The spectra of supersymmetric states in string theory

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4 Black Holes and Multi-Holes

In section 2.2.2 we have introduced the $d = 4$, $\mathcal{N} = 2$ supergravity theory as the low-energy effective theory of type II compactification, together with the four-dimensional supersymmetric algebra and the concept of BPS states. In this section we will discuss the BPS solutions of this supergravity theory. In particular we will focus on stationary solutions, including the supersymmetric black hole and multi-hole solutions. As we will see, these solutions exhibit very interesting properties, which constitute part of the motivations for the research presented in the present thesis. Among them are the black hole attractor mechanism and the phenomenon of walls of marginal stability for multi-hole solutions.

This chapter is organised as follows. In the first section we present the generic stationary BPS solutions in a symplectic-invariant formulation. In the second section we discuss the attractor mechanism for solutions with a single black hole. In section three we focus on multi-hole solutions which contain multiple black holes, and summarise their angular momentum, moduli dependence and other existence criteria. In section four, we first unwrap the equations in the previous sections and rewrite them in components given a basis of the symplectic bundle, so that we obtain the set of equations ready to be used in actual calculations. After that, we specialise in the type IIA setup and present the explicit solutions in details, which will be needed for the next chapter.

4.1 General Stationary Solutions

Here we are mainly interested in the stationary solutions of the $\mathcal{N} = 2$, D=4 supergravity theories described in section 2.2.2, satisfying the BPS bound and preserving some supersymmetries. Using the stationary and flat base space

Ansatz

$$ds^2 = -\frac{\pi}{S(\vec{x})}(dt+\omega)^2 + \frac{S(\vec{x})}{\pi}dx^i dx^i \ , \tag{4.1.1}$$

where $S : \mathbb{R}^3 \to \mathbb{R}^+$ is a positive-definite function and $\omega = \omega_i, dx^i$ is a spatial one-form. The supergravity and vector multiplet part of the action (2.2.25) can be written in a not manifestly spacetime covariant but manifestly symplectic invariant form [54]

$$-16\pi G_{\rm N}^{(4)} L = (\mathcal{G},\mathcal{G}) - 4\sqrt{\frac{\pi}{S}}\,(Q+d\alpha+\frac{\pi}{2S}\star d\omega)\wedge\langle\mathcal{G},{\rm Im}(e^{-i\alpha}\boldsymbol{\Omega})\rangle + \text{ tot. der. } . \tag{4.1.2}$$

This action in a very concise form takes some explanation. First of all, the $\star$ without any subscript refers to the Hodge star with respect to the three-dimensional flat metric of the base space, d refers to the derivative in $\mathbb{R}^3$, and $\alpha : \mathbb{R}^3 \to \mathbb{R}$ is at this point an arbitrary function. Also recall that $\boldsymbol{\Omega}$ is the normalised version of the section of the symplectic bundle $\mathcal{E}$ times the line bundle $\mathcal{L}$ defined in (2.2.4) as

$$\boldsymbol{\Omega} = \frac{\Omega}{\sqrt{i\langle\Omega,\bar{\Omega}\rangle}} = e^{\mathcal{K}/2}\,\Omega \ , \tag{4.1.3}$$

and Q is the connection one-form (2.2.7) for it.

Finally, $\mathcal{G}$ is a spatial two-form taking value in the symplectic bundle $\mathcal{E}$ over $\mathbb{R}^3$. Especially it is a spatial two-form times an element of $H^3(Y)$ in the type IIB and a spatial two-form times an element of $H^{2*}(X)$ in the type IIA setup. It is given by the spatial part $\boldsymbol{\mathcal{F}}$ of the field strength $\mathcal{F}$ in ten dimensions (2.2.26) as

$$\mathcal{G} = \boldsymbol{\mathcal{F}} - 2\sqrt{\frac{S}{\pi}}\,{\rm Im}\star D\,(e^{-i\alpha}\boldsymbol{\Omega}) + 2\sqrt{\frac{\pi}{S}}\,{\rm Re}D\,(e^{-i\alpha}\boldsymbol{\Omega}\,\omega)\,,$$

where

$$D = d + i\left(Q + d\alpha + \frac{\pi}{2S}\star d\omega\right)$$

and the spatial three-form $(\mathcal{G},\mathcal{G})$ is defined as

$$(\mathcal{G},\mathcal{G}) = \frac{\pi}{S}\,\frac{1}{1-\frac{\pi^2}{S^2}\,\omega^2}\int_{\text{C-Y}} \mathcal{G}\wedge\star\mathcal{G} - \frac{\pi^2}{S^2}(\mathcal{G}\wedge\omega)\wedge\star(\mathcal{G}\wedge\omega) + \frac{\pi}{S}\mathcal{G}\wedge\star(\omega\wedge\star\mathcal{G})\ .$$

Finally we remark that, to justify the metric Ansatz (4.1.1) one should actually treat the above action (4.1.2) as an effective action that has to be supplemented by a constraint [76].

From the form of the effective action (4.1.2) we see that a local minimum of the action is given by

$$Q + d\alpha + \frac{\pi}{2S} \star d\omega = \mathcal{G} = 0 \, , \tag{4.1.4}$$

which leads to the following solution of the metric, the scalars and the gauge fields in terms of a set of harmonic functions , namely $H : \mathbb{R}^3 \to H^3(X, \mathbb{R})$ for type IIB and $H : \mathbb{R}^3 \to H^{2*}(X, \mathbb{R})$ for type IIA compactifications:

$$2\sqrt{\frac{S}{\pi}}\, \mathrm{Im}(e^{-i\alpha}\mathbf{\Omega}) = -H \tag{4.1.5}$$

$$d\omega = \star\langle dH, H\rangle \tag{4.1.6}$$

$$\mathcal{A} = 2\sqrt{\frac{\pi}{S}}\, \mathrm{Re}(e^{-i\alpha}\mathbf{\Omega})\,(dt + \omega) + \mathcal{A}_d \tag{4.1.7}$$

where the Dirac part of the gauge field is given by

$$d\mathcal{A}_d = \star dH \, . \tag{4.1.8}$$

The Dirac part of the gauge field is easy to solve since the equation is linear and we have already seen the solution to the single-monopole case in our construction of the Taub-NUT space (1.3.10). In what follows we will therefore refer to this part of the gauge fields simply as $\mathcal{A}_d$ and focus on other more complicated parts of them.

By imposing the condition that metric approaches that of a flat Minkowski space with the usual normalisation and that $\int_{S^2_{(i)}} \mathcal{F} = \Gamma_i$ around a point source of charge Γ_i, we see that the harmonic functions are given by the charges and the asymptotic moduli as

$$H = \sum_i \frac{\Gamma_i}{r_i} + h = \sum_i \frac{\Gamma_i}{|\vec{x} - \vec{x}_i|} - 2\mathrm{Im}(e^{-i\alpha}\mathbf{\Omega})|_\infty \, , \tag{4.1.9}$$

where $\vec{x}_i$ is location of the i-th center in the flat 3d base and "∞" denotes that the expression should be evaluated at spatial infinity. In other words, the constant term in the harmonic function is determined in terms of the asymptotic moduli and the total charges of the solution.

Recalling that the central charge function is defined as (2.2.5)

$$Z(\Gamma; \mathbf{\Omega}) = \langle \Gamma, \mathbf{\Omega}\rangle \tag{4.1.10}$$

for Γ is a combination of three- (even-) forms in the type IIB (IIA) language. From the form of (4.1.5) we now define

$$Z(\Gamma) = Z(\Gamma; \mathbf{\Omega}^*(\Gamma))$$

with $\mathbf{\Omega}^*(\Gamma)$, the so-called attractor moduli, satisfying the equation

$$2\sqrt{\frac{S(\Gamma)}{\pi}}\,\mathrm{Im}(e^{-i\alpha}\mathbf{\Omega}^*(\Gamma)) = -\Gamma\ .$$

Contracting (4.1.5) with $\langle,\mathbf{\Omega}\rangle$ we then get

$$Z(H(\vec{x})) = e^{i\alpha(\vec{x})}\sqrt{\frac{S(H(\vec{x}))}{\pi}}\ , \tag{4.1.11}$$

and from the boundary condition, in particular the correct fall-off of the angular momentum one-form ω, we conclude that

$$e^{i\alpha}|_\infty = \frac{Z(\Gamma;\mathbf{\Omega}_\infty)}{|Z(\Gamma;\mathbf{\Omega}_\infty)|}\ . \tag{4.1.12}$$

Furthermore, from 4.1.11 one can see that the solution saturates the BPS bound

$$M = |Z(\Gamma;\mathbf{\Omega})|_\infty$$

where $\Gamma = \sum_i \Gamma_i$ is the total charge, and therefore preserves four unbroken supersymmetry.

Our exposition here is similar to that in [54]. See also [77] for another construction of these stationary solutions, with R^2 corrections included.

4.2 Extremal Black Holes and Attractor Mechanism

In this section we will focus on the static black hole solutions. Especially we will discuss an important property of them, namely the attraction mechanism for the scalar fields of the theory, in more details.

Using (4.1.11) one can now write (4.1.5) as

$$-2\mathrm{Im}(\bar{Z}(H)\mathbf{\Omega}^*(H)) = H\ . \tag{4.2.1}$$

It's then obvious that the solution of scalar fields is invariant under a rescaling of the harmonic functions, namely

$$\mathbf{\Omega}^*(\lambda H) = \mathbf{\Omega}^*(H) \quad \forall\, \lambda \in \mathbb{R}\,.$$

Especially, considering now a solution with only one center and with arbitrary asymptotic moduli $\mathbf{\Omega}|_\infty$

$$H = \frac{\Gamma}{r} - 2\mathrm{Im}(e^{-i\alpha}\mathbf{\Omega})|_\infty\ .$$

In this case we see that the solution is spherically symmetric and static with $\omega = 0$. From the fact that

$$|Z(H(\vec{x}))|^2 = \frac{S(H(\vec{x}))}{\pi} = -\frac{1}{g_{00}} \to \infty \quad \text{as} \quad r \to 0$$

we also conclude that there is an event horizon at the center $r = 0$. Furthermore one can read out the area of the spherical horizon, which is

$$A = 4S(\Gamma) \ . \tag{4.2.2}$$

Therefore $S(\Gamma) = \frac{A}{4} = \pi |Z(\Gamma)|^2$ is the macroscopic Bekenstein-Hawking entropy of this extremal black hole, a fact that justifies our choice of notation. For the thermodynamical properties of extremal black holes and the semi-classical analysis of them, see for example [78] and references therein.

Near the center $r \to 0$, the equation for the moduli $\mathbf{\Omega}$ (4.2.1) gives

$$-2\mathrm{Im}(\bar{Z}(\Gamma)\mathbf{\Omega}^*(\Gamma)) = \Gamma \ . \tag{4.2.3}$$

The magic of the above equation is, no matter what the asymptotic moduli $\mathbf{\Omega}|_\infty$ is, near the center of a black hole the moduli is always fixed to be at the "attractor point" or "attractor value" given by the above equation $\mathbf{\Omega}|_{r\to 0} = \mathbf{\Omega}^*(\Gamma)$. This is the so-called attractor mechanism for a single black hole solution and (4.2.3) is called the attractor or the stabilisation equation [79].

To understand the attractor mechanism better, it's illuminating to look at the so-called attractor flow equation, which is obtained by taking the derivative of (4.1.5). Using the covariant derivative (2.2.7) we get

$$dH = \sqrt{\frac{S}{\pi}} \{ [Q + d\alpha + \frac{i}{2} d\log(\frac{S}{\pi})] \, e^{-i\alpha} \, \mathbf{\Omega} + i e^{-i\alpha} \, \mathcal{D}\mathbf{\Omega} + \text{ c.c.} \}$$

Contracting the above equation with $\mathcal{D}_A\Omega$, and writing $\mathcal{D}\Omega = dz^B \, \mathcal{D}_B\Omega$ and using the Kähler metric (2.2.10), we get

$$\frac{\partial z^A}{\partial r} = e^{i\alpha} \, \frac{1}{r^2} \sqrt{\frac{\pi}{S}} \, g^{A\bar{B}} \, \bar{\mathcal{D}}_{\bar{B}} \bar{Z} \ .$$

From this we see that the phase of the central charge is constant along the radial evolution, and the radial evolution of the central charge $Z(\Gamma; \mathbf{\Omega})$ is given by

$$\frac{\partial |Z|}{\partial r} = \frac{1}{2|Z|} \left(\bar{Z} \mathcal{D}_r Z + \text{ c.c.} \right) = \frac{1}{r^2} \sqrt{\frac{\pi}{S}} \, |\mathcal{D}Z|^2 \geq 0$$

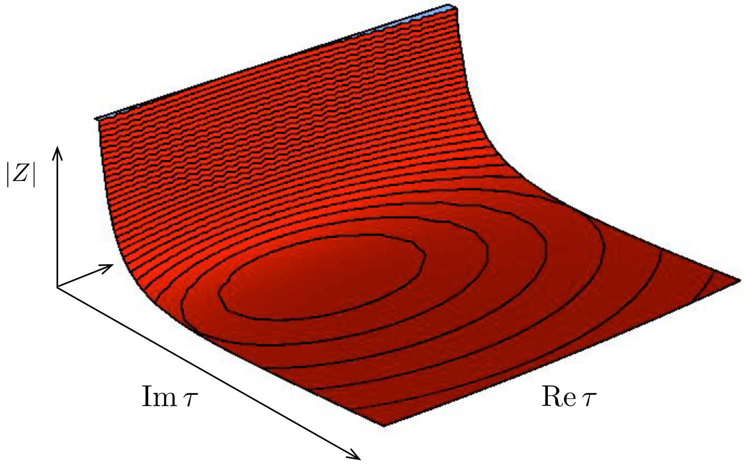

Figure 4.1: The magnitude of the central charge $|Z(\Gamma,\tau)|$ in K3$\times T^2$ compactification as a function of the torus modulus τ, with the rest of the moduli already fixed at the attractor point. The contour lines are the constant $|Z(\Gamma,\tau)|$ line and the function has a unique minimum at the attractor value τ^*.

where $|\mathcal{D}Z|^2 = g^{A\bar{B}}\mathcal{D}_A Z \bar{\mathcal{D}}_{\bar{B}}\bar{Z}$. This means that the value of the central charge always increases when one moves further and further away from the black hole center in the spatial directions, or from the attractor point the in the moduli space. Furthermore, except for the point at infinity $r \to \infty$, the only place where the inequality is saturated is at the attractor point (or at the black hole horizon in the spacetime picture), where $\mathcal{D}Z = 0$. Indeed the attractor equation can be derived by requiring that the moduli renders the central charge to be at a local fixed point. To see this, note that $\mathcal{D}Z = \langle \Gamma, \mathcal{D}\mathbf{\Omega}\rangle = 0$ requires that the moduli must be such that

$$\Gamma = a\mathbf{\Omega} + a^*\,\bar{\mathbf{\Omega}}$$

for some complex number a. Contracting the above equation with $\langle\,,\bar{\mathbf{\Omega}}\rangle$ gives the value of a and then gives the attractor equation (4.2.3). In particular, for the type IIB setup the above equation has a simple geometric explanation that the moduli adjust themselves as one approaches black hole horizon such that the charge $\Gamma \in H^3(X,\mathbb{Z}) \in H^{3,0}(X) \oplus H^{0,3}(X)$.

4.3 Properties of Multi-holes

Many qualitative new features emerge when one considers more general solutions with more than one centers [54, 77, 80]. First of all, the solution will no longer have the $SO(3)$ spherical symmetry and will in general no longer be static. In other words, the rotation one-form ω, determined by (4.1.6) and boundary condition $\omega \to 0$ as $|\vec{x}| \to \infty$, will generically no longer vanish in the interior of the spacetime. As we will see later, this can be understood as due to the fact that, an electron and a monopole at different locations in the space create an electromagnetic field which induces an "intrinsic" angular momentum of the spacetime. Furthermore, there is a integrability condition for ω (4.1.6) [54]

$$d\, d\omega = d \star \langle dH, H \rangle = \langle d \star dH, H \rangle = 0 \;, \tag{4.3.1}$$

which can be regarded as the condition for the absence of closed timelike curves (CTC's) near the line segments connecting pairs of centers. This condition gives constraints on the possible locations of the centers and in turn constraints on the boundary condition for the scalars for a multi-hole solution with a certain charge distribution to exist.

In this section we will discuss the above-mentioned properties of these supersymmetric stationary solutions with multiple centers, which will be important in our future discussion.

4.3.1 Walls of Marginal Stability

As mentioned above, solutions with more than one center has to satisfy the integrability condition for the rotation one-form (4.3.1).

Plugging in the expression for the harmonic function (4.1.9), one obtains one equation for each center

$$\sum_j \frac{\langle \Gamma_i, \Gamma_j \rangle}{r_{ij}} = -\langle \Gamma_i, h \rangle \tag{4.3.2}$$

where $r_{ij} = |\vec{x}_i - \vec{x}_j|$ and $h = -2\mathrm{Im}(e^{-i\alpha}\mathbf{\Omega})|_\infty$, and the sum of the equations for all centers is trivially satisfied

$$\sum_i \sum_j \frac{\langle \Gamma_i, \Gamma_j \rangle}{r_{ij}} = 0 = -\sum_i \langle \Gamma_i, h \rangle \;. \tag{4.3.3}$$

In particular, for the case with two centers there is just one integrability condition which gives

$$r_{12} = |\vec{x}_1 - \vec{x}_2| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2} \frac{M}{\mathrm{Im}(Z_1 \bar{Z}_2)|_\infty} \tag{4.3.4}$$

where

$$M = |Z(\Gamma; \mathbf{\Omega})|_\infty = |Z(\Gamma_1; \mathbf{\Omega}) + Z(\Gamma_2; \mathbf{\Omega})|_\infty$$

is the ADM mass of the solution and $Z_1|_\infty = Z(\Gamma_1;\mathbf{\Omega})|_\infty$. This shows that a solution with two centers with non-zero intersection (Dirac-Zwanziger) product $\langle\Gamma_1,\Gamma_2\rangle \neq 0$ is really a bound state of the two centers, since their distance cannot be adjusted freely but instead is constrained to be a certain fixed value.

Furthermore, an interesting consequence of this is that the solution only exists in some part of the moduli space. More explicitly, from the fact that the distance must always be a positive number, we see that the moduli space is divided into two parts by the wall

$$\mathrm{Im}(Z_1\bar{Z}_2)|_\infty = 0\ , \tag{4.3.5}$$

and it is only possible to have a solution with centers of these charges at the side of the wall which satisfies

$$\langle\Gamma_1,\Gamma_2\rangle\,\mathrm{Im}(Z_1\bar{Z}_2)|_\infty > 0\ . \tag{4.3.6}$$

We will therefore call this co-dimenional one wall (4.3.5) “the wall of marginal stability for charge Γ_1 and Γ_2”.

The presence of the wall of marginal stability can be understood in the following way. A necessary condition for a bound state of two particles to decay is that the mass of the bound state is the same as the sum of the mass of each individual constituent

$$\begin{aligned} M &= M_1 + M_2 \Leftrightarrow |Z_1+Z_2|_\infty = |Z_1|_\infty + |Z_2|_\infty \\ &\Leftrightarrow \mathrm{Im}(Z_1\bar{Z}_2) = 0 \quad , \quad \mathrm{Re}(Z_1\bar{Z}_2) > 0\ . \end{aligned}$$

From this we conclude that the condition of a physical wall of marginal stability should be further implemented by the requirement that the two central charges are aligned rather than anti-aligned.

After determining the location of the walls of marginal stability in the moduli space, we still need to determine which side is the stable and which side is the unstable side. We will provide a heuristic derivation of it here.

Let us consider the scenario that the asymptotic moduli are fixed at the attractor value of the total charge $\mathbf{\Omega}|_\infty = \mathbf{\Omega}^*(\Gamma = \Gamma_1+\Gamma_2)$, then the attractor equation (4.2.3) gives

$$\mathrm{Im}(Z_1\bar{Z}_2)|_\infty = -\frac{1}{2}\langle\Gamma_1,\Gamma_2\rangle \quad \text{when} \quad \mathbf{\Omega}|_\infty = \mathbf{\Omega}^*(\Gamma)\ .$$

For this specific choice of the background moduli, we know that there exists a single black hole solution in which the background moduli is constant throughout the whole space. In other words there is no attractor “flow” in this case. By arguing that this configuration should be more energetically favorable than any other solution with spatial gradient of the scalar fields, including the two-centered solution considered above, we require that the total attractor point must lie on the unstable side of the wall, and derive (4.3.6) as a condition for existence of a specific two-centered solution.

4.3.2 Angular Momentum of the Spacetime

Another qualitatively new feature of the multi-hole solution with mutually non-local centers ($\langle\Gamma_i,\Gamma_j\rangle \neq 0$) is that the spacetime is no longer static but rather stationary with angular momentum. This can be seen from the expression for ω (4.1.6) and can be understood as a consequence of the fact that there are now electrons and monopoles at different points in the space and the electromagnetic fields therefore give contribution to the angular momentum.

Let's now solve for ω. From (4.1.6) and using the integrability constraint one obtains

$$d\omega = \sum_{i,j} \frac{\langle\Gamma_i,\Gamma_j\rangle}{2} \star \left(\frac{dr_i^{-1}}{r_j} - \frac{dr_i^{-1}}{r_{ij}} - (i \leftrightarrow j) \right) .$$

Recall that the star without subscripts denotes the Hodge star in three-dimensional flat base space. It is therefore enough to solve ω pair-wise as

$$\omega = \sum_{i,j} \frac{1}{2}\,\omega_{ij} = \sum_{i<j} \omega_{ij}$$

where

$$d\,\omega_{ij} = \langle\Gamma_i,\Gamma_j\rangle \star \left(\frac{dr_i^{-1}}{r_j} - \frac{dr_i^{-1}}{r_{ij}} - (i \leftrightarrow j) \right) .$$

Using again the elliptic coordinates as we have used for Eguchi-Hanson space (1.3.18) for the two centers $\vec{x}_i$ and $\vec{x}_j$, with now $2a = r_{ij}$, then the above equation together with the boundary condition that $\omega \to 0$ at spatial infinity gives

$$\omega_{ij} = \frac{\langle\Gamma_i,\Gamma_j\rangle}{2r_{ij}} \, \frac{\cosh\eta - 1}{\cosh^2\eta - \cos^2\theta} \, \sin^2\theta \, d\psi \, .$$

From the asymptotic fall-off

$$\omega_{ij} \to \frac{\langle\Gamma_i,\Gamma_j\rangle}{r} \sin^2\theta \, d\psi$$

we conclude that the conserved angular momentum of the spacetime is

$$\vec{J} = \frac{1}{4} \sum_{i,j} \langle\Gamma_i,\Gamma_j\rangle \, \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|} \, .$$

Furthermore, by solving for ω without using the integrability condition, one can show that there will in general be closed timelike curve (CTC) around the line segment $\vec{x}_i - \vec{x}_j$ in the $\mathbb{R}^3$ base, and the length of the line segment such that this does not happen is exactly the length ordained by the integrability condition. One can therefore also interpret the integrability condition (4.3.2) as the physical constraint of the impossibility of time machines.

4.3.3 Split Attractor Flow

In (4.1.5)-(4.1.8) we have seen how to solve for the solution given a configuration of centers with particular charges and background moduli. In (4.3.2) we saw that not all arbitrary configurations permit a solution, because the integrability condition does not always permit a solution for any given background moduli. In fact the situation is even more subtle than that. Namely, the satisfaction of the integrability condition alone is not sufficient to establish the existence of a solution. This is because there is another condition that the solution must satisfy to qualify as a physical solution, namely that the central charge function $Z(H;\mathbf{\Omega})$ does not hit a zero anywhere in the space

$$\frac{S(H(\vec{x}))}{\pi} = |Z(H;\mathbf{\Omega})|^2 > 0 \qquad \text{for all } \vec{x} \in \mathbb{R}^3 \,. \tag{4.3.7}$$

In general, this is of course a very difficult condition to check, because it is a non-local condition in the sense that it has to be satisfied everywhere in the space. In relation to this difficulty we quote here the very useful split attractor flow conjecture, which states that a solution exists if and only if a split attractor flow tree exists in moduli space, which starts at the asymptotic value of the scalars and terminates at the attractor points of each constituent back hole. See [54, 30] for more details.

4.4 In Coordinates

In the previous sections in this chapter, we have written down general stationary supersymmetric solutions of $d = 4$, $\mathcal{N} = 4$ supergravity theories, including the static single-hole solutions and the rotating multi-hole solutions. All this was done in a symplectic-invariant formulation. In practice, when we are working with a specific theory we often need to express the solutions in components, in order to be able to directly interpret them.

In this section we will therefore take up the straightforward task of unwrapping the equations in previous sections and rewriting them in terms of components using a specific basis for the symplectic bundle. These rewritten equations are then ready to be used in actual calculations. Specifically, we work out in detail the solutions for the supergravity theory one gets from type IIA compactifications, which will be needed in the next chapter of this part of the thesis.

As discussed in [80], a solution can be explicitly written in terms of a single function, called the entropy function

$$S(\Gamma) = \pi|Z(\Gamma)|^2 = \pi|Z(\Gamma;\mathbf{\Omega}^*(\Gamma))|^2 \ ,$$

where the attractor moduli $\mathbf{\Omega}^*(\Gamma)$ of charge Γ are given by the attractor equation (4.2.3). Given such an entropy function, the full solution basically follows from just replacing the charge vector Γ with the harmonic functions H. Let's begin with writing down the moduli, the charge, and the harmonic functions in components. Using (2.2.4), (2.2.9) and (2.2.20)

$$\mathbf{\Omega} = e^{\mathcal{K}/2}\left(-X^I\alpha_I + F_I\beta^I\right)$$

and write the charge vector as

$$\Gamma = p^I\alpha_I + q_I\beta^I \tag{4.4.1}$$

and similarly for the harmonic functions

$$H(\vec{x}) = H^I(\vec{x})\alpha_I + H_I(\vec{x})\beta^I \ , \tag{4.4.2}$$

we obtain the magnetic part of the attractor flow equations in components by contracting (4.1.5) with $\langle\,,\beta^I\rangle$

$$H^I = 2e^{\mathcal{K}/2}\,\mathrm{Im}(\bar{Z}X^I) \ .$$

Furthermore, from

$$\frac{1}{\pi}\frac{\partial}{\partial H_I}S(H) = \frac{\partial}{\partial H_I}|Z(H)|^2 = 2e^{\mathcal{K}/2}\,\mathrm{Re}(\bar{Z}X^I)$$

we get the full solution for the scalar fields

$$\begin{aligned} 2e^{\mathcal{K}/2}\bar{Z}X^I(H(\vec{x})) &= iH^I + \frac{1}{\pi}\frac{\partial}{\partial H_I}S \\ \Rightarrow t^A(H(\vec{x})) &= \frac{X^A}{X^0} = \frac{iH^A + \frac{1}{\pi}\frac{\partial}{\partial H_A}S}{iH^0 + \frac{1}{\pi}\frac{\partial}{\partial H_0}S} \ . \end{aligned} \tag{4.4.3}$$

On the other hand, the vector fields are given by contracting (4.1.7) with $\langle\,,-\beta^I\rangle$

$$A^I = \frac{1}{S}\frac{\partial S}{\partial H_I}(dt+\omega) - A^I_d \quad , \quad dA^I_d = \star dH^I \ . \tag{4.4.4}$$

As an illustration and for later use, let's work out the entropy function for the type IIA case (2.2.17) with

$$\mathbf{\Omega} = -e^{\mathcal{K}/2}\, e^{-t} \ .$$

First of all, the electric part of the attractor equation is obtained by contracting (4.2.3) with $\langle\ , \alpha_I\rangle$:

$$\begin{aligned} 2e^{\mathcal{K}/2}\,\mathrm{Im}\left(\bar{Z}\frac{(t^2)_A}{2}\right) &= q_A && (4.4.5)\\ 2e^{\mathcal{K}/2}\,\mathrm{Im}\left(\bar{Z}\frac{(t^3)}{6}\right) &= -q_0\ , && (4.4.6)\end{aligned}$$

where we have used the short hand notation $(t^2)_A \equiv D_{ABC}t^Bt^C$ and $t^3 = D_{ABC}t^At^Bt^C$. Define the variable y^A and L by [81]

$$\begin{aligned} \frac{\partial}{\partial q_0}\big(\frac{S}{\pi}\big)^2 &= 4(p^0)^2L \\ \frac{\partial}{\partial q_A}\big(\frac{S}{\pi}\big)^2 &= 4p^0[-y^A(\frac{y^3}{6}) + p^AL]\ , \end{aligned} \qquad (4.4.7)$$

and using

$$2e^{\mathcal{K}/2}\bar{Z}X^I = ip^I + \frac{1}{\pi}\frac{\partial}{\partial q_I}S \quad , \quad t^A = \frac{X^A}{X^0}\ ,$$

the electric half of the attractor equation (4.4.5) can be written as

$$\begin{aligned} (y^2)_A &= -2q_A + \frac{(p^2)_A}{p^0} \\ L &= -\frac{q_0}{2} - \frac{p\cdot q}{2p^0} + \frac{p^3}{(p^0)^2}\ . \end{aligned} \qquad (4.4.8)$$

Plugging them back into (4.4.7) we finally obtain the expression for the entropy function

$$S(\Gamma) = 2\pi\sqrt{p^0Q^3 - (p^0)^2\,L^2}\ , \qquad (4.4.9)$$

where $Q^3 = (\frac{y^3}{6})^2$.

This entropy function, as promised, gives now the full solution including the scalar fields and vector fields

$$\begin{aligned} t^A(H(\vec{x})) &= \left(\frac{H^A}{H^0} - \frac{L}{Q^{3/2}}y^A\right) + i\frac{S}{2\pi}\frac{y^A}{H^0Q^{3/2}} && (4.4.10)\\ A^0(H(\vec{x})) &= 2\big(\frac{\pi}{S}\big)^2(H^0)^2L(dt+\omega) - A^0_d && (4.4.11)\\ A^A(H(\vec{x})) &= 2\big(\frac{\pi}{S}\big)^2H^0\,[-Q^{3/2}y^A + H^AL](dt+\omega) - A^A_d\ , && (4.4.12)\end{aligned}$$

where $y^A(H)$, $L(H)$ and $S(H)$ are given by (4.4.8) and (4.4.9) but now with the charges p^I, q_I replaced by the corresponding harmonic functions H^I, H_I.

But often we will find the above form of the attractor equation clumsy to use. It is easy to see that the equations (4.4.8) is not ready to be solved when $p^0 = 0$. More generally, from the form of the entropy function (4.4.9), we have to worry about the warp factor of the spacetime $S(H(\vec{x}))$ vanishes near the region $H^0 \to 0$, or from the scalar solution (4.4.10) we have to worry that the Calabi-Yau decompactifies near this region. But actually these are just consequences of the specific variables (4.4.7) we have used. It is therefore sometimes desirable to use a different set of variables to write our solutions with.

Instead of defining variables y^A and L as in (4.4.7), we now define instead ι^A and ℓ by [75]

$$
\begin{aligned}
\frac{\partial}{\partial q_0}(\frac{S}{\pi})^2 &= 4\ell \\
\frac{\partial}{\partial q_A}(\frac{S}{\pi})^2 &= 4[-\mathcal{Q}^{3/2}(p^A + p^0\iota^A) + p^A\frac{\ell}{p^0}] ,
\end{aligned}
\tag{4.4.13}
$$

where $\mathcal{Q}^{3/2}$ is given by ι^A and ℓ to be

$$
\begin{aligned}
\mathcal{Q}^{3/2} &= \ell + (p^0)^2\lambda \\
\lambda &= \frac{q_0}{2} - \frac{\iota\cdot q}{3} - \frac{p\iota^2}{12} .
\end{aligned}
\tag{4.4.14}
$$

Again we have used the shorthand notation $p\iota^2 = D_{ABC}p^A\iota^B\iota^c$.

Now the attractor equations in terms of these new variables ι^A and ℓ are

$$
\begin{aligned}
D_{ABC}p^B\,\iota^C &= -q_A - \frac{p^0}{2}(\iota^2)_A \\
\ell &= -(p^0)^2\,\frac{q_0}{2} - p^0\,\frac{p\cdot q}{2} + \frac{p^3}{6} ,
\end{aligned}
\tag{4.4.15}
$$

and the entropy function becomes

$$
S(\Gamma) = 2\pi\sqrt{2\lambda\ell + (p^0)^2\lambda^2} .
$$

Especially

$$
S(\Gamma) = 2\pi\sqrt{\frac{p^3}{6}\,\hat{q}_0} = 2\pi\sqrt{\frac{p^3}{6}\,(q_0 - \frac{1}{2}D^{AB}q_Aq_B)} \quad \text{when } p^0 = 0, \tag{4.4.16}
$$

where D^{AB} is the inverse of $D_{AB} = -D_{ABC}p^C$.

The full solution for the scalar and the vector fields now reads

$$\begin{aligned} t^A(H(\vec{x})) &= \mathcal{Q}^{-3/2}\{(-\ell\iota^A + H^0H^A\lambda) + i\frac{S}{2\pi}(H^A + H^0\iota^A)\} \\ A^0(H(\vec{x})) &= 2\big(\frac{\pi}{S}\big)^2\ell(dt+\omega) - A^0_d \\ A^A(H(\vec{x})) &= 2\big(\frac{\pi}{S}\big)^2[(-\mathcal{Q}^{3/2}\iota^A - H^0H^A\lambda](dt+\omega) - A^A_d\ , \end{aligned}$$

where ι^A, $\mathcal{Q}^{3/2}$ and ℓ are again solutions to (4.4.15) with the charges p^I, q_I replaced by the harmonic functions H^I, H_I. From the above form of the solution we see that the region with $H^0 \to 0$ is not more prone to singularity than other regions, and verify the claim we made earlier that these are better variables to use in these situations.