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The spectra of supersymmetric states in string theory

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Type IIA and Type IIB Superstring Theory

1.1 The World-Sheet Action

Consider the two-dimensional Ramond-Neveu-Schwarz action

$$S = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right), \quad (1.1.1)$$

where ψ^μ and $\tilde{\psi}^\mu$ are the two components of the super-partner of bosonic fields X^μ .

The conserved currents corresponding to the symmetry of world-sheet translation and the (1,1) world-sheet supersymmetry transformation are, respectively,

$$\begin{aligned} T &= -\frac{1}{2} (\partial X^\mu \partial X_\mu + \psi^\mu \partial \psi_\mu) \\ T_F &= i\psi^\mu \partial X_\mu \end{aligned} \quad (1.1.2)$$

and similarly for the anti-holomorphic part¹. Notice that $T(z)$ and $\bar{T}(\bar{z})$ are the only non-vanishing component of the energy-momentum tensor of this theory. The vanishing of the trace $T_{z\bar{z}}(z, \bar{z})$ of the energy-momentum tensor, which is a property special for the above theory in two dimensions, implies a much larger group of symmetry. Indeed one can check that the Noether current corresponding to any infinitesimal world-sheet transformation $z \rightarrow z + \epsilon(z)$ is conserved, implying the presence of the following conformal symmetry

$$\begin{aligned} X^\mu(z', \bar{z}') &= X^\mu(z, \bar{z}) \\ \psi^\mu(z', \bar{z}') &= \left(\frac{\partial z'}{\partial z}\right)^{-1/2} \psi^\mu(z, \bar{z}), \quad \tilde{\psi}^\mu(z', \bar{z}') = \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-1/2} \tilde{\psi}^\mu(z, \bar{z}), \end{aligned}$$

¹Here and for the rest of this section we have put $\alpha' = 2$ to simplify the equations.

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for any holomorphic function $z'(z)$ of z . The world-sheet supercurrent T_F on the other hand, generates the superconformal transformation

$$\begin{aligned}\delta X^\mu(z, \bar{z}) &= -\epsilon \left(f(z)\psi^\mu(z) + \bar{f}(\bar{z})\tilde{\psi}^\mu(\bar{z}) \right) \\ \delta\psi^\mu(z) &= \epsilon f(z)\partial X^\mu(z) \\ \delta\tilde{\psi}^\mu(\bar{z}) &= \epsilon \bar{f}(\bar{z})\bar{\partial} X^\mu(z).\end{aligned}\tag{1.1.3}$$

In order for the above classical symmetries to be realised at the quantum level, one has to make sure that the path integral is well-defined. To work with the gauged-fixed action (1.1.1), the Jacobian factor of the gauge orbits has to be appropriately taken into account for each gauge symmetry of the theory. This can be done by introducing ghost fields, the so-called Faddeev-Popov ghosts, into the theory. For this particular theory we need ghost fields for both the conformal and the superconformal gauge symmetries. Furthermore, the anomaly associated to the above conformal transformation only vanishes when the total central charge of the full conformal field theory, now with the ghost fields included, vanishes. This condition turns out to imply that the above Ramond-Neveu-Schwarz string theory is only consistent when there are ten of the fields $(X^\mu, \psi^\mu, \tilde{\psi}^\mu)$. In other words, the critical dimension of the target space is ten for these theories. See for example [1] or [2] for further details of the above argument about the critical dimensions.

1.1.1 Canonical Quantisation

Now we would like to quantise this theory in ten-dimensional flat space. We will follow the canonical quantisation formalism. Though arguably not the most elegant way to do it, as opposed to the more systematic way of BRST quantisation, it has the advantage of admitting a simple exposition for our purpose. Since in this approach we put the ghost sector at its ground state at all stages, we will usually avoid writing down the ghost operators. Interested readers can consult, for example, [1] or [2] for a thorough treatment of the topic.

Let's consider the spectrum of the theory (1.1.1) on a cylinder. Without further identification on the target space, the world-sheet scalars X^μ have to return to the same value after circling the cylinder once. The fermions on the other hand, can have two different possibilities. The fermions which return to itself after circling once are said to satisfy the Neveu-Schwarz (NS) boundary condition and those which return to minus itself the Ramond (R) boundary condition.

In other words, upon conformally mapping the cylinder onto a complex

plane with coordinate z , the Laurent expansion of the fields can take the form

$$\begin{aligned}\partial X^\mu(z) &= -i \sum_{n \in \mathbb{Z}} \frac{a_n^\mu}{z^{n+1}} \\ \psi^\mu(z) &= \sum_r \frac{\psi_r^\mu}{z^{r+1/2}},\end{aligned}$$

where one has to choose between the two possible fermion boundary conditions

$$\begin{cases} 2r = 0 \pmod{2} & \text{for R sector} \\ 2r = 1 \pmod{2} & \text{for NS sector,} \end{cases} \quad (1.1.4)$$

and similarly for the right-moving sectors. Notice that all coefficients a_n^μ and ψ_r^μ are conserved quantities, due to the presence of the conformal symmetry. Canonical quantisation therefore gives us infinitely many (anti-)commutation relations

$$\begin{aligned}[a_m^\mu, a_n^\nu] &= m\eta^{\mu\nu} \delta_{m+n,0} \\ \{\psi_r^\mu, \psi_s^\nu\} &= \eta^{\mu\nu} \delta_{r+s,0},\end{aligned} \quad (1.1.5)$$

where $\eta^{\mu\nu} = \text{diag}(-1, 1, \dots, 1)$ is the metric for (9+1)-dimensional flat space-time, and again the same relations hold also for the right-moving sector.

Now one can expand the conformal and the superconformal currents in the same way as

$$\begin{aligned}T &= \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}} \\ T_F &= \sum_{2r \in \mathbb{Z}^\pm} \frac{G_r}{z^{r+3/2}},\end{aligned} \quad (1.1.6)$$

and the generators now have the following expansion

$$\begin{aligned}L_m &= \frac{1}{2} \left(\sum_{n \in \mathbb{Z}} : a_{m-n}^\mu a_{\mu,n} : + \sum_{2r \in \mathbb{Z}^\pm} \left(r - \frac{m}{2}\right) : \psi_{m-r}^\mu \psi_{\mu,r} : \right) + L_m^{\text{gh}} + A_{\text{R,NS}} \delta_{m,0} \\ G_r &= \sum_{n \in \mathbb{Z}} a_n^\mu \psi_{\mu,r-n} + G_r^{\text{gh}},\end{aligned} \quad (1.1.7)$$

where L_m^{gh} and G_r^{gh} denote the contribution from the ghost fields which we won't need, and the $: \dots :$ denotes the normal ordering among the operators.

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From the commutation relations we see that the normal ordering only matters for the generator L_0 , and we have introduced the extra constant “ $A_{R,NS}$ ” in this case to account for this ordering ambiguity. There are various ways to determine this zero point energy, such as the zeta-function regularization for example. The readers can consult various textbooks, for example, [1] or [3] for a careful treatment of this issue. Here we will simply quote the results. These zero-point energies are

$$\begin{cases} A_R = 0 & \text{for R sector} \\ A_{NS} = -\frac{1}{2} & \text{for NS sector .} \end{cases} \quad (1.1.8)$$

Now we are ready to move on to analysing the lowest-lying spectrum of this theory and especially how the tachyonic state can be projected out of the spectrum.

1.1.2 Massless Spectrum

In the canonical quantisation, extra constraints have to be imposed on a physical state. This is because we expect the scattering amplitudes between two physical states to be invariant under the conformal and superconformal transformation. In other words

$$0 = \langle \psi_1 | T(z) | \psi_2 \rangle = \langle \psi_1 | T_F(z) | \psi_2 \rangle \quad (1.1.9)$$

for any two physical states $|\psi_1\rangle$ and $|\psi_2\rangle$.

In the previous two subsections we have seen that the total central charge vanishes, and the currents satisfy the superconformal algebra

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} \\ \{G_r, G_s\} &= 2L_{r+s} \\ [L_m, G_r] &= \frac{m - 2r}{2} G_{m+r} . \end{aligned} \quad (1.1.10)$$

It is therefore self-consistent and sufficient for the purpose of ensuring (1.1.9) to impose

$$L_m |\psi\rangle = G_r |\psi\rangle = 0 \text{ for all } m, r \geq 0 . \quad (1.1.11)$$

Especially, since in the canonical quantisation we are putting the ghost sector to its ground state, now we just have to impose the above condition on the matter part of the Hilbert space. In particular, the L_0 -constraint gives the mass-shell condition

$$L_0 |\psi\rangle = H |\psi\rangle = 0 . \quad (1.1.12)$$

In the above equation, p^μ is the eigenvalue of the center-of-mass mode a_0^μ , $N = \sum_{n>0} a_{-n}^\mu a_{\mu,n} + \sum_{r>0} r \psi_{-r}^\mu \psi_{\mu,r}$ is the oscillation number operator, and the ‘‘Hamiltonian operator’’ is given in terms of them as

$$H = \begin{cases} \frac{1}{2}p^2 + N & \text{for R sector} \\ \frac{1}{2}p^2 + N - \frac{1}{2} & \text{for NS sector .} \end{cases} \quad (1.1.13)$$

Apart from the constraints, there are also equivalence relations among the physical states. Namely

$$|\psi\rangle \sim |\psi\rangle + |\phi\rangle$$

if $\langle\psi'|\phi\rangle = 0$ for all physical states $|\psi'\rangle$, since any two states having the same scattering amplitudes with any other physical state must be equivalent. This happens, for example, when

$$|\phi\rangle = \left(\sum_{m>0} \ell_m L_{-m} + \sum_{r>0} \kappa_r G_{-r} \right) |\phi'\rangle \quad (1.1.14)$$

for some coefficients ℓ_m and κ_r . This equivalence condition together with the constraints will remove for us the two light-cone directions. To see this we will now study the spectra of R-ground states and NS-ground state and their excited states.

Define $|0; p\rangle_{\text{R}}$ to be the R-ground state annihilated by all annihilation operators

$$a_n^\mu |0; p\rangle_{\text{R}} = \psi_n^\mu |0; p\rangle_{\text{R}} = 0 \quad \text{for all } n > 0 \quad \text{and} \quad a_0^\mu |0; p\rangle_{\text{R}} = p^\mu .$$

From the commutation relations of the fermionic zero modes ψ_0^μ , we see that they satisfy the ten-dimensional Clifford algebra. The R-ground states are therefore spacetime fermions. In ten dimensions the smallest representation is the 16-dimensional Weyl-Majorana spinors, which are real and have definite chiralities

$$\Gamma |0; p\rangle_{\text{R}}^\pm = \pm |0; p\rangle_{\text{R}}^\pm ,$$

with Γ anti-commuting with all the ten Gamma matrices. Now the only non-trivial constraints from (1.1.11) are

$$L_0 |0; p\rangle_{\text{R}}^\pm = G_0 |0; p\rangle_{\text{R}}^\pm = 0 .$$

The first condition tells us that the state is massless and the second now gives the massless Dirac equation. The massless Dirac equation further reduces the degrees of freedom of the 16-component spinors in half. We therefore

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open string massless spectrum		
sector	SO(8) rep.	G-parity
R+	8	1
R-	8'	-1
NS+	8_v	1

type IIA massless spectrum		
sector	SO(8) rep.	10d multiplet
(NS+,NS+)	[0]+[2]+(2)	$\Phi, B, g, \psi_{(2/3)}^+, \lambda^+$
(R+,NS+)	8'+56	(Graviton Multiplet)
(NS+,R-)	8+56'	$\psi_{(2/3)}^-, \lambda^-, C^{(1)}, C^{(3)}$
(R+,R-)	[1]+[3]	(Gravitini Multiplet)

type IIB massless spectrum		
sector	SO(8) rep.	10d multiplet
(NS+,NS+)	[0]+[2]+(2)	$\Phi, B, g, \psi_{(2/3)}^+, \lambda^+$
(R+,NS+)	8'+56	(Graviton Multiplet)
(NS+,R+)	8'+56	$C^{(0)}, C^{(2)}, C_+^{(4)}, \psi_{(2/3)}^+, \lambda^+$
(R+,R+)	[0]+[2]+[4] ₊	(Gravitini Multiplet)

Table 1.1: Summary of the massless spectrum of the open, type IIA, type IIB superstring theories.

conclude that the R-ground states transform in the spinor representation **8** or **8'** of $SO(8)$, corresponding to the two possible ten-dimensional chiralities $\Gamma|0; p\rangle_{\mathbb{R}}^{\pm} = \pm|0; p\rangle_{\mathbb{R}}^{\pm}$. In this case there is no state of this form which can be created by operators of the form as in (1.1.14) and the equivalence relation does not impose further conditions.

Next we turn to the NS sector. Now there is a unique tachyonic ground state. The first excited states are the ones obtained by acting with the lowest lying fermionic creating operators

$$|v; p\rangle_{\text{NS}} = v^{\mu} \psi_{\mu, -1/2} |0; p\rangle_{\text{NS}}$$

satisfying

$$N|v; p\rangle_{\text{NS}} = \frac{1}{2}|v; p\rangle_{\text{NS}}.$$

The non-trivial constraint $L_0|v; p\rangle_{\text{NS}} = G_{1/2}|v; p\rangle_{\text{NS}} = 0$ now gives the mass-shell and the orthogonality condition $p^2 = p^{\mu} v_{\mu} = 0$. But here a state of the

same form can also be produced by acting on the tachyonic ground state by $G_{-1/2}$:

$$G_{-1/2}|0; p\rangle_{\text{NS}} = |p; p\rangle_{\text{NS}} ,$$

and we are led to further impose the equivalence condition (1.1.14) $v \sim v + \mathbb{R}p$. Therefore the first excited states of the NS sector are massless and transform in the vector representation $\mathbf{8}_v$ of $\text{SO}(8)$. While the degrees of freedom along the light-cone directions for the massless states of the R-sector are forbidden by the constraint which is equivalent to the massless Dirac equation, in the NS sector they are frozen by the constraint together the equivalence relation. This is yet another way to see why the zero point energy of the NS sector has to be $-1/2$: $\text{SO}(8)$ is the little group of the massless particles in ten dimensions. For consistency the excited states transforming as a massless photon should indeed be massless [4].

After seeing that the dynamics in light-cone directions are unphysical, we now concentrate on the transversal degrees of freedom and define a parity operator

$$G = \begin{cases} \Gamma(-1)^{\sum_{n=1}^{\infty} \psi_{-n}^i \psi_{i,n}} & \text{for R sector } i = 1, \dots, 8 \\ (-1)^{\sum_{n=1}^{\infty} \psi_{-(n-1/2)}^i \psi_{i, (n-1/2)+1}} & \text{for NS sector} \end{cases} . \quad (1.1.15)$$

The massless spectrum of R and NS open string, together with their eigenvalue under the above parity operator, is summarised in Table 1.1.

Notice that by projecting the NS states onto the positive-G-parity states we eliminate the tachyonic ground state. Another merit of this projection, the so-called GSO (Gliozzi-Scherk-Olive) projection, is that it yields a closed string spectrum with equal number of (spacetime) bosonic and fermionic fields when left- and right-moving copies of the massless fields are combined. In other words, it is a projection which yields a consistent conformal field theory with spacetime supersymmetry. It is also possible to derive this projection by various consistency requirement, for example the modular invariance of the loop amplitudes, but we will not do it here. The full GSO projection leaves us with two consistent theories in ten dimensions, corresponding to whether the left- and right-moving R-ground states are combined with the same or the opposite parities. By carefully combing the left- and right-moving components of the NS- and R-massless states discussed above, we obtain the massless spectrum of these two theories as listed in Table 1.1. The difference between IIA and IIB is that the spin-1/2 fields in the type IIB case are of the same chirality and those of IIA are of opposite chiralities. We therefore call type IIB string theory a *chiral* theory and IIA a non-chiral one. Each theory contains one graviton

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and one gravitini supermultiplets, and the massless spectrum of type IIA and IIB string theory is therefore identical to that of type IIA and IIB $\mathcal{N} = 2$ supergravity. This is not surprising because the large amount of supersymmetries highly constrains the structure of the fields². In fact, the supersymmetry does not only fix the field content but also the allowed action; there are two possible supergravity theory with sixteen supercharges in ten dimensions, the type IIA and IIB supergravity, which are simply the low-energy effective theory of type IIA and IIB string theory. We will see the explicit form of these low-energy effective actions in the following section.

1.1.3 T-Duality

In the previous subsection we introduced two kinds of superstring theories, namely the type IIA and IIB theories. But in fact the two superstring theories discussed above are not so independent from each other as they may seem. They are related by the so-called T-duality, which involves reflecting the space-time parity along one (or any odd number in general) spatial direction on one of the two sides (right- or left-moving) of the world-sheet.

For concreteness let's choose

$$\begin{aligned} X^1(\bar{z}) &\rightarrow -X^1(\bar{z}) \ ; \ \tilde{\psi}^1(\bar{z}) \rightarrow -\tilde{\psi}^1(\bar{z}) \\ X^1(z) &\rightarrow X^1(z) \ ; \ \psi^1(z) \rightarrow \psi^1(z) . \end{aligned}$$

Upon this transformation, the eigenvalues of the parity operator defined in (1.1.15) flip signs for the right-moving R-sector states and remain unchanged for the NS-sector states. Consulting the table 1.1, we then see that T-duality, a duality which is stringy by nature, exchanges the chiral (IIB) and the non-chiral (IIA) theories. In this sense type IIA and type IIB string theory are really the same theory.

For later use we would like to have an explicit map between the bosonic massless degrees of freedom under T-duality. We will now derive it based on the canonical quantisation approach we followed earlier. Let's begin first with the NS-NS sector as its field content is shared by both the IIA and the IIB theory. As we saw earlier in Table 1.1, these are the spacetime fields Φ, G_{ij}, B_{ij} corresponding to the massless states of the form $\tilde{\psi}^i \psi^j |0, \tilde{0}\rangle$ in the world-sheet theory. Here we remind the readers that the total Hilbert space is the tensor product of left- and right-moving Hilbert spaces, and our notation really means $|0, \tilde{0}\rangle = |0\rangle \otimes |\tilde{0}\rangle$. Now, instead of considering them all in a

²Strictly speaking, because we begin with the Ramond-Neveu-Schwarz superstring action (1.1.1) as opposed to the Green-Schwarz one, it is therefore not a priori clear that the theory has spacetime supersymmetry.

flat background as in the previous sections, we would like to derive a map between backgrounds. For this purpose it will be more convenient to turn to the vielbein frame, in other words we will consider the spacetime fields $e_{\hat{j}}^i$ and $B_{\hat{k}\hat{\ell}} := B_{ij}e_{\hat{k}}^ie_{\hat{\ell}}^j$, and operators $\psi_{\hat{i}}$ etc, where the hatted indices denote the orthonormal indices. Matching the representation under the rotation group we obtain a map between the operators and the perturbations of the spacetime fields under consideration

$$\begin{aligned}\tilde{\psi}_{(\hat{i}\psi_{\hat{j}})} &\longrightarrow e_{(\hat{i}}^ke_{\hat{j})}^\ell \delta G_{k\ell} = \frac{1}{2}e_{\hat{i}}^ke_{\hat{j}}^\ell \delta G_{k\ell} \\ \tilde{\psi}_{[\hat{i}\psi_{\hat{j}}]} &\longrightarrow e_{[\hat{i}}^ke_{\hat{j}]}^\ell \delta B_{k\ell} \\ \tilde{\psi}_{\hat{i}}\psi^{\hat{i}} &\longrightarrow 2\delta\Phi.\end{aligned}$$

Now consider a T-duality transformation along the 1st direction, with $e_1^i\partial_i$ being an isometry (a Killing vector) of the background. For any such background metric we can always choose the vielbein such that $e_1^{\hat{a}} = e_1^a = 0$, with $a = 2, \dots, 8$ being the directions transversal to the light-cone and the T-dual directions. In other words, we can now write the metric in the form

$$\begin{aligned}ds^2 &= G_{ij}dx^i dx^j = G_{1,1}(dx^1 + A_a dx^a)^2 + g_{ab}dx^a dx^b \\ &= \theta^{\hat{1}} \otimes \theta^{\hat{1}} + \theta^{\hat{a}} \otimes \theta^{\hat{a}}\end{aligned}$$

where $\theta^{\hat{i}} = e_{\hat{j}}^i dx^j$.

In this frame we can rewrite the above relations as follows.

$$\begin{aligned}\tilde{\psi}_{(\hat{1}\psi_{\hat{1}})} &\rightarrow \frac{1}{2}\delta(\log G_{1,1}) & \tilde{\psi}_{(\hat{a}\psi_{\hat{b}})} &\rightarrow \frac{1}{2}\delta g_{ab} \\ \tilde{\psi}_{(\hat{1}\psi_{\hat{a}})} &\rightarrow \frac{1}{2}\sqrt{G_{1,1}}e_a^b\delta A_b & \tilde{\psi}_{[\hat{1}\psi_{\hat{a}}]} &\rightarrow \frac{1}{2}\frac{1}{\sqrt{G_{1,1}}}e_a^b\delta B_{1b} \\ \tilde{\psi}_{[\hat{a}\psi_{\hat{b}}]} &\rightarrow e_{[\hat{a}}^ce_{\hat{b}]}^d(\delta B_{ab} - 2A_a\delta B_{1b}) & \tilde{\psi}_{\hat{i}}\psi^{\hat{i}} &\rightarrow 2\delta\Phi = \frac{1}{2}g^{ab}\delta g_{ab} + \frac{1}{2}\delta(\log G_{1,1})\end{aligned}\tag{1.1.16}$$

When we now reflect the right-moving side along the first direction in the orthonormal frame, namely $\tilde{\psi}_{\hat{1}} \rightarrow -\tilde{\psi}_{\hat{1}}$, it's easy to see that it's equivalent to a field redefinition as listed in Table 1.2. This table rewritten in terms of the full metric G_{ij} is the usual Buscher rules. Finally we also have to set

$$\Phi \rightarrow \Phi - \frac{1}{2}\log G_{1,1}.\tag{1.1.17}$$

before \longleftrightarrow after	
$G_{1,1}$	$\frac{1}{G_{1,1}}$
g_{ab}	g_{ab}
A_a	B_{1a}
B_{1a}	A_a
$B_{ab} - A_{[a}B_{1b]}$	$B_{ab} + A_{[a}B_{1b]}$

Table 1.2: T-duality (Buscher Rules)

To understand this map between the dilaton fields under T-duality, first recall that not all the metric fluctuations are physical because of the diffeomorphism invariance $G_{ij} \sim G_{ij} + \nabla_{(i}v_{j)}$, the trace mode here gives rise to the dilaton fluctuation. Of course now the choice of coefficient in (1.1.17) is just a matter of convention. But it is chosen in such a way that later the world-sheet action (1.2.7) in arbitrary consistent NS background will stay conformally invariant after a T-duality transformation. This choice also renders the nine-dimensional Newton's constant invariant when reduced on the T-circle invariant under T-duality.

When the 1st direction is a circle, by studying the massive spectrum of the “compactified” theory before and after the above transformation, one concludes that T-duality also changes the radius of the circle as

$$R_1 \rightarrow \frac{\alpha'}{R_1}, \quad (1.1.18)$$

where α' is the coupling constant appearing in the world-sheet action (1.1.1). This is indeed consistent with the world-sheet dictionary that $G_{1,1} \leftrightarrow 1/G_{1,1}$.

We can now do the same analysis for the R-R sector. It's a straightforward exercise involving first inserting p Gamma matrices to make p -form fields out of a (spacetime) spinor bilinear, and then flipping the chirality along one spatial direction. Since the R-R gauge fields are different in type IIA and IIB string theory, we expect T-duality to also exchange the objects charged under these higher-form fields.

Furthermore, if there are also open strings in the theory, which have either Neumann or Dirichlet boundary condition at the end points, it's not hard to see that the T-duality transformation exchanges the two boundary conditions. In other words, T-duality together with the presence of open strings forces onto us other kinds of objects on which the open strings can end. We will introduce these Dirichlet branes (D-branes) from the point of view of the low-energy effective theory in the following sections. As we will see, these are

exactly the objects charged under the R-R fields which get exchanged under T-duality accordingly.

1.2 Low Energy Effective Action

1.2.1 Supergravity Theory in Eleven and Ten Dimensions

In the previous section we derived the massless spectrum of type IIA and IIB superstring theory. In this section we would like to describe the interactions of these massless modes, which is constrained by supersymmetry to be described by the type IIA and IIB supergravity theories in ten dimensions. It will nevertheless turn out to be a rewarding path to begin with the eleven dimensional supergravity theory. Supersymmetry ensures that this theory is unique. Furthermore the IIA ten-dimensional supergravity has to be the dimensional reduction of this higher-dimensional theory, since the two theories have the same supersymmetry algebras.

The field content of this theory is rather simple: for the bosons there are just graviton with $\frac{9 \times 10}{2} - 1 = 44$ components and a three-form potential with $\frac{9 \times 8 \times 7}{3!} = 84$ components, in representation of the SO(9) little group of massless particles in eleven dimensions. There is also the gravitino with its 16×8 degrees of freedom, in representation of the covering group $Spin(9)$. This is indeed the same number as the number of massless degrees of freedom of the type II string theory as recorded in Table 1.1. The bosonic part of the action is

$$(16\pi G_N^{(11)}) S^{(11)} = \int d^{11}x \sqrt{-G} \left(R - \frac{1}{2} |F^{(4)}|^2 \right) - \frac{1}{3!} \int A^{(3)} \wedge F^{(4)} \wedge F^{(4)}, \quad (1.2.1)$$

where $F^{(4)}$ is the field strength of the three-form potential $F^{(4)} = dA^{(3)}$ and we use the notation

$$|F^{(n)}|^2 = \frac{F^{(n)} \wedge \star F^{(n)}}{\text{volume form}}$$

for the kinetic term of a (n-1)-form potential. Here and most of the time in this thesis we will avoid writing down the fermionic part of the action. This is because we are interested in supersymmetric solutions with zero fermionic fields, which of course have vanishing action for the fermionic part.

To dimensionally reduce it, write the eleven-dimensional metric as

$$G_{MN} = e^{-\frac{2}{3}\Phi} \begin{pmatrix} g_{\mu\nu} + e^{2\Phi} A_\mu A_\nu & e^{2\Phi} A_\mu \\ e^{2\Phi} A_\nu & e^{2\Phi} \end{pmatrix}, \quad (1.2.2)$$

where we use $M, N, \dots = 0, 1, \dots, 10$ to denote the eleven-dimensional and $\mu, \nu, \dots = 0, 1, \dots, 9$ the ten-dimensional directions. The above choice of defin-

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ing the ten-dimensional fields will be justified in subsection 1.2.2 when we make a detailed comparison with the world-sheet theory.

We also reduce the three-form potential $A_{MNP}^{(3)}$ as $A_{\mu\nu\rho}$ when it has no “leg” in the 11-th direction and as $A_{MN10}^{(3)} = B_{\mu\nu}$ and $H^{(3)} = dB^{(2)}$ when it does.

Under this field redefinition, and truncating all the dependence on the eleventh direction, the action reduces to

$$\begin{aligned}
 S^{(\text{IIA})} &= S_{\text{NS}} + S_{\text{R}}^{(\text{IIA})} + S_{\text{C-S}}^{(\text{IIA})} \\
 2\kappa^2 S_{\text{NS}} &= \int d^{10} \sqrt{-g} e^{-2\Phi} \left(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |H_3|^2 \right) \\
 2\kappa^2 S_{\text{R}}^{(\text{IIA})} &= -\frac{1}{2} \int d^{10} x \left(|F^{(2)}|^2 + |\tilde{F}^{(4)}|^2 \right) \\
 2\kappa^2 S_{\text{C-S}}^{(\text{IIA})} &= -\frac{1}{2!} \int B^{(2)} \wedge F^{(4)} \wedge F^{(4)}, \tag{1.2.3}
 \end{aligned}$$

where $F^{(2)}$ is the field strength of the Kaluza-Klein gauge field $A^{(1)}$ and

$$\tilde{F}^{(4)} = dA^{(3)} + A^{(1)} \wedge H^{(3)} \tag{1.2.4}$$

is the field strength modified by the Chern-Simons term. This is the bosonic action for the type IIA supergravity that we want to construct. The 10d gravitational coupling constant κ will be discussed in the following subsection.

Type IIB supergravity, on the other hand, cannot be obtained by dimensionally reducing the eleven-dimensional supergravity. Although in principle it is related to the IIA supergravity by T-dualise the IIA string theory and then take the low-energy limit, it is actually not at all a straightforward task to write down a classical action for the field content recorded in Table 1.1. This is because in $d=2 \pmod{4}$ dimensions there is no straightforward way to incorporate in the action the self-duality condition on a middle rank (*i.e.*, $(\frac{d}{2})$ -form) field strength. Recall that in type IIB string theory this is indeed the case at hand, since the R-R field $C_+^{(4)}$ is constrained to have self-dual field strength. Here we shall write down an action analogous to the IIA version, while the self-duality condition should be imposed as an additional constraint.

The NS sector bosonic action is identical to the type IIA case, as expected from our notation, while the rest reads

$$\begin{aligned}
 S^{(\text{IIB})} &= S_{\text{NS}} + S_{\text{R}}^{(\text{IIB})} + S_{\text{C-S}}^{(\text{IIB})} \\
 2\kappa^2 S_{\text{R}}^{(\text{IIB})} &= -\frac{1}{2} \int d^{10}x \left(|F^{(1)}|^2 + |\tilde{F}^{(3)}|^2 + \frac{1}{2} |\tilde{F}^{(5)}|^2 \right) \\
 2\kappa^2 S_{\text{C-S}}^{(\text{IIB})} &= -\frac{1}{2!} \int C^{(4)} \wedge H^{(3)} \wedge F^{(3)} ,
 \end{aligned} \tag{1.2.5}$$

where $F^{(1)}$ is the field strength of the R-R zero form potential $C^{(0)}$ and

$$\begin{aligned}
 \tilde{F}^{(3)} &= dC^{(2)} - C^{(0)} \wedge H^{(3)} \\
 \tilde{F}^{(5)} &= dC^{(4)} - \frac{1}{2} C^{(2)} \wedge H^{(3)} + \frac{1}{2} B^{(2)} \wedge F^{(3)}
 \end{aligned}$$

are again the field strength with Chern-Simons term, while the self-duality constraint

$$\tilde{F}^{(5)} = \star \tilde{F}^{(5)} , \tag{1.2.6}$$

must be imposed by hand additionally.

1.2.2 Couplings of String Theory

In the last section we introduced the type IIA and IIB superstring theory, and in the last subsection the type IIA and IIB supergravity theory. Furthermore we have observed that the massless spectrum of the two sets of theories are the same. We therefore conclude that, for the supergravity theories to be the low-energy effective description of the superstring theories, the dynamics of these massless modes must be explained by both theories. In this subsection we will establish this connection, and furthermore spell out the relation between the coupling constants of the ten- and eleven-dimensional supergravity and the various quantities of string theory .

Let us focus on the NS part of the action (1.2.3), which is common for both type IIA and type IIB supergravity³. Then the equations of motion obtained from S_{NS} is the same as requiring the absence of the conformal anomaly in the following world-sheet theory

$$S_{\text{world-sheet}} = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{h} \left((h^{ab} g_{\mu\nu} + i\epsilon^{ab} B_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu + \alpha' \Phi(X) R(h) \right) , \tag{1.2.7}$$

where h^{ab} is the world-sheet metric, $R(h)$ its Ricci scalar, and ϵ^{ab} is an anti-symmetric tensor normalised such that the term involving the B -field simply

³Although possible, here we will not attempt to explain the effects caused by a source of the higher-form fields from a world-sheet point of view.

equal to $2\pi i \int B^{(2)}$. Here and in the rest of the chapter we will not make a difference in notation for the pull-back fields when it is clear from the context.

It might be surprising that the bulk Einstein equation appears as the requirement of conformal invariance of the world-sheet theory. This indeed requires some justification. Strictly speaking, we have obtained the massless spectrum of type II strings by quantising it in the flat background with B-field and “dilaton” Φ turned off. How do we know that the same theory is also consistent in other backgrounds, except for the hint from the supergravity theory as a low-energy effective theory? Indeed what we just saw is that a consistent NS background for the world-sheet theory at the one-loop level is also automatically a solution to the equation of motion of the proposed low-energy effective theory. This connection justifies our choice of frame for dimension reduction (1.2.2). This choice of scaling of the ten-dimensional metric will therefore be called the “string frame”, since this is the target space metric which shows up in the string world-sheet action. This “string frame” is different from the usual “Einstein frame”, in which the curvature term in the action has no pre-factor $e^{-2\Phi}$ in front.

Now we will comment on the different roles of and relations among various coupling constants in the ten- and eleven-dimensional spacetime and the various quantities in the world-sheet theory. First of all we have α' which sets the length scale in the world-sheet action (1.2.7). We therefore call the string length

$$\ell_s \sim \sqrt{\alpha'} .$$

From the quantisation of the superstring we see that the mass-shell condition gives

$$m^2 \sim \frac{1}{\alpha'} N \sim \frac{1}{\ell_s^2} N ,$$

where N is the oscillator number above the massless level. From this we see that the low-energy effective action, in which we truncate the fields to only the massless ones, is valid when one is only interested in physics much larger than the string length.

Furthermore, comparing the world-sheet and the supergravity action (1.2.3) we conclude that the gravity coupling constant is related to the string scale by

$$\kappa^2 \sim \ell_s^8 \sim \alpha'^4 .$$

But this is not yet the ten-dimensional Planck length. To discuss that we should first understand the role played by the “dilaton” field $\Phi(X)$. From the world-sheet action we see that it controls the scattering between strings. For

example, when $\Phi(X) = \Phi_0$ is constant in spacetime, $e^{-S_{\text{world-sheet}}}$ has a factor

$$e^{-\Phi\chi} = e^{-\Phi(2-2g)} \equiv g_s^{2g-2},$$

where χ is the Euler characteristic and g the genus of the world-sheet (see (A.0.8) for Gauss-Bonnet theorem which relates the two quantities). In other words, the dilaton field Φ controls the genus expansion, the string theory counter-part of the loop expansion in particle physics. We therefore identify as the string coupling's constant

$$g_s = e^{\Phi}.$$

We should emphasize here that this is not just a parameter but really a dynamical field of the theory.

Now we are ready to identify the ten-dimensional gravitational coupling. By going to the Einstein frame in which the Einstein action takes the standard form, we see that

$$G_N^{(10)} \sim (\ell_P^{(10)})^8 \sim \kappa^2 e^{2\Phi_0} \sim \ell_s^8 g_s^2 \sim \alpha'^4 g_s^2, \quad (1.2.8)$$

where Φ_0 is now the asymptotic value of the dilaton field. Finally we work out the relation between the radius R_M of the eleventh-dimensional circle on which we reduce the eleven-dimensional supergravity to obtain the type IIA supergravity, and the eleven-dimensional Planck length. From (1.2.1) and (1.2.2) we get

$$\frac{R_M}{\ell_P^{(11)}} \sim (e^{4\Phi/3})^{1/2} \sim g_s^{2/3} \quad \text{and} \quad \frac{G_N^{(11)}}{R_M} \sim \frac{(\ell_P^{(11)})^9}{R_M} \sim G_N^{(10)} \sim \ell_s^8 g_s^2,$$

in other words

$$\ell_P^{(11)} \sim \ell_s g_s^{1/3} \quad \text{and} \quad R_M \sim \ell_s g_s. \quad (1.2.9)$$

Here we see an interesting phenomenon, namely that the radius of the eleventh-dimensional circle in string unit becomes large when the strings are strongly interacting. When the string coupling constant is large, the perturbative string theory we discussed in the last section should not be trusted.

There is a similar breaking down of the validity of the ten-dimensional theory on the supergravity side. In the ‘‘decompactification limit’’ in which the Kaluza-Klein circle becomes larger and larger, the momentum modes along the Kaluza-Klein direction becomes lighter and lighter. As a result, the truncation of the spectrum to a lower-dimensional one eventually becomes invalid. In other words,

$$\text{type IIA supergravity} \begin{array}{c} \xleftarrow{\text{decompactify}} \\ \xrightarrow{\text{compactify}} \end{array} \text{11d supergravity} \quad (1.2.10)$$

In subsection 1.2.1 we have used the eleven-dimensional supergravity just as a convenient starting point to write down the ten-dimensional supergravity action. But if we take this reduction a step further, it seems to suggest that the ten-dimensional supergravity is only a valid low-energy description of the full non-perturbative theory at small ℓ_s and small g_s . At large g_s the eleven-dimensional theory becomes a better description. Indeed, later in section 1.3.1 we will see that there are dynamical objects other than the fundamental strings which become light at strong coupling and which are captured by the eleven-dimensional supergravity but not by the ten-dimensional one.

1.3 Non-Perturbative Aspects

In the last section we introduced superstring theory as a perturbative theory. But in fact the theory is much richer than that. In particular, for the purpose of studying the supersymmetric spectrum, especially the spectrum which is responsible for the existence of black hole entropy, the non-perturbative aspects of the theory will play a crucial role in our understanding of the problem.

While in general the non-perturbative aspects of string theory is very difficult to study, there are regions in the moduli space that are fortunately accessible to us. The key word here is “dual perspective”. An example of which we have seen earlier is the T-duality relating type IIA and IIB string theory, stating that while the two descriptions look different, they offer “dual perspectives” on the same theory.

A duality is especially useful if this theory offers a complementary range of computational accessibility. In this section we will introduce a few dualities like this, mapping non-perturbative physics on one side to perturbative physics on the other side of the duality. We will also introduce the solitonic objects of the theory, which are generically called “branes”. These objects can often be described from two dual perspectives, a fact that makes black hole counting in string theory possible and motivates an extremely important gauge/gravity duality.

1.3.1 M-theory

In the last section, we have just saw the interesting possibility that an eleven-dimensional supergravity might be the low-energy effective action for the type IIA string theory at *strong* coupling. In this section we will explore this possibility further.

Historically, eleven-dimensional supergravity theory is interesting because eleven is the highest dimension in which Minkowski signature with Poincaré and supersymmetry invariance is possible. But one should keep in mind that, just as type IIA and IIB supergravity should only be seen as an effective theory at low energy but not a complete theory because of its non-renormalisability, the eleven-dimensional supergravity can only at best be a low-energy description of a consistent theory. We will refer to this complete theory as “M-theory”, whose non-perturbative description is unfortunately not yet fully developed and out of the scope of the present thesis.

From the above discussion we see that this “M-theory”, no matter of what nature it actually is, must have the following relationship with type IIA string theory

$$\text{IIA string theory} \xrightarrow{g_s \gg 1} \text{M-theory}$$

as suggested by the low-energy relation (1.2.10), where the identification of the compactification radius is given by (1.2.9). We will later refer to this relation as the “M-theory lift”.

Without really knowing the non-perturbative definition of M-theory, we will now use its low-energy effective theory as a guide to explore the structure of the theory. As we will see, it will turn out to be a fruitful path towards a simple understanding of many of the non-perturbative features of the type II string theory.

1.3.2 Branes

Let us begin by finding supersymmetric classical solutions to the eleven dimension supergravity. Since the three-form potential $A^{(3)}$ is the only bosonic degree of freedom besides the gravitons, from the experience with the usual Maxwell-Einstein theory, we expect to find objects that are charged under these fields. A straightforward generalisation of the Wilson line coupling to one-form potential of a charged point particle is

$$\int_{\text{world-line}} A^{(1)} \longrightarrow \int_{\text{world-volume}} C^{(n)},$$

which leads us to expect an object with a $(2 + 1)$ -dimensional world-volume which plays the role of the “electron” for $A^{(3)}$. Indeed there is a 1/2-BPS solution (a supersymmetric solution with half of the supersymmetry unbroken) which has a non-zero Noether charge

$$Q = \int_{S^7} \star F^{(4)} - \frac{1}{2} A^{(3)} \wedge F^{(4)} \quad (1.3.1)$$

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for the three-form field. Notice that the expression of the Noether charge gets modified in the presence of a Chern-Simons term in the action. The solution reads

$$\begin{aligned} ds_{M2}^2 &= f_{M2}^{-2/3} ds_{3,L}^2 + f_{M2}^{1/3} ds_{8,E}^2 \\ A^{(3)} &= f_{M2}^{-1} dV_{3,L} \\ f_{M2} &= 1 + a_2 Q \left(\frac{1}{r}\right)^6. \end{aligned} \tag{1.3.2}$$

It also has a magnetic cousin which looks like

$$\begin{aligned} ds_{M5}^2 &= f_{M5}^{-1/3} ds_{6,L}^2 + f_{M5}^{2/3} ds_{5,E}^2 \\ A^{(6)} &= f_{M5}^{-1} dV_{6,L}, \quad dA^{(3)} = \star dA^{(6)} \\ f_{M5} &= 1 + a_5 P \left(\frac{1}{r}\right)^3, \end{aligned} \tag{1.3.3}$$

and satisfies

$$P = \int_{S^4} F^{(4)}. \tag{1.3.4}$$

In the above equations $ds_{n,L}^2$ denotes the usual metric of a flat Lorentzian space with mostly positive signature $(1, n-1)$ and $dV_{n,L}$ its volume form, and

$$ds_{n,E}^2 = dr^2 + r^2 d\Omega_{(n-1)}^2$$

is the metric of a flat Euclidean space. The constants a_2 , a_5 are chosen such that (1.3.1), (1.3.4) are satisfied.

We see that the above solutions carrying electric and magnetic charges have $(2+1)$ and $(5+1)$ ‘‘tangent’’ directions respectively. We will therefore call them the M2 and the M5 brane solutions. In general, from Hodge duality we see that a $(p+1)$ -dimensional object in D dimensions must be electric-magnetic dual to another object with $(D-p-4)$ spatial directions, when both objects are required to have a time-span.

Furthermore, as analogous to the Maxwell case, the Dirac quantisation, namely the well-definedness of an electron wave-function in a monopole background, will impose on us⁴

$$QP \in 2\pi\mathbb{Z}. \tag{1.3.5}$$

Notice that this condition cannot be seen from studying the supergravity action alone and is therefore a strictly quantum effect.

⁴This Dirac quantisation condition holds when (gravitational) anomaly effects can be neglected. See [5] for a discussion about the correction of the charge quantisation condition in M-theory due to gravitational effects, and section 6.2 of the present thesis for an explicit example in which the charge quantisation condition is modified.

11d field	sources (elec/mag)	10d field	sources (elec/mag)
	$G_{\parallel\parallel}$	Φ	
G_{MN}	$G_{\perp\perp}$ everything	$g_{\mu\nu}$	everything
	$G_{\parallel\perp}$ KK-monopole	$C^{(1)}$	D0 D6
$A^{(3)}$	$A_{\parallel}^{(3)}$ M2 $_{\parallel}$ M5 $_{\perp}$	$B^{(2)}$	F1 NS5
	$A_{\perp}^{(3)}$ M2 $_{\perp}$ M5 $_{\parallel}$	$C^{(3)}$	D2 D4

Table 1.3: Dimensional reduction from M-theory branes to type IIA branes.

So far we have discovered the two charge-carrying fundamental objects of M-theory, called M2 and M5 branes. In the full theory they should be dynamical objects, but the quantisation of them is not as developed of that of fundamental strings and will not be discussed here.

Since the only scale of this theory is the eleven-dimensional Planck length, we conclude that their tensions are

$$\tau_{M2} \sim (\ell_P^{(11)})^{-3} \quad ; \quad \tau_{M5} \sim (\ell_P^{(11)})^{-6} \quad \Rightarrow \quad \tau_{M2}\tau_{M5} \sim (\ell_P^{(11)})^{-9} \sim \frac{1}{G_N^{(11)}} \quad , \quad (1.3.6)$$

This can also be checked from an explicit calculation using the gravity solution.

Now we would like to explore what they mean in type IIA string theory when we perform the dimensional reduction to ten dimensions, an operation valid when $g_s \ll 1$. From the map of the dimensional reduction (1.2.2) between the field contents, we can deduce a map between charged objects of the two theories. This is recorded in Table 1.3.

First we will explain the notations in the above table. The subscript “ \parallel ” denotes the field components or the M-theory branes with a leg (legs) or extent in the M-theory circle direction along which we dimensionally reduce the theory. Similar for the transversal direction “ \perp ”. Note that we leave the sources for the size of the circle direction ($G_{\parallel,\parallel}$) empty, since the asymptotic size of the eleventh-direction, or equivalently the ten-dimensional Newton’s constant, is a parameter of the theory from the ten-dimensional point of view.

Now we will explain the objects that appear in this table.

First we begin with the KK (Kaluza-Klein) monopole. It is the magnetic monopole with respect to the Kaluza-Klein gauge field $A_{\mu} = G_{\mu,\parallel}$, which is

a geometry having the structure as the product of a seven-dimensional flat Minkowski space and a four-dimensional Taub-NUT gravitational instanton. Its metric is

$$\begin{aligned} ds_{\text{T-N}}^2 &= V(\vec{x})d\vec{x} \cdot d\vec{x} + R^2 V^{-1}(\vec{x})(d\phi_{10} + \omega^0)^2 \\ V(\vec{x}) &= 1 + \frac{R}{|\vec{x}|} \\ d\omega^0 &= \star_3 dV \quad ; \quad \phi_{10} \sim \phi_{10} + 4\pi . \end{aligned} \tag{1.3.7}$$

This solution has a self-dual curvature two-form just like the usual Yang-Mills instantons, and therefore the name “gravitational instanton”⁵. The magnetic charge corresponding to this solution is given by

$$\int_{S^2} dA = - \int_{S^2} \star_3 dV = 1 . \tag{1.3.8}$$

The above structure can be easily generalised to obtain multi-instanton solutions. We will now digress to discuss them since they will also be needed in the subsequent parts of the thesis. But the reader can safely skip this part and return at any time.

Digression Some Gravitational Instantons⁶

Theorem 1.3.1 *Any hyper-Kähler four-manifold (four-dimensional Riemannian manifold with $Sp(1) \sim SU(2)$ holonomy) with a triholomorphic Killing vector, namely any Ricci-flat Riemannian four-manifolds with a Killing vector which preserves all three complex structures, must be of the following Gibbons-Hawking form [7, 8, 9]*

$$\begin{aligned} ds_{G-H}^2 &= H(\vec{x})d\vec{x} \cdot d\vec{x} + H^{-1}(\vec{x})(dx_5 + \omega^0)^2 \\ H(\vec{x}) &= h + \sum_{a=1}^n \frac{q_a}{|\vec{x} - \vec{x}_a|} \\ d\omega^0 &= \star_3 dH \quad . \end{aligned}$$

⁵Note that this is just an analogy. The term “gravitational instanton” is also used sometimes to refer to any four-dimensional Cauchy Riemannian manifold that is a solution to the vacuum Einstein’s equation, even if it does not satisfy the self-duality condition.

⁶Please see the next chapter for some of the background knowledge about classical geometry. An excellent review on the present subject of can be found in [6].

It has a self-dual curvature two-form and the anti-self-dual hyper-Kähler (three complex) structure

$$J^{(i)} = (dx_5 + \omega^0) \wedge dx^i - \frac{1}{2} \epsilon_{ijk} H dx^j \wedge dx^k . \quad (1.3.9)$$

Example Taub-NUT space

The above Taub-NUT metric (1.3.7) can be obtained by taking the special case

$$H(\vec{x}) = \frac{1}{R^2} \left(1 + \frac{1}{|\vec{x}|} \right) \quad (1.3.10)$$

and rescale the coordinates appropriately. The coordinate identification comes from requiring the absence of any Dirac-string-like singularity.

To make the isometry of this space manifest, it will be useful to introduce the SU(2) left- and right-invariant one-forms on the three sphere S^3 .

First observe that, parametrising \mathbb{C}^2 using the coordinates

$$\begin{aligned} z_1 &= \rho \cos \frac{\theta}{2} e^{i \frac{\psi + \phi}{2}} & ; & & z_2 &= \rho \sin \frac{\theta}{2} e^{i \frac{\psi - \phi}{2}} \\ \pi &\in [0, \pi], \quad \phi \in [0, 2\pi), \quad \psi \in [0, 4\pi) , \end{aligned}$$

the flat metric reads

$$ds_{\mathbb{R}^4}^2 = dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2 = d\rho^2 + \frac{\rho^2}{4} (d\theta^2 + d\phi^2 + d\psi^2 + 2 \cos \theta d\phi d\psi) .$$

A general SU(2) rotation takes the matrix form

$$U(\theta, \phi, \psi) = \frac{1}{\rho} \begin{pmatrix} z_1 & \bar{z}_2 \\ -z_2 & \bar{z}_1 \end{pmatrix} .$$

Furthermore, SU(2) acts on itself by left- and right- multiplication. We can decompose the left-invariant variation $U^{-1}dU$ and similarly the right-invariant variation $dU U^{-1}$ in the basis of Pauli-matrices and get the following right- and left-one forms

$$\begin{aligned} \sigma_{1,L} &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi \\ \sigma_{2,L} &= \cos \psi d\theta + \sin \psi \sin \theta d\phi \\ \sigma_{3,L} &= d\psi + \cos \theta d\phi \end{aligned} \quad (1.3.11)$$

and

$$\begin{aligned} \sigma_{1,R} &= \sin \phi d\theta - \cos \phi \sin \theta d\psi \\ \sigma_{2,R} &= \cos \phi d\theta + \sin \phi \sin \theta d\psi \\ \sigma_{3,R} &= d\phi + \cos \theta d\psi \end{aligned} \quad (1.3.12)$$

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satisfying

$$\begin{aligned} d\sigma_i^L &= \frac{1}{2}\epsilon_{ijk}\sigma_j^L \wedge \sigma_k^L \\ d\sigma_i^R &= -\frac{1}{2}\epsilon_{ijk}\sigma_j^R \wedge \sigma_k^R . \end{aligned}$$

We now see that the above metric of the flat \mathbb{R}^4 can be written as

$$\begin{aligned} ds^2 &= d\rho^2 + \frac{\rho^2}{4} (\sigma_{1,R}^2 + \sigma_{2,R}^2 + \sigma_{3,R}^2) \\ &= d\rho^2 + \frac{\rho^2}{4} (\sigma_{1,L}^2 + \sigma_{2,L}^2 + \sigma_{3,L}^2) . \end{aligned} \quad (1.3.13)$$

With the above form of the metric, it becomes manifest that \mathbb{R}^4 has a $SU(2)_L \times SU(2)_R$ symmetry generated by the following dual vectors of the above one-forms

$$\begin{aligned} \xi_{1,L} &= -\cot\theta \cos\psi \partial_\psi - \sin\psi \partial_\theta + \frac{\cos\psi}{\sin\theta} \partial_\phi \\ \xi_{2,L} &= -\cot\theta \sin\psi \partial_\psi + \cos\psi \partial_\theta + \frac{\sin\psi}{\sin\theta} \partial_\phi \\ \xi_{3,L} &= \partial_\psi \end{aligned} \quad (1.3.14)$$

and

$$\begin{aligned} \xi_{1,R} &= \cot\theta \cos\phi \partial_\phi + \sin\phi \partial_\theta - \frac{\cos\phi}{\sin\theta} \partial_\psi \\ \xi_{2,R} &= -\cot\theta \sin\phi \partial_\phi + \cos\phi \partial_\theta + \frac{\sin\phi}{\sin\theta} \partial_\psi \\ \xi_{3,R} &= \partial_\phi . \end{aligned} \quad (1.3.15)$$

Let's return to the Taub-NUT space. Now we can rewrite the metric (1.3.7) as

$$ds_{\text{T-N}}^2 = \left(1 + \frac{R}{r}\right) (dr^2 + r^2(\sigma_{1,L}^2 + \sigma_{2,L}^2)) + \left(1 + \frac{R}{r}\right)^{-1} R^2 \sigma_{3,L}^2 . \quad (1.3.16)$$

In this form it is manifest that the Taub-NUT space has an $U(2) = U(1)_L \times SU(2)_R$ symmetry generated by $\xi_{i,R}$ and $\xi_{3,L}$. The $\xi_{3,L}$ isometry has a single fixed point, called a “nut”, at $|\vec{x}| = 0$.

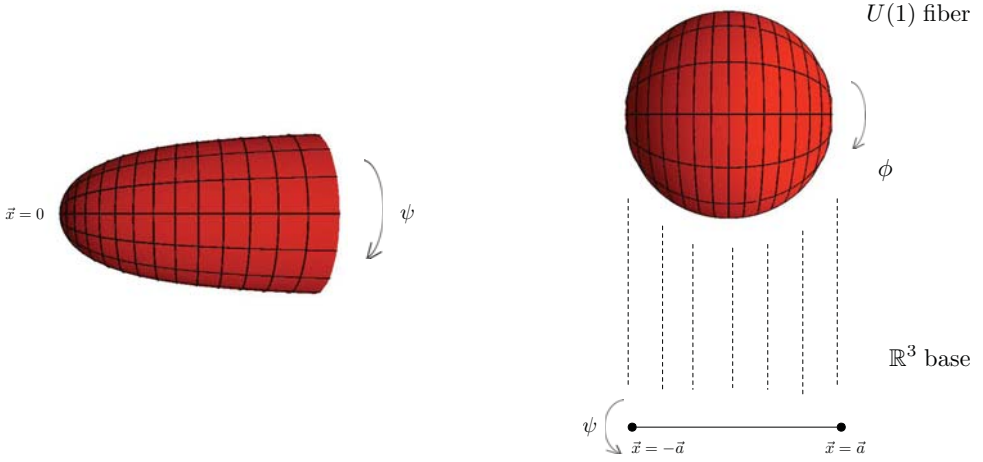


Figure 1.1: (a) The Taub-NUT geometry, with each point representing a two-sphere. (b) The two-cycle (the bolt) given by the fixed point of a $U(1)$ symmetry of the two-centered Gibbons-Hawking space, also known as the Eguchi-Hanson space.

Example Eguchi-Hanson space

Take the harmonic function in the Gibbons-Hawking Ansatz to be of a two-centered form

$$H(\vec{x}) = \frac{1}{|\vec{x} - \vec{a}|} + \frac{1}{|\vec{x} + \vec{a}|} \quad , \quad \vec{a} = a\hat{z} . \quad (1.3.17)$$

Using the elliptic coordinates

$$\begin{aligned} x &= a \sinh \eta \sin \theta \cos \psi \\ y &= a \sinh \eta \sin \theta \sin \psi \\ z &= a \cosh \eta \cos \theta , \end{aligned} \quad (1.3.18)$$

the harmonic function and the metric for the flat \mathbb{R}^3 base becomes

$$\begin{aligned} H &= 2 \frac{\cosh \eta}{\cosh^2 \eta - \cos^2 \theta} \\ a^{-2} ds_{\mathbb{R}^3}^2 &= (\cosh^2 \eta - \cos^2 \theta) (d\eta^2 + d\theta^2) + \sinh^2 \eta \sin^2 \theta d\psi^2 . \end{aligned}$$

The solution for ω^0 can then be solved to be

$$\omega^0 = 2 \frac{\sinh^2 \eta \cos \theta}{\cosh^2 \eta - \cos^2 \theta} d\psi .$$

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Define now $R = \sqrt{8a}$ and the new coordinates

$$\begin{aligned}\rho &= R \cosh^{1/2} \eta \\ x_5 &= 2\phi ,\end{aligned}$$

the metric takes the familiar form

$$ds_{\text{E-H}}^2 = \left(1 - \frac{R^4}{\rho^4}\right)^{-1} d\rho^2 + \frac{\rho^2}{4} (\sigma_{1,L}^2 + \sigma_{2,L}^2) + \frac{\rho^2}{4} \left(1 - \frac{R^4}{\rho^4}\right) \sigma_{3,L}^2 \quad (1.3.19)$$

$$R \leq \rho < \infty \quad \theta \in [0, \pi] \quad ; \quad \psi, \phi \in [0, 2\pi].$$

It has again a $U(1)_L \times SU(2)_R$ symmetry generated by $\xi_{i,R}$ and $\xi_{3,L}$. Actually the Eguchi-Hanson and the Taub-NUT spaces are the only non-flat, half-flat, asymptotically locally flat spaces with $U(2)$ symmetries. But unlike the Taub-NUT case, the $\xi_{3,L}$ isometry has now a S^2 surface of fixed point at $\rho = R$, called a “bolt”. The unusual identification of the ψ coordinate comes from requiring that the space approaches $\mathbb{R}^2 \times S^2$ without conical singularity near the fixed point $\rho = R$. Comparing the asymptotic form of the metric when $\rho \rightarrow \infty$ with the flat metric (1.3.13), we see that it is an asymptotically locally Euclidean (ALE) space, only locally because of the presence of the above-mentioned \mathbb{Z}_2 identification (A_1 in terms of the A-D-E classification).

After discussing the KK monopole, we now turn to the reduction of the M2 brane along the M-theory circle. From the table 1.3 we see that they must be the electric source of the anti-symmetric B-field. Recall that there is an electric coupling between the anti-symmetric B-field and the string world-sheet (1.2.7), we conclude that the M2 brane wrapping the M-theory circle must be reduced to the fundamental string we began with.

A circle-wrapping M2 brane is electric-magnetic dual to an M5 brane that is transversal to the M-theory circle. From the preservation of the electric-magnetic duality after the dimensional reduction, we know that such an M5 brane must reduce to the magnetic dual of the fundamental string. In type II string theory, the (5+1)-dimensional object dual to the fundamental string is called an NS5 brane, the name because it couples to the degrees of freedom coming from quantising the NS-NS sector of the type II strings. We therefore conclude that an M5 brane transversal to the Kaluza-Klein circle becomes a type IIA NS5 brane upon dimensional reduction.

Finally we turn to the Dp -branes, the official nickname for the $(p + 1)$ -dimensional Dirichlet branes. In the earlier discussion about T-duality we noted that this IIA-IIB duality and the presence of open strings implies the existence of some objects which couple to the Ramond-Ramond fields and on which the open string can end. Here we see we indeed get these objects in the spectrum, this time purely from the spacetime point of view. The Dp -branes obtained by dimensionally reducing the M-theory objects are exactly what we need.

In this table we leave out the end-of-the-world M-theory nine branes and the corresponding eight branes in type IIA. Although important for introducing gauge interactions into the theory, we will nowhere need them in the present thesis. The same will be true for the type IIB D7-branes. Albeit fascinating objects, they will play no role in our future discussions.

From the tension of the M-theory branes (1.3.6) and the map between ten- and eleven-dimensional units (1.2.9) and by carefully following the reduction procedure, we arrive at the following results for the tension of our newly discovered objects:

$$\tau_{Dp\text{-brane}} \sim g_s^{-1} \ell_s^{-(1+p)} \quad ; \quad \tau_{F1} \sim \ell_s^{-2} \quad (1.3.20)$$

and the rest follows from the relation

$$\tau_{(\text{object})} \tau_{(\text{E-M dual object})} \sim \frac{1}{G_N^{(10)}} \sim \ell_s^{-8} g_s^{-2}. \quad (1.3.21)$$

Now we have seen yet another reason why type IIA supergravity is not a good description when strings couplings are large. In this case all other objects are lighter (smaller tension) compared to the fundamental string, and it is therefore also not surprising that the degrees of freedom coming from quantising the fundamental string alone are not sufficient to account for the physics in that regime.

1.3.3 D-brane World-Volume Action

Besides the closed-string world-sheet action we wrote down earlier (1.2.7), for open strings we can add an extra boundary term

$$i \int_{\partial\Sigma} A^{(1)}$$

to the world-sheet action, since the world-sheet has in the open string case a (connected or disconnected) boundary. As we said before, the boundaries of open strings lie on the D-branes that we just introduced, the presence of

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this boundary coupling suggests that there is an $U(1)$ gauge field living on the D-branes. We would like to understand how the dynamics of D-branes, including the dynamics of this world-volume $U(1)$ gauge field, can be described by a world-volume action on the D-brane, in parallel with the way the string dynamics is captured by the string world-sheet action.

The action for D-branes is a very rich subject and as we won't need too much detail of it later, it will suffice just to have a pauper's account of the D-brane world-volume action here (not poor man's because K-theory, the framework needed to discuss the subject properly [10, 11] and which we will not introduce here, is a "poor man's derived category" [12]).

The basic strategy is to first find the right action for the massless open string modes limited on the D-brane world-volume. Let's first begin with the gravitational part. The gravitational coupling of the world-sheet action we used (1.2.7)

$$\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{h} h^{ab} g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$$

seems pretty hard to be generalised to a higher-dimensional world-volume. But in fact, we could have equally well begun with the Nambu-Goto string action

$$\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-\det(g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu)}, \quad (1.3.22)$$

whose equivalence with the original action can be shown by eliminating the non-dynamical world-sheet metric h_{ab} using its Euler-Lagrange equation. The Nambu-Goto action, on the other hand, can be generalised easily to higher dimensional object as

$$- \tau_p \int d^{p+1}\sigma e^{-(\Phi-\Phi_0)} \sqrt{\det G_{ab}}, \quad (1.3.23)$$

where $\Phi_0 = \log g_s$ is the asymptotic value of the dilaton, for we have absorbed this factor in the physical string tension τ_p derived earlier. The quantity G_{ab} is the pull-back of the metric under the embedding map of the D-brane. This action clearly has the geometric interpretation as the size of the D-brane given a specific embedding.

Next we turn to the gauge coupling. Consider the B-field coupling term

$$\frac{1}{2\pi\alpha'} \int_{\Sigma} B^{(2)}. \quad (1.3.24)$$

in the string world-sheet action (1.2.7), in the presence of world-sheet boundary, the usual gauge transformation

$$B^{(2)} \rightarrow B^{(2)} + d\epsilon^{(1)}$$

does not seem to be a symmetry anymore. But this can be repaired by a simultaneous gauge transformation of the $U(1)$ world-volume field

$$A^{(1)} \rightarrow A^{(1)} - \frac{1}{2\pi\alpha'} \epsilon^{(1)} .$$

Now we see that the gauge-invariant field combination is really

$$\mathcal{F} = B + 2\pi\alpha' F \tag{1.3.25}$$

and is therefore the only combination in which the B-field and the $U(1)$ field strength can appear in the D-brane world-volume action. This leads us to the following so-called Dirac-Born-Infeld action

$$S_{\text{D-B-I}} = -\tau_p \int d^{p+1} \sigma e^{-(\Phi - \Phi_0)} \sqrt{\det(G_{ab} + \mathcal{F}_{ab})} . \tag{1.3.26}$$

But this is obviously not the whole story. As we mentioned earlier, a higher-dimensional version of the Wilson line coupling

$$q_p \int C^{(p)} \tag{1.3.27}$$

should also be included.

But this is again not the full answer for the R-R coupling. Because the world-volume theory has to furthermore be anomaly-free. From the fact that D-branes act as sources for the gravitational and the p -form fields, in general we might expect there to be gauge and gravitational anomalous coupling. By considering two intersecting branes and requiring anomaly cancellation when the open-string zero-modes along the intersection sub-manifold are included, we get the following full ‘‘Chern-Simons’’ term in the world-volume action [10, 13]

$$q_p \int C \wedge ch(\mathcal{F}) \wedge \frac{\sqrt{\hat{A}(R_T)}}{\sqrt{\hat{A}(R_N)}} , \tag{1.3.28}$$

where $C = \sum_p C^{(p+1)}$ is the sum of all p -form fields in the theory, and the Chern character and the A-roof genus are defined and explained in (A.0.2) and (A.0.4). $R_{T,N}$ refers to the tangent and normal bundle of the world-volume manifold respectively. All this is of course only the bosonic part of the action, the supersymmetric action can be built by replacing the above bosonic fields by the appropriate superfields.

One very interesting consequence of this anomalous coupling is, a Dp -brane is not only the source of $C^{(p+1)}$ but can possibly also source other lower form fields.

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All this is for one single D-brane. In the case of N -coincident D-branes, things again become more complicated. Now both the gauge potential $A^{(1)}$ and the transverse coordinate X^i become $N \times N$ matrices and it's now not anymore so clear how one should pull-back the bulk fields in this non-commutative geometry. There is a generalisation of both the Dirac-Born-Infeld and the Chern-Simons part of the world-volume action, see for example [14, 15]. For the purpose of our discussion we will only need the leading in α' terms of the non-Abelian Dirac-Born-Infeld action, which reads

$$-\tau_p \frac{(2\pi\alpha')^2}{4} \int d^{p+1} \sigma e^{-(\Phi-\Phi_0)} \text{Tr} \left(F_{ab} F^{ab} + 2D^a X^i D_a X_i + [X^i, X^j]^2 \right) .$$

From this we see conclude that we have $U(N)$ but not just $U(1)^N$ world-volume field theory for N -coincident branes, with gauge coupling

$$g_{\text{Y-M}}^2 \sim g_s \tau_p^{-1} \alpha'^{-2} \sim g_s (\ell_s)^{p-3} . \quad (1.3.29)$$

1.3.4 Gauge/Gravity Correspondence

It is absolutely out of the scope of the present thesis to give a full account of the AdS/CFT correspondence. We will just sketch the ideas we will need later. Please see [16, 17, 18, 19] for reviews of the basic ideas (as opposed to the applications) of the correspondence.

As we mentioned earlier, from dimensional reducing (and taking the T-dual of) the M2- and M5-brane solutions in M-theory (1.3.2), (1.3.3) we obtain various extremal Dp -brane solutions of type IIA (IIB) string theory. From these solutions it is then not hard to see that the metric reduces to that of $AdS_{p+2} \times S^{D-p-2}$ for p -brane solutions in the theory of total spacetime dimensions D , when we zoom in the region $r \rightarrow 0$ near the location of the brane.

Let's take the D3-brane solution in type IIB string theory for example. This case is especially simple since the coupling constant of the D-brane world-volume (open string) theory is dimensionless (1.3.29), or, equivalently, that the dilaton of the spacetime solution is constant everywhere. There are apparently two different ways to describe the physics of this system; one of string theory and one of the D-brane theory. First of all, as we have seen earlier, each of them has its "low-energy" description, namely the supergravity and the $U(N)$ gauge theory for N coincident D-branes respectively. We would like to know when each of them is a valid description.

Let's begin with the gravitation side. First we note that, because of the infinite redshift factor $\sqrt{g_{tt}}$ near the horizon, classically the modes near the horizon $r \rightarrow 0$ never climb up the gravitational potential well and therefore

decouple from the rest of the spacetime. From the gravitational point of view it is thus valid to take the “decoupling limit” and focus on the near horizon geometry $AdS_5 \times S^5$. Secondly, from the relation between the radius of curvature and the string length and the ten-dimensional Planck length

$$\frac{R}{\ell_s} \sim \lambda^{1/4} \quad , \quad \lambda \equiv g_s N \sim g_{\text{Y-M}}^2 N$$

$$\frac{R}{\ell_{\text{P}}^{(10)}} \sim N^{1/4} \quad ,$$

we see that the α' corrections and the quantum gravitational effects are controlled by parameters λ and N respectively, and that the supergravity is a valid description if

$$\lambda \gg 1 \quad , \quad N \gg 1 .$$

On the open string side, for the gauge theory description to be valid we need $\alpha' \rightarrow 0$ while keeping the W-boson mass, proportion to r/α' , fixed. This leads us again to the near-horizon limit $r \rightarrow 0$ where we can consistently truncate the D-brane world-volume theory to $SU(N)$ gauge theory.⁷ Furthermore, it can be shown that the perturbative analysis of the $SU(N)$ Yang-Mills theory is valid when

$$\lambda \ll 1 .$$

Therefore, for large N , the supergravity theory on the $AdS_5 \times S^5$ background and the $SU(N)$ super-Yang-Mills theory discussed above are two effective theories describing the system at complementary regimes: the former valid when $\lambda \gg 1$ and the latter when $\lambda \ll 1$. This motivates the AdS/CFT conjecture, which in this specific case of D3 branes states that the ten-dimensional type IIB supergravity theory on the $AdS_5 \times S^5$ background is *dual* to the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory.

More generally, this conjecture says that the closed string theory on a $AdS_{p+2} \times K$ background is dual to a conformal field theory living on the conformal boundary $\partial(AdS_{p+2})$ of AdS_{p+2} . Or, a little bit more concretely, it states

$$Z_{\text{string}}(\phi_0) = \langle e^{\int \phi_{0,i} \mathcal{O}^i} \rangle ,$$

where ϕ_0 denotes the boundary condition of the fields on the conformal boundary and \mathcal{O}^i denote the dual operators in the CFT.

Of course this is an account of the conjecture at the level of caricature. There is first of all the issue of regularisation of the AdS space on the LHS of

⁷Notice that we have omitted the $U(1)$ part of the $U(N)$, which correspond to the center-of-mass degree of freedom and decouples from the rest of the theory.

the above equation. Secondly there are various interesting and useful generalisations of the above conjecture. This is by itself a vast subject and much more than what we will need later.

Instead we will simply remark that, first of all, the most remarkable aspect of this conjecture is that it relates a gravitational theory to a theory without gravity. In this sense having a theory of quantum gravity is not too different from solving the field theory. Secondly, this is the first full-blown example of the principle of holography [20, 21], motivated by the black hole thermodynamics, stating that the degrees of freedom of a $(d + 1)$ -dimensional gravitational theory is encoded in the d -dimensional boundary. In the AdS/CFT setting the extra dimension turns out to be the *scale* dimension.

1.3.5 S-duality

As some alert readers might have noticed, as we introduced the branes in subsection 1.3.2 by first presenting the two-brane and five-brane solutions in eleven-dimensional supergravity and then dimensionally reducing them, we actually haven't explicitly discussed the extended objects in the type IIB superstring theory. In this section we will study the type IIB branes by T-dualising Table 1.3 that appeared when we discussed the reduction of M-theory branes to type IIA objects, and show how a non-perturbative string duality can be revealed in this way.

As explained in subsection 1.1.3, the perturbative string T-duality maps type IIA string theory on a circle to type IIB string theory on a dual circle. Furthermore, it exchanges the Neumann with the Dirichlet boundary condition for the open strings, and therefore exchange D-branes of odd and those of even dimensions. Following the world-sheet discussion on the T-duality earlier it's not hard to see that it indeed maps the p -form field potentials under which the D-branes are charged accordingly.

We have also learned that M-theory compactified on a small circle is dual to type IIA string theory at weak coupling. Applying subsequently a T-duality, one is led to the conclusion that M-theory compactified on a torus is dual to type IIB theory on a circle. Following the Kaluza-Klein reduction and T-duality rules we can then trace the charged objects of the three different theory in a straightforward way. See Table 1.4 for the map under this duality chain. Here we call $S_{(1)}^1$ the M-theory circle and $S_{(2)}^1$ the T-duality circle. Branes wrapping at least one of the two circles will be labeled with the number in the parenthesis and “-” means they extend only in the directions transversal to both $S_{(1)}^1$ and $S_{(2)}^1$. The label (i,j) behind the KK-monopole denotes whether the solution is homogeneous along one of the circle directions (i) and under which Kaluza-Klein gauge field they are charged (j).

Table 1.4: From M- to type IIB theory. Extended charged objects.

M-theory	reduce on $S^1_{(1)} \rightarrow$	IIA	T-dualise along $S^1_{(2)} \rightarrow$	IIB
M5(1,2)		D4(2)		D3 (-)
M2(-)		D2(-)		D3(2)
M2(1,2)		F1(2)		p(2)
M2(1)		F1 (-)		F1(-)
M2(2)		D2(2)		D1(-)
p(2)		p(2)		F1(2)
p(1)		D0(-)		D1(2)
M5(2)		NS5(2)		NS5(2)
M5(1)		D4(-)		D5 (2)
KK(1,2)		KK(-,2)		NS5(-)
KK(2,1)		D6 (2)		D5(-)

Of course, if both circles are small, nothing can stop us from exchanging the two circles $S^1_{(1)}$ and $S^1_{(2)}$, which means we now first reduce along the second and then T-dualise along the first circle. From Table 1.4 we see something rather amusing: this simply exchange and fundamental with the D-string, and NS5 and D5 branes, while leaving D3 branes untouched! This exchange is actually a part of a much larger duality group, namely the modular group $PSL(2, \mathbb{Z})$ of the torus on which we compactify M-theory on. This means, apart from exchanging the two cycles, we can also consider an arbitrary change of basis. Let's begin with a torus, described as the complex plane \mathbb{C}^1 with the following identification

$$z \sim z + v_1 \sim z + v_2 . \quad (1.3.30)$$

A linear change of basis will take the form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} . \quad (1.3.31)$$

To preserve the identification (1.3.30) we should consider only integral changes of basis, and to keep the volume and orientation invariant we should have $\vec{v}_1 \times \vec{v}_2 = \text{Im}(\bar{v}_1 v_2)$ invariant. These conditions show that the modular group of a torus is

$$SL(2, \mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbb{Z} \right\} . \quad (1.3.32)$$



Figure 1.2: Three different models for the same hyperbolic space: the upper-half plane, the hyperboloid and the Poincaré disk. The red region drawn in the upper-half plane is a fundamental domain $\mathcal{H}_1/PSL(2, \mathbb{Z})$ under the modular group $PSL(2, \mathbb{Z})$.

It takes a point in the upper-half plane $\tau \in \mathcal{H}_1 = \{z \in \mathbb{C} | \text{Im}z > 0\}$ to another point in the upper-half plane by

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad (1.3.33)$$

where $\tau = \frac{\int_B dz}{\int_A dz}$ encodes the angle between the two one-cycles which we call the A - and the B -cycle, or in other words the complex structure of the torus. If we want to be more precise, notice that γ and $-\gamma = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ give the same map $\mathcal{H}_1 \rightarrow \mathcal{H}_1$, therefore the modular group of the complex structure of a torus is really $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/(\gamma \sim -\gamma)$.

Considering the mapping of the type IIB fields under the exchange of the two circles $S^1_{(1)}$ and $S^1_{(2)}$, which is the so-called S-transformation corresponding to the following $SL(2, \mathbb{Z})$ element

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we are led to the guess that the two-component field has to transform as

$$B = \begin{pmatrix} C^{(2)} \\ B^{(2)} \end{pmatrix} \rightarrow \gamma B$$

when the circles are changed as (1.3.31), while the chiral four-form potential remains invariant. Indeed, the low energy type IIB supergravity has an even larger symmetry which is broken at the quantum level to $SL(2, \mathbb{Z})$. To see

this, defining also the field combination, the “axion-dilaton”, as

$$\lambda = \lambda_1 + i\lambda_2 = C_0 + ie^{-\Phi} \quad , \quad \mathcal{T} = \frac{1}{\lambda_2} \begin{pmatrix} |\lambda|^2 & \lambda_1 \\ \lambda_1 & 1 \end{pmatrix} . \quad (1.3.34)$$

Then it’s not hard to see that the IIB supergravity action (1.2.5) can be rewritten as

$$\begin{aligned} 2\kappa^2 S^{(\text{IIB})} &= \int d^{10}x \sqrt{-g} \left(R - \frac{1}{12} H_{\mu\nu\rho}^T H^{\mu\nu\rho} + \frac{1}{4} \text{Tr}(\partial^\mu \mathcal{T} \partial_\mu \mathcal{T}^{-1}) - \frac{1}{4} |\tilde{F}^{(5)}|^2 \right) \\ &\quad - \frac{1}{4} \int C^{(4)} \wedge B^T \wedge SB \end{aligned} \quad (1.3.35)$$

where we have gone to the Einstein frame by rescaling the metric as $g_{\mu\nu} \rightarrow e^{-\Phi/2} g_{\mu\nu}$ to isolate the Φ dependence in the axion-dilaton combination λ . From

$$\mathcal{T} \rightarrow \gamma \mathcal{T} \gamma^T \quad \text{when} \quad \lambda \rightarrow \frac{a\lambda + b}{c\lambda + d} , \quad (1.3.36)$$

and the fact that $SL(2, \mathbb{Z}) \cong Sp(2, \mathbb{Z})$, namely $\gamma S \gamma^T = S$, we see that the above action is manifestly invariant under the S-duality transformation

$$\begin{aligned} B &\rightarrow \gamma B \quad , \quad \lambda \rightarrow \frac{a\lambda + b}{c\lambda + d} \\ g_{\mu\nu} &\rightarrow g_{\mu\nu} \quad , \quad C^{(4)} \rightarrow C^{(4)} . \end{aligned}$$

Note that at this level we don’t have any reason to require $\gamma \in SL(2, \mathbb{Z})$. Any $\gamma \in SL(2, \mathbb{R})$ is sufficient to ensure the invariance of the above supergravity action. But since we have seen the geometric origin of this symmetry from our M-theory derivation, we conclude that the real symmetry group should be the discrete torus modular group $PSL(2, \mathbb{Z})$. Or, another way to see this is the Dirac quantisation condition (1.3.5) which has to be satisfied by branes and strings. We simply cannot map one D5 brane to “0.32 D5 + 6.7292 NS5” branes without destroying the Dirac quantisation.

Notice that, unlike the T-duality, this “S-duality” is non-perturbative by nature since we can map from small g_s to the large coupling regime. Especially, from the brane tensions (1.3.20) and (1.3.21) we see that D1 string becomes the light degrees of freedom instead of the fundamental string after the S-transformation. And similarly for NS5 and D5 branes. This duality suggests that various different objects in string theory should probably be treated at the equal footing, and string theory is really about “strings” only at a corner of the moduli space of the theory.

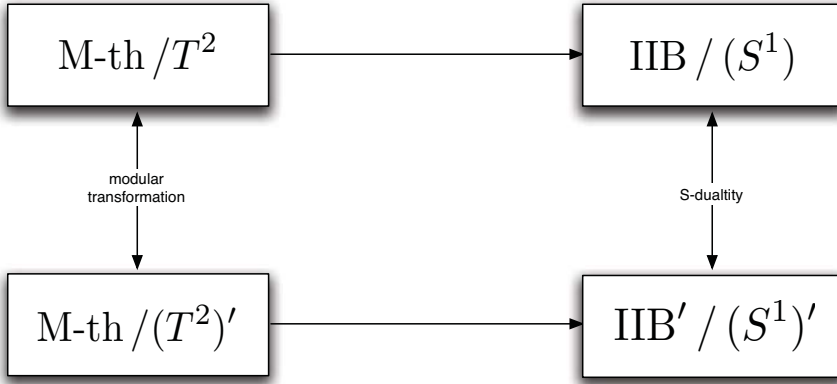


Figure 1.3: S-duality as the modular group of M-theory torus.

Digression Upper Half-Plane

The upper-half plane is defined in the obvious way as

$$\mathcal{H}_1 = \{ \tau \in \mathbb{C} \mid \text{Im}\tau > 0 \} . \tag{1.3.37}$$

As depicted in Figure 1.2, it is equivalent to the hyperbolic space, namely the Euclidean AdS_2 space defined as

$$-T^2 + X_1^2 + X_2^2 = -1 \ , \ T > 0 \ , \tag{1.3.38}$$

and to the Poincaré Disk

$$\{ z \in \mathbb{C} \mid |z|^2 < 1 \} \tag{1.3.39}$$

and also to the coset space

$$\frac{SL(2, \mathbb{R})}{U(1)} . \tag{1.3.40}$$

This can be seen using the map

$$\begin{aligned} \mathcal{T} &= \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} T + X_1 & X_2 \\ X_2 & T - X_1 \end{pmatrix} \\ &= \frac{2}{\sqrt{3}} \frac{1}{1 - |z|^2} \begin{pmatrix} |z + e^{\frac{5\pi i}{6}}|^2 & \sqrt{3}\text{Re}z - \frac{1}{2}|z - i|^2 \\ \sqrt{3}\text{Re}z - \frac{1}{2}|z - i|^2 & |z - i|^2 \end{pmatrix} \\ &= \xi^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xi \ , \ \xi \sim \xi_0(\theta)\xi \in SL(2, \mathbb{R}) \ , \end{aligned} \tag{1.3.41}$$

$$\tag{1.3.42}$$

where

$$\xi_0(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (1.3.43)$$

is any element of the $U(1)$ group that stabilises the point $\tau = i$. Notice that the above map is of course not unique, since one can use the $SO(2, 1)$ symmetry of the space to obtain other equivalent maps. For instance, the complicated-looking map given above between the Poincaré disk and the upper half plane is in fact simply the Möbius transformation

$$z = i \left(\frac{\tau + e^{-\frac{i\pi}{3}}}{\tau + e^{\frac{i\pi}{3}}} \right),$$

but any other Möbius transformation of the form

$$z = e^{i\phi} \left(\frac{\tau + \bar{\tau}_0}{\tau + \tau_0} \right), \quad \tau_0 \in \mathcal{H}_1 \quad (1.3.44)$$

will equally do the job.

All these four different models for the space \mathcal{H}_1 will later make their appearances in different part of the thesis.
