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The spectra of supersymmetric states in string theory

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Calabi-Yau Compactifications

In this chapter we will discuss superstring theories compactified on Calabi-Yau three-folds, leading to $\mathcal{N} = 2$ supersymmetry in four dimensions. Some discussions of the basic properties of these manifolds can be found in appendix A. We will begin with a world-sheet analysis, meaning studying the (2,2) superconformal theory, describing a string moving in the Calabi-Yau space. Along the way we will introduce various concepts useful for studying the ground state of a supersymmetric conformal theory, which we will often rely on for the studying of supersymmetric spectrum of a black hole system in string theory.

On the other hand, the geometric intuition will also be indispensable for understanding the compactified string theory. We will therefore switch to a spacetime perspective after basic concepts have been introduced from a world-sheet viewpoint. In particular we will discuss the structure of the geometric moduli space of the Calabi-Yau manifolds in details.

After that we are ready to introduce the low energy effective actions in lower dimensions, and discuss their range of validity.

2.1 (2,2) Superconformal Field Theory

In the beginning part of the thesis we have introduced the superstring theory as a two-dimensional conformal theory, considered to have critical central charge equals to 15 and thus correspond to a total of ten spacetime dimensions. So-called compactification, can therefore be thought of having a product CFT with a factor with central charge $c = 6$ (four spacetime dimensions) and an “internal” factor with central charge $c = 9$. Furthermore, as we have seen in the superstring example, world-sheet supersymmetries are intimately linked to spacetime supersymmetries. Since we would like to end up with a lower-dimensional theory with unbroken spacetime supersymmetry, as we will see shortly it turns out that choosing the internal CFT to have (2,2) world-sheet

supersymmetry will serve the purpose. In other words we have a total string theory of $M_4 \times [(2, 2), c = 9]$ in mind and will now concentrate on the latter “internal” part.

2.1.1 $\mathcal{N} = 2$ Superconformal Algebra

The two-dimensional $\mathcal{N} = 2$ superconformal algebra is rather similar to the $\mathcal{N} = 1$ version we have seen in (1.1.10). The extra supersymmetry means the presence of the second superconformal current G , and an R-current J under which they are charged. In terms of their Fourier modes the algebra reads

$$\begin{aligned}
 [L_m, L_n] &= (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \\
 [J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0} \\
 [L_n, J_m] &= -mJ_{m+n} \\
 [L_n, G_r^\pm] &= \left(\frac{n}{2} - r\right)G_{r+n}^\pm \\
 [J_n, G_r^\pm] &= \pm G_{r+n}^\pm \\
 \{G_r^+, G_s^-\} &= 2L_{r+s} + (r - s)J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}
 \end{aligned}$$

and as before we have two possible periodic conditions for the fermions

$$\begin{cases} 2r = 0 \pmod{2} & \text{for R sector} \\ 2r = 1 \pmod{2} & \text{for NS sector.} \end{cases} \quad (2.1.1)$$

Example Consider the *non-linear sigma model* with action

$$\begin{aligned}
 \mathcal{S} &= \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2z \left(\frac{1}{2}g_{\mu\nu}(X)\partial X^\mu\bar{\partial}X^\nu + g_{\mu\nu}(X)(\tilde{\psi}^\mu\bar{D}\tilde{\psi}^\nu + \psi^\mu D\psi^\nu) \right. \\
 &\quad \left. + \frac{1}{4}R_{\mu\nu\rho\sigma}\tilde{\psi}^\mu\tilde{\psi}^\nu\psi^\rho\psi^\sigma \right), \quad (2.1.2)
 \end{aligned}$$

where D and \bar{D} is the holomorphic and anti-holomorphic pull-back of the covariant derivative with respect to the metric $g_{\mu\nu}$.

For this action to have (2,2) supersymmetry, the kinetic term and its supersymmetric partner must be able to be written in the superspace form as

$$\mathcal{S} = -\frac{1}{8\pi\alpha'} \int d^2z d^4\theta K(\Phi^i, \bar{\Phi}^{\bar{i}}), \quad (2.1.3)$$

where Φ^i and $\bar{\Phi}^{\bar{i}}$ are the chiral and anti-chiral superfields, satisfying

$$\mathcal{D}_+ \Phi^i = \bar{\mathcal{D}}_+ \bar{\Phi}^{\bar{i}} = 0$$

and the opposite R-charge counterpart for $\bar{\Phi}^{\bar{i}}$. And we have used here the convention

$$\mathcal{D}_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + \theta^{\mp} \partial \quad ; \quad \mathcal{Q}_{\pm} = \frac{\partial}{\partial \theta^{\pm}} - \theta^{\mp} \partial$$

and their holomorphic counterpart.

In the superspace form it is immediately clear that the presence of (2,2) supersymmetry imposes the Kählerity condition on the target space. The action (2.1.2) is then indeed equivalent to (2.1.3) with the Kähler metric $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$.

Moreover, as in (1.2.7) we can include in the action the topological coupling to the B-field

$$2\pi i \int B^{(2)}. \quad (2.1.4)$$

We say this coupling is topological because the action only depends on the cohomology classes of $B^{(2)}$. Furthermore, since an action always appears in the path integral in the form of e^{-S} , we conclude that shifting the B-field by an element of the integral cohomology classes of the target space must be a symmetry of the theory.

Apart from supersymmetry we would like to require conformal symmetry as well. As we have seen in the superstring case, at the leading order of α' , the vanishing of the beta function imposes that the target space is Ricci flat, at least in the absence of dilaton gradient or the H-flux, which we will not consider in the present thesis. Recalling the fact that a compact manifold admitting a Ricci flat metric must be a Calabi-Yau manifold (A.0.13), the conformal symmetry of the superconformal field theory imposes the Calabi-Yau condition on the target space.

We would like to stress that this theory is by far not the only possible “internal CFT” on one can compactify the superstring on. But it is certainly an obvious candidate and indeed leads to rich structure and analytic control.

2.1.2 Chiral Ring

To build up a representation for the above algebra, just like in the $\mathcal{N} = 1$ case we are especially interested in the “highest weight state” annihilated by all the positive modes

$$L_m |\phi\rangle = J_n |\phi\rangle = G_r^{\pm} |\phi\rangle = 0 \quad \text{for all } n, m, r > 0. \quad (2.1.5)$$

The reason for this is that we can build a representation by acting with creation operators on these highest weight states. We say they have conformal weight h and charge q if they have eigenvalues $L_0|\phi\rangle = h|\phi\rangle$, $J_0|\phi\rangle = q|\phi\rangle$ under the zero index operators. In the context of state-field correspondence, a highest weight state is said to be created by a “primary field” ϕ , such that $|\phi\rangle = \phi|0\rangle$.

Analogous to the case of superstring, more care should be taken for the Ramond sector, because in this case there exist zero index fermionic modes G_0^\pm . We will call a state an R-ground state if $G_0^\pm|\phi\rangle = 0$. From the $\{G_0^+, G_0^-\}$ commutation relation we see that R-ground states always have the weight

$$h(\text{Ramond ground state}) = \frac{c}{24}. \quad (2.1.6)$$

Furthermore, from the hermiticity condition $(G_r^\pm)^\dagger = G_{-r}^\mp$, we see that the above is also a sufficient condition that the state is an R-ground state. For the NS sector, it will turn out to be useful to further refine the concept of primary fields into chiral primary fields. A field is called a chiral primary field if it is a primary field which satisfies the condition

$$G^+(z)\phi_c(w) \sim \text{regular}, \quad (2.1.7)$$

or, using the mode expansion $G^\pm(z) = \sum_r G_r^\pm z^{-r-\frac{3}{2}}$, in the operator language the above equation is equivalent to

$$G_{-1/2}^+|\phi_c\rangle = 0. \quad (2.1.8)$$

Furthermore, from the $\{G_{-1/2}^+, G_{1/2}^-\}$ and $\{G_{-3/2}^+, G_{3/2}^-\}$ commutation relations, we see that the conformal weight and the R-charge of a chiral primary satisfy

$$0 \leq h_c = \frac{q_c}{2} \leq \frac{c}{6}. \quad (2.1.9)$$

From the OPE between two chiral primaries and the conservation of the R-charge, it's not hard to see that the product also satisfies the chiral primary condition $h = \frac{q}{2}$. This suggests that chiral primary fields form a ring, called the “chiral ring”, with the Yukawa coupling given by

$$\phi_{i,c}\phi_{j,c} = C^k{}_{ij}\phi_{k,c}. \quad (2.1.10)$$

Similarly, there is also an anti-chiral ring, with anti-chiral primaries defined as

$$G_{-1/2}^-|\phi_a\rangle = 0. \quad (2.1.11)$$

and satisfies

$$0 \leq h_a = -\frac{q_a}{2} \leq \frac{c}{6}. \quad (2.1.12)$$

Combining the left- and right-moving sector, we have the following four rings (c,c), (a,c), (c,a), (a,a) in a (2,2) superconformal field theory. Their significance for us will be illustrated in the following non-linear sigma model example.

Example It is a usual phenomenon that the supersymmetric ground states of a theory are given by the cohomology of a relevant space, at least in the limit in which the string tension is large in the case of 2-d CFT. We will now see how this comes about in our non-linear sigma model example. We have seen in the last subsection that the consistency of the CFT requires the target space to be Calabi-Yau, and will therefore assume that this is the case in the following discussion.

As in the case of superstring theory we have seen before, the Ramond ground states are spacetime fermions with definite chirality. In the non-linear sigma model example, using the Kählerity of the target space M , or equivalently $\mathcal{N} = 2$ supersymmetry, we have an extra grading on these spacetime fermions. In other words, writing out the action (2.1.3) in components and from the supersymmetry transformation we read out the action of the zero modes of the world-sheet current

$$G_0^+ = \psi^i D_i \quad ; \quad G_0^- = \psi^{\bar{i}} D_{\bar{i}} \quad (2.1.13)$$

and similar for the right-moving part. From the chiral and anti-chiral multiplet structure it is easy to see that the fermionic fields carry the following R-charges under (J, \tilde{J}) :

$$\begin{aligned} \psi^i & (1, 0) \\ \psi^{\bar{i}} & (-1, 0) \\ \tilde{\psi}^i & (0, 1) \\ \tilde{\psi}^{\bar{i}} & (0, -1) . \end{aligned} \quad (2.1.14)$$

From quantising the fermions

$$\{\psi^\mu, \psi^\nu\} = g^{\mu\nu} , \quad (2.1.15)$$

we can choose ψ^i to be the creation and $\psi^{\bar{i}}$ the annihilation operators on the left-moving side. This choice amounts to a choice of the chirality of the Weyl spinors. Similarly one can now choose the right-moving ground states to have the same or the opposite chirality, namely, apart from $\psi^{\bar{i}}|0; 0_\pm\rangle = 0$ we also impose $\tilde{\psi}^{\bar{i}}|0; 0_+\rangle = 0$ or $\tilde{\psi}^i|0; 0_-\rangle = 0$.

In the first case we see that the ground states correspond to the cohomology class $H^{n-r,s}(M)$

$$\Omega_{\bar{1}\dots\bar{n}} f^{\bar{j}_1\dots\bar{j}_r} \tilde{\psi}_{i_1\dots i_s} \tilde{\psi}_{\bar{j}_1} \dots \tilde{\psi}_{\bar{j}_r} \psi^{i_1} \dots \psi^{i_s} |0; 0_+\rangle . \quad (2.1.16)$$

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where we have used the unique harmonic $(0, n)$ form of the Calabi-Yau (A.0.14) to lower the indices. From the index structure one can see that these states have the same sign for the R-charges on the left- and right-moving sides.

We will see in the next subsection that there is a symmetry of the superconformal algebra which relates R-ground states to NS chiral primary fields. In particular, the R-ground states discussed above correspond to (c, c) fields with conformal weights and R-charges equal to

$$(2h, 2\tilde{h}) = (q, \tilde{q}) = (s, r) . \quad (2.1.17)$$

Similarly, there are also ground states of the following form

$$f_{\bar{j}_1 \dots \bar{j}_r i_1 \dots i_s} \tilde{\psi}^{\bar{j}_1} \dots \tilde{\psi}^{\bar{j}_r} \psi^{i_1} \dots \psi^{i_s} |0; 0_- \rangle . \quad (2.1.18)$$

They correspond to the cohomology class $H^{r, s}(M)$ and the corresponding NS-NS fields are (c, a) fields with

$$(2h, -2\tilde{h}) = (q, \tilde{q}) = (s, -r) . \quad (2.1.19)$$

Notice that since $c = 3n$ for n -complex-dimensional target space, the (anti)chiral primary condition $|q| \leq \frac{c}{3} = n$ (2.1.9), (2.1.12) is indeed in accordance with the correspondence between the CFT chiral ring and the cohomology ring of the target space.

Let's now focus on the case of Calabi-Yau three-folds. We are especially interested in the ring elements with $h + \tilde{h} = 1$. This is because, by combining them with the appropriate superconformal currents G , we can build from them marginal operators with total conformal weight 2 and which are neutral under R-symmetry. These marginal operators can be then used to deform the superconformal theory. Here we shall remind the readers that marginal operators are the operators that don't become trivial nor dominant when flowing to the IR fixed point, and can therefore be thought of as taking a conformal field theory to another "nearby" CFT.

From the above analysis we see that in the CY three-fold case, the marginal operators are given by the elements in the (c, c) ring corresponding to elements in $H^{2, 1}(M)$ and those in (c, a) ring corresponding to elements in $H^{1, 1}(M)$. The former ones correspond to deforming the complex structure (the shape) of the Calabi-Yau manifold, and the second one the Kähler form (the size) of it. This can be seen from the expression for their corresponding harmonic forms (2.1.16) and (2.1.18) as follows. Roughly speaking, they correspond to the complex structure and the Kähler part of the metric deformation:

$$f_{ij} \sim \delta g_{ij} \quad ; \quad f_{i\bar{j}} \sim \delta g_{i\bar{j}} . \quad (2.1.20)$$

ring	cohomology	deformation
(c,c)	$H^{2,1}(M)$	complex structure
(c,a)	$H^{1,1}(M)$	Kähler form

Table 2.1: Summary of the relation between chiral rings, marginal deformation and the cohomology class of the target space.

It will turn out to be important to have some further knowledge about the structure of the space of all deformations, namely the moduli space of the theory. In principle we can now compute the moduli space metric from the OPE's of the marginal operators (the Zamolodchikov metric). First of all it's easy to show that the space of the (c,c) and the (c,a) part of the deformation is locally a direct product. Namely

$$\mathcal{M} = \mathcal{M}_{\text{complex}} \times \mathcal{M}_{\text{Kähler}} \quad (\text{locally}) .$$

Furthermore, it can be shown that, by employing the tt^* equation for example [22], the moduli spaces are of the special Kähler kind which we defined in (A.0.12). But since we will need the geometric picture repeatedly in later analysis, we will postpone the derivation and later derive it in a way that makes its geometric meaning directly manifest.

2.1.3 Spectral Flow

Another important property of the $\mathcal{N} = 2$ superconformal algebra is that it has an inner automorphism, which means that the algebra remains the same under the following redefinition

$$\begin{aligned} L_n &\rightarrow L_n + \eta J_n + \eta^2 \frac{c}{6} \delta_{n,0} \\ J_n &\rightarrow J_n + \eta \frac{c}{3} \delta_{n,0} \\ G_r^\pm &\rightarrow G_{r \pm \eta}^\pm . \end{aligned} \tag{2.1.21}$$

An isomorphism of the algebra implies that of a representation. Namely, when the transformation of the operators is induced by a similarity transformation $\hat{\mathcal{O}} \rightarrow U \hat{\mathcal{O}} U^{-1}$, then there is a corresponding transformation of the representation $|\phi\rangle \rightarrow U|\phi\rangle$.

Now we will transform the states in the way mentioned above by means of a vertex operator insertion

$$U_\eta = e^{-i\eta \sqrt{\frac{c}{3}} H} , \tag{2.1.22}$$

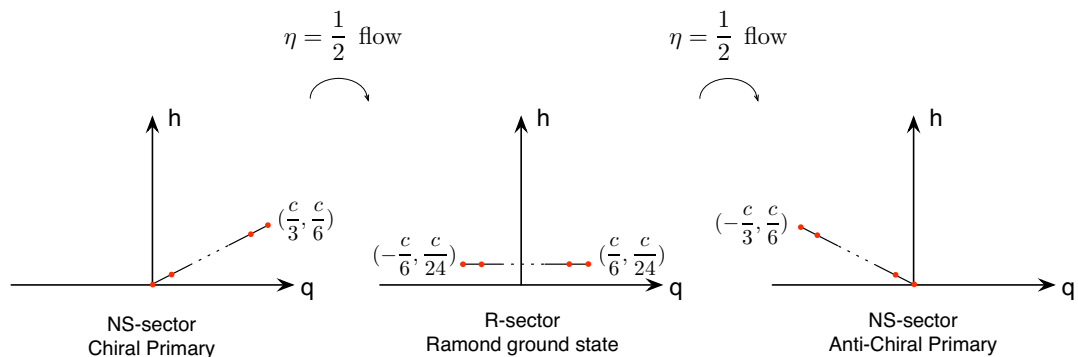


Figure 2.1: The relationship between the NS- and the R- sector ground states under the spectral flow[23].

where H is the free boson from bosonising the R-current

$$J(z) = i\sqrt{\frac{c}{3}}\partial H. \quad (2.1.23)$$

The net effect is just to shift the $U(1)$ charge of every state by $-\eta\frac{c}{3}$, and to change the periodicity of the fermionic current $G^\pm(z)$.

To sum up, beginning with a state $|\phi\rangle$ with weight h and R-charge q , the state $U_\eta|\phi\rangle$ transformed by $U_\eta : \mathcal{H} \rightarrow \mathcal{H}_\eta$ will have

$$\begin{aligned} h_\eta &= h - \eta q + \frac{c}{6}\eta^2 \\ q_\eta &= q - \frac{c}{3}\eta. \end{aligned} \quad (2.1.24)$$

This operation is called the “spectral flow” of $\mathcal{N} = 2$ CFT relating different representations of the algebra. In particular, as promised before, this symmetry relates the R-ground states to the NS sector (anti-)chiral primary states. Indeed, from the above relation (2.1.24) it’s easy to see that a $\eta = 1/2$ flow takes chiral primary states to Ramond ground states, and another $\eta = 1/2$ flow takes them again to anti-chiral primary states, and vice versa for the $\eta = -1/2$ flows. In this sense there is really a unique notion of “ground states” in $\mathcal{N} = 2$ superconformal field theories.

2.1.4 Topological String Theory

In this part of the discussion, we will concentrate on our main example, namely the Calabi-Yau sigma model with $c = 9$. We will be very schematic on this

subject, since this is not our main topic of interest and also because there exists already a fair amount of excellent summary and review literature on the topic. See for example [24, 25, 26].

As we have just discussed, a field $\phi_{(c,a)}$ in the (c,a) chiral-anti-chiral ring satisfies

$$(G^+ + \tilde{G}^-)\phi_{(c,a)} \sim 0 \quad (2.1.25)$$

and accounts for the deformation of the Kähler moduli in the target Calabi-Yau space. Furthermore the operator which annihilates it satisfies the nilpotency condition $(G^+ + \tilde{G}^-)^2 \sim 0$. Similarly a chiral-chiral primary field $\phi_{(c,c)}$ satisfies

$$(G^+ + \tilde{G}^+)\phi_{(c,c)} \sim 0 \quad (2.1.26)$$

with $(G^+ + \tilde{G}^+)^2 \sim 0$ and accounts for the deformation of the complex structure moduli.

A question one might ask now is: since we will be mainly interested in this part of the theory, why not use the nilpotent operators as BRST operators and focus on the BRST cohomology? But we need one more step before this can be done, since G^\pm and \tilde{G}^\pm have conformal weights 3/2 but we need fields of weight 1 so that we can integrate them around a loop on the world-sheet to get a conserved charge. This is where the spectral flow property of the theory comes to help. By introducing a coupling term

$$\pm \frac{1}{2} \int_{\Sigma} \omega J = \pm \frac{i\sqrt{3}}{2} \int_{\Sigma} \omega \partial H , \quad (2.1.27)$$

with ω being the spin connection, we can “twist” the theory by

$$T \rightarrow T \mp \frac{1}{2} \partial J , \quad (2.1.28)$$

and especially

$$L_0 \rightarrow L_0 \pm \frac{1}{2} J_0 . \quad (2.1.29)$$

This twist shifts one of the two G^\pm to have dimension (1,0) and the other dimension (2,0), depending on the sign of the twist. Choosing the opposite (same) sign for the left- and right-movers, we obtain the so-called A- (B-) model topological string theory, with the original (c,a) ((c,c)) ring as now the BRST cohomology. This is summarised in Table 2.2.

It is not difficult to check that the energy-momentum tensor is Q-exact, which means the theory is invariant under a continuous change of world-sheet metric and therefore the name “topological”. But this is not yet the whole story. In order to get an interesting theory we still have to couple it

top. string	cohomology	deformation
B-model	$H^{2,1}(M)$	complex structure
A-model	$H^{1,1}(M)$	Kähler form

Table 2.2: Summary of the relation between BRST cohomology of the A- and B-model topological string theory and the cohomology class of the target space.

to “topological gravity” on the world-sheet [27], namely to sum over classes of conformally inequivalent metrics. For concreteness let’s now focus on a $L_0 \rightarrow L_0 - \frac{1}{2}J_0$ twist. In this case the BRST-charge $Q = \oint G^+$ and G^- satisfies $T \sim \{Q, G^-\}$ and $JG^\pm = \pm G^\pm$, therefore the U(1) current now plays the role of the conserved current for the ghost number and G^- that of the anti-ghost. Coupling to topological gravity in this case then follows in close analogy with the procedure of computing higher genera amplitudes of bosonic string theory. We refer the reader who needs more background on this topic to, for example, [1]. Recall that the higher-genus vacuum has anomalous ghost number $-3\chi = 6g - 6$ [28] due to the presence of a non-trivial moduli space for the genus- g Riemann surface, which has $\dim_{\mathbb{C}}\mathcal{M}_g = 3g - 3$. We need therefore $3g - 3$ insertions of anti-ghost on each (left- and right-moving) sector to produce a ghost-neutral amplitude. Or said in another way, to produce the correct measure factor for the moduli space. Therefore we define the genus- g amplitude for topological strings to be

$$F_g = \int_{\mathcal{M}_g} \left\langle \prod_{i=1}^{3g-3} G^-(\mu_i) \tilde{G}^\pm(\mu_i) \right\rangle, \quad (2.1.30)$$

where the plus (minus) sign corresponds to the A- (B-)model and μ_i stands for the Beltrami differential. Now we are ready to define the (perturbative) topological strings partition function as

$$Z_{\text{top}} := \exp(F_{\text{top}}) = \exp\left(\sum_{g=0}^{\infty} g_{\text{top}}^{2g-2} F_g\right), \quad (2.1.31)$$

where g_{top} will be referred to as the topological string coupling constant.

For future use let’s also discuss here the expansion of the above partition function. As we have mentioned above, the A-model partition function is, loosely speaking, a function of the Kähler moduli $t^A = B^A + iJ^A$, in a notation that will be explained in more detail later (2.2.14). Around the semi-classical limit $t \rightarrow \infty$, or the target space large-volume limit, the A-model free energy

has the following expansion[29, 30]¹

$$F_{\text{top}} = F_{\text{pert}} + F_{\text{GW}}^{(0)} + F_{\text{GW}} \quad (2.1.32)$$

$$F_{\text{pert}} = -i \frac{(2\pi)^3}{6g_{\text{top}}^2} D_{ABC} t^A t^B t^C - \frac{2\pi i}{24} c_{2A} t^A \quad (2.1.33)$$

$$F_{\text{GW}}^{(0)} = -\frac{1}{2} \chi \log[M(e^{-g_{\text{top}}})]$$

$$\begin{aligned} F_{\text{GW}} &= \sum_{\beta \in H^2(X, \mathbb{Z})} \sum_{g \geq 0} N_{g, \beta} g_{\text{top}}^{2g-2} e^{2\pi i \beta \cdot t} = F_{\text{GV}} = \log Z_{\text{GV}} \\ &= \sum_{\beta \in H^2(X, \mathbb{Z})} \sum_{g \geq 0} \sum_{m \in \mathbb{N}} \alpha_g^\beta \frac{1}{m} \left(2i \sinh\left(\frac{m g_{\text{top}}}{2}\right) \right)^{2g-2} e^{2\pi i m \beta \cdot t} \\ &= \log Z_{\text{DT}} \end{aligned}$$

$$Z_{\text{DT}} = \sum_{\beta \in H^2(X, \mathbb{Z})} \sum_{m \in \mathbb{Z}} n_{\text{DT}}(\beta, m) (-e^{-g_{\text{top}}})^m e^{2\pi i \beta \cdot t},$$

where

$$M(q) = \prod_{n \geq 1} (1 - q^n)^n \quad (2.1.34)$$

is the MacMahon function and $\chi = \chi(X)$ is the Euler characteristic of the target Calabi-Yau space X . $N_{g, \beta}$ is called the Gromov-Witten invariants which are rational numbers counting holomorphic curves, while the α_g^β and $n_{\text{DT}}(\beta, m)$, called the Gopakumar-Vafa [31, 32] and the Donaldson-Thomas invariants respectively, have the physical interpretation of counting wrapped M2 branes and D6-D2-D0 bound states respectively.

Especially, one can show from the above formula that the Gopakumar-Vafa partition function Z_{GV} takes the following suggestive product form [32, 33]

$$\begin{aligned} Z_{\text{GV}}(g_{\text{top}}, t) &= \prod_{\beta \in H^2(X, \mathbb{Z})} \left(\prod_{r=1}^{\infty} \left(1 - e^{-r g_{\text{top}}} e^{2\pi i \beta \cdot t} \right)^{r \alpha_0^\beta} \right. \\ &\quad \left. \times \prod_{g=1}^{\infty} \prod_{\ell=0}^{2g-2} \left(1 - e^{-(g-\ell-1)g_{\text{top}}} e^{2\pi i \beta \cdot t} \right)^{(-1)^{g+\ell} \binom{2g-2}{\ell} \alpha_g^\beta} \right), \end{aligned} \quad (2.1.35)$$

and renders itself intelligible as the partition function of second quantised M2 branes. Finally we remark that the equivalence of Gromov-Witten partition function with the Donaldson-Thomas partition function is a partially proven conjecture. See [34, 35, 36, 37] for relevant discussions.

¹Note that we omit all the perturbative terms and MacMahon factors in our definition of Z_{DT} and Z_{GV} .

2.1.5 Elliptic Genus and Vector-Valued Modular Forms

Elliptic genus is a useful tool to obtain structured and controllable information about the spectrum of a conformal theory with (2, 2) world-sheet supersymmetry[38, 39]. It is defined as

$$\chi(\tau, z) = \text{Tr}_{\text{RR}} (-1)^F e^{2\pi iz J_0} e^{2\pi i\tau(L_0 - \frac{c}{24})} e^{-2\pi i\bar{\tau}(\tilde{L}_0 - \frac{c}{24})}, \quad (2.1.36)$$

where $(-1)^F = e^{\pi i(J_0 + \tilde{J}_0)}$, and the subscript denotes the fact that the trace should be taken with the R-R boundary condition. It has also the interpretation as a path integral of the theory on the torus with appropriate U(1) coupling to “label” the left-moving R-charge. This Wilson line coupling is absent on the right-moving side. From the $[\tilde{J}_0, \tilde{G}_0^\pm]$ and $[\tilde{L}_0, \tilde{G}_0^\pm]$ commutation relations we see that only states annihilated by \tilde{G}_0^\pm , namely states with the right-movers at their R-ground states, contribute to the index. Supersymmetry therefore guarantees its rigidity property which often renders it computable. A special case of this is the Witten index $\chi(\tau, 0) = \text{Tr}_{\text{RR}} (-1)^F$.

Furthermore, the invariance of the spectrum under the spectral flow has interesting implications for the elliptic genus. To shorten the equations we will use in this part of the discussion the symbols

$$\begin{aligned} \hat{c} &= \frac{c}{3} \\ q &= e^{2\pi i\tau} \quad ; \quad y = e^{2\pi iz} \\ \mathbf{e}[x] &= e^{2\pi ix} . \end{aligned}$$

A short manipulation of (2.1.36) using the spectral flow relation (2.1.24) shows that the elliptic genus has the following two properties. First of all

$$\chi(\tau, z + \ell\tau + m) = e^{-\pi i\hat{c}(\ell^2\tau + 2\ell z)} \chi(\tau, z), \quad (2.1.37)$$

and secondly

$$\chi(\tau, z) = \sum_{\mu = -\frac{\hat{c}}{2}}^{\frac{\hat{c}}{2}-1} h_\mu(\tau) \theta_\mu(\tau, z) \quad (2.1.38)$$

$$h_\mu(\tau) = \sum_{n \in \mathbb{Z}_+} c_\mu(n) q^{-\frac{1}{2\hat{c}}\mu^2 + n} = \sum_{n \in \mathbb{Z}_+} c(n - \frac{\mu^2}{2\hat{c}}) q^{n - \frac{1}{2\hat{c}}\mu^2} \quad (2.1.39)$$

$$\theta_\mu(\tau, z) = \sum_{\ell \in \mathbb{Z}} q^{\frac{\hat{c}}{2}(\ell + \frac{\mu}{\hat{c}})^2} y^{(\mu + \hat{c}\ell)}. \quad (2.1.40)$$

The restriction $\mu \in (-\frac{\hat{c}}{2}, \frac{\hat{c}}{2}]$ can be understood as a consequence of the fact that one unit of spectral flow, namely $\eta = 1$ in (2.1.24), shifts the U(1) charge by a unit of \hat{c} .

From the above equation we see that the elliptic genus for a unitary (2,2) CFT has the following form

$$\chi(\tau, z) = \sum_{n \in \mathbb{Z}_+, \ell + \frac{\hat{c}}{2} \in \mathbb{Z}} c(n, \ell) q^n y^\ell = \sum_{n \in \mathbb{Z}_+, \ell + \frac{\hat{c}}{2} \in \mathbb{Z}} c(n - \frac{\ell^2}{2\hat{c}}) q^n y^\ell. \quad (2.1.41)$$

Notice that $L_0 - \frac{1}{2\hat{c}} J_0^2$ is indeed the (up to a multiplicative factor and addition of a constant) unique combination linear in L_0 that is invariant under the spectral flow (2.1.24). Especially, from Figure 2.1.3 and recalling that $n = L_0 - \frac{c}{24}$, we see that the coefficients $c(n - \frac{\ell^2}{2\hat{c}}) = 0$ for $n - \frac{\ell^2}{2\hat{c}} < -\frac{c}{24}$. The functions $h_{\mu = -\frac{\hat{c}}{2}}(\tau)$ has therefore q -expansion beginning from $q^{-\frac{c}{24}}$. We will call the part of the elliptic genus with negative arguments for $c(n - \frac{\ell^2}{2\hat{c}})$ the *polar part*

$$\chi^-(\tau, z) = \sum_{n - \frac{\ell^2}{2\hat{c}} < 0} c(n - \frac{\ell^2}{2\hat{c}}) q^n y^\ell, \quad (2.1.42)$$

or equivalently

$$h_{\mu}^-(\tau) = \sum_{n - \frac{\ell^2}{2\hat{c}} < 0} c(n - \frac{\ell^2}{2\hat{c}}) q^{n - \frac{1}{2\hat{c}} \mu^2}. \quad (2.1.43)$$

The physical and mathematical significance of this special “polar part” of the elliptic genus will be seen later when we discuss the elliptic genus for the black hole CFT in section 6.4.4.

This nice structure brought to us by the spectral flow, or related to it the spacetime supersymmetry, can be made more transparent by thinking about the action of an one-dimensional free Abelian group, namely an one-dimensional lattice, endowed with a \mathbb{Z} -valued bilinear form $(\ell, \ell) = \hat{c}\ell^2$ (recall that \hat{c} is an integer for a (2,2) non-linear sigma model on a Calabi-Yau manifold of any dimension). Let’s call this lattice Λ . The dual lattice Λ^* is then defined as the lattice of all vectors in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ with integral inner products with all vectors in the original lattice Λ . In the case at hand, in the integral basis of the one-dimensional lattice Λ^* , we see that the lattice Λ is the lattice of the points $\hat{c}\mathbb{Z}$. In this basis in which all points in Λ^* have integral coefficients, the bilinear form becomes

$$(\mu|\mu) = \frac{1}{\hat{c}} \mu^2. \quad (2.1.44)$$

We can now rewrite (2.1.38)-(2.1.40) as

$$\chi(\tau, z) = \sum_{\mu \in \Lambda^*/\Lambda} h_\mu(\tau) \theta_\mu(\tau, z) \quad (2.1.45)$$

$$\theta_\mu(\tau, z) = \sum_{\lambda \in \mu + \Lambda} q^{\frac{(\lambda|\lambda)}{2}} y^\lambda \quad (2.1.46)$$

$$= \mathbf{e}[-\frac{\hat{c} z^2}{2\tau}] \sum_{\lambda \in \mu + \Lambda} \mathbf{e}[\frac{\tau}{2}(\lambda + \frac{z}{\tau} \hat{c} |\lambda + \frac{z}{\tau} \hat{c})]. \quad (2.1.47)$$

From the last expression and using the Poisson resummation formula we get the modular transformation of the above theta-function

$$\theta_\mu(-\frac{1}{\tau}, \frac{z}{\tau}) = \mathbf{e}[\frac{\hat{c} z^2}{2\tau}] \sqrt{\frac{1}{\hat{c}}} \sqrt{-i\tau} \sum_{\mu' \in \Lambda^*/\Lambda} \mathbf{e}[(\mu|\mu')] \theta_{\mu'}(\tau, z). \quad (2.1.48)$$

We would like to know the modular property of the full elliptic genus as well. The z -independent part of the transformation must be compensated by the transformation of $h_\mu(\tau)$, since the Witten index $\chi(\tau, 0) = (-1)^F = \chi(X)$ is clearly modular invariant. Using this fact and the R-charge conjugation symmetry of the CFT, we get the following transformation rule

$$\chi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \mathbf{e}[\frac{\hat{c} cz^2}{2c\tau + d}] \chi(\tau, z). \quad (2.1.49)$$

This can also be understood as the modular transformation of the path integral with an extra coupling to the U(1) current. The property (2.1.37), (2.1.41), (2.1.49) is exactly the definition of a weak Jacobi form of index $\hat{c}/2$ and weight zero. The elliptic genus of a (2,2) SCFT therefore enjoys many special properties of a weak Jacobi form. Here we will list one important fact that will be needed later.

The space $J_{2*,*}$ of all weak Jacobi forms of even weight and any index is known to be a ring of all polynomials in the following four functions $\phi_{0,1}(\sigma, z)$, $\phi_{-2,1}(\sigma, z)$, $E_4(\tau), E_6(\tau)$ [40, 41], where E_4, E_6 are the usual Eisenstein series and $\phi_{0,1}, \phi_{-2,1}$ are the weak Jacobi forms of index one and weight 0 and -2 given by

$$\begin{aligned} \phi_{0,1}(\tau, z) &= \frac{\phi_{12,1}(\tau, z)}{\Delta(\tau)} = y^{-1} + 10 + y + \mathcal{O}(q) \\ \phi_{-2,1}(\tau, z) &= \frac{\phi_{10,1}(\tau, z)}{\Delta(\tau)} = y^{-1} - 2 + y + \mathcal{O}(q) \\ \phi_{12,1}(\tau, z) &= \frac{1}{144} (E_4^2 E_{4,1} - E_6 E_{6,1}) \\ \phi_{10,1}(\tau, z) &= \frac{1}{144} (E_6 E_{4,1} - E_4 E_{6,1}) \end{aligned}$$

and the discriminant is

$$\Delta = \eta^{24} = \frac{E_4^3 - E_6^2}{1728}. \quad (2.1.50)$$

This gives great constraint of the form the elliptic genus can take. For example, the space of weak Jacobi forms of weight zero and index one is one-dimensional. Together with the fact that $\chi(\tau, z = 0) = \chi(K3) = 24$ for the case of Calabi-Yau two-fold this completely fixes the K3 elliptic genus to be

$$\chi_{K3}(\tau, z) = 2\phi_{0,1}(\tau, z) = \sum_{n \in \mathbb{Z}_+, \ell \in \mathbb{Z}} c(4n - \ell^2) q^n y^\ell = 2y^{-1} + 20 + 2y + \mathcal{O}(y). \quad (2.1.51)$$

For the properties of elliptic genus with Calabi-Yau manifolds as the target space, see [42, 39]. See also [43, 42] for a geometric definition of the elliptic genus without reference to a superconformal theory.

The above lattice formulation can be readily generalized to higher dimensional lattices, with possibly not positive definite bilinear forms. For future use we now digress briefly to discuss them.

For a lattice Λ with a non-degenerate bilinear form of signature (σ^+, σ^-) we can define a modular form with values in the group ring $\mathbb{C}[\Lambda^*/\Lambda]$. In other words, in analogy with the concept of a modular form for an one-dimensional lattice, we have vector-valued modular forms in the higher-dimensional cases. Here we will only discuss them in terms of their components h_μ for each vector $\mu \in \Lambda^*/\Lambda$.

The novel property is that, for a negative-definite lattice, convergence of the series requires that the theta-functions can be written as a sum of $\mathbf{e}[\frac{\bar{\tau}}{2}(x|x)]$ instead of $\mathbf{e}[\frac{\tau}{2}(x|x)]$. For a lattice with mixed signature we therefore have both holomorphic and anti-holomorphic couplings.

Definition For a lattice Λ with a non-degenerate bilinear form (|) and signature (σ^+, σ^-) , given a maximally positive definite subspace in $\Lambda \otimes \mathbb{R}$, namely, given an element in the Grassmannian $v \in G(\sigma^+, \sigma^-)$, we define the (Siegel or Siegel-Narain) theta-function as

$$\theta_\mu(\tau; \alpha, \beta) = \sum_{\lambda \in \mu + \Lambda} \mathbf{e}\left[\frac{\tau}{2}(\lambda + \beta)_+^2 + \frac{\bar{\tau}}{2}(\lambda + \beta)_-^2 - (\lambda + \frac{\beta}{2}|\alpha)\right] \quad (2.1.52)$$

for $\mu \in \Lambda^*/\Lambda$, $\alpha, \beta \in \Lambda \otimes \mathbb{R}$,

where $x^2 := (x|x)$ and notice that the projection into the positive and negative part depends on $v \in G(\sigma^+, \sigma^-)$.

Using the higher-dimensional version of Poisson resummation, one can show

that the modular property of the above theta-function is

$$\theta_{\mu}\left(-\frac{1}{\tau}; -\beta, \alpha\right) = \frac{1}{\sqrt{|\Lambda^*/\Lambda|}} (\sqrt{-i\tau})^{\sigma^+} (\sqrt{i\bar{\tau}})^{\sigma^-} \sum_{\nu} \mathbf{e}[-(\mu|\nu)] \theta_{\nu}(\tau; \alpha, \beta), \quad (2.1.53)$$

where the pre-factor $\sqrt{|\Lambda^*/\Lambda|} = |\text{Vol}(\Lambda)|$ is the volume of the unit cell of the lattice. For the details of this computation and the generalisation to degenerate lattices, see [44].

2.1.6 Mirror Symmetry and Non-perturbative Effects

The alert readers might have already noticed that, the differentiation between the (c,c) and the (c,a) ring is rather *ad hoc* in our analysis of the chiral ring structure in the Calabi-Yau non-linear sigma model. And in fact also as a warning to the reader, the conventions do vary in the literature. There is of course nothing to stop us from flipping the sign of the R-current on one (let's say the right-moving) side and exchanging what we call G^+ and G^- while keeping the left-movers untouched. Although a trivial isomorphism from the world-sheet point of view, it implies something rather drastic on the geometric side. Namely it exchanges what we call the $H^{n-r,s}$ and $H^{r,s}$ cohomology classes, where n is again the complex dimension of the Calabi-Yau manifold. In particular, the Euler character changes its sign for three-folds, as $\chi(\text{CY}_3) = 2(h^{1,1} - h^{1,2}) \rightarrow -\chi$.

But since the world-sheet theory remains the same, there must be a pair of Calabi-Yau three-folds with exchanging $H^{3-r,s}$ and $H^{r,s}$. This symmetry is called the “mirror symmetry”. Geometrically, the easiest way to think about this symmetry is to think of the Calabi-Yau pair as a T^3 fibration over a three-real-dimensional base space, with possibly singular fibre at various points. The mirror symmetry is then implemented by doing three T-dualities along the fibre directions. As we have discussed in the previous chapter, a T-duality exchanges type IIA with type IIB superstring theory. This mirror symmetry must therefore also exchange type IIA and IIB strings living on the Calabi-Yau space. In other words, given a mirror pair (X, Y) of Calabi-Yau three-folds with

$$h^{2,1}(X) = h^{1,1}(Y) \quad ; \quad h^{2,1}(Y) = h^{1,1}(X)$$

and a pair of string theories IIA and IIB, there are only two instead of four independent theories one can write down. They are, schematically

$$(\text{IIB} / X) \quad \simeq \quad (\text{IIA} / Y)$$

and

$$(\text{IIB} / Y) \simeq (\text{IIA} / X).$$

Later we will also see that there is indeed a corresponding relationship between the spacetime low-energy effective theories.

Combining with different properties of the complex structure and the Kähler moduli space, mirror symmetry has been very useful in predicting properties of the theory in different parts of the moduli space, relating classical and quantum geometry. In particular, it predicts that the conformal field theory is well-behaved on the special submanifold in the moduli space where classical geometric intuitions fail, namely when the internal manifold goes through the so-called flop transitions. This is where we again see how strings are superior to point particles as probes for spacetime. But in the following analysis we will concentrate on parts of the moduli space where these non-perturbative effects do not occur. In particular, from now on we will stay well inside the Kähler cone and away from conifold points. In other words, we will only consider the part of the moduli space where all the homology cycles are “large” in the string unit.

2.2 Spacetime Physics

2.2.1 Moduli Space and Special Geometry

As we discussed earlier, the (2,2) SCFT's naturally come in families, related to each other by marginal deformations of the theory. We have also seen that these deformations, in the non-linear sigma model case, have the interpretation of deforming the target Calabi-Yau space. Indeed Calabi-Yau manifolds also come in families, meaning we can continuously deform the size and the shape of these internal manifolds without changing the topological properties of the space. As we have noticed in our discussion of the chiral ring, we can separate these deformations into two kinds. First is the shape, or the complex structure deformation, corresponding to $\delta g_{ij} \sim f_{i\bar{k}\bar{\ell}} \Omega^{\bar{k}\bar{\ell}}_j$, given by the harmonic three-form $f^{(3)} \in H^{1,2}(X, \mathbb{C})$ and using the (up to a factor) unique (3,0)-form of the Calabi-Yau X to contract the indices. Second is the size, or the Kähler deformation. Combining it with the NS-NS two-form potential we get $\delta B_{i\bar{j}} + i\delta g_{i\bar{j}} \sim f_{i\bar{j}}$, where $f^{(2)} \in H^{1,1}(X, \mathbb{C})$. The requirement that the deformation should be given by harmonic forms can be understood as the preservation of the Ricci-flatness of the metric. We have also argued that the moduli space is locally a direct product of these two separate moduli spaces. We will therefore study them separately now.

Complex Structure Moduli

First we look at the complex structure moduli. A change in the complex structure means a different decomposition of the tangent (or equivalently the cotangent) bundle into holomorphic and anti-holomorphic part. Especially, under an infinitesimal change of complex structure, the Calabi-Yau (3,0) form Ω becomes a linear combination of a (3,0) and a (2,1) form. We can therefore think of Ω as a section of a $H^3(X, \mathbb{C})$ bundle over the moduli space $\mathcal{M}_{\text{complex}}$. We will call it the Hodge bundle \mathcal{E}^{H} . This bundle is naturally endowed with the following symplectic structure $\langle \cdot, \cdot \rangle : H^3(X, \mathbb{C}) \times H^3(X, \mathbb{C}) \rightarrow \mathbb{C}$, given by

$$\langle \Gamma_1, \Gamma_2 \rangle = -\langle \Gamma_2, \Gamma_1 \rangle = \int_X \Gamma_1 \wedge \Gamma_2, \quad (2.2.1)$$

which has the geometric interpretation as the intersection number of the dual three-cycles of Γ_1 and Γ_2 . Furthermore, it defines a natural hermitian metric on $H^3(X, \mathbb{C})$

$$(\Gamma_1, \Gamma_2) = i\langle \Gamma_1, \bar{\Gamma}_2 \rangle = (\Gamma_2, \Gamma_1)^*.$$

But this is not yet the full story. Since the complex structure Ω is only defined up to a constant, or said differently, a rescaling of it will change the section in \mathcal{E}^{H} while it really doesn't mean a change of the complex structure of X , we should introduce a line bundle \mathcal{L} to account for the redundancy. Using the above hermitian metric, it is now natural to define the metric on \mathcal{L} to be $e^{\mathcal{K}}$, where

$$\mathcal{K} = -\log(\Omega, \Omega) = -\log(i\langle \Omega, \bar{\Omega} \rangle).$$

In other words, due to the extra rescaling symmetry, the complex structure three-form Ω should be thought of a section of $\mathcal{E}^{\text{H}} \otimes \mathcal{L}$.

When the (3,0)-form is rescaled as $\Omega \rightarrow e^f \Omega$ with a local holomorphic function f , we see that the ‘‘Kähler potential’’ \mathcal{K} scales like $\mathcal{K} \rightarrow \mathcal{K} - f - \bar{f}$. In particular, this means that a Kähler metric invariant under a local rescaling of the (3,0)-form can be defined on the moduli space $\mathcal{M}_{\text{complex}}$ using \mathcal{K} as the Kähler potential. See appendix A for properties of Kähler manifolds. The resulting Kähler metric is sometimes called the Weil-Peterson metric.

Furthermore, the connection

$$\nabla = \partial + \partial \mathcal{K} \quad (2.2.2)$$

satisfies the desired property $\nabla(e^f \Omega) = e^f \nabla \Omega$. In particular, it should take value in $H^{2,1}(M, \mathbb{C})$. Finally, it follows from the pairing of the cohomology classes that

$$\langle \partial \Omega, \Omega \rangle = 0, \quad (2.2.3)$$

since $\partial\Omega$ has only a $(3, 0)$ - and a $(2, 1)$ -form part. These properties mean that $\mathcal{M}_{\text{complex}}$ is a (local) special Kähler manifold defined in (A.0.12).

Since a rescaling of Ω does not have any physical significance, it will be convenient to define a “unit vector”

$$\mathbf{\Omega} = \frac{\Omega}{\sqrt{i\langle\Omega, \bar{\Omega}\rangle}} = e^{\mathcal{K}/2} \Omega, \quad (2.2.4)$$

satisfying

$$(\mathbf{\Omega}, \mathbf{\Omega}) = i\langle\mathbf{\Omega}, \bar{\mathbf{\Omega}}\rangle = 1,$$

and a “central charge” function $Z : H^3(X, \mathbb{Z}) \times \mathcal{M}_{\text{complex}} \rightarrow \mathbb{C}$, whose name will be justified later, as

$$Z(\Gamma; \Omega) = \langle\Gamma, \mathbf{\Omega}\rangle, \quad (2.2.5)$$

which in the present case is just $\int_{\Gamma} \mathbf{\Omega}$, where we have used the same symbol Γ for the Poincaré dual of the three-form Γ and $\mathbf{\Omega}$ for the pull-back of Ω .

From the behaviour of $\mathbf{\Omega}$ under a Kähler transformation, we see that $\mathbf{\Omega}$, and therefore Z , are sections of $\mathcal{L}^{1/2} \otimes \bar{\mathcal{L}}^{-1/2}$ and have therefore the following covariant derivatives

$$\mathcal{D}\mathbf{\Omega} = (\partial + iQ)\mathbf{\Omega}, \quad (2.2.6)$$

where the connection is given by

$$iQ = \frac{1}{2} (\partial\mathcal{K} - \bar{\partial}\mathcal{K}) = i\text{Im}(\partial\mathcal{K}). \quad (2.2.7)$$

In Coordinates

After this rather abstract derivation of the special Kähler properties of $\mathcal{M}_{\text{complex}}$, to have a better feeling of what is really going on let’s now un-package the information by choosing a local coordinates on $\mathcal{M}_{\text{complex}}$. Of course, writing things out in components in a local coordinate system does not bring new information. The main reason for the following formulation is really to make it easier for the readers to connect to the existing literature.

For this purpose we have to choose a basis for the middle-cohomology $H^3(X, \mathbb{Z})$ of the Calabi-Yau space X , namely a coordinate system for the fibre of the Hodge bundle \mathcal{E}^H . From the anti-symmetric intersection of the homology $H_3(X, \mathbb{Z})$ we see that we can always choose a real basis (α_I, β^J) for $H^3(X, \mathbb{Z})$, and the corresponding basis (A^I, B_J) of $H_3(X, \mathbb{Z})$, which is or-

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thonormal in the following sense

$$\begin{aligned} \langle \alpha_I, \beta^J \rangle &= -\langle \beta^J, \alpha_I \rangle = \int_X \alpha_I \wedge \beta^J = \delta_I^J = \#(A^I \cap B_J) & (2.2.8) \\ \int_{A^I} \alpha_J &= -\int_{B_J} \beta^I = \delta_J^I \quad ; \quad \int_{A^I} \beta^J = \int_{B_J} \alpha_I = 0 \\ & I, J = 0, \dots, h^{2,1} . \end{aligned}$$

We can then write the (3,0)-form in terms of these coordinates as²

$$\Omega = -X^I \alpha_I + F_I \beta^I . \quad (2.2.9)$$

These coordinates then determine the complex structure. Actually it overdetermines it. To see this, recall we said earlier that an infinitesimal change of complex structure turns the (3,0)-form into a linear combination of a (3,0)-part and a (2,1)-part, and the dimension of the moduli space is therefore $\dim_{\mathbb{C}} \mathcal{M}_{\text{complex}} = h^{2,1}$ and not $2h^{2,1} + 2$. Together with the fact that the overall scale does not have a physical significance, which is the reason we why introduced the line bundle \mathcal{L} in the above construction, we can regard X^I 's as the homogeneous coordinates for the projective space of infinitesimal variation of the complex structure, and F_I as $F_I(X)$. Locally, where X^0 does not vanish, we can then use the coordinates, called the “special coordinates”, defined as

$$t^A = \frac{X^A}{X^0} \quad ; \quad A = 1, \dots, h^{2,1} ,$$

as the coordinates of the complex structure moduli space $\mathcal{M}_{\text{complex}}$. But as we will see later, it will often be useful to work with the homogeneous coordinates X^I instead.

In these coordinates, the Kähler potential reads

$$e^{-\mathcal{K}} = 2\text{Im}(X^I \bar{F}_I) ,$$

and the Kähler metric is given by

$$g_{I\bar{J}} = \partial_I \bar{\partial}_{\bar{J}} \mathcal{K} = -\frac{i \langle \nabla_I \Omega, \bar{\nabla}_{\bar{J}} \bar{\Omega} \rangle}{i \langle \Omega, \bar{\Omega} \rangle} = -i \langle \mathcal{D}_I \Omega, \bar{\mathcal{D}}_{\bar{J}} \bar{\Omega} \rangle . \quad (2.2.10)$$

The condition (2.2.3) which follows from the cohomology pairing and which is a part of our definition of special Kähler geometry (A.0.12), gives in these coordinates

$$\langle \partial_I \Omega, \Omega \rangle = -F_I + X^J \partial_J F_I = 0 ,$$

²Warning: Some authors use the normalized three-form $\Omega = X^I \alpha_I - F_I \beta^I$ as the definition for $(\frac{X^I}{F_I})$, corresponding to the “gauge choice” $2\text{Im}(X^I \bar{F}_I) = 1$.

which implies that F_I is homogeneous of degree 1 as a function of X 's. We can therefore define

$$F(X) = \frac{1}{2} X^I F_I,$$

and it's easy to check that it satisfies

$$\begin{aligned} \partial_I F(X) &= F_I \\ X^I \partial_I F(X) &= 2F(X). \end{aligned} \quad (2.2.11)$$

This homogeneous function F of degree 2 is the so-called ‘‘prepotential’’. Specifically, for later use we define

$$F_{IJ} = \partial_I \partial_J F(X),$$

then

$$F_I = F_{IJ} X^J. \quad (2.2.12)$$

Notice that our choice of basis (2.2.8) is only fixed up to a symplectic transformation. To see this, note that in our ‘‘orthonormal’’ basis, the symplectic form on $H^3(X) \simeq H_3(X)$ in the matrix representation is just

$$\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix},$$

where $\mathbb{1}$ is the $(h^{1,2} + 1) \times (h^{1,2} + 1)$ unit matrix. Then an integral change of basis given by $Sp(2h^{2,1} + 2, \mathbb{Z})$, namely matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with A, B, C, D being $(h^{2,1} + 1) \times (h^{2,1} + 1)$ matrices satisfying

$$\begin{aligned} A^T C - C^T A &= 0 \\ B^T D - D^T B &= 0 \\ A^T D - C^T B &= \mathbb{1}, \end{aligned}$$

always leaves the above symplectic matrix invariant.

Under the above change of basis, the coordinates X^I and F_I transform as a symplectic vector

$$V = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X^I \\ F_I \end{pmatrix}.$$

From the fact that $F(X) = \frac{1}{4} V^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} V$ we see that the prepotential is not invariant under the symplectic transformation. This is another reason why we chose to define the special geometry without reference to such a prepotential.

Kähler Moduli

After discussing the complex moduli we now turn to the Kähler Moduli of the Calabi-Yau three-fold. As we mentioned earlier in section 2.1.2, the Kähler moduli space of a Calabi-Yau X is given by its cohomology class $H^{1,1}(X)$. Recall that in the case of complex structure, we double the space $H^{2,1}(X)$ into $H^{2,1}(X) \oplus H^{1,2}(X)$ and further enlarge it with $H^{3,0}(X) \oplus H^{0,3}(X)$ to construct the symplectic bundle. Here we will do a similar thing. To construct the symplectic bundle for the Kähler moduli space $\mathcal{M}_{\text{Kähler}}$, we first double $H^{1,1}(X)$ into $H^{1,1}(X) \oplus H^{2,2}(X)$ and further enlarge it with $H^{0,0}(X) \oplus H^{3,3}(X)$. In other words, we consider a $H^{2*}(X, \mathbb{C})$ bundle over $\mathcal{M}_{\text{Kähler}}$.

Again similar to the complex structure case, we want to employ the symplectic pairing of $H^{2*}(X, \mathbb{C})$ and thereby see that $\mathcal{M}_{\text{Kähler}}$ is again a special Kähler manifold. But this time the geometric meaning of such a pairing will not be as clear. What must stay true is that the pairing should be between the cohomology classes that are Hodge dual to each other. In order for the pairing to be symplectic, we now define a map

$$\check{\Gamma} = (-1)^n \Gamma \quad \text{for } \Gamma \in H^{(n,n)}(X, \mathbb{C})$$

and requires it to act component-wise on a general $\Gamma \in H^{2*}(X, \mathbb{C})$. The symplectic product $\langle, \rangle : H^{2*}(X, \mathbb{C}) \times H^{2*}(X, \mathbb{C}) \rightarrow \mathbb{C}$ is then given by

$$\langle \Gamma_1, \Gamma_2 \rangle = -\langle \Gamma_2, \Gamma_1 \rangle = \int_X \Gamma_1 \wedge \check{\Gamma}_2. \quad (2.2.13)$$

The next step will be to find a section which gives the Kähler potential (A.0.12). Since a deformation in the Kähler moduli changes the complexified (with the B-field) Kähler form

$$t = B + iJ \in H^{1,1}(X, \mathbb{C}) \quad (2.2.14)$$

to a nearby vector in $H^{1,1}(X, \mathbb{C})$, a natural way to build such a section Ω in the $H^{2*}(X, \mathbb{C})$ bundle will be

$$\Omega = -e^t = -\left(1 + t + \frac{1}{2!}t \wedge t + \frac{1}{3!}t \wedge t \wedge t\right). \quad (2.2.15)$$

Then the Kähler potential

$$e^{-\mathcal{K}} = i\langle \Omega, \bar{\Omega} \rangle = \frac{4}{3} \int J \wedge J \wedge J = 8 \text{vol}(X) \quad (2.2.16)$$

is given by the Calabi-Yau volume, and the normalised section is

$$\Omega = -e^{\mathcal{K}/2} e^t = \frac{-e^t}{\sqrt{\frac{4}{3}J^3}} \quad (2.2.17)$$

Finally, another condition $\langle \partial\Omega, \Omega \rangle = 0$ for special Kähler manifold is now obviously true, from the simple fact that $\partial(-e^t) = \partial t \Omega$. We therefore conclude that moduli space $\mathcal{M}_{\text{Kähler}}$ is again a special Kähler manifold, and the same properties (2.2.2)-(2.2.7) we discussed for $\mathcal{M}_{\text{complex}}$ carry straightforwardly to this case.

As the reader might have noticed, for convenience we have actually already used the special coordinates, given by $t \in H^{1,1}(X, \mathbb{C})$, in the definition of (2.2.15). To formulate the special geometry in homogeneous coordinates as before, we introduce the basis α_A for $H^{1,1}(X, \mathbb{Z})$ and the dual basis β^A for $H^{2,2}(X, \mathbb{Z})$ defined by $\int_X \alpha_A \wedge \beta^B = \delta_A^B$, with $A = 1, \dots, h^{1,1}$. For this basis we then define the triple-intersection number of the dual four-cycles as

$$\int_X \alpha_A \wedge \alpha_B \wedge \alpha_C = D_{ABC} . \quad (2.2.18)$$

Furthermore we write $\alpha_0 = 1$ and $\beta^0 = -\frac{J \wedge J \wedge J}{\int_X J \wedge J \wedge J}$ as the basis for $H^{0,0}(X, \mathbb{Z})$ and $H^{3,3}(X, \mathbb{Z})$.

In other words, for the special geometry of the Kähler moduli we use

$$\begin{aligned} \alpha_0 &= 1 \\ \alpha_A &\in H^{1,1}(X, \mathbb{Z}) \\ \beta^A &\in H^{2,2}(X, \mathbb{Z}) \\ \beta^0 &= -\frac{J \wedge J \wedge J}{\int_X J \wedge J \wedge J} \end{aligned} \quad (2.2.19)$$

satisfying

$$\langle \alpha_I, \beta^J \rangle = \delta_I^J \quad , \quad I, J = 0, \dots, h^{1,1}$$

as the symplectic basis of $H^{2*}(X, \mathbb{Z})$.

Again, for the convenience of the reader we would like to make contact with the convention involving the prepotential used in a large proportion of the existing literature.

Using the projective coordinates X^I , we can introduce the following section of the even cohomology bundle

$$\Omega = -X^0 e^{\frac{1}{X^0} X^A \alpha_A} = -X^I \alpha_I + F_I \beta^I , \quad (2.2.20)$$

which reduces to the definition (2.2.15) when the special coordinates

$$\frac{X^A}{X^0} = t^A \quad ; \quad X^0 = 1$$

is used.

In these coordinates the Kähler potential is again given by

$$e^{-\mathcal{K}} = 2\text{Im}(X^I \bar{F}_I),$$

and one can check that the prepotential reads

$$F(X) = \frac{1}{2}F^I X_I = \frac{1}{6}D_{ABC} \frac{X^A X^B X^C}{X^0}, \quad (2.2.21)$$

which is again a homogeneous function of degree two.

As we will comment more on it later, while the metric of the complex structure moduli space is exact, that of the Kähler moduli space receives perturbative and non-perturbative corrections. Here we can see the resemblance of the leading prepotential for the Kähler moduli space (2.2.21) and the leading perturbative topological strings amplitude (2.1.33). As we will see later, the rest of F_{top} will also have its role in the prepotential. Specifically, the symmetry of the shift of the B-field

$$\text{Re } t^A \rightarrow \text{Re } t^A + \text{constant},$$

known as the PQ (Peccei-Quinn) symmetry, constrains the form of the perturbative correction to the prepotential. But non-perturbatively this symmetry is broken and not observed by non-perturbative corrections.

This finished our discussion about the moduli space of Calabi-Yau three-folds. For a point of entry into the literature on special geometry, we refer to [45, 46, 47, 48], or various reviews [49, 50, 51, 29].

2.2.2 Four- and Five-Dimensional Low Energy Supergravity Theory

When one considers a Calabi-Yau three-fold as a part of the spacetime, from the decomposition of spinors under the rotation group $Spin(6) \simeq SU(4)$ into representations of the holonomy group $SU(3)$ as $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$, we see that the manifold admits a “Killing spinor” satisfying $\nabla_k \eta = 0$. In the absence of a dilaton gradient or a background $H^{(3)} = dB^{(2)}$ flux, the existence of such a Killing spinor means unbroken spacetime supersymmetry. Furthermore, from the above decomposition we see that a quarter of the supersymmetry is preserved. Starting from a ten- or eleven-dimensional theory with 32 supercharges, this leads to eight remaining supercharges after compactification to four- or five-dimensions.

Four Dimensional Supergravity

Let's first look at the supersymmetry algebra in four dimensions. The minimal supersymmetry algebra reads

$$\{Q_\alpha, \bar{Q}_\beta\} = -2P_\mu \Gamma_{\alpha\beta}^\mu \quad , \quad [P^\mu, Q_\alpha] = 0 ,$$

where Q 's are Majorana fermions with four degrees of freedom, $\bar{Q} = Q^\dagger \Gamma^0$, Γ^μ 's are gamma matrices furnishing a representation of the Clifford algebra, P_μ is the spacetime momentum vector and α, β are the spinor indices. By writing out the gamma matrices explicitly, one can easily see that the Q anti-commutators give one pair of fermionic creation and annihilation operators when P_μ is massless (lightlike), while the physical fields must be annihilated by the rest two of the Q 's. On the other hand, when P_μ is massive (timelike) the Q anti-commutators give two pairs of fermionic annihilation and creation operators and the physical fields are annihilated by none of the Q 's.

For the case of $4\mathcal{N}$ supercharges, the generalisation of the above supersymmetry algebra is

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = -2\delta^{AB} P_\mu \Gamma_{\alpha\beta}^\mu \quad , \quad [P^\mu, Q_\alpha^A] = 0 \quad A, B = 1, \dots, \mathcal{N} .$$

Just as before, when P_μ is massless, Q^A gives a pair of fermionic creation and annihilation operators for each $A \in \{1, \dots, \mathcal{N}\}$. As mentioned above, for the Calabi-Yau compactification we have $\mathcal{N} = 2$ and therefore two pairs of fermionic creation and annihilation operators. As a result, an $\mathcal{N} = 2$ massless multiplet has the following helicities

$$j, j + \frac{1}{2}, j + \frac{1}{2}, j + 1$$

and accompanied by their CPT conjugate if the multiplet is not conjugate to itself. In particular, as will be seen shortly, there are three massless multiplets, corresponding to $j = -\frac{1}{2}, 0, 1$, which will be relevant for the fields content of the low energy effective theory obtained by compactifying type II string theory on Calabi-Yau three-folds. These are

$$\begin{array}{ll} \text{hypermultiplet} & (-\frac{1}{2}, 0^2, \frac{1}{2}) + (-\frac{1}{2}, 0^2, \frac{1}{2}) \\ \text{vector multiplet} & (-1, -\frac{1}{2}^2, 0) + (0, \frac{1}{2}^2, 1) \\ \text{supergravity multiplet} & (-2, -\frac{3}{2}^2, -1) + (1, \frac{3}{2}^2, 2) \end{array}$$

To obtain the massless spectra of the lower-dimensional theory, we have to compactify the massless spectra of type II superstring listed in Table 1.1. Using the fact that the Laplacian factorizes into

$$\nabla_{10d} = \nabla_{4d} + \nabla_{\text{C-Y}} ,$$

IIA/CY massless spectrum		
number	multiplet	bosonic field content
1	supergravity multiplet	$G_{\mu\nu}, (C_\mu)$
$h^{1,1}$	vector multiplet	$(C_{\mu i\bar{j}}), G_{i\bar{j}}, B_{i\bar{j}}$
$h^{2,1}$	hypermultiplet	$C_{i\bar{j}k}, G_{ij}$
1	(universal) hypermultiplet	$C_{ijk}, \Phi, B_{\mu\nu}$

IIB/CY massless spectrum		
number	multiplet	bosonic field content
1	supergravity multiplet	$G_{\mu\nu}, C_{\mu i\bar{j}k}$
$h^{2,1}$	vector multiplet	$C_{\mu i\bar{j}k}, G_{ij}$
$h^{1,1}$	hypermultiplet	$C_{\mu\nu i\bar{j}}, C_{i\bar{j}}, G_{i\bar{j}}, B_{i\bar{j}}$
1	(universal) hypermultiplet	$C, \Phi, B_{\mu\nu}, C_{\mu\nu}$

M-theory/CY massless spectrum		
number	multiplet	bosonic field content
1	supergravity multiplet	$G_{\mu\nu}, (A_{\mu i\bar{j}})$
$h^{1,1} - 1$	vector multiplet	$\phi^a, (A_{\mu i\bar{j}})$
$h^{2,1}$	hypermultiplet	$G_{ij}, A_{ij\bar{k}}$
1	(universal) hypermultiplet	\mathcal{V}, A_{ijk}

Table 2.3: Summary of the massless spectrum of the type IIA, type IIB superstring theories compactified on Calabi-Yau three-folds. The parenthesis denotes the fact that the gauge field in the supergravity multiplet, the graviphoton field, is actually a linear combination of the $(1+h^{1,1})$ gauge fields in the parenthesis. Similarly, in the case of M-theory compactification, the $h^{1,1}$ vectors $A_{\mu i\bar{j}}$ split into one supergravity and $(h^{1,1} - 1)$ vector multiplet gauge fields upon dimensional reduction.

we see that the four-dimensional massless spectrum is given by the cohomology classes of the internal manifold. These massless fields, grouped in terms of the $\mathcal{N} = 2$ multiplets, is given in Table 2.3. Note that we have used the self-duality of the $C_+^{(4)}$ of type IIB theory and the fact that a two-form is dual to a scalar field in four dimensions through $\star_4 dC^{(2)} = d\phi$ in obtaining this table.

But we are not done yet with the supersymmetry algebra. For the cases of extended supersymmetry with $\mathcal{N} > 1$, it's possible to have central extensions of the above algebra. Without breaking the Lorentz invariance, namely without incorporating extended sources, the most general form is

$$\{Q_\alpha^A, \bar{Q}_\beta^B\} = -2\delta^{AB} P_\mu \Gamma_{\alpha\beta}^\mu - 2iZ^{AB} \delta_{\alpha\beta} \quad , \quad [P^\mu, Q_\alpha^A] = [Z, Q] = [Z, P] = 0 \quad . \quad (2.2.22)$$

By taking the charge conjugation of the above anti-commutation relation and use the Majorana condition $\bar{Q} = Q^\dagger \Gamma^0 = Q^T C$ we see that the central charge matrix is anti-symmetric,

$$Z^{AB} = -Z^{BA}.$$

In the case that \mathcal{N} is even, the central charge matrix can therefore be written as a block-diagonal form with the i -th block being

$$\begin{pmatrix} 0 & Z_i \\ -Z_i & 0 \end{pmatrix}, \quad i = 1, \dots, \frac{1}{2}\mathcal{N}.$$

Suppose now that the momentum vector is timelike with mass M , the anti-commutation relation implies that the eigenvalues of the central charge matrix satisfies

$$M \geq |Z_i|, \quad i = 1, \dots, \frac{1}{2}\mathcal{N}. \quad (2.2.23)$$

This is called the BPS (Bogomolny-Prasad-Sommerfield) bound on the mass respective to the charges.

Analogous to the case without central extensions, the $\{Q, Q\}$ anti-commutator gives two pairs of fermionic creation and annihilation operators for each $A \in \{1, \dots, \mathcal{N}\}$ if none of the BPS bound is saturated, just as in the massive case in the algebra without central extensions. Now for each i for which the BPS bound is saturated, two pairs of fermionic creation and annihilation operators are removed. When all of the $\frac{1}{2}\mathcal{N}$ BPS bound are saturated there is just one pair of fermionic creation and annihilation operators for each $A \in \{1, \dots, \mathcal{N}\}$, and we have the same representation of this algebra as the massless one in the case with no central charges. To sum up, the relationship between the BPS bound and the unbroken supersymmetry in four dimensions is that the saturation of each BPS bound implies four preserved supersymmetry.

For example, when $\mathcal{N} = 4$ we have two BPS bounds

$$M \geq |Z_1| \geq |Z_2|. \quad (2.2.24)$$

When only one of the BPS bounds are saturated, we have six fermionic creation operators and the multiplet therefore contains 2^6 states. Especially each state is annihilated by 4 of the total 16 supercharges and is therefore called 1/4-BPS. When both of the bounds are saturated, a multiplet contains 2^4 states just like in the massless case and the states are said to be 1/2-BPS for obvious reasons.

Back to the $\mathcal{N} = 2$ case at hand, now there is only one BPS bound

$$M \geq |Z_1|,$$

and in this case the central charge is given by the graviphoton charge, namely the gauge field in the supergravity multiplet.

72 2. Calabi-Yau Compactifications

After discussing the massless field content of the type II string theory compactified on a Calabi-Yau, we are now ready to reduce the 10-dimensional low energy supergravity action (1.2.3), (1.2.5) and obtain the 4-dimensional low energy effective action. The resulting bosonic action for the supergravity and vector multiplets is

$$\begin{aligned}
 16\pi G_N^{(4)} L &= L_{\text{Eins}} + L_{\text{scalar}} + L_{\text{vector}} \\
 L_{\text{Eins}} &= R \star_4 1 \\
 L_{\text{scalar}} &= -g_{A\bar{B}} dt^A \wedge \star_4 d\bar{t}^{\bar{B}} & A = 1, \dots, n \\
 L_{\text{vector}} &= -\frac{1}{2} \mathcal{F}^I \wedge \mathcal{G}_I & I = 0, \dots, n,
 \end{aligned} \tag{2.2.25}$$

where \star_4 denotes the Hodge dual in four dimensions and n is the number of the vector multiplet fields which is given in Table 2.3 in terms of the topological data of the internal Calabi-Yau manifolds. Furthermore, \mathcal{G}_I is given by the requirement that

$$\mathcal{F} = \mathcal{F}^I \otimes \alpha_I - \mathcal{G}_I \otimes \beta^I = \star_{10} \mathcal{F} \tag{2.2.26}$$

when the Hodge dual in the Calabi-Yau space is taken to be

$$\begin{aligned}
 \star_{\text{C-Y}} \bar{\Omega} &= i\Omega \\
 \star_{\text{C-Y}} \nabla_I \Omega &= i\bar{\nabla}_{\bar{I}} \bar{\Omega}.
 \end{aligned} \tag{2.2.27}$$

³ From the expression of the ten-dimensional gravity coupling constant in terms of the string theory data (1.2.8) and following the standard Kaluza-Klein procedure, we conclude that the four-dimensional gravity coupling constant is given by

$$G_N^{(4)} \sim (\ell_P^{(4)})^2 \sim \frac{(\ell_s g_s)^2}{\mathcal{V}^{(s)}(\text{CY})},$$

where $\mathcal{V}^{(s)}(\text{CY}) = \text{Vol}(\text{CY})/(\ell_s)^6$ is the volume of the internal manifold in string unit.

Not surprisingly, this action is exactly the tree-level action of $\mathcal{N} = 2$, D=4 supergravity action, constructed using the superconformal tensor calculus. See for superconformal supergravity [52, 48] and [53] for the dimensional reduction. See also, for example, [49] and references therein for more details.

Again, for completeness we will now rewrite the above action, written in the form as in [54], in a probably more familiar form in terms of the coordinates and prepotential.

³Using this definition, one also has to take the complex conjugate of the coefficient, $\star a \bar{\Omega} = i a^* \Omega$ for example, in order to have the the bilinear $\int \Gamma \wedge \star \Gamma$ positive definite.

For the scalar part of the vector-multiplet action, using the homogeneous property of the prepotential (2.2.11)-(2.2.12), we can show that the scalar field metric (2.2.10) can be explicitly written as

$$g_{I\bar{J}} = e^{\mathcal{K}} N_{IJ} + e^{2\mathcal{K}} \bar{X}_{\bar{I}} X_J$$

where

$$N_{IJ} \equiv 2\text{Im}F_{IJ}, \quad X_I \equiv X^J N_{IJ}$$

satisfies

$$X \cdot \bar{X} \equiv X^I \bar{X}_I = \bar{X}^I X_I = -e^{-\mathcal{K}}.$$

Notice that this metric given in terms of the projective coordinates X^I has one degenerate direction, namely $g_{I\bar{J}} X^I \bar{X}^{\bar{J}} = 0$. The reader should remember that this indeed has to be the case, because the moduli space is really only parametrized by the n -scalars t^A and therefore we have to project out the unphysical direction corresponding to rescaling $\Omega \rightarrow \lambda \Omega$.

Put the above equations together, given a prepotential $F(X)$, the scalar action is

$$L_{\text{scalar}} = \frac{1}{2} \left(\frac{N_{IJ}}{X \cdot \bar{X}} + \frac{\bar{X}_{\bar{I}} X_J}{(X \cdot \bar{X})^2} \right) dX^I \wedge \star_4 d\bar{X}^{\bar{J}}.$$

As for the vector part of the action, define the ‘‘coupling matrix’’ \mathcal{N}_{IJ} such that

$$\begin{aligned} F_I &= \mathcal{N}_{IJ} X^J \\ \bar{\nabla}_{\bar{K}} \bar{F}_{\bar{I}} &= \mathcal{N}_{IJ} \bar{\nabla}_{\bar{K}} \bar{X}^{\bar{J}}, \end{aligned} \quad (2.2.28)$$

which is solved to be

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i \frac{X_I X_J}{X \cdot \bar{X}}$$

with $X \cdot X = X^I X_I$.

Using its property (2.2.28), the Hodge star relation can be written in terms of coupling matrix \mathcal{N}_{IJ} as

$$\star_{\text{C-Y}} (\alpha_I - \mathcal{N}_{IJ} \beta^J) = i (\alpha_I - \mathcal{N}_{IJ} \beta^J).$$

Now it’s straightforward to solve the self-duality condition (2.2.26) and rewrite the vector part of the action as

$$\begin{aligned} L_{\text{vector}} &= \frac{1}{2} (\text{Re} \mathcal{N}_{IJ} \mathcal{F}^I \wedge \mathcal{F}^J + \text{Im} \mathcal{N}_{IJ} \mathcal{F}^I \wedge \star_4 \mathcal{F}^J) \\ &= \frac{1}{2} (\mathcal{N} \mathcal{F}^+ \wedge \mathcal{F}^+ + \bar{\mathcal{N}} \mathcal{F}^- \wedge \mathcal{F}^-), \end{aligned}$$

where we have split the field strength into the self-dual and the anti-self dual part

$$\star_4 \mathcal{F}^\pm = \pm i \mathcal{F}^\pm ,$$

and the reality of the field strength implies $\mathcal{F}^- = (\mathcal{F}^+)^*$.

We will not discuss the hypermultiplet part of the action since we will not need it later. Let's just remark that it decouples from the supergravity multiplet and vector multiplet part of the action discussed above, in the sense that the hypermultiplet action, including the coupling constants of it, only depends on hypermultiplet fields. Furthermore, the scalar manifold of the hypermultiplet action is not special Kähler but the so-called quaternionic Kähler manifold. The relationship between the scalar manifold of the vector- and the hypermultiplet sectors as predicted by mirror symmetry has to be seen by further compactifying down to three dimensions, using the so-called c-map.

Five-dimensional Supergravity

We have just discussed the four-dimensional low-energy effective action of type II string theory compactified on Calabi-Yau three-folds, obtained by compactifying the ten-dimensional type IIA and IIB supergravity theories to four dimensions. We can also consider the five-dimensional low energy effective action of M-theory compactified on Calabi-Yau manifold. This can be done by Kaluza-Klein reduce the eleven-dimensional supergravity action (1.2.1) to five dimensions, since the eleven-dimensional supergravity is supposed to be the low-energy description of M-theory. Not surprisingly, the result is the same as the action of the $\mathcal{N} = 1, d=5$ supergravity [55].

First let's look at the massless spectrum of the theory. Again splitting the eleven-dimensional spacetime indices into the internal ones (i, j, \bar{i}, \bar{j}) and the five-dimensional ones (μ, ν) , we get the five-dimensional spectrum as recorded in Table 2.3. Note that the scalar fields ϕ^a are now real instead of complex, since there is no B-field in M-theory. More specifically, the $h^{1,1}$ scalars given by the Kähler moduli

$$J = J^A \alpha_A \quad , \quad \alpha_A \in H^{1,1}(X, \mathbb{Z})$$

is now divided into the volume factor $\mathcal{V} = \frac{1}{3!} D_{ABC} J^A J^B J^C$ and $(h^{1,1} - 1)$ scalars ϕ^a , $a = 1, \dots, h^{1,1} - 1$, which are coordinates of the co-dimension one hypersurface inside the Kähler moduli space satisfying $\mathcal{V}(\phi) = 1$. The former goes in the (universal) hypermultiplet while the latter make up the $(h^{1,1} - 1)$ real scalars of the $(h^{1,1} - 1)$ vector multiplets of the theory. In particular, since the hypermultiplet part of the action decouples from the rest, the volume of

Calabi-Yau space in eleven-dimensional Planck unit plays no important role in the physical solution.

The vector and supergravity multiplets part of the bosonic action is [56]

$$\begin{aligned}
 16\pi G_N^{(5)} L &= L_{\text{Eins}} + L_{\text{scalar}} + L_{\text{vector}} + L_{\text{C-S}} \\
 L_{\text{Eins}} &= R \star_5 1 \\
 L_{\text{scalar}} &= -h_{ab} d\phi^a \wedge \star_5 d\phi^b & a, b = 1, \dots, h^{1,1} - 1 \\
 L_{\text{vector}} &= -\frac{1}{2} a_{AB} F^A \wedge \star_5 F^B & A, B = 1, \dots, h^{1,1} \\
 L_{\text{C-S}} &= \frac{1}{3!} D_{ABC} A^A \wedge F^B \wedge F^C, &
 \end{aligned} \tag{2.2.29}$$

where \star_5 denotes the Hodge dual in five dimensions. To understand the scalar metric h_{ab} and the gauge coupling a_{AB} , let's consider the natural metric on the J^A -space

$$g_{AB} = \frac{\partial}{\partial J^A} \frac{\partial}{\partial J^B} \mathcal{K} = \int_{CY} \alpha_A \wedge \star \alpha_B, \tag{2.2.30}$$

where \mathcal{K} is again given by (2.2.16):

$$e^{-\mathcal{K}} = \frac{4}{3} \int J \wedge J \wedge J = 8\mathcal{V}(J).$$

Then h_{ab} and a_{AB} are given by, up to a convention-dependent coefficient, the induced metric on the hypersurface $\mathcal{V} = 1$ and the restriction of g_{AB} on the same hypersurface respectively.

4D-5D Connection

As we discussed in the previous chapter, M-theory compactified on a circle is dual to type IIA string theory. Taking the low-energy limit on both sides compactified on a Calabi-Yau manifold, it implies that a solution in the above five-dimensional supergravity theory gives rise to a solution in the four-dimensional supergravity theory when reduced on a circle. Of course, the above statement can also be understood just using the usual Kaluza-Klein reduction of the five-dimensional supergravity theory without reference to string or M-theory. Specifically, a four-dimensional solution can be “lifted” to a five-dimensional solution with a U(1) symmetry, with the presence of this U(1) isometry assuring the absence of higher Kaluza-Klein modes. To see how it works, let's do some dimensional analysis first.

Recall the relation between the string length, the eleven-dimensional Planck length and the radius of the M-theory circle (1.2.9), and the usual Kaluza-Klein

relation

$$\ell_{\text{P}}^{(5)} \sim \frac{\ell_{\text{P}}^{(11)}}{(\mathcal{V}^{(\text{M})})^{1/3}} \quad , \quad \ell_{\text{P}}^{(4)} \sim \frac{\ell_{\text{P}}^{(10)}}{(\mathcal{V}^{(\text{s})})^{1/2}} \quad ,$$

and the expression of the ten-dimensional Planck length (1.2.8). From the above relations together with the fact that

$$\mathcal{V}^{(\text{s})} (\ell_{\text{s}})^6 = \mathcal{V}^{(\text{M})} (\ell_{\text{P}}^{(11)})^6 \quad , \quad (2.2.31)$$

where $\mathcal{V}^{(\text{s})}$ and $\mathcal{V}^{(\text{M})}$ are the Calabi-Yau volume in string and M-theory units respectively, we conclude that

$$R_{\text{M}} \sim (\mathcal{V}^{(\text{s})})^{1/3} \ell_{\text{P}}^{(5)} \quad , \quad \ell_{\text{P}}^{(5)} \sim (\mathcal{V}^{(\text{s})})^{1/6} \ell_{\text{P}}^{(4)} \quad , \quad (2.2.32)$$

with the volume \mathcal{V} denotes the volume at the spatial infinity.

Being careful with the coefficients not listed above, this suggests that a four-dimensional solution $ds_{4\text{D}}^2$, $A_{4\text{D}}^A$, $A_{4\text{D}}^0$, t^A gives a five-dimension solution

$$\begin{aligned} ds_{5\text{D}}^2 &= 2^{2/3} \mathcal{V}^{2/3} (d\psi - A_{4\text{D}}^0)^2 + 2^{-1/3} \mathcal{V}^{-1/3} ds_{4\text{D}}^2 \\ A_{5\text{D}}^A &= A_{4\text{D}}^A + \text{Ret}^A (d\psi - A_{4\text{D}}^0) \\ Y^A &= \frac{\text{Im}t^A}{\mathcal{V}^{1/3}} \end{aligned} \quad (2.2.33)$$

where the right-hand side of the equations are given by four-dimensional quantities, for example $\mathcal{V} = \mathcal{V}^{(\text{s})}$. This can be checked by a careful comparison of five- and four-dimensional action.

This is the so-called 4D-5D connection reported in [57, 58], see also related earlier work [59, 60]. Very often this connection turns out to be a useful way to generate new BPS solutions in five dimensions, by simply uplifting the known four-dimensional BPS solutions. See for example the discussions in chapter 5. Nevertheless, it should be stressed that, of course this procedure only gives solutions with at least one U(1) isometries in five dimensions.

2.2.3 Range of Validity and Higher Order Corrections

It is important to consider when the low-energy effective action discussed above is actually “effective”, namely a good description of the physics occurred. In particular, we would like to know when the classical BPS solutions give a reliable account of the system. For this purpose there are a few scales that are relevant, and we will discuss them beginning with type II compactification.

First of all, before compactification we want the D-branes not to be too light in the ten-dimensional Planck units. In other words, we have to consider

the correction by the creation and annihilation of virtual D-branes unless the D-brane tension (1.3.20) in ten-dimensional Planck unit (1.2.8)

$$\tau_{\text{D}p\text{-brane}} (\ell_{\text{P}}^{(10)})^{1+p} \sim g_s^{\frac{p-3}{4}} \quad (2.2.34)$$

is large. Secondly, to suppress the four-dimensional quantum gravitational effect, we need the typical length scale, the radius of curvature near the horizon for example, to be large in the four-dimensional Planck unit. This gives a condition on the charges

$$S(\Gamma) \gg 1$$

and thus require that we consider *large charges* Γ , since the horizon area scales as charge squared. Thirdly, the α' stringy correction of the 4d Lagrangian is controlled by the size of the internal manifold in string unit. As a consequence, in order to suppress them we need to stay in the regime where the following is true,

$$\mathcal{V}^{(s)} \gg 1,$$

namely the *large radius* regime. Finally, for the suppression of the stringy effect on the spacetime scale, in particular at the scale of the black hole horizon, we cannot go all the way to the decompactification limit $\mathcal{V}^{(s)} \rightarrow \infty$ either. In other words, we need the horizon to be very large in the string scale. In formulas this means that we require

$$\frac{S(\Gamma) (\ell_{\text{P}}^{(4)})^2}{\ell_s^2} \sim S(\Gamma) \frac{g_s^2}{\mathcal{V}^{(s)}} \gg 1,$$

and also because going to the decompactification limit means bad spectrum contamination from the KK-modes.

After discussing the range of validity for various kinds of corrections, let's now take a look at the nature of the corrections. As mentioned before, away from the singularities, local supersymmetry dictates a decoupling between the hypermultiplets Lagrangian from the rest, namely the supergravity and vector multiplets degrees of freedom. Specifically, as we have seen implicitly in the derivation of the BPS solutions, these solutions are specified by the supergravity and vector multiplets degrees of freedom alone and the hypermultiplets can have any vev in these solutions.

In both type IIA and IIB compactifications, the dilaton field sits in the universal hypermultiplet, as recorded in Table 2.3. Therefore the good news is that we don't have to worry about g_s -corrections in the supergravity and vector multiplets Lagrangian. In string theories we have to consider α' -corrections as well. In the compactification setting, the α' -corrections to the lower-dimensional effective theory is controlled by the size of the internal manifold

in string unit $\mathcal{V}^{(s)} = \frac{1}{6} D_{ABC} J^A J^B J^C$, which is given by the Kähler moduli. The conclusion one can draw from this fact is the non-renormalizability property that the effective action of the type IIB compactification is exact in both α' and g_s , while the effective action of the type IIA compactification receives both the perturbative world-sheet loop α' -corrections and the non-perturbative world-sheet instanton α' -corrections. This gives us the possibility to compute the type IIA higher-order in α' -corrections employing the mirror symmetry discussed in the last section.

There is a special family of corrections, the F-term corrections, which is independent of the hypermultiplet fields and is therefore relevant for the correction of black hole entropies. These are studied in [61, 62, 63]. It can be shown that this family of corrections is computed by the topological string theory discussed in the last section [64, 24]. In particular, as discussed above (2.2.34), at the strong string coupling regime of type IIA string theory, which is better described in the language of M-theory, the correction to the four-dimensional F-term has its ten- (or eleven-) dimensional origin as the loop integral of virtual D2-D0 bound states which are light in the large g_s limit [31, 32]. This identification gives the expression of the topological strings free energy in terms of the so-called ‘‘Gopakumar-Vafa invariants’’ enumerating D2-D0 bound states as mentioned in (2.1.32).

Furthermore, the relation between the F-term correction and the topological strings leads to a natural conjecture between black hole BPS degeneracies and the topological strings, called the OSV conjecture [65], which in our convention reads

$$Z_{BH}(p^I; \phi^I) := \sum_{q_I} D(p^I, q_I) e^{-\pi \phi^I q_I} = |Z_{\text{top}}(t^A, g_{\text{top}})|^2 \quad (2.2.35)$$

$$t^A = \frac{ip^A + \phi^A}{ip^0 + \phi^0} \quad , \quad g_{\text{top}} = \frac{4\pi}{ip^0 + \phi^0} \quad ,$$

where the right-hand side is the topological strings partition function defined in (2.1.31). We will not go into the details about the higher-order derivatives nor OSV conjectures, since there are already many excellent reviews in the literature. See for example, [49, 50, 29] and references therein.