\overline{D}-D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1-2\overline{L^D}\right)\end{array}\right]*l_1^Y\left(l_1^Y+\tau^y\right)\varepsilon_{t+1}u''- \\ u'''*\left[l_1^D l_1^Y+l_1^Y\rho+\tau^y\rho-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^Y\right]*\left(l_1^Y+\tau^y\right)\varepsilon_{t+1} \\ -u''*\left(\begin{array}{c}\left(l_{11}^D l_1^Y+l_{11}^{DY}\rho\right)l_1^\sigma+l_1^D l_{11}^{Y\sigma}+\left(l_{11}^{DY}l_1^Y+l_{11}^Y\rho\right)\varepsilon_{t+1}+l_{11}^{D\sigma}l_1^Y+l_{11}^{Y\sigma}\rho \\ -k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_{11}^{Y\sigma}\end{array}\right)\end{array}\right].$$

Then, grouping the terms of the equation above and multiplying it by $\widetilde{Y}_t\sigma_\varepsilon$, we have

$$\begin{aligned} & p_{Y\sigma_\varepsilon}\widetilde{Y}_t\sigma_\varepsilon \\ = & \left\{\begin{array}{c} l_{11}^{Y\sigma}\left[\begin{array}{c}\left[1+k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)-l_1^D-\rho\right]u''- \\ \beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]u'\end{array}\right] \\ -l_{11}^{D\sigma}l_1^Y u''-l_{11}^{DY}l_1^Y\varepsilon_{t+1}u''-l_{11}^Y\rho\varepsilon_{t+1}u''- \\ \left(l_1^Y+\tau^y\right)\left[\begin{array}{c} l_1^Y\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]u'' \\ +\left[\begin{array}{c} l_1^D l_1^Y+l_1^Y\rho+\tau^y\rho- \\ k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^Y\end{array}\right]u'''\end{array}\right]\varepsilon_{t+1}\end{array}\right\}\widetilde{Y}_t\sigma_\varepsilon \\ = & \left(l_{11}^{Y\sigma}-l_{11}^{D\sigma}l_1^Y u''+\varepsilon_{t+1}\right)\widetilde{Y}_t\sigma_\varepsilon.\end{aligned}$$

Finally, taking expectations

$$\mathrm{E}_t\left[p_{Y\sigma_\varepsilon}\widetilde{Y}_t\sigma_\varepsilon\right]=E_t\left[\left(l_{11}^{Y\sigma}-l_{11}^{D\sigma}l_1^Y u''+\varepsilon_{t+1}\right)\widetilde{Y}_t\sigma_\varepsilon\right]=\left(l_{11}^{Y\sigma}-l_{11}^{D\sigma}l_1^Y u''\right)\widetilde{Y}_t\sigma_\varepsilon. \qquad (2.124)$$

Derivation of $p_{\sigma_\varepsilon\sigma_\varepsilon}$ Differentiate p_{σ_ε} with respect to σ_ε around the point $\left(\overline{D},\overline{Y},0\right)$ and also use (2.100):

$$p_{\sigma_\varepsilon\sigma_\varepsilon}=\left[\begin{array}{c} \frac{\partial}{\partial\sigma_\varepsilon}\left(u''\left(G_t\right)\right)*\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)+u''*\frac{\partial}{\partial\sigma_\varepsilon}\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)- \\ \beta_g k^D*\left[2\frac{\partial}{\partial\sigma_\varepsilon}\left(\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)+\frac{\partial}{\partial\sigma_\varepsilon}\left(\left(D_{t+1}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)\right]*u' \\ -\beta_g k^D\left[2\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}+\left(\overline{D}-D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right]*\frac{\partial(u'(G_{t+1}))}{\partial Y_t} \\ -\beta_g k^D*\frac{\partial}{\partial\sigma_\varepsilon}\left(L_{t+1}^D+\left(D_{t+1}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\right)*u''* \\ \left(\frac{\partial D_{t+2}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)- \\ \frac{\partial}{\partial\sigma_\varepsilon}\left(u''\left(G_{t+1}\right)\right)*\left(\frac{\partial D_{t+2}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right) \\ -u''*\frac{\partial}{\partial\sigma_\varepsilon}\left(\frac{\partial D_{t+2}}{\partial\sigma_\varepsilon}+\tau^y\varepsilon_{t+1}-k^D L_{t+1}^D\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}-k^D\left(D_{t+1}-D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)\end{array}\right].$$

Using that $\frac{\partial D_{t+1}}{\partial \sigma_\varepsilon} = l_1^\sigma = 0$ and the first- and second-order derivatives of D_{t+1} and D_{t+2} with respect to σ_ε, we write the last expression as

$$p_{\sigma_\varepsilon\sigma_\varepsilon} = \left[\begin{array}{c} u''\frac{\partial^2 D_{t+1}}{\partial^2\sigma_\varepsilon} - \\ \left[\begin{array}{c} 2\left(\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}} \left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)^2 + \frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial^2\sigma_\varepsilon} \right) + \frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)^2 + \\ \left(\overline{D} - D^c\right)\left(\frac{\partial^3 L_{t+1}^D}{\partial^3 D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)^2 + \frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial^2\sigma_\varepsilon} \right) \end{array} \right] \beta_g k^D u' \\ - \left[\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon} + \frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\frac{\partial L_{t+1}^D}{\partial D_{t+1}} + \left(\overline{D} - D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial D_{t+1}^2}\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon} \right] * \\ \left(l_1^D l_1^\sigma + l_1^Y\varepsilon_{t+1} + l_1^\sigma + \tau^y\varepsilon_{t+1}\right)\beta_g k^D u'' \\ -u''' * \left(l_1^D l_1^\sigma + l_1^Y\varepsilon_{t+1} + l_1^\sigma + \tau^y\varepsilon_{t+1}\right)^2 - \\ \left(\begin{array}{c} \frac{\partial^2 D_{t+2}}{\partial^2\sigma_\varepsilon} - k^D\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)^2 - k^D\overline{L^D}\frac{\partial^2 D_{t+1}}{\partial^2\sigma_\varepsilon} - k^D\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)^2 \\ -k^D\left(\overline{D} - D^c\right)\frac{\partial^2 L_{t+1}^D}{\partial^2 D_{t+1}}\left(\frac{\partial D_{t+1}}{\partial\sigma_\varepsilon}\right)^2 - k^D\left(\overline{D} - D^c\right)\frac{\partial L_{t+1}^D}{\partial D_{t+1}}\frac{\partial^2 D_{t+1}}{\partial^2\sigma_\varepsilon} \end{array} \right) u'' \end{array} \right].$$

Simplifying once more, and using the first- and second-order derivatives of D_{t+1} and D_{t+2}, it follows that

$$p_{\sigma_\varepsilon\sigma_\varepsilon} = \left[\begin{array}{c} +u'' l_{11}^\sigma - u'''\left(\left(l_1^Y + \tau^y\right)\varepsilon_{t+1}\right)^2 - \\ \left[2\gamma\overline{L^D}(1-\overline{L^D}) + \left(\overline{D} - D^c\right)\gamma^2\overline{L^D}(1-\overline{L^D})\left(1 - 2\overline{L^D}\right)\right] l_{11}^\sigma\beta_g k^D u' - \\ \left(\begin{array}{c} \left(l_{11}^D l_1^\sigma + l_{11}^{DY}\varepsilon_{t+1} + 2l_{11}^{D\sigma}\right) l_1^\sigma + \left(l_{11}^{DY} l_1^\sigma + 2l_{11}^{Y\sigma}\right)\varepsilon_{t+1} + l_{11}^Y\varepsilon_{t+1}^2 + \left(1 + l_1^D\right) l_{11}^\sigma \\ -k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1-\overline{L^D})\right) l_{11}^\sigma \end{array} \right) u'' \end{array} \right].$$

Grouping terms and multiplying by σ_ε^2, we obtain

$$\begin{aligned} p_{\sigma_\varepsilon\sigma_\varepsilon}\sigma_\varepsilon^2 &= \left\{ \begin{array}{c} l_{11}^\sigma \left[\begin{array}{c} \left[1 + k^D\overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma(1-\overline{L^D})\right) - \left(1 + l_1^D\right)\right] u'' \\ -\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2 + \left(\overline{D} - D^c\right)\gamma\left(1 - 2\overline{L^D}\right)\right] u' \end{array} \right] \\ -2l_{11}^{Y\sigma}\varepsilon_{t+1}u'' - \left[\left(l_1^Y + \tau^y\right)^2 u''' + l_{11}^Y u''\right]\varepsilon_{t+1}^2 \end{array} \right\} \sigma_\varepsilon^2 \\ &= \left[e l_{11}^\sigma - 2l_{11}^{Y\sigma}\varepsilon_{t+1}u'' - \left(\left(l_1^Y + \tau^y\right)^2 u''' + l_{11}^Y u''\right)\varepsilon_{t+1}^2\right]\sigma_\varepsilon^2. \end{aligned}$$

Finally, taking expectations, we obtain

$$\begin{aligned} \mathrm{E}_t\left[p_{\sigma_\varepsilon\sigma_\varepsilon}\sigma_\varepsilon^2\right] &= \mathrm{E}_t\left\{\left[e l_{11}^\sigma - 2l_{11}^{Y\sigma}\varepsilon_{t+1}u'' - \left(\left(l_1^Y + \tau^y\right)^2 u''' + l_{11}^Y u''\right)\varepsilon_{t+1}^2\right]\sigma_\varepsilon^2\right\} \\ &= \left[e l_{11}^\sigma - \left(\left(l_1^Y + \tau^y\right)^2 u''' + l_{11}^Y u''\right)\sigma_\varepsilon^2\right]\sigma_\varepsilon^2. \end{aligned} \tag{2.125}$$

Second-order Taylor expansion

Substituting (2.120), (2.121), (2.122), (2.123), (2.124) and (2.125), we can write the second-order Taylor expansion of 2.29 as

$$\begin{array}{r} \mathrm{E}_t\left\{p\left(D_t, Y_t, \sigma_\varepsilon\right)\right] \simeq \\ \frac{1}{2}\left\{ \left[\begin{array}{c} \left(\Lambda l_{11}^D + \Xi\right)\widetilde{D}_t^2 + 2\left(\Pi l_{11}^{DY} + \Upsilon l_{11}^D + \Phi\right)\widetilde{D}_t\widetilde{Y}_t + \\ 2\Psi l_{11}^{D\sigma}\widetilde{D}_t\sigma_\varepsilon + \left(\digamma l_{11}^Y - 2\rho l_1^Y l_{11}^{DY}u'' - \left(l_1^Y\right)^2 l_{11}^D u'' + \yen\right)\widetilde{Y}_t^2 + \\ 2\left(l_{11}^{Y\sigma} - l_{11}^{D\sigma}l_1^Y u''\right)\widetilde{Y}_t\sigma_\varepsilon + \left\{e l_{11}^\sigma - \left[\left(l_1^Y + \tau^y\right)^2 u''' + l_{11}^Y u''\right]\sigma_\varepsilon^2\right\}\sigma_\varepsilon^2 \end{array} \right] \right\} = \end{array} \tag{2.126}$$

Because this expression has to hold for all combinations $\left\{\widetilde{D}_t^2, \widetilde{D}_t\widetilde{Y}_t, \widetilde{D}_t\sigma_\varepsilon, \widetilde{Y}_t^2, \widetilde{Y}_t\sigma_\varepsilon, \sigma_\varepsilon^2\right\}$, we obtain six linear equations in six unknowns. Setting the coefficient of $\widetilde{D}_t^2$ to zero, we obtain

$$l_{11}^D = -\frac{\Xi}{\Lambda},$$

since $\Lambda \neq 0$. Then, l_{11}^D becomes

$$l_{11}^D = \frac{\left[\begin{array}{c} \left[l_1^D - k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right]^2\left(\left(l_1^D\right)^2-1\right)u'''+ \\ \beta_g k^D\gamma^2\overline{L^D}(1-\overline{L^D})\left[3\left(1-2\overline{L^D}\right)+\left(\overline{D}-D^c\right)\gamma\left(1-6\overline{L^D}+6\left(\overline{L^D}\right)^2\right)\right]\left(l_1^D\right)^2u' \\ -k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\left[-1+3\left(l_1^D\right)^2-2\beta_g\left(l_1^D\right)^3\right]u'' \end{array}\right]}{\left[\begin{array}{c} \left(1+k^D\overline{L^D}\left[1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right]-\left(l_1^D\right)^2-l_1^D\right)u''- \\ \beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]u' \end{array}\right]}. \tag{2.127}$$

With the value of l_{11}^D, we solve for l_{11}^{DY} by setting the coefficient of $\widetilde{D}_t\widetilde{Y}_t$ to zero:

$$l_{11}^{DY} = -\frac{\Upsilon l_{11}^D + \Phi}{\Pi},$$

or

$$l_{11}^{DY} = \frac{\left[\begin{array}{c} l_{11}^D l_1^D l_1^Y u''+ \\ \beta_g k^D\gamma^2\overline{L^D}(1-\overline{L^D})\left[3\left(1-2\overline{L^D}\right)+\left(\overline{D}-D^c\right)\gamma\left(1-6\overline{L^D}+6\left(\overline{L^D}\right)^2\right)\right]l_1^D l_1^Y u' \\ -\left[l_1^D - k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)\right] * \\ \left[\begin{array}{c} l_1^Y+\tau^y-\left(l_1^D\right)^2 l_1^Y - l_1^D l_1^Y\rho - l_1^D\tau^y\rho+ \\ k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^D l_1^Y \end{array}\right]u'''+ \\ k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\left[\begin{array}{c} 2\beta_g\left(l_1^D\right)^2 l_1^Y + \beta_g l_1^D l_1^Y\rho \\ +\beta_g l_1^D\tau^y\rho - 3l_1^D l_1^Y \end{array}\right]u'' \end{array}\right]}{\left\{\begin{array}{c} \left[1+k^D\overline{L^D}\left[1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right]-(1+\rho)\,l_1^D\right]u'' \\ -\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]u' \end{array}\right\}}, \tag{2.128}$$

Similarly, we obtain the solution of l_{11}^Y as

$$l_{11}^Y = \frac{2\rho l_1^Y l_{11}^{DY}u'' + \left(l_1^Y\right)^2 l_{11}^D u'' - \yen}{F},$$

or

$$l_{11}^{Y}=\frac{\left[\begin{array}{c}2u''\rho l_1^Y l_{11}^{DY}+u''\left(l_1^Y\right)^2 l_{11}^D+\\ u'\beta_g k^D\gamma^2\overline{L^D}(1-\overline{L^D})\left[\begin{array}{c}3\left(1-2\overline{L^D}\right)+\\ \left(\overline{D}-D^c\right)\gamma\left(1-6\overline{L^D}+6\left(\overline{L^D}\right)^2\right)\end{array}\right]\left(l_1^Y\right)^2-\\ u''k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\left[\begin{array}{c}3\left(l_1^Y\right)^2-2\beta_g l_1^D\left(l_1^Y\right)^2\\ -2\beta_g\left(l_1^Y\right)^2\rho-2\beta_g l_1^Y\tau^y\rho\end{array}\right]\\ -u'''\left\{\left(l_1^Y+\tau^y\right)^2-\left[l_1^D l_1^Y+l_1^Y\rho+\tau^y\rho-k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)l_1^Y\right]^2\right\}\end{array}\right]}{\left[\begin{array}{c}u''\left[1+k^D\overline{L^D}\left(1+\left(\overline{D}-D^c\right)\gamma(1-\overline{L^D})\right)-\left(l_1^D+\rho^2\right)\right]\\ -u'\beta_g k^D\gamma\overline{L^D}(1-\overline{L^D})\left[2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right]\end{array}\right]}. \tag{2.129}$$

In addition, (2.126) implies

$$l_{11}^{D\sigma} = 0 \Rightarrow \tag{2.130}$$

$$l_{11}^{Y\sigma} = 0. \tag{2.131}$$

Finally, given (2.112) and (2.129), we obtain the last unknown coefficient l_{11}^{σ}:

$$l_{11}^{\sigma}=\frac{\left[\left(l_1^Y+\tau^y\right)^2u'''+l_{11}^Y u''\right]\sigma_\varepsilon^2}{e}. \tag{2.132}$$

Second-order approximation of D_{t+1}

The second-order approximation of the true non-linear solution of D_{t+1} around the point $\left(\overline{D},\overline{Y},0\right)$ can be written as

$$D_{t+1}\approx\left[\begin{array}{c}l\left(\overline{D},\overline{Y},0\right)+l_1^D\widetilde{D}_t+l_1^Y\widetilde{Y}_t+l_1^\sigma\sigma_\varepsilon+\\ \frac{1}{2}\left[l_{11}^D\widetilde{D}_t^2+2l_{11}^{DY}\widetilde{D}_t\widetilde{Y}_t+2l_{11}^{D\sigma}\widetilde{D}_t\sigma_\varepsilon+l_{11}^Y\widetilde{Y}_t^2+2l_{11}^{Y\sigma}\widetilde{Y}_t\sigma_\varepsilon+l_{11}^\sigma\sigma_\varepsilon^2\right)\end{array}\right].$$

Using (2.111), (2.112), (2.109), (2.127), (2.128), (2.129), (2.130), (2.131) and (2.132), this becomes

$$D_{t+1}\approx\left\{\begin{array}{c}l\left(\overline{D},\overline{Y},0\right)+l_1^D\widetilde{D}_t+l_1^Y\widetilde{Y}_t+\\ \frac{1}{2}\left[l_{11}^D\widetilde{D}_t^2+2l_{11}^{DY}\widetilde{D}_t\widetilde{Y}_t+l_{11}^Y\widetilde{Y}_t^2+l_{11}^\sigma\sigma_\varepsilon^2\right]\end{array}\right\}. \tag{2.133}$$

Using (2.8), we can rewrite (2.133) further as

$$B_{t+1}-(1+r)B_t\approx\left\{\begin{array}{c}l\left(\overline{D},\overline{Y},0\right)+l_1^D*\left[B_t-(1+r)B_{t-1}-\overline{D}\right]+l_1^Y*\left(Y_t-\overline{Y}\right)\\ +\frac{1}{2}\left[\begin{array}{c}l_{11}^D*\left[B_t-(1+r)B_{t-1}-\overline{D}\right]^2+\\ 2l_{11}^{DY}*\left[B_t-(1+r)B_{t-1}-\overline{D}\right]*\left(Y_t-\overline{Y}\right)\\ +l_{11}^Y*\left(Y_t-\overline{Y}\right)^2+l_{11}^\sigma\sigma_\varepsilon^2\end{array}\right]\end{array}\right\}. \tag{2.134}$$

2.G.3 Unconditional expectation of the primary deficit

The unconditional expectation of (2.133) is:

$$\mathrm{E}\left[D_{t+1}\right] \simeq \left\{ \begin{array}{c} \overline{D} + l_1^D \left(\mathrm{E}\left[D_t\right] - \overline{D}\right) + l_1^Y \left(\mathrm{E}\left[Y_t\right] - \overline{Y}\right) + \\ \frac{1}{2} \left[\begin{array}{c} l_{11}^D \mathrm{E}\left[\left(D_t - \overline{D}\right)^2\right] + 2 l_{11}^{DY} \mathrm{E}\left[\left(D_t - \overline{D}\right)\left(Y_t - \overline{Y}\right)\right] \\ + l_{11}^Y \mathrm{E}\left[\left(Y_t - \overline{Y}\right)^2\right] + l_{11}^{\sigma} \sigma_{\varepsilon}^2 \end{array} \right] \end{array} \right\}.$$

Define

$$\widehat{D} \equiv \mathrm{E}\left(D_t\right) = \mathrm{E}\left[D_{t+1}\right],$$

and work out the terms in the expression for $\mathrm{E}[D_{t+1}]$:

$$\mathrm{E}\left(Y_t\right) - \overline{Y} = 0,$$

$$\mathrm{E}\left[\left(Y_t - \overline{Y}\right)^2\right] = \mathrm{Var}\left(Y_t\right) = \frac{\sigma_{\varepsilon}^2}{1-\rho^2},$$

$$\begin{aligned} \mathrm{E}\left[\left(D_t - \overline{D}\right)^2\right] &= \mathrm{E}\left\{\left[\left(D_t - \widehat{D}\right) + \left(\widehat{D} - \overline{D}\right)\right]^2\right\} \\ &= \mathrm{E}\left[\left(D_t - \widehat{D}\right)^2\right] + 2\mathrm{E}\left[\left(D_t - \widehat{D}\right)\left(\widehat{D} - \overline{D}\right)\right] + \mathrm{E}\left[\left(\widehat{D} - \overline{D}\right)^2\right] \\ &= \mathrm{E}\left[\left(D_t - \widehat{D}\right)^2\right] + \left(\widehat{D} - \overline{D}\right)^2 = \mathrm{Var}\left(D_t\right) + \left(\widehat{D} - \overline{D}\right)^2, \end{aligned}$$

$$\begin{aligned} \mathrm{E}\left[\left(D_t - \overline{D}\right)\left(Y_t - \overline{Y}\right)\right] &= \mathrm{E}\left\{\left[\left(D_t - \widehat{D}\right) + \left(\widehat{D} - \overline{D}\right)\right]\left(Y_t - \overline{Y}\right)\right\} \\ &= \mathrm{E}\left[\left(D_t - \widehat{D}\right)\left(Y_t - \overline{Y}\right)\right] + \mathrm{E}\left[\left(\widehat{D} - \overline{D}\right)\left(Y_t - \overline{Y}\right)\right] \\ &= \mathrm{E}\left[\left(D_t - \widehat{D}\right)\left(Y_t - \overline{Y}\right)\right] = \mathrm{Cov}\left(D_t, Y_t\right). \end{aligned}$$

Substitute these terms into the expression for $\mathrm{E}[D_{t+1}]$, to give:

$$\begin{aligned} \widehat{D} - \overline{D} &\simeq l_1^D \left(\widehat{D} - \overline{D}\right) + \frac{1}{2} \left[\begin{array}{c} l_{11}^D \left(\mathrm{Var}\left(D_t\right) + \left(\widehat{D} - \overline{D}\right)^2\right) + 2 l_{11}^{DY} \mathrm{Cov}\left(D_t, Y_t\right) \\ + l_{11}^Y \frac{\sigma_{\varepsilon}^2}{1-\rho^2} + l_{11}^{\sigma} \sigma_{\varepsilon}^2 \end{array} \right] \Rightarrow \\ \widehat{D} &\simeq \overline{D} + \frac{1}{2\left(1 - l_1^D\right)} \left[l_{11}^D \mathrm{Var}\left(D_t\right) + 2 l_{11}^{DY} \mathrm{Cov}\left(D_t, Y_t\right) + \left(\frac{l_{11}^Y}{1-\rho^2} + l_{11}^{\sigma}\right) \sigma_{\varepsilon}^2 \right]. \end{aligned} \tag{2.135}$$

where we have made use of the fact that $\left(\widehat{D}-\overline{D}\right)^2$ is fourth-order, respectively. We have

$$\begin{aligned}
\text{Cov}\left(D_t, Y_t\right) &= \text{Cov}\left(D_{t+1}, Y_{t+1}\right) = \text{E}\left[\left(D_{t+1}-\widehat{D}\right)\left(Y_{t+1}-\overline{Y}\right)\right] \\
&= \text{E}\left[\left(l_1^D \widetilde{D}_t + l_1^Y \widetilde{Y}_t + \overline{D} - \widehat{D}\right)\left(\rho \widetilde{Y}_t + \sigma_\varepsilon \varepsilon_{t+1}\right)\right] \\
&= \rho l_1^D \text{E}\left[\widetilde{D}_t \widetilde{Y}_t\right] + \rho l_1^Y \text{E}\left[\widetilde{Y}_t^2\right] \\
&= \rho l_1^D \text{E}\left[\widetilde{D}_t \widetilde{Y}_t\right] + \rho l_1^Y \frac{\sigma_\varepsilon^2}{1-\rho^2}.
\end{aligned}$$

Hence,

$$\text{E}\left[\widetilde{D}_t \widetilde{Y}_t\right] = \frac{\rho l_1^Y}{1-\rho l_1^D} \frac{\sigma_\varepsilon^2}{1-\rho^2}.$$

Further, taking unconditional expectation of the square of the first-order approximation of $D_t + 1$, or

$$\begin{aligned}
\text{Var}\left(D\right) &= \text{E}\left[\left(D_{t+1}-\widehat{D}\right)^2\right] \simeq \text{E}\left[\left(l_1^D \widetilde{D}_t + l_1^Y \widetilde{Y}_t\right)^2\right] \\
&= \text{E}\left[\left(l_1^D\right)^2 \left(D_t - \overline{D}\right)^2 + 2 l_1^D l_1^Y \left(D_t - \overline{D}\right)\left(Y_t - \overline{Y}\right) + \left(l_1^Y\right)^2 \left(Y_t - \overline{Y}\right)^2\right] \\
&= \left(l_1^D\right)^2 \text{E}\left[\left(D_t - \overline{D}\right)^2\right] + 2 l_1^D l_1^Y \text{E}\left[\left(D_t - \overline{D}\right)\left(Y_t - \overline{Y}\right)\right] + \left(l_1^Y\right)^2 \text{E}\left[\left(Y_t - \overline{Y}\right)^2\right] \\
&= \left(l_1^D\right)^2 \text{E}\left[\left(D_t - \overline{D}\right)^2\right] + 2 l_1^D l_1^Y \frac{\rho l_1^Y}{1-\rho l_1^D} \frac{\sigma_\varepsilon^2}{1-\rho^2} + \left(l_1^Y\right)^2 \frac{\sigma_\varepsilon^2}{1-\rho^2} \\
&= \left(l_1^D\right)^2 \text{E}\left[\left(D_t - \overline{D}\right)^2\right] + \left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right)\left(l_1^Y\right)^2 \frac{\sigma_\varepsilon^2}{1-\rho^2},
\end{aligned}$$

where we have eliminated terms that are of higher order than two. Hence,

$$\text{Var}\left(D\right) \simeq \left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) \frac{\left(l_1^Y\right)^2}{\left(1-\left(l_1^D\right)^2\right)\left(1-\rho^2\right)} \sigma_\varepsilon^2. \tag{2.136}$$

Substitute into (2.135), to arrive at:

$$\widehat{D} \simeq \overline{D} + \frac{\sigma_\varepsilon^2}{2\left(1-l_1^D\right)}\left[\left(\frac{l_{11}^D \left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) + 2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D} + l_{11}^Y\right)\frac{1}{1-\rho^2} + l_{11}^\sigma\right]. \tag{2.137}$$

From $D_{t+1} = B_{t+1} - (1+r) B_t$, we have $\text{E}[D_{t+1}] = \text{E}[B_{t+1}] - (1+r)\text{E}[B_t]$. Hence,

Solution 2.3 *The unconditional expectation or "stochastic steady state" value of the debt in the primary deficit-based sanction case is given by*

$$\widehat{B} \simeq -\frac{1}{r}\widehat{D}. \tag{2.138}$$

2.G.4 Stochastic steady state of government expenditure with deficit-based sanction

Now, we approximate government expenditure G_t. For that, we use (2.21), where in period t the only unknown variable is D_{t+1}. Nevertheless, we have just obtained a second-order approximation of that variable (2.133). So, inserting it on (2.21) for $s = t$ and isolating G_t, allow us to approximate it as

$$G_t \simeq \left\{ \begin{array}{c} \overline{D} + l_1^D \widetilde{D}_t + l_1^Y \widetilde{Y}_t + \frac{1}{2}\left[l_{11}^D \widetilde{D}_t^2 + 2l_{11}^{DY}\widetilde{D}_t\widetilde{Y}_t + l_{11}^Y \widetilde{Y}_t^2 + l_{11}^\sigma \sigma_\varepsilon^2\right] \\ +\tau^y Y_t - k^D \left(D_t - D^c\right) L_t^D \end{array} \right\}. \tag{2.139}$$

First-order approximation of G_t

The first-order Taylor expansion of (2.27) around the point $\left(\overline{D}, \overline{Y}, 0\right)$ is

$$G_t \simeq j\left(\overline{D}, \overline{Y}, 0\right) + j_1^D \widetilde{D}_t + j_1^Y \widetilde{Y}_t + j_1^\sigma \sigma_\varepsilon,$$

where $j_1^D \equiv \frac{\partial G_t}{\partial D_t}$, $j_1^Y \equiv \frac{\partial G_t}{\partial Y_t}$ and $j_1^\sigma \equiv \frac{\partial G_t}{\partial \sigma_\varepsilon}$ evaluated at the point $\left(\overline{D}, \overline{Y}, 0\right)$. Hence, using (2.139) we can find these derivatives as:

- First order partial derivative of (2.139) with respect to D_t:

$$\frac{\partial G_t}{\partial D_t} = l_1^D + \frac{1}{2}\left[2l_{11}^D\left(D_t - \overline{D}\right) + 2l_{11}^{DY}\left(Y_t - \overline{Y}\right)\right] - k^D L_t^D - k^D\left(D_t - D^c\right)\frac{\partial L_t^D}{\partial D_t}.$$

  If we evaluate this derivative at the point $\left(\overline{D}, \overline{Y}, 0\right)$ and use (2.98), we arrive to

$$j_1^D = l_1^D - k^D \overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma\left(1 - \overline{L^D}\right)\right). \tag{2.140}$$

- First order partial derivative of (2.139) with respect to Y_t:

$$\frac{\partial G_t}{\partial Y_t} = l_1^Y + \frac{1}{2}\left[2l_{11}^{DY}\left(D_t - \overline{D}\right) + 2l_{11}^Y\left(Y_t - \overline{Y}\right)\right] + \tau^y \Rightarrow j_1^Y = l_1^Y + \tau^y, \tag{2.141}$$

  since we evaluate this derivative at the point $\left(\overline{D}, \overline{Y}, 0\right)$.

- First order partial derivative of (2.139) with respect to σ_ε:

$$\frac{\partial G_t}{\partial \sigma_\varepsilon} = \frac{1}{2}\left[2l_{11}^\sigma \sigma_\varepsilon\right] \Rightarrow j_1^\sigma = 0, \tag{2.142}$$

  since in the point that we evaluate at $\left(D_t, Y_t, \sigma_\varepsilon\right) = \left(\overline{D}, \overline{Y}, 0\right)$.

Substituting the above derivatives as well as

$$\overline{G} \equiv f\left(\overline{D}, \overline{Y}, 0\right) = \tau^y \overline{Y} + \overline{D} - k^D\left(\overline{D} - D^c\right)\overline{L^D}, \tag{2.143}$$

into the first-order Taylor expansion of G_t yields:

$$G_t \simeq \overline{G} + \left[l_1^D - k^D \overline{L^D}\left(1 + \left(\overline{D} - D^c\right)\gamma\left(1 - \overline{L^D}\right)\right)\right] * \widetilde{D}_t + \left(l_1^Y + \tau^y\right) * \widetilde{Y}_t. \tag{2.144}$$

Second-order approximation of G_t

The second-order Taylor expansion of (2.27) around the point $\left(\overline{D}, \overline{Y}, 0\right)$ is:

$$G_t \approx \left\{ \begin{array}{c} j\left(\overline{D}, \overline{Y}, 0\right) + j_1^D \widetilde{D}_t + j_1^Y \widetilde{Y}_t + j_1^\sigma \sigma_\varepsilon + \\ \frac{1}{2}\left[j_{11}^D \widetilde{D}_t^2 + 2 j_{11}^{DY} \widetilde{D}_t \widetilde{Y}_t + 2 j_{11}^{D\sigma} \widetilde{D}_t \sigma_\varepsilon + j_{11}^Y \widetilde{Y}_t^2 + 2 j_{11}^{Y\sigma} \widetilde{Y}_t \sigma_\varepsilon + j_{11}^\sigma \sigma_\varepsilon^2 \right] \end{array} \right\},$$

where, for any generic variables X and Z, $j_{11}^X \equiv \frac{\partial^2 G_t}{\partial^2 X_t}$ and $j_{11}^{XZ} \equiv \frac{\partial^2 G_t}{\partial X_t \partial Z_t}$.

- Computation of j_{11}^D: Differentiating $\frac{\partial G_t}{\partial D_t}$ with respect to D_t, we obtain:

$$\frac{\partial^2 G_t}{\partial^2 D_t} = \frac{1}{2}\left[2 l_{11}^D\right] - k^D \frac{\partial L_t^D}{\partial D_t} - k^D \frac{\partial L_t^D}{\partial D_t} - k^D \left(D_t - D^c\right) \frac{\partial^2 L_t^D}{\partial^2 D_t}.$$

If we evaluate this derivative at the point $\left(\overline{D}, \overline{Y}, 0\right)$ and use (2.105), we arrive at

$$j_{11}^D = l_{11}^D - k^D \gamma \overline{L^D}\left(1 - \overline{L^D}\right)\left(2 + \left(\overline{D} - D^c\right)\gamma\left(1 - 2\overline{L^D}\right)\right). \tag{2.145}$$

- Computation of j_{11}^{DY}: Differentiating $\frac{\partial G_t}{\partial Y_t}$ with respect to D_t, we obtain:

$$\frac{\partial^2 G_t}{\partial D_t \partial Y_t} = \frac{1}{2}\left[2 l_{11}^{DY}\right] \Longrightarrow j_{11}^{DY} = l_{11}^{DY}. \tag{2.146}$$

- Computation of $j_{11}^{D\sigma}$ and $j_{11}^{Y\sigma}$: Differentiating $\frac{\partial G_t}{\partial D_t}$ and $\frac{\partial G_t}{\partial Y_t}$ with respect to σ_ε, we obtain:

$$j_{11}^{D\sigma} = j_{11}^{Y\sigma} = \frac{\partial^2 G_t}{\partial \sigma_\varepsilon \partial D_t} = \frac{\partial^2 G_t}{\partial \sigma_\varepsilon \partial Y_t} = 0. \tag{2.147}$$

- Computation of j_{11}^Y: Differentiating $\frac{\partial G_t}{\partial Y_t}$ with respect to Y_t, we obtain:

$$\frac{\partial^2 G_t}{\partial Y_t^2} = \frac{1}{2}\left[2 l_{11}^Y\right] \Longrightarrow j_{11}^Y = l_{11}^Y. \tag{2.148}$$

- Computation of j_{11}^σ: Differentiating $\frac{\partial G_t}{\partial \sigma_\varepsilon}$ with respect to σ_ε, we obtain:

$$\frac{\partial^2 G_t}{\partial \sigma_\varepsilon^2} = \frac{1}{2}\left[2 l_{11}^\sigma\right] \Longrightarrow j_{11}^\sigma = l_{11}^\sigma. \tag{2.149}$$

Upon substitution, the second-order approximation for G_t becomes

$$G_t \simeq \left\{ \begin{array}{c} \overline{G} + \left[l_1^D - (1/\beta_g)\right] \widetilde{D}_t + \left(l_1^Y + \tau^y\right) \widetilde{Y}_t + \\ \frac{1}{2}\left\{ \begin{array}{c} \left[l_{11}^D - k^D \gamma \overline{L^D}\left(1 - \overline{L^D}\right)\left(2 + \left(\overline{D} - D^c\right)\gamma\left(1 - 2\overline{L^D}\right)\right)\right] \widetilde{D}_t^2 \\ + 2 l_{11}^{DY} \widetilde{D}_t \widetilde{Y}_t + l_{11}^Y \widetilde{Y}_t^2 + l_{11}^\sigma \sigma_\varepsilon^2 \end{array} \right\} \end{array} \right\}, \tag{2.150}$$

where we have also used (2.100).

Unconditional expectation of government spending with primary-deficit-based sanction

Recall

$$\mathrm{E}\left[\widetilde{D}_t\widetilde{Y}_t\right] = \frac{\rho l_1^Y}{1-\rho l_1^D}\frac{\sigma_\varepsilon^2}{1-\rho^2}.$$

Further,

$$\begin{aligned}
\mathrm{E}\left[\widetilde{D}_t^2\right] &= \mathrm{E}\left[\left(D_t - \widehat{D} + \widehat{D} - \overline{D}\right)\left(D_t - \widehat{D} + \widehat{D} - \overline{D}\right)\right] \\
&= \mathrm{E}\left[\left(D_t - \widehat{D}\right)^2\right] + \left(\widehat{D} - \overline{D}\right)^2 \\
&\simeq \mathrm{E}\left[\left(D_t - \widehat{D}\right)^2\right] = \mathrm{Var}\left(D_t\right).
\end{aligned}$$

Hence, taking unconditional expectations of (2.150) and using (2.5), we obtain:

$$\widehat{G} \simeq \left\{ \begin{array}{c} \overline{G} + \left[l_1^D - (1/\beta_g)\right]\left(\widehat{D} - \overline{D}\right) + \frac{1}{2}\left(\frac{l_{11}^Y}{1-\rho^2} + l_{11}^\sigma\right)\sigma_\varepsilon^2 + \frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D}\frac{\sigma_\varepsilon^2}{1-\rho^2} + \\ \frac{1}{2}\left\{\left[l_{11}^D - k^D\gamma\overline{L^D}\left(1-\overline{L^D}\right)\left(2 + \left(\overline{D} - D^c\right)\gamma\left(1 - 2\overline{L^D}\right)\right)\right]\mathrm{Var}\left(D_t\right)\right\} \end{array} \right\}.$$

Finally, substituting (2.136), and (2.137) into this expression, we obtain

Solution 2.4 *The unconditional expectation or "stochastic steady state" value of the government expenditures in the primary deficit-based sanction case is given by*

$$\begin{aligned}
\widehat{G} &\simeq \left\{ \begin{array}{c} \overline{G} + \frac{\left[l_1^D - (1/\beta_g)\right]\sigma_\varepsilon^2}{2\left(1-l_1^D\right)}\left[\left(\frac{l_{11}^D\left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) + 2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D} + l_{11}^Y\right)\frac{1}{1-\rho^2} + l_{11}^\sigma\right] + \frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D}\frac{\sigma_\varepsilon^2}{1-\rho^2} \\ +\frac{1}{2}\left[\frac{\left[l_{11}^D - k^D\gamma\overline{L^D}\left(1-\overline{L^D}\right)\left(2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right)\right]\left(l_1^Y\right)^2\sigma_\varepsilon^2}{\left(1-\left(l_1^D\right)^2\right)\left(1-\rho^2\right)}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) + \left(\frac{l_{11}^Y}{1-\rho^2} + l_{11}^\sigma\right)\sigma_\varepsilon^2\right] \end{array} \right\} \\
&\simeq \left\{ \begin{array}{c} \overline{D} + \tau^y\overline{Y} - k^D\left(\overline{D} - D^c\right)\overline{L^D} + \\ \frac{\sigma_\varepsilon^2}{2}\left\{ \begin{array}{c} \left[\frac{l_1^D - (1/\beta_g)}{1-l_1^D}\right]\left[\left(\frac{l_{11}^D\left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) + 2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D} + l_{11}^Y\right)\frac{1}{1-\rho^2} + l_{11}^\sigma\right] + \\ 2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D}\frac{1}{1-\rho^2} + \frac{\left[l_{11}^D - k^D\gamma\overline{L^D}\left(1-\overline{L^D}\right)\left(2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right)\right]\left(l_1^Y\right)^2}{\left(1-\left(l_1^D\right)^2\right)\left(1-\rho^2\right)}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) \\ +\frac{l_{11}^Y}{1-\rho^2} + l_{11}^\sigma \end{array} \right\} \end{array} \right\}
\end{aligned} \quad (2.151)$$

2.H Welfare analysis

Now we compute expected social welfare under both types of constraints. Taking a second-order Taylor expansion of $u\left(G_t\right)$, we obtain:

$$u\left(G_t\right) \simeq u\left(\overline{G}\right) + u'\left(\overline{G}\right) * \left(G_t - \overline{G}\right) + \tfrac{1}{2}u''\left(\overline{G}\right) * \left(G_t - \overline{G}\right)^2.$$

Hence,

$$\mathrm{E}\left[u\left(G_t\right)\right] \simeq u\left(\overline{G}\right) + u'\left(\overline{G}\right) * \left(\widehat{G} - \overline{G}\right) + \tfrac{1}{2}u''\left(\overline{G}\right) * \mathrm{Var}\left(G_t\right),$$

where $\widehat{G}$ has been computed above and we can compute Var(G) in a way analogous to our computation of Var(B) and Var(D). Hence, expected social welfare is equal to (2.30).

2.H.1 Welfare in the debt-based sanction case

Substituting (2.88) and (2.96) in (2.30), we obtain

$$U_w^B \approx \frac{\left[\begin{array}{c} u\left(\tau^y \overline{Y} - r\overline{B} - k^B\left(\overline{B} - B^c\right)\overline{L^B}\right) + \\ u'\left(\tau^y \overline{Y} - r\overline{B} - k^B\left(\overline{B} - B^c\right)\overline{L^B}\right) * \\ \frac{\sigma_\varepsilon^2}{2}\left\{\begin{array}{c} \frac{h_1^B - (1/\beta_g)}{1-h_1^B}\left[\frac{h_{11}^B\left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)(1-\rho^2)}\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) + 2\frac{\rho h_{11}^{BY} h_1^Y}{1-\rho h_1^B}\frac{1}{1-\rho^2} + \frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma\right] \\ + \left[\begin{array}{c} \frac{\left[h_{11}^B - k^B\gamma\overline{L^B}\left(1-\overline{L^B}\right)\left(2+\left(\overline{B}-B^c\right)\gamma\left(1-2\overline{L^B}\right)\right)\right]\left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)(1-\rho^2)}\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) + \\ 2\left(\frac{\rho h_1^Y h_{11}^{BY}}{1-\rho h_1^B}\right)\frac{1}{1-\rho^2} + \frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma \end{array}\right] \end{array}\right\} \\ +\frac{1}{2}u''\left(\tau^y \overline{Y} - r\overline{B} - k^B\left(\overline{B} - B^c\right)\overline{L^B}\right)\mathrm{Var}\left(G_t\right) \end{array}\right]}{1-\beta_w}, \tag{2.152}$$

where U_w^B is social welfare under the debt-based sanction. We still need to determine $\mathrm{Var}(G_t)$. Recall from (2.84) that $G_t \simeq \overline{G} + f_1^B \widetilde{B}_t + f_1^Y \widetilde{Y}_t$ (because $f_1^\sigma = 0$). Recall also $\mathrm{E}\left[\widetilde{B}_t\widetilde{Y}_t\right] \simeq \left(\frac{\rho h_1^Y}{1-\rho h_1^B}\right)\frac{\sigma_\varepsilon^2}{1-\rho^2}$. Hence,

$$\begin{aligned}
\mathrm{Var}\left(G_t\right) &\simeq \mathrm{E}\left[\left(\overline{G} + f_1^B\widetilde{B}_t + f_1^Y\widetilde{Y}_t - \widehat{G}\right)^2\right] \\
&\simeq \mathrm{E}\left[\left(f_1^B\widetilde{B}_t + f_1^Y\widetilde{Y}_t\right)^2\right] + 2\left(\overline{G} - \widehat{G}\right)\mathrm{E}\left[f_1^B\widetilde{B}_t + f_1^Y\widetilde{Y}_t\right] + \mathrm{E}\left[\left(\overline{G} - \widehat{G}\right)^2\right] \\
&\simeq \mathrm{E}\left[\left(f_1^B\right)^2 * \widetilde{B}_t^2 + 2f_1^B f_1^Y\widetilde{B}_t\widetilde{Y}_t + \left(f_1^Y\right)^2 * \widetilde{Y}_t^2\right] \\
&\simeq \left(f_1^B\right)^2 * \mathrm{Var}\left(B_t\right) + \left[2f_1^B\left(\frac{\rho h_1^Y}{1-\rho h_1^B}\right) + f_1^Y\right] f_1^Y\frac{\sigma_\varepsilon^2}{1-\rho^2} \\
&\simeq \left\{\begin{array}{c} \left(h_1^B - 1/\beta_g\right)^2\mathrm{Var}\left(B_t\right) + \\ \left[2\left(h_1^B - 1/\beta_g\right)\left(\frac{\rho h_1^Y}{1-\rho h_1^B}\right) + \left(\tau^y + h_1^Y\right)\right]\left(\tau^y + h_1^Y\right)\frac{\sigma_\varepsilon^2}{1-\rho^2} \end{array}\right\},
\end{aligned}$$

where we have ignored higher-than-second-order terms. Using (2.80), we obtain:

$$\begin{aligned}
\mathrm{Var}\left(G_t\right) &\approx \left\{\begin{array}{c} \left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right)\frac{\left(h_1^B - 1/\beta_g\right)^2\left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)(1-\rho^2)}\sigma_\varepsilon^2 + \\ \left[2\left(h_1^B - 1/\beta_g\right)\left(\frac{\rho h_1^Y}{1-\rho h_1^B}\right) + \left(1 + h_1^Y\right)\right]\frac{\left(1+h_1^Y\right)\sigma_\varepsilon^2}{1-\rho^2} \end{array}\right\} \Rightarrow \\
\mathrm{Var}\left(G_t\right) &\approx \left\{\frac{\sigma_\varepsilon^2}{(1-\rho^2)}\left\{\begin{array}{c} \left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right)\left(h_1^Y\right)^2\frac{\left(h_1^B - 1/\beta_g\right)^2}{1-\left(h_1^B\right)^2} + \\ \left[2\left(\frac{\rho h_1^Y\left(h_1^B - 1/\beta_g\right)}{1-\rho h_1^B}\right) + \left(1 + h_1^Y\right)\right]\left(1 + h_1^Y\right) \end{array}\right\}\right\}.
\end{aligned} \tag{2.153}$$

Therefore, inserting (2.153) in (2.152) gives us welfare when the fiscal constraint is imposed on the debt:

$$U_w^B \approx \frac{\left[\begin{array}{c} u\left(\tau^y\overline{Y} - r\overline{B} - k^B\left(\overline{B} - B^c\right)\overline{L^B}\right) + \\ u'\left(\tau^y\overline{Y} - r\overline{B} - k^B\left(\overline{B} - B^c\right)\overline{L^B}\right) * \\ \frac{\sigma_\varepsilon^2}{2}\left\{\begin{array}{c} \frac{h_1^B-(1/\beta_g)}{1-h_1^B}\left[\frac{h_{11}^B\left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)(1-\rho^2)}\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) + 2\frac{\rho h_{11}^{BY}h_1^Y}{1-\rho h_1^B}\frac{1}{1-\rho^2} + \frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma\right] \\ + \left[\begin{array}{c} \frac{\left[h_{11}^B - k^B\gamma\overline{L^B}\left(1-\overline{L^B}\right)\left(2+\left(\overline{B}-B^c\right)\gamma\left(1-2\overline{L^B}\right)\right)\right]\left(h_1^Y\right)^2}{\left(1-\left(h_1^B\right)^2\right)(1-\rho^2)}\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right) \\ +2\left(\frac{\rho h_1^Y h_{11}^{BY}}{1-\rho h_1^B}\right)\frac{1}{1-\rho^2} + \frac{h_{11}^Y}{1-\rho^2} + h_{11}^\sigma \end{array}\right] \end{array}\right\} \\ +\frac{1}{2}u''\left(\tau^y\overline{Y} - r\overline{B} - k^B\left(\overline{B} - B^c\right)\overline{L^B}\right) * \\ \frac{\sigma_\varepsilon^2}{(1-\rho^2)}\left\{\left(\frac{1+\rho h_1^B}{1-\rho h_1^B}\right)\left(h_1^Y\right)^2\frac{\left(h_1^B-1/\beta_g\right)^2}{1-\left(h_1^B\right)^2} + \left[2\left(\frac{\rho h_1^Y\left(h_1^B-1/\beta_g\right)}{1-\rho h_1^B}\right) + \left(\tau^y + h_1^Y\right)\right]\left(\tau^y + h_1^Y\right)\right\} \end{array}\right]}{1-\beta_w}. \tag{2.154}$$

2.H.2 Welfare in the primary deficit-based sanction case

Substituting from (2.151) into (2.30), we get welfare U_w^D under the primary deficit constraint

$$U_w^D \approx \frac{\left\{\begin{array}{c} u\left(\overline{D} + \tau^y\overline{Y} - k^D\left(\overline{D} - D^c\right)\overline{L^D}\right) + u'\left(\overline{D} + \tau^y\overline{Y} - k^D\left(\overline{D} - D^c\right)\overline{L^D}\right) * \\ \frac{\sigma_\varepsilon^2}{2}\left\{\begin{array}{c} \left[\frac{l_1^D-(1/\beta_g)}{1-l_1^D}\right]\left[\left(\frac{l_{11}^D\left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) + 2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D} + l_{11}^Y\right)\frac{1}{1-\rho^2} + l_{11}^\sigma\right] \\ +2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D}\frac{1}{1-\rho^2} + \frac{\left[l_{11}^D - k^D\gamma\overline{L^D}\left(1-\overline{L^D}\right)\left(2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right)\right]\left(l_1^Y\right)^2}{\left(1-\left(l_1^D\right)^2\right)(1-\rho^2)}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) \\ +\frac{l_{11}^Y}{1-\rho^2} + l_{11}^\sigma \end{array}\right\} \\ +\frac{1}{2}u''\left(\overline{D} + \overline{Y} - k^D\left(\overline{D} - D^c\right)\overline{L^D}\right) * \operatorname{Var}(G_t) \end{array}\right\}}{1-\beta_w}. \tag{2.155}$$

Using that $\mathrm{E}\left[\widetilde{D}_t\widetilde{Y}_t\right] = \frac{\rho l_1^Y}{1-\rho l_1^D}\frac{\sigma_\varepsilon^2}{1-\rho^2}$, we have

$$\begin{aligned} \operatorname{Var}(G_t) &= \mathrm{E}\left[\left(G_t - \overline{G}\right)^2\right] \simeq \mathrm{E}\left[\left(j_1^D\widetilde{D}_t + j_1^Y\widetilde{Y}_t\right)^2\right] \\ &\simeq \left(j_1^D\right)^2\operatorname{Var}(D_t) + 2j_1^Dj_1^Y\mathrm{E}\left[\widetilde{D}_t\widetilde{Y}_t\right] + \left(j_1^Y\right)^2\operatorname{Var}(Y_t) \\ &\simeq \left\{\begin{array}{c} \left(l_1^D - 1/\beta_g\right)^2\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right)\frac{\left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2}\frac{\sigma_\varepsilon^2}{1-\rho^2} + \\ 2\left(l_1^D - 1/\beta_g\right)\frac{\rho l_1^Y\left(l_1^Y+\tau^y\right)}{1-\rho l_1^D}\frac{\sigma_\varepsilon^2}{1-\rho^2} + \left(l_1^Y + \tau^y\right)^2\frac{\sigma_\varepsilon^2}{1-\rho^2} \end{array}\right\}. \end{aligned}$$

Hence,

$$\operatorname{Var}(G_t) \simeq \left[\begin{array}{c} \left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right)\frac{\left(l_1^D-1/\beta_g\right)^2\left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2} + \\ 2\frac{\rho l_1^Y\left(l_1^Y+\tau^y\right)\left(l_1^D-1/\beta_g\right)}{1-\rho l_1^D} + \left(l_1^Y + \tau^y\right)^2 \end{array}\right]\frac{\sigma_\varepsilon^2}{1-\rho^2}. \tag{2.156}$$

So, inserting (2.156) in (2.155) gives us the value of welfare when the fiscal constraint is imposed on the value of the primary deficit:

$$U_w^D \approx \frac{\left\{\begin{array}{c} u\left(\overline{D}+\tau^y\overline{Y}-k^D\left(\overline{D}-D^c\right)\overline{L^D}\right)+u'\left(\overline{D}+\tau^y\overline{Y}-k^D\left(\overline{D}-D^c\right)\overline{L^D}\right)* \\ \frac{\sigma_\varepsilon^2}{2}\left\{\begin{array}{c} \left[\frac{l_1^D-(1/\beta_g)}{1-l_1^D}\right]\left[\left(\frac{l_{11}^D\left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right)+2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D}+l_{11}^Y\right)\frac{1}{1-\rho^2}+l_{11}^\sigma\right] \\ +2\frac{\rho l_1^Y l_{11}^{DY}}{1-\rho l_1^D}\frac{1}{1-\rho^2}+\frac{\left[l_{11}^D-k^D\gamma\overline{L^D}\left(1-\overline{L^D}\right)\left(2+\left(\overline{D}-D^c\right)\gamma\left(1-2\overline{L^D}\right)\right)\right]\left(l_1^Y\right)^2}{\left(1-\left(l_1^D\right)^2\right)(1-\rho^2)}\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right) \\ +\frac{l_{11}^Y}{1-\rho^2}+l_{11}^\sigma \end{array}\right\} \\ +\frac{1}{2}u''\left(\overline{D}+\tau^y\overline{Y}-k^D\left(\overline{D}-D^c\right)\overline{L^D}\right)* \\ \left[\left(\frac{1+\rho l_1^D}{1-\rho l_1^D}\right)\frac{\left(l_1^D-1/\beta_g\right)^2\left(l_1^Y\right)^2}{1-\left(l_1^D\right)^2}+2\frac{\rho l_1^Y\left(l_1^Y+\tau^y\right)\left(l_1^D-1/\beta_g\right)}{1-\rho l_1^D}+\left(l_1^Y+\tau^y\right)^2\right]\frac{\sigma_\varepsilon^2}{1-\rho^2} \end{array}\right\}}{1-\beta_w}. \tag{2.157}$$