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### The spectra of supersymmetric states in string theory

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# 6 A Farey Tail for Attractor Black Holes

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## 6.1 Introduction

One of the main successes of string theory has been the microscopic explanation of black hole entropy. The microstates for extremal BPS black holes are well understood in theories with 16 or more supercharges. This includes the D1-D5-P system in type IIB theory on  $K3 \times S^1$  for which the microstates are represented by the elliptic genus of a (4,4) CFT with target space given by a symmetric product of  $K3$  [113, 43]. The elliptic genus for this target space can be explicitly computed, leading to a concrete and exact expression for the number of BPS-states.

The D1-D5-P system has a well understood dual description in terms of type IIB theory on  $K3 \times AdS_3 \times S^3$ . A rather remarkable result is that the elliptic genus has an exact asymptotic expansion, which has a natural interpretation as a sum over semi-classical contributions of saddle-point configurations of the dual supergravity theory. This exact asymptotic expansion, together with its semi-classical interpretation, has been coined the Black Hole Farey Tail [112, 111]<sup>1</sup>. Although the Farey tail was first introduced in the context of the D1-D5 system, it applies to any system that has a microscopic description in terms of a (decoupled) 2d conformal field theory and has a dual description as a string/supergravity theory on a spacetime that contains an asymptotically  $AdS_3$ .

The aim of this part of the thesis is to apply the generalised Rademacher formula [111] to black holes in theories with eight supercharges and in this way

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<sup>1</sup>In the paper [108] we based our analysis on the Farey Tail expansion developed in [112], which was later improved in [111]. In this part of the thesis I will use the “Modern Farey Tail” of [111] without any derivation and refer the reader to the paper and the PhD thesis of fellow student Jan Manschot for further details.

extend the Farey Tail to  $\mathcal{N} = 2$  (or “attractor” in a slight abuse of language) black holes. Specifically, we consider M-theory compactified on a Calabi-Yau three-fold  $X$  and study the supersymmetric bound states of wrapped M5-branes with M2-branes. These states correspond to extremal four dimensional black holes after further reduction on a circle. For this situation a microscopic description was proposed quite a while ago by Maldacena, Strominger and Witten (MSW) [95], who showed that the black hole microstates are represented by the supersymmetric ground states of a (0,4) conformal field theory. These states are counted by an appropriately defined elliptic genus of the (0,4) CFT.

The interest in attractor black holes has been revived in recent years due to the connection with topological string theory discovered in [61, 65] and subsequently studied by many different authors. As was discussed in (2.2.35), it was conjectured by Ooguri, Strominger and Vafa (OSV) in [65] that the mixed partition function of 4d BPS black holes is given by the absolute value squared of the topological string partition function. Earlier, in a separate development, a different connection between BPS states and topological strings was discovered by Gopakumar and Vafa (GV) [32], who showed that topological string theory computes the number of five-dimensional BPS-invariants of wrapped M2 branes in M-theory on a Calabi-Yau. The GV-result differs from the OSV-conjecture (2.2.35) in the sense that the topological string coupling constant appears in an S-dual way. Recently, this aspect of the OSV conjecture have been considerably clarified in the work of Gaiotto, Strominger, and Yin [114]. These authors used the CFT approach of MSW to show that the elliptic genus of the (0,4) CFT has a low temperature expansion which (approximately) looks like the square of the GV-partition function. The OSV conjecture then follows from the modular invariance of the elliptic genus, which at the same time naturally explains the different appearances of the coupling constant.

In this part of the thesis we will show that elliptic genus of the (0,4) SCFT can be written as a generalised Rademacher series (or the “Modern Farey tail” expansion) similar to that of the previously studied (4,4) case.<sup>2</sup> An important property of the SCFT is the presence of a spectral flow that relates states with different charges, and implies that the elliptic genus can be expanded in terms of theta functions. The presence of these theta functions signal the presence of a set of chiral scalars in the SCFT, while from a spacetime point of view their appearance naturally follows from the Chern-Simons term in the effective action. We find that the (modern) Farey tail expansion contains subleading contributions to each saddle point that can be interpreted as being

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<sup>2</sup>Some related results were obtained independently in [30, 110, 109].

due to a virtual cloud of BPS-particles (actually, wrapped M2-branes) that are “light” enough such that they do not form a black hole by themselves. The degeneracies of these particles are, in the large central charge limit, given in terms of the Gopakumar-Vafa invariants. In this way we see that the results of [114] naturally fit in and to some extent follow from our Attractor Farey Tail.

The outline of this part of the thesis is as follows: in section 6.2 we review the bound states of wrapped M5 and M2 branes in M-theory on a Calabi-Yau three-fold and explain the emergence of the spectral flow. We then discuss the decoupling limit and the near horizon geometry and describe the dimensionally reduced effective action on  $AdS_3$ . In section 6.3 we turn to the M5 brane world-volume theory and its reduction to the (0,4) SCFT. Here we also define the generalised elliptic genus which counts the graded black hole degeneracies. In section 6.5 we interpret our result from the dual supergravity perspective and discuss its relation to the OSV conjecture. Finally in section 6.6 we conclude by summarising this part of the thesis and raise some open questions.

## 6.2 Wrapped M-branes and the Near Horizon Limit

To establish the notation, in this section we describe the BPS bound states of wrapped M5 and M2 branes in M-theory on a Calabi-Yau from a spacetime point of view. We will derive a spectral flow symmetry relating states with different M2 and M5 brane charges, first from an eleven-dimensional perspective and subsequently in terms of the effective three dimensional supergravity that appears in the near horizon limit.

### 6.2.1 Wrapped Branes on Calabi-Yau and the Spectral Flow

Consider M-theory on a Calabi-Yau threefold  $X$  and a circle  $\mathbb{R}^{3,1} \times X \times S^1$ , and an M5-brane wrapping a 4-cycle  $\mathcal{P}$  with  $[\mathcal{P}] = p^A S_A$  in the Calabi-Yau three-fold  $X$ . Here  $\{S_{A=1, \dots, h^{1,1}}\}$  is a basis of integral 4-cycles  $H_4(X, \mathbb{Z})$  in  $X$ , where  $h^{1,1}(X)$  is the second Betti number of the Calabi-Yau manifold. In order for this five-brane to be supersymmetric, the 4-cycle  $\mathcal{P}$  has to be realized as a positive divisor. Therefore we will assume that  $\mathcal{P}$  is a smooth ample divisor so that classical geometry is a valid tool for our analysis. There is a line bundle  $\mathcal{L}$  with

$$c_1(\mathcal{L}) = [\mathcal{P}] = p^A \alpha_A := P \tag{6.2.1}$$

associated to this divisor, where  $\alpha_A \in H^2(X, \mathbb{Z})$  is a basis of harmonic 2-forms Poincaré dual to the four-cycles  $S_A$ , such that the position of the four-cycle  $\mathcal{P}$  can be thought of as the zero locus of a section of this line bundle.

The wrapped M5-brane reduces to a string in the remaining five dimensions. In addition there are five-dimensional particles corresponding to M2-branes wrapping a two-cycle  $[\Sigma] = q_A \Sigma^A$ , where  $\Sigma^A$  is a basis of  $H_2(X, \mathbb{Z})$  dual to  $\{S_A\}$ , i.e.  $\Sigma^A \cap S_B = \delta_B^A$ . These particles carry charges  $q_A$  under the  $U(1)$  gauge fields  $A^A$  which arise from the dimensional reduction of the M-theory 3-form  $A^{(3)}$ , as summarised in Table 2.3,

$$A^{(3)} = \sum_A A^A \wedge \alpha_A . \quad (6.2.2)$$

Such an ensemble of strings and particles can form a BPS bound state which leaves four of the eight supersymmetries unbroken.

Eventually we are interested in the BPS states of the 4d black hole that is obtained by further compactifying the string along an  $S^1$ . These states carry an additional quantum number  $q_0$  related to the Kaluza-Klein momentum along the string. From the four dimensional perspective, the quantum numbers  $(p^A, q_A, q_0)$  are the D4, D2 and D0 brane charges in the type IIA compactification on  $X$ . In this part of the thesis we will be switching back and forth between a spacetime perspective from eleven (M-theory), ten (type IIA), five (M-th/CY), four (IIA/CY), or even three dimensions ( $AdS_3$ ), and a world-volume perspective of the M5-brane or its reduction to a world-sheet.

Before going to the world-volume description of the M5-brane and its reduction to a string, let us describe the spectral flow symmetry of BPS states from the spacetime perspective.

First recall that, as discussed in section 2.2.2 and 4, a supergravity solution with  $U(1)$  isometry of the low-energy effective action obtained from compactifying M-theory on a Calabi-Yau space  $X$  can be thought of as the “lift” of a four-dimensional solution of the  $\mathcal{N} = 2$ ,  $d = 4$  supergravity theory and is specified by  $2h^{1,1} + 2$  harmonic functions  $H^I, H_I : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,  $I = 0, \dots, h^{1,1}$ . There is a symmetry between different solutions whose corresponding set of harmonic functions are related by [75, 115]

$$\begin{aligned} H^0 &\rightarrow H^0 \\ H^A &\rightarrow H^A - H^0 k^A \\ H_A &\rightarrow H_A - D_{ABC} H^B k^C + \frac{H^0}{2} D_{ABC} k^B k^C \\ H_0 &\rightarrow H_0 + k^A H_A - \frac{1}{2} D_{ABC} H^A k^B k^C + \frac{H^0}{6} D_{ABC} k^A k^B k^C , \end{aligned} \quad (6.2.3)$$

where  $D_{ABC}$  is the triple-intersection number for the basis  $\alpha_A$  introduced in (2.2.18). From (4.4.17) or equivalently (4.4.10), and (2.2.33), we see that the

above transformation in particular leaves the geometry part of the solution invariant and induces a shift in the five-dimensional vector field by

$$A^A \rightarrow A^A - k^A d\psi, \quad (6.2.4)$$

where  $\psi \in [0, 1)$  is the coordinate of the M-theory circle  $S^1$ . In particular, for the case at hand we have  $H^0 = 0$  and the above transformation corresponds to the following shift of charges

$$p^A \rightarrow p^A$$

$$q_A \rightarrow q_A - D_{ABC} k^B p^C \quad (6.2.5)$$

$$q_0 \rightarrow q_0 + k^A q_A - \frac{1}{2} D_{ABC} k^A k^B p^C, \quad (6.2.6)$$

which leaves the geometry part of the solution invariant.

From M-theory point of view, the above symmetry among solutions of the five-dimensional supergravity can be understood in the following way. The low energy action of M-theory (1.2.1) contains the Chern-Simons coupling

$$S_{CS} = -\frac{1}{3!} \int A^{(3)} \wedge F^{(4)} \wedge F^{(4)}, \quad (6.2.7)$$

where  $F^{(4)} = dA^{(3)} = F^A \wedge \alpha_A$  is the four-form field strength. As a result, the M2-brane charge is defined as (here we work in 11D planck units)

$$q_A = \int_{S^2 \times S^1 \times D_A} (*F + C \wedge F). \quad (6.2.8)$$

The charge thus contains a Chern-Simons type contribution depending explicitly on the  $A^{(3)}$ -field. This term can be written as a volume integral of  $F \wedge F$  and hence is invariant under small gauge transformations that vanish at infinity. However, it can still change under large gauge transformations corresponding to shifts in  $A^{(3)}$  by a closed and integral three-form

$$A^{(3)} \rightarrow A^{(3)} - \sum_{A=1, \dots, h^{1,1}} k^A d\psi \wedge \alpha_A, \quad (6.2.9)$$

which gives exactly the shift of the lower-dimensional gauge field (6.2.4) upon dimensional reduction. This transformation should be an exact symmetry of M-theory. The value of the charge  $q_A$ , though, is not invariant but instead receives an extra contribution proportional to the M5-brane charge  $p^A$ . Namely, using

$$\int_{S_A \times S^2 \times S^1} d\psi \wedge \alpha_B \wedge F = D_{ABC} \int_{S^2} F^C = D_{ABC} p^C, \quad (6.2.10)$$

one finds that

$$q_A \rightarrow q_A - D_{ABC} k^B p^C \quad (6.2.11)$$

under a large gauge transformation of the  $A^{(3)}$  field.

Alternatively, this shift of the conserved charges can also be understood from a type IIA point of view. From Table 2.3 we see that the integral shift of the three-form fields now translates into a shift of internal Neveu-Schwarz B-field

$$B^{(2)} \rightarrow B^{(2)} - \sum_{A=1, \dots, h^{1,1}} k^A \alpha_A, \quad (6.2.12)$$

which should be an exact symmetry of string theory. From the Wess-Zumino part of the D-brane action (1.3.28) of a D4 brane wrapping the four-cycle  $\mathcal{P}$  with  $[\mathcal{P}] = p^A S_A$ , we see that the presence of the factor  $ch(\mathcal{F}) = e^{F+B}$  causes the shift of the induced D2, D0 brane charges (6.2.5, 6.2.6) when the B-field is shifted<sup>3</sup>.

It will turn out to be convenient to introduce the symmetric bilinear form

$$D_{AB} = -D_{ABC} p^C = - \int_{\mathcal{P}} \alpha_A \wedge \alpha_B, \quad (6.2.13)$$

where  $\alpha_A$  should be understood as the pullback of the harmonic 2-forms from the ambient Calabi-Yau  $X$  to the 4-cycle  $\mathcal{P}$ . Notice that the extra minus sign in the definition implies that the anti-self-dual directions now have positive signatures.

By the Lefschetz hyperplane theorem, which states that the inclusion map  $H^2(\mathcal{P}) \rightarrow H^2(X)$  is surjective and  $H^1(X)$  and  $H^1(\mathcal{P})$  are isomorphic, this form is non-degenerate. In fact, according to the Hodge index theorem it has signature  $(h^{1,1} - 1, 1)$ . Thus, for every positive divisor  $\mathcal{P}$  we obtain a natural metric  $D_{AB}$  on  $H^2(X, \mathbb{Z})$  which turns it into a Lorentzian lattice  $\Lambda = \Lambda^{h^{1,1}-1,1}$ .

Generically this lattice is not self-dual (or equivalently, unimodular), namely that the inverse metric  $D^{AB}$  is not integral. We will call the dual lattice, defined as the set of all vectors  $v^A \in \mathbb{R}^{h^{1,1}-1,1}$  such that the inner product with all vectors in  $\Lambda$  take integral values,  $\Lambda^*$ .

This dual lattice can be naturally identified with the lattice  $H^4(X, \mathbb{Z})$  and the Dirac quantization condition suggests that M2-brane charges takes value in  $\Lambda^*$ , whose bilinear form  $D^{AB}$  is given by the inverse of  $D_{AB}$ . However, due to the presence of the Freed-Witten anomaly the M2 brane charge gets shifted and the above statement is no longer true.

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<sup>3</sup>Notice that our convention defines the D0 brane to be the objects that couples to  $-C^{(1)}$

The shift of the charges can be understood as the fact that when the divisor is not spin, namely when the second Whitney class is non-vanishing or equivalently when  $[\mathcal{P}] = c_1(\mathcal{L})$  is odd, the  $U(1)$  gauge field on it has to have a non-trivial holonomy in order for it to have a  $Spin^c$  structure, and as a result the M2-brane charge does not satisfy the usual Dirac quantization condition, but rather [116, 10]

$$q_A \in \frac{1}{2}D_{ABCP}{}^B p^C \oplus \Lambda^* , \quad \text{or just} \quad q \in \Lambda^* + \frac{p}{2}. \quad (6.2.14)$$

In terms of the bilinear form  $D_{AB}$  the flow equations (6.2.5) and (6.2.6) read

$$q_A \rightarrow q_A + D_{AB}k^B, \quad (6.2.15)$$

$$q_0 \rightarrow q_0 + k^A q_A + \frac{1}{2}D_{AB}k^A k^B. \quad (6.2.16)$$

In this form one can see explicitly that the following combination of charges

$$\hat{q}_0 = q_0 - \frac{1}{2}D^{AB}q_A q_B \quad (6.2.17)$$

is the unique combination (up to additive constant and multiplication factors) that is invariant under the combined spectral flow of  $q_A$  and  $q_0$ . This phenomenon is familiar from our study of the spectral flow symmetry of the  $\mathcal{N} = 2$  superconformal algebra in section 2.1.3 and 2.1.5, where the flow-invariant combination  $L_0 - \frac{1}{2\bar{c}}J_0^2$  plays a similar role.

From (6.2.5) we see that the spectral flow transformation amounts to shifting the vector  $q$  by an element  $k \in \Lambda$ . Note that due to the integrality of the symmetric bilinear  $D_{AB}$  one has  $\Lambda \subset \Lambda^*$ . In general,  $\Lambda$  is a proper subset of  $\Lambda^*$ , which means that not all charge configurations ( $\Lambda^*$ ) are related to each other by spectral flow ( $\Lambda$ ). Explicitly, any given M2 charge vector  $q \in \Lambda^* + p/2$  has a unique decomposition

$$q = \mu + k + \frac{p}{2} \quad , \quad \mu \in \Lambda^*/\Lambda \quad , \quad k \in \Lambda \quad (6.2.18)$$

where  $\mu \neq 0$  for generic charges.

From the above argument we conclude that the combined spectral flow transformations (6.2.15) and (6.2.16) constitute a symmetry of M-theory/string theory. This gives a non-trivial prediction on the BPS degeneracies that the number of BPS states  $d_{\mathcal{P}}(q_A, q_0)$  should be invariant under these transformations.

We can now compare this microscopic prediction with the macroscopic result and see that it indeed passes the consistency check. The leading macroscopic entropy of the 4d black hole with charges  $p^A$ ,  $q_A$  and  $q_0$  is given by (4.4.16)

$$S = 2\pi\sqrt{\hat{q}_0 D} \quad (6.2.19)$$

where

$$6D \equiv \int_X P \wedge P \wedge P = D_{ABC} p^A p^B p^C. \quad (6.2.20)$$

is the norm of the vector  $p \in \Lambda$ . From this we see that the entropy formula is indeed consistent with our prediction that the entropy must be invariant under the spectral flow.

Finally, we would like to point out that the spectral flow (6.2.15) can be induced spontaneously by the nucleation of a M5/anti-M5 brane pair with magnetic charge  $k^A$ , where the M5 loops through the original (circular) M5 brane before annihilating again with the anti-M5 brane. We will make use of this comment in the next section where this same process is translated to the near horizon geometry.

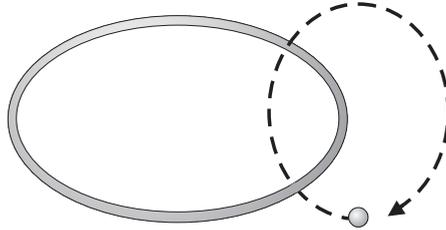


Figure 6.1: An M5 brane loops through the original (circular) M5 brane and then annihilates again with an anti-M5 brane.

## 6.2.2 The Near-Horizon Geometry and Reduction to Three Dimensions

In the decoupling near-horizon limit, the spacetime physics can be entirely captured by the world-volume theory of the brane. In this limit the 11-dimensional geometry becomes

$$X \times AdS_3 \times S^2, \quad (6.2.21)$$

with the Kähler moduli  $J = J^A \alpha_A \in H^{1,1}(X)$  of the Calabi-Yau fixed by the attractor mechanism to be proportional to the charge vector  $P = p^A \alpha_A$ .

More explicitly, the attractor equation reads (2.2.33), (4.4.17)

$$\frac{J^A}{\mathcal{V}^{1/3}} = \frac{p^A}{D^{1/3}},$$

where  $\mathcal{V}$  denotes the volume of the Calabi-Yau in the eleven-dimensional Planck unit

$$\mathcal{V} = \frac{1}{6} \int_X J \wedge J \wedge J = \frac{1}{6} D_{ABC} J^A J^B J^C . \quad (6.2.22)$$

As explained in section 2.2.2, the volume  $\mathcal{V}$  sits in the universal hypermultiplet and is not fixed by the attractor equation. Furthermore, from the relation between the eleven- and five-dimensional Newton's constant and the expression for the five-dimensional solution (2.2.33), (4.4.17)

$$\ell_{\text{p}}^{(5)} \sim \ell_{\text{p}}^{(11)} \mathcal{V}^{-1/3} \quad , \quad \ell \sim \ell_{\text{p}}^{(5)} D^{1/3} , \quad (6.2.23)$$

we see the ratio  $\mathcal{V}/D$  turns out to be related to the curvature radius  $\ell$  of the  $AdS_3$  and  $S^2$  as

$$\frac{\ell}{\ell_{\text{p}}^{(11)}} \sim \left( \frac{D}{\mathcal{V}} \right)^{1/3} . \quad (6.2.24)$$

Therefore, for the five-dimensional supergravity to be a valid description we need the universal hypermultiplet scalar to satisfy  $D \gg \mathcal{V} \gg 1$ . For our purpose it will be useful to consider a further reduction along the compact  $S^2$  to a three dimensional theory on the non-compact  $AdS_3$  spacetime. In the low energy limit, this theory contains the metric and the  $U(1)$  gauge fields  $A^A$  as the massless bosonic fields.

From the five-dimensional perspective, the five-brane flux of M-theory background gets translated into a magnetic flux  $F^A = dA^A$  of the  $U(1)$  gauge fields through the  $S^2$ :

$$\int_{S^2} F^A = p^A .$$

The eleven-dimensional Chern-Simons term of the  $A^{(3)}$ -field can therefore be reduced in two steps. First to five dimensions, where it takes the form (2.2.29)

$$16\pi G_{\text{N}}^{(5)} S_{CS} = \frac{1}{3!} \int_{AdS^3 \times S^2} D_{ABC} A^A \wedge F^B \wedge F^C ,$$

and subsequently, by integrating over the  $S^2$ , to three dimensions, where it turns into the usual (Abelian) Chern-Simons action for the gauge fields on  $AdS_3$ . In combination with the standard kinetic terms, we get

$$16\pi G_{\text{N}}^{(3)} S = \int d^3x \sqrt{g} \left( R - \frac{2}{\ell^2} \right) - \frac{a_{AB}}{2} \int F^A \wedge \star F^B + \int D_{ABA} A^A \wedge dA^B$$

as the terms in the bosonic action relevant for our discussion, where  $a_{AB} = \int_X \alpha_A \wedge \star \alpha_B$  is given in (2.2.30). The 3d Newton constant is given by

$$\frac{1}{G_{\text{N}}^{(3)}} \sim \frac{\ell^2}{(\ell_{\text{p}}^{(5)})^3} \sim \frac{D}{\ell} . \quad (6.2.25)$$

We will end this section by some discussions about the spectral flow in the setting of the attractor geometry. First we note that the spectral flow argument in the previous section can be carried to the three-dimensional setting by dimensional reduction. The M2-brane charge  $q_A$  is defined now as an integral over a circle at spatial infinity of the  $AdS_3$  as

$$q_A = \int_{S^1} \left( -a_{AB} \star F^B + D_{AB} A^B \right).$$

Again one easily verifies that it changes as in (6.2.15) as a result of a large gauge transformation  $A^A \rightarrow A^A - k^A d\psi$  in three dimensions. The charge  $q_0$  is related to the angular momentum in  $AdS_3$ . To understand the shift in  $q_0$  under spectral flow, one has to determine the contribution to the three-dimensional stress energy tensor due to the gauge field.

As mentioned above, the spectral flow has a nice physical interpretation in terms of the nucleation of an M5/anti-M5 brane pair. Let us now describe this process in the near horizon geometry. The following argument is most easily visualized by suppressing the (Euclidean) time direction and focusing on a spatial section of  $AdS_3$ , which can be thought of as a copy of Euclidean  $AdS_2$  and hence is topologically a disk (1.3.5). Together with the  $S^2$  it forms a four dimensional space. First, recall that a wrapped M5 brane appears as a string-like object in this four dimensional space. Since an M5-brane is magnetically charged under the five-dimensional gauge fields  $A^A$ , it creates a "Dirac surface" of  $A^A$ . Of course, the location of the Dirac surface is unphysical and can be moved by a gauge transformation. Now suppose at a certain time an M5/anti-M5-brane pair nucleates in the center of  $AdS_2$  in a way that the M5 and the anti-M5 branes both circle the equator of the  $S^2$ . Subsequently, the M5 and the anti-M5 branes move in opposite directions on the  $S^2$ , say the M5 brane to the north pole and anti-M5 to the south pole. In this way the M5 and anti-M5 brane pair creates a Dirac surface that stretches between them. Eventually both branes slip off and self-annihilate on the poles of the  $S^2$ . What they leave behind now is a Dirac surface that wraps the whole  $S^2$  and still sits at the origin of the  $AdS_2$ . To remove it one literally has to move it from the center and take it to the spatial infinity. Once it crosses the boundary circle, its effect is to perform a large gauge transformation that is determined by the charge  $k^A$  of the M5 brane of the nucleated pair. We conclude that spectral flow can thus be induced by the nucleation of pairs of M5 and anti-M5 branes.

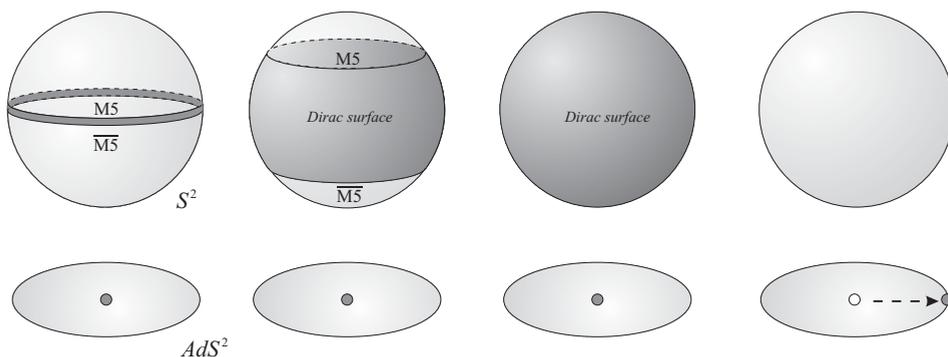


Figure 6.2: A large gauge transformation: (i) An M5-anti-M5 pair wrapping the equator of the  $S^2$  nucleates at the center of the  $AdS_2$ . (ii) The M5 and anti-M5 begin to move in the opposite directions in the  $S^2$ , while still stay at the center of the  $AdS_2$ . (iii) A Dirac surface wrapping the whole  $S^2$  is formed. (iv) Finally one moves the Dirac surface from the bulk of the  $AdS_2$  towards the spatial infinity across the boundary.

### 6.3 The (0,4) Superconformal Field Theory

The existence of the bound states of M2-branes to the M5-brane can be seen in an elegant way from the point of the view of the five-brane world-volume theory. This world-volume theory is a six-dimensional (0,2) superconformal field theory whose field content are five scalars, two Weyl fermions, and a tensor field with self-dual 3-form field strength  $H$  [117]. The spacetime  $A^{(3)}$ -field couples to  $H$  through the term

$$\int_W A^{(3)} \wedge H \quad (6.3.1)$$

where  $W = \mathcal{P} \times S^1 \times \mathbb{R}_t$  denotes the world-volume of the five-brane.

In a bound state the M2-brane charges are dissolved into fluxes of  $H$  in the following way: the self-dual tensor  $H$  that carries the charges  $q_A$  has spatial components

$$H_\psi = -D^{AB} q_A \alpha_B \wedge d\psi, \quad (6.3.2)$$

with  $\psi$  being again the coordinate of the  $S^1$  and the two-cycles  $\alpha_A$  should be understood as their pull-back from the Calabi-Yau  $X$  to the divisor  $\mathcal{P}$ . The timelike components follow from the self-duality condition. Combining the formulas (6.2.2) and (6.3.2), one sees that this produces the right coupling

$$\int_W A^{(3)} \wedge H = q_A \int_{\mathbb{R}_t} A^A$$

of the  $U(1)$  gauge fields  $A^A$  to the charges  $q_A$ .

When we take the scale of the Calabi-Yau to be much smaller than the radius of the M-theory circle, the M5 world-volume theory naturally gets reduced along the 4-cycle  $\mathcal{P}$  to a two-dimensional conformal field theory with  $(0, 4)$  supersymmetry. As usual the superconformal symmetries are identified with the supersymmetric isometries of the  $AdS_3 \times S^2$  manifold. In this case the right-moving superconformal algebra contains “small”  $\mathcal{N} = 4$  superconformal algebra (3.1.1) as a sub-algebra. In particular the  $SU(2)$  R-symmetry corresponds to the rotations of the  $S^2$  factor.

### 6.3.1 Counting the Degrees of Freedom

Let us now find the degrees of freedom of the CFT by dimensionally reducing the massless fields of the five-brane theory on the divisor  $\mathcal{P}$  in the Calabi-Yau which the M5 brane wraps. Our treatment here follows closely that of [95, 118]. Here we assume some familiarity with basic algebraic geometry. Especially we will use some formulas which we haven’t introduced before, including the adjunction formula, Kodaira vanishing theorem, and Lefschetz hyperplane theorem, which can be found in the chapter (I), section two of [119].

First let’s begin with the five scalar fields, which in the original five-brane theory correspond to the locations of the five-brane in the remaining five spatial dimensions normal to the world-volume. Upon dimension reduction, three of them correspond to the center of mass location of the string in the three non-compact directions transversal to it, and simply reduce to three scalar fields of the CFT which have both left- and right-moving components. The remaining two scalars, let’s call them  $X^1$  and  $X^2$ , now correspond to the position of the cycle  $\mathcal{P}$  inside the Calabi-Yau. In other words, they correspond to the deformations of  $\mathcal{P}$  in the Calabi-Yau while keeping the homology class invariant. In order to reduce the complex scalar  $X^1 + iX^2$  we have to know the space of deformations of  $\mathcal{P}$ .

Since the four-cycle can be thought of as the zero locus of a section of the line bundle  $\mathcal{L}$ , this space is  $\mathbb{P}H^0(X, \mathcal{L})$ . From the Hirzebruch-Riemann-Roch theorem (A.0.11), which gives

$$\begin{aligned} w &= \sum_k (-1)^k \dim H^k(X, \mathcal{L}) = \int_X ch(\mathcal{L}) Td(X) \\ &= \int_X e^P \left(1 + \frac{1}{12} c_2(X)\right) = D + \frac{1}{12} c_2 \cdot P, \end{aligned}$$

and the fact that  $\dim H^k(X, \mathcal{L}) = 0$  for  $k > 0$  from the Kodaira’s vanishing theorem (recall that we have assumed  $\mathcal{P}$  to be a positive divisor in section 6.2.1),

we obtain the complex dimension of the space of infinitesimal deformations of  $\mathcal{P}$

$$\dim_{\mathbb{C}} \mathbb{P}H^0(X, \mathcal{L}) = D + \frac{1}{12}c_2 \cdot P - 1 := N, \quad (6.3.3)$$

and notice that the space is projective because sections related by a complex multiplication have the same zero locus. We therefore conclude that the complex scalar  $X^1 + iX^2$  reduces to  $N$  complex scalars which are both left- and right-moving.

As the next step we dimensionally reduce the chiral two-form on the five-brane to the string world-sheet. Rewriting the two-form field using the basis  $\{w_I\}$  of  $H^2(\mathcal{P})$

$$b^{(2)} = \phi^I w_I, \quad I = 1, \dots, b^2(\mathcal{P}), \quad (6.3.4)$$

then the self-duality condition on  $db^{(2)} = H$  implies that the scalars  $\phi^I$  is left-moving when  $w_I$  is anti-self-dual and right-moving if  $w_I$  is self-dual.

To compute the dimension of the self-dual and anti-self-dual part of  $H^2(\mathcal{P})$  we have to first collect a few facts about the topology of the divisor  $\mathcal{P}$ . First of all, from the adjunction formula we have

$$T\mathcal{P} = \frac{TX|_{\mathcal{P}}}{\mathcal{L}|_{\mathcal{P}}}, \quad (6.3.5)$$

which combines with the composition rule for Chern classes gives the Chern classes for the divisor  $\mathcal{P}$

$$c(T\mathcal{P}) = \frac{1 + c_2(X)}{1 + P} = 1 - P + c_2(X) + P^2, \quad (6.3.6)$$

where we have used the fact that  $X$  is a Calabi-Yau and therefore has vanishing first Chern class. Explicitly, this gives

$$c_1(\mathcal{P}) = -c_1(\mathcal{L}) = -P, \quad c_2(\mathcal{P}) = c_2(X) + P^2. \quad (6.3.7)$$

On the other hand, the Lefschetz hyperplane theorem tells us that  $b_1(\mathcal{P}) = b_1(X) = 0$ . Combining these facts with the Gauss-Bonnet theorem (A.0.8) and the signature index theorem (A.0.10), we conclude that the numbers of right- and left-moving bosons obtained by reducing the chiral two-form to the two dimensional world-sheet, which are the same as the dimensions of the self- and anti-self-dual part of  $H^2(\mathcal{P})$ , are

$$\begin{aligned} b_2^R &= \frac{1}{2}(\chi + \sigma) - 1 = 2D + \frac{1}{6}c_2 \cdot P - 1 = 2N + 1 \\ b_2^L &= \frac{1}{2}(\chi - \sigma) - 1 = 4D + \frac{5}{6}c_2 \cdot P - 1. \end{aligned}$$

Especially, the direction proportional to the Kähler class of the divisor  $\mathcal{P}$  is the only direction in  $H^{1,1}(\mathcal{P})$  that is self-dual, a statement that can be checked by using the Hirzebruch-Riemann-Roch theorem (A.0.11) with the bundle  $V$  taken to be the bundle of  $(q, 0)$  forms on  $\mathcal{P}$  and get

$$\begin{aligned} h^{2,0}(\mathcal{P}) &= N = D + \frac{1}{12}c_2 \cdot P - 1 \\ h^{1,1}(\mathcal{P}) &= b_2^L + 1 = 4D + \frac{5}{6}c_2 \cdot P. \end{aligned} \quad (6.3.8)$$

Finally we will reduce the two Weyl spinors. By decomposing the spin bundle on the ambient Calabi-Yau space  $X$  into the product of the spin bundles on  $\mathcal{P}$  and its normal bundle [118], we see that the fermionic zero modes are all right-moving and given by zero-forms and holomorphic two-forms on  $\mathcal{P}$ . Namely that there are

$$4(h^{2,0}(\mathcal{P}) + 1) = 4(N + 1) \quad (6.3.9)$$

real right-moving fermionic degrees of freedom and no left-moving ones.

This ends our derivation of the massless fields of the two-dimensional conformal field theory. In particular, putting all the bosons and fermions together we get the following counting of the left- and right-moving central charges of the CFT

$$\begin{aligned} c_R &= 6D + \frac{1}{2}c_2 \cdot P \\ c_L &= 6D + c_2 \cdot P = \chi(\mathcal{P}), \end{aligned}$$

where the equality between the left-moving central charge and the Euler characteristic follows from the expression for the Chern classes of the divisor  $\mathcal{P}$  (6.3.7) and the Gauss-Bonnet theorem (A.0.8).

### 6.3.2 The Universal Sigma Model

For the discussion of the BPS states of the CFT it will be useful to separate the CFT into two factors of heterotic sigma models. The first factor, which we call the universal sigma model following [118], is the heterotic sigma model obtained by reducing the five-brane theory on the part of cohomology classes of  $\mathcal{P}$  which are images of the injective map  $H^2(X) \rightarrow H^2(\mathcal{P})$  induced by the inclusion map  $\mathcal{P} \rightarrow X$ . This separation is useful because the universal factor is the one containing the information about conserved charges. One way to understand this is the following. Although the two-form field  $b^{(2)}$  reduces also on the two-cycles in  $\mathcal{P}$  which do not correspond to any element in  $H_2(X)$

(6.3.4), the corresponding charges are not conserved due to the existence of the membrane instantons which wraps the three-ball in  $X$  that have the two-cycle in  $\mathcal{P}$  as boundary. Notice that as  $\dim H^2(\mathcal{P})$  is generically much larger than  $\dim H^2(X)$ , the universal part of the CFT actually accounts for only a small portion of the central charges.

The five five-brane world-volume scalars give rise to three left- and right-moving bosons in the universal sector, corresponding to the position of the brane in the three non-compact transversal directions. On the chiral two-form side, from the fact that the Kähler form direction, which is given by  $J \sim p^A \alpha_A$  at the attractor point (6.2.2), is the only self-dual direction inside  $H^{1,1}(\mathcal{P})$ , we conclude that the chiral two-form field  $b^{(2)}$  on the world-volume reduces to a single right-moving boson and  $h^{1,1}(X) - 1$  left-moving bosons. Explicitly, following (6.3.2), the M2 brane charges  $q_A$  can be thought of as the H-fluxes through the following two-cycles

$$\begin{aligned} -q_A D^{AB} \alpha_B &= \frac{p \cdot q}{6D} (p^B \alpha_B) + (-q_A D^{AB} - \frac{p \cdot q}{6D} p^B) \alpha_B \\ &= q_- \alpha_- + q_+^a \alpha_{+,a}, \quad a = 1, \dots, h^{1,1}(X) - 1 \quad (6.3.10) \\ q^2 &= D^{AB} q_A q_B = q_+^2 - q_-^2. \end{aligned}$$

Notice that we have separated the charges, which can now be thought of the charges under the  $U(1)$  gauge fields on the world-sheet, into the right-moving ( $q_-$ ) and left-moving ( $q_+$ ) parts. The way to separate them, namely the choice of a vector inside the Grassmannian  $O(h^{1,1} - 1, 1)/O(1) \times O(h^{1,1})$ , is independent of the moduli of the divisor  $\mathcal{P}$ . In general it will depend on the Kähler moduli of the Calabi-Yau but for convenience we have fixed it to be at the attractor value given by the total M5 charges.

Finally we will look at the fermions. From the decomposition of the spin bundle (6.3.9), we see that the spinor which is (projectively) covariantly constant along  $\mathcal{P}$  is independent of the moduli of  $\mathcal{P}$  and therefore belongs to the universal factor of the sigma model.

Putting everything together, we conclude that the field content of the universal sigma model we are interested in includes three left- and right-moving scalars from the non-compact direction,  $(h^{1,1} - 1, 1)$  compact scalars from reducing the chiral two-form fields, and four right-moving fermionic zero modes. Especially the right-moving degrees of freedom form a  $\mathcal{N} = 4$  scalar multiplet, which we will sometimes refer to as the “universal” hypermultiplet. We can therefore interpret this universal part of the CFT as a (0,4) heterotic sigma model, with the target space being  $\mathbb{R}^3 \times S^1$  and with the left-moving Narain model of gauge group  $(U(1))^{h^{1,1}-2}$ .

The presence of the four fermion zero modes, which can be thought of as

the Goldstinos of the broken supersymmetry, makes the criterion for a state to be supersymmetric somewhat more involved. To see this we have to know more about the superconformal algebra of the right-moving (supersymmetric) side of the CFT. Now with the presence of the universal scalar multiplet, the superconformal algebra is not just the “small”  $\mathcal{N} = 4$  SCA written down in (3.1.1) but is enlarged into the  $\mathcal{A}_{\kappa,\infty}$  SCA [120, 110, 66, 121]. Especially, in the  $\vec{\pi} = 0$  sector for the center of mass degrees of freedom in  $\mathbb{R}^3$ , we have on top of (3.1.1) an extra piece of superconformal algebra which includes

$$\begin{aligned}
[J_n^i, Q_r^{\alpha+}] &= \sigma_{\alpha\beta}^i Q_{r+n}^{\beta+} \quad , [J_n^i, Q_r^{\alpha-}] = -Q_{r+n}^{\beta-} \sigma_{\beta\alpha}^i \\
[U_n, U_m] &= n\delta_{n+m,0} \\
[U_n, G_r^{\pm\pm}] &= nQ_{n+r}^{\pm\pm} \\
\{Q_r^{\alpha+}, Q_s^{\beta-}\} &= \delta^{\alpha\beta} \delta_{r+s,0} \quad \text{and other } \{Q, Q\} = 0 \\
\{Q_r^{\alpha+}, G_s^{\beta-}\} &= \delta^{\alpha\beta} U_{r+s} \quad \text{and other } \{Q, G\} = 0 \quad , \quad (6.3.11)
\end{aligned}$$

where  $Q^{\pm\pm}(\bar{z})$  are now the fermionic currents and  $U(\bar{z})$  is the  $U(1)$  current corresponding to the right-moving part of the M2 charge (6.3.10). Notice that, here and in the following discussion we write all the operators without the tilde’s for the readability of the formulas, despite of the fact that they are the right-movers. We hope that the fact that only the right-moving sector is supersymmetric will prevent any possible confusion.

In the sector in which all states have not only  $\vec{\pi} = 0$  but also  $U_0|0\rangle = q_-|0\rangle = 0$ , the  $G$ ’s and the  $Q$ ’s decouple and the states that preserve unbroken supersymmetries are those annihilated by  $G_0^{\pm\pm}$  as well as by all positive modes. This is in direct analogy with the Ramond ground states we discussed in (2.1.6), except for now the four Goldstino’s  $Q^{\alpha\beta}$  act non-trivially as two pairs of creation and annihilation operators and produce a short multiplet with four BPS states. It’s easy to check that this short  $\mathcal{N} = 4$  multiplet does not contribute to the  $\mathcal{N} = 2$  elliptic genus defined in (2.1.36) but does contribute to the modified version of it when we insert the factor  $F^2 = (J_0^3)^2$  in the trace, where  $J^3$  is the  $U(1)$  current of the R-symmetry group  $SU(2)_R$ . We will therefore insert this factor when we later define the generalised elliptic genus for the present theory.

Now we turn to the more interesting cases with non-vanishing right-moving M2 charge  $q_-$ . If we “bosonize” the  $U(1)$  current and write  $U(\bar{z}) = i\bar{\partial}\varphi$ , then such a state can be thought of as being created from a state with  $U_0|0\rangle = 0$  by adding a vertex operator

$$|q_-\rangle = e^{iq_-\varphi}|0\rangle . \quad (6.3.12)$$

Then using the  $[U, G]$  commutator we see that the supersymmetry condition

gets modified into

$$0 = e^{iq_-\varphi} G_0^{\pm\pm} |0\rangle = (G_0^{\pm\pm} - q_- Q_0^{\pm\pm}) |q_-\rangle, \quad (6.3.13)$$

while the other four combination of  $G_0^{\pm\pm}$  and  $Q_0^{\pm\pm}$  generate the four-component short multiplet which contributes to the modified elliptic genus.

Another consequence of the modification of the supersymmetry condition by the presence of the right-moving charges is a change of the value of the conformal weight of the BPS states. Instead of (2.1.6)

$$\frac{1}{4} \delta^{\alpha\beta} \{G_0^{\alpha+}, G_0^{\beta-}\} |0\rangle = (\bar{L}_0 - \frac{c_R}{24}) |0\rangle = 0, \quad (6.3.14)$$

we now have

$$\frac{1}{4} \delta^{\alpha\beta} \{G_0^{\alpha+} - q Q_0^{\alpha+}, G_0^{\beta-} - q Q_0^{\beta-}\} |q_-\rangle = (\bar{L}_0 - \frac{c_R}{24} - \frac{1}{2} q_-^2) |q_-\rangle = 0, \quad (6.3.15)$$

where

$$q_-^2 = \int_{\mathcal{P}} q_- \alpha_- \wedge q_- \alpha_- = \frac{(p \cdot q)^2}{6D}. \quad (6.3.16)$$

The above relation between the conformal weight and the right-moving charges will be important when we discuss the modular properties of the modified elliptic genus later.

In the general cases in which we also have charges  $\vec{\pi} \neq 0$  in the non-compact  $\mathbb{R}^3$  directions, there are terms other than those listed in (6.3.11) in the  $\mathcal{A}_{\kappa, \infty}$  superconformal algebra which will play a role. They can be found in, for example, [121, 66]. Incorporating these extra charges and repeating exactly the same analysis as above, we conclude that for a BPS state with right-moving charge  $q_-$ ,  $\vec{\pi}$  the following relation is satisfied

$$(\bar{L}_0 - \frac{c_R}{24} - \frac{1}{2} q_-^2 - \frac{1}{2} \vec{\pi}^2) |q_-, \vec{\pi}\rangle_R = 0. \quad (6.3.17)$$

Similarly there is also a  $\frac{1}{2} \vec{\pi}^2$  contribution to the  $L_0$  eigenvalue.

The spectral flow relations (6.2.15) (6.2.16) are implemented in the (0,4) CFT as a symmetry of the superconformal algebra. It is given by

$$\begin{aligned} \tilde{L}_n &\rightarrow \tilde{L}_n + k_- U_n + \frac{1}{2} k_-^2 \delta_{n,0} \\ U_n &\rightarrow U_n + k_- \delta_{n,0} \end{aligned} \quad (6.3.18)$$

for the bosonic part of the right-moving side, and

$$\begin{aligned} L_n &\rightarrow L_n + k_+^a A_{n,a} + \frac{1}{2} k_+^2 \delta_{n,0} \\ A_{n,a} &\rightarrow A_{n,a} + k_+^b D_{ab} \delta_{n,0} \quad , \quad a, b = 1, \dots, h^{1,1}(X) - 1 \end{aligned} \quad (6.3.19)$$

for the left-moving side. In the above formulae, the projection of the large gauge transformation parameter  $k$  into the left- and right-moving component is again given by (6.3.10), with  $q_A$  now replaced by  $D_{AB}k^B$ , and the metric

$$\int_{\mathcal{P}} \alpha_{+,a} \wedge \star \alpha_{+,b} = - \int_{\mathcal{P}} \alpha_{+,a} \wedge \alpha_{+,b} = D_{ab} \quad (6.3.20)$$

is given by the restriction of  $D_{AB}$  onto the hypersurface in  $H^2(X)$  orthogonal to the Kähler form. Note that these transformations leave  $L_0 - \frac{1}{2}A_0^2$  and  $\bar{L}_0 - \frac{1}{2}U_0^2$  invariant.

## 6.4 A Generalised Elliptic Genus

The generating function of BPS bound states for a fixed M5-brane charge  $p^A$  can be identified with a generalised elliptic genus of the CFT (see also [30, 110, 109]). More precisely, we want to compute the partition function

$$Z'_{\mathcal{P}}(\tau, y) = \text{Tr}_R \left[ F^2(-1)^F e^{\pi i p \cdot q} e^{2\pi i \tau (L_0 - \frac{c_L}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c_R}{24})} e^{2\pi i y \cdot q} \right], \quad (6.4.21)$$

where  $y \in \Lambda \otimes \mathbb{R}$  can be thought of as being the “potential” for the M2 charges  $q$ ,  $F = J_0^3$  and the  $F^2$  insertion is needed to absorb the four leftover right-moving fermionic zero modes as we explained in section 6.3.2.

The purpose of this subsection is to discuss the following three properties of the generalised elliptic genus that we will need in order to give the microstates contributing to this index a gravitational interpretation.

### 6.4.1 The Modified Fermion Number

First we will explain the necessity for the extra  $e^{\pi i p \cdot q}$  phase insertion. The presence of it is closely connected to the presence of the Freed-Witten anomaly we discussed before. In particular, both effects are absent if  $[\mathcal{P}] = c_1(\mathcal{L}) = p^A \alpha_A$  is even. We will understand it in terms of the consistency of the conformal theory OPE, while other explanations can be found in [30, 122]. Later we will also see explicitly that the inclusion of this factor is crucial for the modular properties of the generalised elliptic genus.

Note that the shift (6.2.14) only affects the right-moving part of the charges  $q_-$ , we will therefore concentrate our analysis on them. Consider the spectral flow vertex operators  $e^{ik-\varphi}$  acting on the state  $|q_-\rangle$ , from the OPE between  $e^{ik-\varphi(\bar{z}')} and  $e^{iq-\varphi(\bar{z})}$  one has$

$$e^{ik-\varphi(\bar{z})} |q_-\rangle = \bar{z}^{(k \cdot q)_-} |q_- + k_-\rangle. \quad (6.4.22)$$

The OPE will pick up a phase  $\exp(2\pi i(k \cdot q)_-) = \exp(2\pi i k \cdot q) = (-1)^{k \cdot p}$  when  $z$  circles around the origin. Locality of the OPE requires projection onto the states with even  $k \cdot p$ , which explains why the elliptic genus needs to contain a factor  $(-1)^{p \cdot k}$  for it to be modular invariant. For convenience we include however a factor  $e^{\pi i p \cdot q} = e^{\pi i p \cdot \mu} e^{\pi i p \cdot k}$  in our definition of the elliptic genus, where  $\mu$  and  $k$  are given by the unique decomposition of the M2 charge vector (6.2.18). The term  $(p \cdot \mu)$  could be interpreted as an additional overall phase or as a fractional contribution to the fermion number.

### 6.4.2 The Modular Properties

Eventually our aim is to show that the partition function has an asymptotic expansion in terms of semi-classical saddle-points of the three-dimensional supergravity theory. A crucial ingredient for this to work is the property of  $Z'_{\mathcal{P}}(\tau, y)$  that it is a modular form. To be more precise, it is a modular form of weight  $(0, 2)$ . To show this, let us introduce a generalised partition function

$$W_{\mathcal{P}}(\tau, y, z) = \text{Tr}_R \left[ e^{2\pi i z F} e^{\pi i p \cdot q} e^{2\pi i \tau (L_0 - \frac{c_L}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c_R}{24})} e^{2\pi i y \cdot q} \right] \quad (6.4.23)$$

such that

$$Z'_{\mathcal{P}}(\tau, y) = -\frac{1}{4\pi^2} \partial_z^2 W_{\mathcal{P}}(\tau, y, z)|_{z=1/2}.$$

The function  $W_{\mathcal{P}}(\tau, y, z)$  can be thought of as a generalised partition function of the  $(0, 4)$  CFT on a torus with Wilson lines parametrized by  $y$  and  $z$ . It should be independent of the choice of cycles on the torus, and hence be of weight  $(0, 0)$  under the modular transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \bar{\tau} \rightarrow \frac{a\bar{\tau} + b}{c\bar{\tau} + d}, \quad y_+ \rightarrow \frac{y_+}{c\tau + d}, \quad y_- \rightarrow \frac{y_-}{c\bar{\tau} + d}, \quad z \rightarrow \frac{z}{c\bar{\tau} + d}.$$

In the above formula  $y_+$ ,  $y_-$  are the Wilson line parameters coupling to the left- and right-moving charges respectively and are again given by the projection of the vector  $y \in \Lambda \otimes \mathbb{R}$  into the positive- and negative-definite part as in (6.3.10). This together with the fact that  $\partial_z$  has weight one proves that  $Z'_{p^A}(\tau, y^A)$  has weight  $(0, 2)$ .

As mentioned above, the partition function  $Z'_{\mathcal{P}}(\tau, y)$  contains a continuous degeneracy in the BPS states due to the zero-modes in the  $\mathcal{N} = (0, 4)$  “universal” multiplet, which can be thought of the momenta in the  $\mathbb{R}^3$  part of the  $S^1 \times \mathbb{R}^3$  target space of the universal sigma model. Macroscopically they correspond to the center of the mass degrees of freedom of the M5 brane which decouple from the rest of the degrees of freedom. We wish to extract this

degeneracy by defining

$$Z'_{\mathcal{P}}(\tau, y) = Z_{\mathcal{P}}(\tau, y) \int d^3\vec{\pi} (e^{2\pi i\tau} e^{-2\pi i\bar{\tau}})^{\frac{1}{2}\vec{\pi}^2} \quad (6.4.24)$$

where  $Z_{\mathcal{P}}(\tau, y)$  can be thought of as the trace among the BPS states with charges  $\vec{\pi} = 0$ . The Gaussian integral is proportional to  $\text{Im}(\tau)^{-3/2}$  and therefore has weight  $(\frac{3}{2}, \frac{3}{2})$ . We can therefore conclude that  $Z_{\mathcal{P}}(\tau, \bar{\tau}, y)$  has weight  $(-\frac{3}{2}, \frac{1}{2})$ .

### 6.4.3 The Theta-Function Decomposition

The spectral flow symmetry discussed in section 6.2.1, or equivalently the isomorphism of the superconformal algebra discussed in (6.3.18) and (6.3.19), implies the presence of extra structures in the generalised elliptic genus. As we shall see, similar to the case of the elliptic genus of a (2, 2) CFT (2.1.45), these structures are most manifest when we write the generalised elliptic genus in a decomposed form in terms of the theta functions.

First of all, since we have argued that the elliptic genus  $Z_{\mathcal{P}}(\tau, y)$  only receives contribution from BPS states with  $\vec{\pi} = 0$ , using (6.3.15) we can rewrite it as

$$Z_{\mathcal{P}}(\tau, y) = \text{Tr}_R \left[ F^2(-1)^F \mathbf{e} \left[ \tau(L_0 - \frac{c_L}{24}) - \frac{\bar{\tau}}{2} q_-^2 + (q|y + \frac{p}{2}) \right] \right] \quad (6.4.25)$$

in the shorthand notation introduced in section 2.1.5. Recall that the bilinear  $(|) : (\Lambda \otimes \mathbb{R}) \times (\Lambda \otimes \mathbb{R}) \rightarrow \mathbb{R}$ , first defined in (6.2.13), is given by  $-\int_{\mathcal{P}} * \wedge *$  and of Lorentzian signature. A direct consequence of this is that the partition function depends on  $\bar{\tau}$  in a specific way, namely the anti-holomorphic part is entirely captured by the “heat equation”

$$\left[ \partial_{\bar{\tau}} + \frac{1}{4\pi i} \partial_{y_-}^2 \right] Z_{\mathcal{P}}(\tau, \bar{\tau}, y) = 0. \quad (6.4.26)$$

In particular, this implies that the anti-holomorphic part of  $Z_{\mathcal{P}}(\tau, y)$  contains redundant information which is already encoded in the Wilson line ( $y$ )-dependence of it. It will turn out that by decomposing the elliptic genus in terms of the theta-functions we can indeed isolate the holomorphic factor factor which is all we need to determine the degeneracies of BPS states.

Secondly, we are interested in the BPS degeneracies  $d_{\mathcal{P}}(q_A, q_0)$ , but on the other hand the spectral flow symmetry implies a relation among those degeneracies for different charges  $(q_A, q_0)$ . A generating function for BPS degeneracies in terms of spectral flow invariant combinations of charges is therefore desirable. It will turn out that such generating functions are exactly the coefficients  $h_{\mu}(\tau)$  of the generalised elliptic genus  $Z_{\mathcal{P}}(\tau, y)$  in its theta-function decomposition.

To begin let us write the type IIA D0 brane charge, proportional to the five- (or three-)dimensional angular momentum along the M-theory circle, as

$$\begin{aligned} q_0 &= \left(L_0 - \frac{c_L}{24}\right) - \left(\bar{L}_0 - \frac{c_R}{24}\right) \\ &= Q_0 + \frac{1}{2}q^2 - \frac{\chi}{24}, \end{aligned}$$

where we have used  $c_L = \chi(\mathcal{P})$ . From the CFT point of view,  $\frac{1}{2}q^2$  is the contribution of the universal factor  $U(1)$  currents to the  $S^1$  momentum, and central charge contribution corresponds to the ground state energy of  $AdS_3$ , while  $Q_0$  is the contribution from the bosonic zero-modes of the part of the CFT which is not “universal”, together with the contribution from the left-moving excitations,. On the other hand, from the type IIA point of view this is also a natural split. Considering the D2 brane charges as fluxes on the D4 brane we recognise the second term as induced by the  $ch(\mathcal{F})$  factor in the anomalous brane coupling (1.3.28), while the third one induced by the A-roof genus curvature factor, leaving  $Q_0$  being the number of pointlike D0 branes together with the contribution from the part of the world-volume flux which does not correspond to conserved D2 brane charges.

In terms of the spectral flow invariant combination  $\hat{q}_0$  (6.2.17), we have

$$L_0 - \frac{c_L}{24} = \hat{q}_0 + \frac{1}{2}q_+^2. \quad (6.4.27)$$

Using this and the decomposition of the M2 charges (6.2.18), we can again rewrite the generalised elliptic genus (6.4.25) as

$$\begin{aligned} Z_{\mathcal{P}}(\tau, y) &= \\ \text{Tr}_R \left[ F^2(-1)^F \mathbf{e}[\tau\hat{q}_0] \mathbf{e}\left[\frac{\tau}{2}(\mu+k+\frac{p}{2})_+^2 - \frac{\bar{\tau}}{2}(\mu+k+\frac{p}{2})_-^2 + (\mu+k+\frac{p}{2})\left(y+\frac{p}{2}\right)\right] \right] \\ \mu \in \Lambda^*/\Lambda \quad , \quad k \in \Lambda. \end{aligned} \quad (6.4.28)$$

The statement of spectral flow that each  $k \in \Lambda$  gives the same contribution to  $Z_{\mathcal{P}}(\tau, y)$ , can be now translated into the following statement

$$Z_{\mathcal{P}}(\tau, y) = \sum_{\mu \in \Lambda^*/\Lambda} \Theta_{\mu}(\tau, y) h_{\mu}(\tau) \quad (6.4.29)$$

in which

$$\begin{aligned}
 h_\mu(\tau) &= \text{Tr}_{q=\mu+p/2} [F^2(-1)^F \mathbf{e}[\tau\hat{q}_0]] = \sum_{\hat{q}_0 + \frac{q_L}{24} \geq 0} d_\mu(\hat{q}_0) e^{2\pi i \tau \hat{q}_0} \\
 \Theta_\mu(\tau, y) &= \sum_{q \in \mu + \Lambda + \frac{p}{2}} \mathbf{e}\left[\frac{\tau}{2} q_+^2 - \frac{\bar{\tau}}{2} q_-^2 + (q|y + \frac{p}{2})\right] \\
 &= \mathbf{e}\left[\frac{p}{4}(y + \frac{p}{2})\right] \theta_\mu(\tau; -y - \frac{p}{2}, \frac{p}{2}), \tag{6.4.30}
 \end{aligned}$$

where  $\theta_\mu(\tau; \alpha, \beta)$  is the Siegel theta function we introduced in (2.1.5).

As promised, now we see that the entire information about BPS degeneracies contained in the generalised elliptic genus is encapsulated in the holomorphic modular forms  $h_\mu(\tau)$ , whose Fourier coefficient  $d_\mu(\hat{q}_0)$  depends only on the spectral invariant combination  $(\mu, \hat{q}_0)$  of the charges  $(q_0, q_A)$ .

In order to evaluate the saddle-point contribution to  $h_\mu(\tau)$  we will need to know its modular property. The modular transformation of  $\Theta_\mu(\tau, y)$  can be computed by Poisson resummation, which has been performed in (2.1.53) and gives

$$\Theta_\mu\left(-\frac{1}{\tau}, \frac{y}{\tau}\right) = \frac{1}{\sqrt{|\Lambda^*/\Lambda|}} (\sqrt{-i\tau})^{h^{1,1}-1} (\sqrt{i\bar{\tau}}) \times \tag{6.4.31}$$

$$\mathbf{e}\left[\frac{1}{2\tau} y_+^2 - \frac{1}{2\bar{\tau}} y_-^2\right] \mathbf{e}\left[-\frac{p^2}{4}\right] \sum_{\nu \in \Lambda^*/\Lambda} \mathbf{e}[-(\mu|\nu)] \Theta_\nu(\tau, y). \tag{6.4.32}$$

While the extra exponential factors involving  $\mathbf{e}\left[\frac{y^2}{2\tau}\right]$  are expected as given by the modular transformation of the generalised elliptic genus in analogy with the elliptic genus case (2.1.49), the  $\tau$ -prefactors together with the knowledge that  $Z_{\mathcal{P}}(\tau, y)$  has weight  $(-\frac{3}{2}, \frac{1}{2})$  shows that  $\{h_\mu(\tau)\}$  transforms as a vector-valued modular form of weight  $-\frac{h^{1,1}+2}{2}$ . See section 2.1.5 for a small introduction of these the vector-valued modular forms. More explicitly, it transforms as

$$h_\mu\left(-\frac{1}{\tau}\right) = \sqrt{|\Lambda^*/\Lambda|} (\sqrt{-i\tau})^{-(h^{1,1}+2)} \mathbf{e}\left[\frac{p^2}{4}\right] \sum_{\nu \in \Lambda^*/\Lambda} \mathbf{e}[-(\mu|\nu)] h_\nu(\tau, y), \tag{6.4.33}$$

where  $\sqrt{|\Lambda^*/\Lambda|} = |\text{Vol}(\Lambda)|$  is the volume of a unit cell of the lattice  $\Lambda$ .

The T-transformation of the theta-functions can also be computed. Recall that the lattice  $\Lambda$  is not necessarily even and this makes the computation a bit more involved than usual. Representing a lattice vector  $k \in \Lambda$  as an integral two-cycle imbedded in the hypersurface  $\mathcal{P}$ , then the adjunction formula

together with the Riemann-Roch theorem gives

$$Q \cdot Q + Q \cdot P = 2g - 2, \quad (6.4.34)$$

and it turns out that  $k^2 - (p|k) = 0 \pmod{2}$ . A direct computation then shows

$$\Theta_\mu(\tau + 1, y) = e^{\left[\frac{(\mu + p/2)^2}{2}\right]} \Theta_\mu(\tau, y), \quad (6.4.35)$$

which implies the T-transformation for the vector-valued modular form  $h_\mu(\tau)$

$$h_\mu(\tau + 1) = e^{\left[-\frac{(\mu + p/2)^2}{2}\right]} h_\mu(\tau). \quad (6.4.36)$$

#### 6.4.4 The (Modern) Farey Tail Expansion

After decomposing the BPS-states-counting elliptic genus of the low-energy CFT into combinations of theta-functions, we are now ready to employ some important mathematical properties of our vector-valued modular functions  $h_\mu(\tau)$  in order to give the elliptic genus a spacetime interpretation. The treatment of this part of the story in the original paper [108] of the present author is not completely correct in the most general cases and was later improved in the publication [111]. We will refer to this paper and the PhD thesis of fellow student Jan Manschot for further details and simply summarise the results we need here.

Suppose we have a weight  $w$  vector-valued modular form  $f_\mu(\tau)$ , where  $\mu \in \Lambda^*/\Lambda$  for some lattice  $\Lambda$ , which transforms under  $\Gamma = PSL(2, \mathbb{Z})$  modular transformation as

$$f_\mu\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w M(\gamma)_\mu^\nu f_\nu(\tau) \quad , \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}). \quad (6.4.37)$$

Given the Fourier expansion

$$f_\mu(\tau) = \sum_{n \geq 0} D_\mu(n) q^{n - \Delta_\mu}, \quad (6.4.38)$$

recall that the polar part of  $f_\mu(\tau)$  is given by (2.1.43)

$$f_\mu^-(\tau) = \sum_{0 \leq n \leq \Delta_\mu} D_\mu(n) q^{n - \Delta_\mu}. \quad (6.4.39)$$

The special property which will be useful for us is that the full vector-valued modular form is determined by its polar part alone.

This property will give us an expression for the BPS partition function which looks like, at the cartoon level and ignoring regularisation,

$$Z_{\mathcal{P}}(\tau, y) \sim \sum_{\Gamma_{\infty} \setminus \Gamma} Z_{\mathcal{P}}^{-}\left(\frac{a\tau + b}{c\tau + d}, \frac{y}{c\tau + d}\right), \quad (6.4.40)$$

where the coset  $\Gamma_{\infty} \setminus \Gamma$  denotes  $\tau \sim \tau + 1$ , with each term in the sum lending itself to a natural interpretation in terms of the dual gravitational theory.

The full expression including all the details, however, is substantially more involved. It reads

$$\begin{aligned} f_{\mu}(\tau) = & \frac{1}{2} D_{\mu}(\Delta_{\mu}) + \frac{1}{2} \sum_{0 \leq n \leq \Delta_{\mu}} \lim_{K \rightarrow \infty} \sum_{(\Gamma_{\infty} \setminus \Gamma)_K} \left( \right. \\ & (c\tau + d)^{-w} e^{i \left[ \frac{a\tau + b}{c\tau + d} (n - \Delta_{\nu}) \right]} R_w \left( \frac{2\pi i |n - \Delta_{\nu}|}{c(c\tau + d)} \right) \\ & \left. \times M^{-1}(\gamma)_{\mu}^{\nu} D_{\nu}(n) \right), \end{aligned} \quad (6.4.41)$$

where

$$R_w(x) = \frac{1}{\Gamma(1-w)} \int_0^x e^{-z} z^{-w} dz \quad (6.4.42)$$

and

$$\sum_{(\Gamma_{\infty} \setminus \Gamma)_K} = \sum_{\substack{|c|, |d| \leq K \\ (c, d) = 1}}. \quad (6.4.43)$$

Please see [111] and the PhD thesis of Jan Manschot for the derivation of the above formula.

Now we can apply this formula on our character  $h_{\mu}(\tau)$  of the elliptic genus, where the transition matrix  $M(\gamma)$  can be read off from the T- and S-transformation of  $h_{\mu}(\tau)$  that we calculated in (6.4.36) and (6.4.33). From this procedure and combining again with the appropriate theta-functions we obtain an expression for the generalised elliptic genus  $Z_{\mathcal{P}}(\tau, y)$  which is in its spirit given by (6.4.40). In the following section we will give this expansion a physical meaning in terms of geometries.

## 6.5 Spacetime Interpretation of the Attractor Farey Tail

So far the generalised Rademacher formula appears to be just a mathematical result. What makes it interesting is that it has a very natural interpretation from the point of view of a dual gravitational theory. In this section we

discuss the interpretation of the Farey tail expansion first in terms of the effective supergravity action, and subsequently from an M-theory/string theory perspective. We will first discuss the gravitational interpretation of the general formula presented in section 6.4.4, and then turn to the present case of the  $\mathcal{N} = 2$  black holes.

### 6.5.1 Gravitational Interpretation of the Generalised Rademacher formula

Microscopic systems described by a 2d CFT have a dual description in terms of a string- or M-theory on a space that contains  $AdS_3$  as the non-compact directions. This is because  $AdS_3$  is the unique space whose isometry group is identical to the 2d conformal group. The miracle of AdS/CFT is that the dual theory contains gravity, which suggests that the partition function of the 2d CFT somehow must have an interpretation as a sum over geometries in the classical limit. The full dual theory is defined on a space that is 10- or 11-dimensional, but except for the three directions of  $AdS_3$  are the only non-compact dimensions. Hence, by performing a dimensional reduction along the compact directions we find that the dual theory can be represented as a (super-)gravity theory on  $AdS_3$ . The effective action therefore contains the Einstein action for the 3d metric

$$S_E = \frac{1}{16\pi G_3} \int_{AdS} \sqrt{g} (R - \frac{2}{\ell^2}) + \frac{1}{8\pi G_3} \int_{\partial(AdS)} \sqrt{h} (K - \frac{1}{\ell})$$

where we have included the Gibbons-Hawking boundary term. Here  $\ell$  represents the AdS-radius. According to the AdS/CFT dictionary, the 3d Newton constant  $G_3$  is related to the central charge  $c$  of the CFT by [123]

$$\frac{3\ell}{2G_3} = c. \tag{6.5.1}$$

The dictionary also states that the partition function  $Z(\tau)$  of the CFT is equal to that of the dual gravitational theory on (a quotient of)  $AdS_3$ , whose boundary geometry coincides with the 2d torus on which the CFT is defined. The shape of the torus is kept fixed and parametrized by the modular parameter  $\tau$ .

The rules of quantum gravity tell us to sum over all possible geometries with the same asymptotic boundary conditions. For the case at hand, this means that we have to sum over all possible three dimensional geometries with the torus as the asymptotic boundary. Semi-classically, these geometries satisfy the equations of the motion of the supergravity theory, and hence are

locally  $AdS_3$ . There indeed exists an Euclidean three geometry with constant curvature which has  $T^2$  with modular parameter  $\tau$  as its boundary. It is the BTZ black hole, which is described by the Euclidean line element

$$ds^2 = N^2(r)dt_E^2 + \ell^{-2}N^{-2}(r)dr^2 + r^2(d\phi + N_\phi(r)dt_E)^2$$

with

$$N^2(r) = \frac{(r^2 - \tau_2^2)(r^2 + \tau_1^2)}{r^2}, \quad N_\phi(r) = \frac{\tau_1\tau_2}{r^2}.$$

Here  $\tau = \tau_1 + i\tau_2$  is the modular parameter of the boundary torus. Using (6.5.1), one can compute the Euclidean action of this solution and obtain [124]

$$S = -\frac{\pi c}{6} \text{Im} \frac{1}{\tau}.$$

For the present purpose of counting BPS states, one needs to consider extremal BTZ black holes. With the Minkowski signature this means that its mass and angular momentum are equal. After analytic continuation to a Euclidean complexified geometry, one finds that the action has become complex and equals  $i\pi \frac{c}{12} \tau$ .

Note that a torus with modular parameter  $\tau$  is equivalent to a torus with parameter  $\frac{a\tau+b}{c\tau+d}$ , since they differ only by a relabelling of the  $A$ - and  $B$ -cycles. But the Euclidean BTZ solution labelled by  $\frac{a\tau+b}{c\tau+d}$  in general differs from the one labelled by  $\tau$ , with the difference being that these three-dimensional geometries fill up the boundary torus in distinct ways. Namely, for the above BTZ solution the torus is filled in such a way that its  $A$ -cycle is contractible. After a modular transformation, this would become the  $\gamma(A) = cA + dB$  cycle. In fact, the BTZ black hole is related to thermal  $AdS_3$  with metric

$$ds^2 = (r^2 + \ell^2)dt_E^2 + \frac{dr^2}{r^2 + \ell^2} + r^2d\phi^2 \quad (6.5.2)$$

after interchanging the  $A$ - and  $B$ -cycles and with  $t_E$  and  $\phi$  periodically identified as

$$t_E \equiv t_E + 2\pi n\tau_2, \quad \phi \equiv \phi + 2\pi n\tau_1.$$

In this case the  $B$ -cycle is non-contractible, while the  $A$ -cycle is now contractible. Notice that in this metric it is manifest that  $\tau \rightarrow \tau + 1$  gives the same geometry. The Euclidean action for this geometry is  $S = i\pi \frac{c}{12} \tau$ .

The classical geometries with a given boundary torus with modular parameter  $\tau$  can now be obtained from either the thermal AdS background or the BTZ background by modular transformations. For definiteness let us take

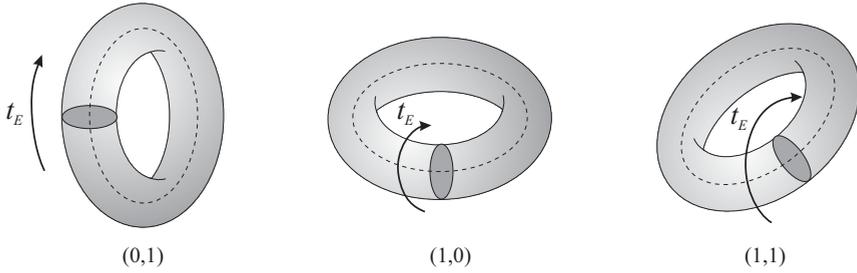


Figure 6.3: (i) Thermal  $AdS_3$ , with the B-cycle being contractible. (ii) BTZ black hole, with the A-cycle being contractible. (iii) The geometry with the (A+B)-cycle being contractible.

the thermal AdS as our reference point, so that the classical action for the geometry obtained by acting with an element  $\gamma$  of the modular group is

$$S = i\pi \frac{c}{12} \left( \frac{a\tau + b}{c\tau + d} \right).$$

One easily recognizes that these solutions precisely give all the leading contributions in the Farey tail expansion corresponding to  $e^{-\frac{c_L}{24} \frac{a\tau+b}{c\tau+d}}$  factor in (6.4.41). In fact, these terms occur with a multiplicity one since they represent the vacuum of the SCFT. The other terms in the expansion should then be regarded as dressing the Euclidean background with certain contributions that change the energy of the vacuum. These contributions were already given an interpretation as coming from virtual particles that circle around the non-contractible cycle in the previous work on black hole Farey tails [112]. In fact, in the next section we will give a further justification of this interpretation for the case of the attractor black holes. Specifically, by using the arguments of Gaiotto et al.[114], we find that the subleading contributions are due to a gas of wrapped M2-branes which carry quantum numbers corresponding to the charges and the spin in the  $AdS_3$  geometry. The truncation to the polar terms can in turn be interpreted as imposing the restriction that the gas of particles are not heavy enough yet to form a black hole. The  $AdS_3$  geometry carries a certain negative energy which allows a certain amount of particles to be present without causing gravitational collapse. However, when the energy surpasses a certain bound then a black hole will form through a Hawking-Page transition. In the case of the  $AdS_3/CFT_2$  correspondence such an interpretation was first proposed by Martinec [125].

Recall that the way we obtain the generalised Rademacher formula for the five-brane CFT is essentially to apply it (6.4.41) on the characters  $h_\mu(\tau)$ , while the theta-functions are then combined with the resulting expression and

are therefore present in each thermal or BTZ background. The origin of the presence of the theta functions lies in the chiral bosons on the world-sheet of the string reduction of the M5-brane. From a spacetime perspective, on the other hand, the theta functions arise due to the presence of gauge fields and in particular the presence of the Chern-Simons term in the action, as discussed in section 6.2.1. In fact, the partition function of a spacetime effective theory that includes precisely such Chern-Simons terms in addition to the usual Yang-Mills action was analyzed in detail by Gukov et al. in [126]. These authors showed that the partition function indeed decomposes into a sum of Siegel-Narain theta functions. There the  $\bar{\tau}$ -dependence arises because one of the gauge field components is treated differently from the others, to ensure that the partition function indeed converges. In this part of the thesis we will not give further details of this calculation. A more recent discussion in which parts of this calculation were carefully worked out is [109].

### 6.5.2 Wrapped M2-branes

In the previous subsection, we have interpreted the sum over modular orbits of the most polar term as the sum over gravitational background, and the appearance of the theta functions as the effect of the spectral flow symmetry induced by the gauge Chern-Simons term in the action. In this subsection we would like to further give a spacetime interpretation to the rest of the polar terms

$$\sum_{\hat{q}_0 < 0} d_\mu(\hat{q}_0) \mathbf{e}[\hat{q}_0 \frac{a\tau + b}{c\tau + d}] R_w \left( \frac{2\pi i |n - \Delta_\nu|}{c(c\tau + d)} \right) \quad (6.5.3)$$

in (6.4.41) as a dilute gas of wrapped M2 branes. In particular one would like to give a more detailed accounting of those polar states from the point of view of string theory on  $\text{CY} \times S^2 \times \text{AdS}_3$ . In fact, a nice physical picture of a large class of these states was given in [114] in terms of M2 and anti-M2 branes which fill up Landau levels near the north and south pole of the  $S^2$  respectively. In a dilute gas approximation, their macroscopic computation gives rise to the following contribution to the elliptic genus  $Z$ :

$$Z_{\text{gas}}(\tau, y) = \mathbf{e}[-\frac{c_L}{24}] Z_{\text{sugra}}(\tau) Z_{GV}(\tau, \frac{1}{2}p\tau + y) Z_{GV}(\tau, \frac{1}{2}p\tau - y). \quad (6.5.4)$$

Let's now explain the different factors in the above formula. First of all, the presence of the factor  $\mathbf{e}[-\frac{c_L}{24}]$  is due to the fact that the supergravity partition function should be computed with the NS boundary condition, since the  $\text{AdS}_3$  circle is contractible. Second,  $Z_{\text{sugra}}(\tau)$  denotes the contribution of the supergravity modes, which is basically given by the MacMahon function

(2.1.34) as [114, 109, 127]

$$Z_{\text{sugra}}(\tau) = M(q)^{-\chi} \quad (6.5.5)$$

where  $\chi$  is the Euler number of the Calabi-Yau manifold in question (A.0.16). Finally, the two (reduced) Gopakumar-Vafa partition functions (2.1.35) account for the contribution from the wrapped M2 and anti-M2 charges, since it's known that the Gopakumar-Vafa invariants  $\alpha_g^q$  counts wrapped membranes with charge  $q \in H_2(X, \mathbb{Z})$  when the non-compact directions are flat five-dimensional Minkowski space [31, 32]. Using the relationship between the Gopakumar-Vafa partition function and topological string partition function discussed in section 2.1.4, together with the modular invariance of the elliptic genus  $Z_{\mathcal{P}}(\tau, y)$ , the expression (6.5.4) provides a justification of the OSV conjecture (2.2.35) when the gas approximation is to be trusted [114].

As is clear from the form of the Gopakumar-Vafa partition function (2.1.35), what it counts are non-interacting wrapped membranes. From the point of view of the world-sheet SCFT, the M2/anti-M2 brane gas describes a collection of states that is freely generated by a collection of chiral vertex operators. It is clear that (6.5.4) suffers from various kinds of limitations. The dilute gas approximation will eventually break down, there could be other BPS configurations that contribute, the Landau levels can start to fill out the entire  $S^2$ , the  $SU(2)$  quantum numbers are bounded by the level of the  $SU(2)$  current algebra, etc. Furthermore, it also does not exhibit the right behavior under spectral flow.

Therefore, the gas expression (6.5.4) is really only valid in the limit in which the five-branes charges are large and membranes charges are small. In this limit, the factor

$$d_\mu(\hat{q}_0) \mathbf{e}\left[\hat{q}_0 \frac{a\tau + b}{c\tau + d}\right] \quad , \quad \hat{q}_0 < 0 \quad (6.5.6)$$

can be thought of as counting the degeneracies of a gas of charged particles in the appropriate thermal  $AdS_3$  or BTZ background, which are made of membranes wrapping internal cycles and are not heavy enough to form a black hole. The latter statement can be seen directly from the expression for the black hole entropy (6.2.19).

Finally, the extra factor  $R_w\left(\frac{2\pi i|n-\Delta_\nu|}{c(c\tau+d)}\right)$  in (6.5.3) should be thought of as a regularising factor for the gravitational path integral. Again we refer to the PhD thesis of fellow student Jan Manschot for further details.

## 6.6 Summary and Conclusion

In this part of the thesis we present the Farey tail expansion for  $\mathcal{N} = 2$  D4-D2-D0 black holes. The central idea of this expansion is to first truncate the partition function so that it includes only particular low excitation states, and then sum over all images of it under the modular group. Each term can be interpreted as representing the contribution of a particular (semi-)classical background. The formula can thus be regarded as partly microscopic (as the states counted in the "tail") as well as macroscopic (as the sum over classical backgrounds). We would like to emphasize that in this expansion, there is no one to one correspondence between microstates and gravitational backgrounds and the major part of the entropy is carried by one particular black hole background.

The supergravity interpretation of the Farey expansion involves a natural complete collection of backgrounds of a given type. It is natural to ask whether the expansion can be refined by including more general macroscopic backgrounds. It is indeed likely that such refinements exist, but one expects that their contribution will follow a similar pattern: one has to truncate the microscopic spectrum even further and replace the contribution of the omitted states by certain classical backgrounds. Here one can think of various type of backgrounds, such as multi-centered solutions, bubbling solutions that deform the horizon geometry, black rings...etc. A large class of such solutions is known, but the list is presumably incomplete, and it remains an interesting problem to use them in a systematic manner.