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TECHNICAL NOTE

A Note on Parametric Analysis in Linear Assignment

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A classic application of the linear assignment problem is the assignment of people to jobs (or jobs to people). In this context, it is interesting to measure competition for jobs and to generate a suitable list of jobs from which a person can choose; the length of the list is a parameter. A known list-generation procedure is based on an interior-point method followed by a parametric analysis. We describe a more efficient procedure, exploiting linear assignment theory and shortest-path computations. Further, we propose an alternative list-generation procedure, based on a special type of dual values for the linear assignment problem.

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1. Introduction

A classic application of the linear assignment problem (LAP) is the assignment of jobs to people. Holder (2005) considered a personnel-planning problem arising in the United States Navy. In this motivating application, each person is assigned at least one job and each job at most one person (instead of the opposite); we follow this exchange of jobs and people. The problem is to generate a list of jobs so that a person can choose one job from this list. This problem is treated as a parametric-analysis problem and is applicable in many situations. Holder (2005) further proposed a procedure based on a path-following interior-point algorithm. We describe a more efficient procedure that exploits linear assignment theory and is based on optimizing an extended LAP.

We introduce some notation and then the mathematical model. Although we present it in terms of assigning people to jobs, the parametric analysis we consider is valid for any LAP. Let J be the set of jobs (index j) and S be the set of people (index i and $|S| = n$) to be assigned to a job. Define

$$\mathcal{A} = \{(i, j): \text{person } i \text{ can do job } j\}, \quad |\mathcal{A}| = m,$$

$$\mathcal{A}(i) = \{j: (i, j) \in \mathcal{A}\} \quad \text{for each } i \in S,$$

$$\mathcal{A}'(j) = \{i: (i, j) \in \mathcal{A}\} \quad \text{for each } j \in J.$$

For each $(i, j) \in \mathcal{A}$, variable x_{ij} is a 0-1 decision variable that indicates if person i is assigned job j . The cost coefficients c_{ij} give the (nonnegative) costs of such an assignment.

The basic problem considered is

$$(AS) \quad \min \sum_{(i, j) \in \mathcal{A}} c_{ij} x_{ij} \quad (1)$$

$$\text{subject to } \sum_{j \in \mathcal{A}(i)} x_{ij} \geq 1, \quad i \in S, \quad (2)$$

$$\sum_{i \in \mathcal{A}'(j)} x_{ij} \leq 1, \quad j \in J, \quad (3)$$

$$x_{ij} \in \{0, 1\}, \quad (i, j) \in \mathcal{A}. \quad (4)$$

Constraint (2) assures that each person is assigned at least one job and constraint (3) assures that each job is assigned to at most one person.

Without loss of generality, we assume for ease of presentation that the problem AS always has a feasible solution, i.e., any set of people $S' \subseteq S$ obeys $|S'| \leq |\bigcup_{i \in S'} \mathcal{A}(i)|$.

In §2, we present the LAP approach and the parametric results, and in §3 we suggest an alternative list-generation procedure, using special dual values. We finish with some remarks and conclusions in §4.

2. The LAP Approach

Holder (2005) introduced a parametric model to generate the job list for a person, say i^o , with the desired list length given by the parameter θ :

$$(ASP_\theta) \quad \min \sum_{(i, j) \in \mathcal{A}} c_{ij} x_{ij} = f^*(i^o, \theta)$$

$$\text{subject to } \sum_{j \in \mathcal{A}(i)} x_{ij} \geq 1, \quad i \in S \setminus \{i^o\},$$

$$\begin{aligned} \sum_{j \in \mathcal{A}(i^\circ)} x_{i^\circ j} &\geq \theta, \\ \sum_{i \in \mathcal{A}'(j)} x_{ij} &\leq 1, \quad j \in J, \\ x_{ij} &\in \{0, 1\}, \quad (i, j) \in \mathcal{A}. \end{aligned}$$

If we relax $x_{ij} \in \{0, 1\}$ to $x_{ij} \geq 0$ and assume that θ is a real number, the function $f^*(i^\circ, \theta)$ is piecewise linear, continuous, and convex in θ (see, e.g., Gal 1995). From now on we consider θ to be integer and we restrict θ to values such that $\theta \leq |\mathcal{A}(i^\circ)|$, as otherwise (ASP _{θ}) has no feasible solution.

We explain how to efficiently solve ASP _{θ} by introducing a suitable LAP model and applying known linear assignment properties to it. First, note that given nonnegative c -coefficients, ASP _{θ} always has an optimal solution for which the second constraint holds with equality. As a result, ASP _{θ} with $\theta > 1$ can be modified to a standard LAP by introducing *duplicate 0-1 variables* for person i° , say y_{kj} ($k = 1, \dots, \theta - 1, j \in \mathcal{A}(i^\circ)$), that indicate whether the k th duplicate of person i° can be assigned job j with the corresponding objective coefficients all equal to $c_{i^\circ j}$. The y -variables must obey similar constraints as the x -variables, i.e.,

$$\sum_{j \in \mathcal{A}(i^\circ)} y_{kj} = 1 \quad \text{for } k = 1, \dots, \theta - 1.$$

The related LAP, denoted LAP _{θ} , is equivalent to ASP _{θ} and formulated as

$$\begin{aligned} (\text{LAP}_\theta) \quad \min \quad & \sum_{(i, j) \in \mathcal{A}} c_{ij} x_{ij} + \sum_{k=1}^{\theta-1} \sum_{j \in \mathcal{A}(i^\circ)} c_{i^\circ j} y_{kj} = f^*(i^\circ, \theta) \\ \text{subject to} \quad & \sum_{j \in \mathcal{A}(i)} x_{ij} \geq 1, \quad i \in S \setminus \{i^\circ\}, \\ & \sum_{j \in \mathcal{A}(i^\circ)} x_{i^\circ j} = 1, \\ & \sum_{j \in \mathcal{A}(i^\circ)} y_{kj} = 1, \quad k = 1, \dots, \theta - 1, \\ & \sum_{i \in \mathcal{A}'(j)} x_{ij} + \sum_{k=1}^{\theta-1} y_{kj} \leq 1, \quad j \in J, \\ & x_{ij} \in \{0, 1\}, \quad (i, j) \in \mathcal{A}, \\ & y_{kj} \in \{0, 1\}, \quad (i^\circ, j) \in \mathcal{A}, k = 1, \dots, \theta - 1. \end{aligned}$$

Problem LAP _{θ} also has optimal criterion value $f^*(i^\circ, \theta)$ because an optimal solution of LAP _{θ} can be simply transformed into a solution to ASP _{θ} , say w_{ij} , by choosing

$$\begin{aligned} w_{ij} &= x_{ij}, \quad i \in S \setminus \{i^\circ\} \text{ or } (i, j) = (i^\circ, j^\circ), \\ w_{i^\circ j} &= \sum_{k=1}^{\theta-1} y_{kj}, \quad j \in \mathcal{A}(i^\circ) \setminus \{j^\circ\}, \end{aligned}$$

where j° is the index with $x_{i^\circ j^\circ} = 1$ in the optimal solution of LAP _{θ} .

Consider, in general, LAP _{t} , $t = 2, \dots, \theta$. Observe that LAP _{t} is an extended version of LAP _{$t-1$} that contains (at most) $|J|$ extra duplicate y -variables for person i° and one extra constraint in these variables. Given an optimal solution of LAP _{$t-1$} , one can determine an optimal solution of LAP _{t} as follows. Substitute the optimal solution of LAP _{$t-1$} into LAP _{t} and extend this partial solution of LAP _{t} to a (complete) optimal one of LAP _{t} by solving a shortest-path subproblem as in any shortest-path-based LAP algorithm (see, e.g., Ahuja et al. 1993). The computation of the shortest path has complexity $O(n^2)$.

The *linearity intervals* (or the *break points*) of f^* are of interest because they indicate where the rate of the objective function value changes if another assignment is made. The intervals can be found by successively solving the problem LAP _{θ} for $\theta = 1, 2, \dots$ as long as a solution exists and the value of θ is relevant for the instance at hand. The $f^*(\theta)$ values are then also available. A break point of f^* is found if (see Table 1)

$$f^*(\theta) - f^*(\theta - 1) < f^*(\theta + 1) - f^*(\theta), \quad \theta \geq 2.$$

Holder (2005) showed that the first break point of f^* roughly measures the competition for jobs.

To compare the complexity of the two parametric-analysis methods, we observe that they both are dominated by solving ASP₁. The interior-point method as promoted by Holder (2005) has a complexity of $O(Lm^3/\ln m)$, with L denoting the input length of the given problem, see Anstreicher (1999). To avoid distracting details, we assume that $O(\theta) = O(n)$. Without this restriction, both related complexities have to be modified because they both depend on θ . We consider two cases for the number $|J|$ of jobs.

In the *general case*, $O(|J|) = O(n)$, i.e., $O(m) = O(n^2)$ and $O(L) = O(m)$; thus, the complexity of the interior-point method is $O(n^8/\ln n)$, in contrast to the complexity of $O(n^3)$ for the linear assignment problem, (see, e.g., Dell'Amico and Toth 2000).

In the *special case* that $O(|\mathcal{A}(i)|)$ is constant for each i , i.e., $O(m) = O(n)$, the interior-point method has a complexity of $O(n^4/\ln n)$ in contrast to the complexity of $O(n^2 + n^2 \ln n)$ for the LAP as implemented according to

Table 1. An algorithm to calculate λ break points of f^* .

<i>Step 1: initialisation</i> {Abbreviate $f^*(i^\circ, \theta)$ as $f^*(\theta)$ }	
$\theta := 2$; $l := \lambda$; determine $f^*(1)$ and $f^*(2)$;	
$\Delta f_2^* := f^*(2) - f^*(1)$;	
<i>Step 2: iteration</i>	
$\theta := \theta + 1$; determine $f^*(\theta)$; $\Delta f_\theta^* := f^*(\theta) - f^*(\theta - 1)$;	
<i>Step 3: new break point</i>	
if $\Delta f_{\theta-1}^* < \Delta f_\theta^*$ then	
a new break point $(\theta - 1)$ of f^* has	
been found, $l := l - 1$;	
<i>Step 4: termination</i>	
if $l > 0$ then go to Step 2 else stop.	

Gabow and Tarjan (1989). The implementation in this case is more demanding because it exploits Fibonacci heaps to solve shortest-path steps in complexity $O(m + n \ln n)$ (see Fredman and Tarjan 1987).

In both cases, the number of duplicate variables can be reduced. The jobs can be renumbered such that $j \in \mathcal{A}(i^\circ)$ for $j = 1, \dots, \theta$; then the set of duplicate variables in the described procedure can obviously be reduced to

$$y_{11}, y_{22}, \dots, y_{\theta-1, \theta-1} \quad \text{and} \quad y_{lj} \\ \text{for } l = 1, \dots, \theta - 1, j = \theta, \dots, |\mathcal{A}(i^\circ)|$$

because each assignment of θ jobs to person i° can be represented by one x -variable and $(\theta - 1)$ y -variables equal to one, whereas the other variables corresponding to i° are equal to zero.

3. An Alternative Procedure

We present an alternative for the list-generation procedure that is based on a suggestion of Holder (2005) and exploits $\mathcal{A}^*$, a collection defined as follows. Let $\mathcal{A}^*$ be the collection of all person-job assignments that appear in some optimal solution of ASP₁, i.e., $(i, j) \in \mathcal{A}^*$ if and only if there is an optimal solution to ASP₁, where person i is assigned to job j . Holder's suggestion to generate a list of jobs for person i is to find every job such that $(i, j) \in \mathcal{A}^*$.

To find the set $\mathcal{A}^*$, one first solves ASP₁, where $(\mathbf{u}^*, \mathbf{v}^*)$ is a dual optimal solution with u_i^* ($i \in S$) and v_j^* ($j \in J$) being the dual variables corresponding to (2) and (3), respectively. Then, the set $\{(i, j): c_{ij} - u_i^* - v_j^* = 0\}$ can be transformed into $\mathcal{A}^*$ by checking for the existence of a cycle for each (i, j) with $c_{ij} - u_i^* - v_j^* = 0$; these checks can be done in $O(m)$ all together (see, e.g., Uno 2001). If such a cycle does not exist for (i, j) , then $(i, j) \notin \mathcal{A}^*$.

Figure 1 shows an example matrix C to illustrate the construction of $\mathcal{A}^*$. The optimal dual variables are given by $\mathbf{u} = (3, 3, 5, 4)$ and $\mathbf{v} = (0, -4, 0, 2)$, resulting in the matrix of reduced costs C_{red} . The set $\mathcal{A}^* = \{(1, 3), (2, 1), (2, 4), (3, 2), (4, 1), (4, 4)\}$.

Holder (1995) suggested as the job list for person i all the jobs with $(i, j) \in \mathcal{A}^*$; he noted as a shortcoming that this method provides little control over the list length. The LAP approach gives the opportunity to overcome this shortcoming in some sense by exploiting the hard dual values h as introduced by Kindervater et al. (1985). For an arbitrary pair (i, j) , the value $h(i, j)$ is defined as the increment

of the optimal criterion value under the extra constraint given by $x_{ij} = 1$. The value $h(i, j)$ is associated with the constraint $x_{ij} \geq 0$ or parametrized $x_{ij} \geq \tau_{ij}$ (τ_{ij} a parameter, $0 \leq \tau_{ij} \leq 1$); the value gives the increase of the criterion value if the right-hand side parameter grows from zero to one. Kindervater et al. (1985) show that for each $i \in S$, a dual solution $(\mathbf{u}^i, \mathbf{v}^i)$ can be computed with the property that

$$h(i, j) = c_{ij} - u_i^i - v_j^i \quad (j \in \mathcal{A}(i)).$$

The computation costs $O(n^2)$ time for a fixed index i by solving a shortest-path problem. Given the h -values, we can alternatively find $\mathcal{A}^*$ as

$$\mathcal{A}^* = \{(i, j): h(i, j) = 0\}.$$

It is easy to check whether job j° is only optimal for person i , as for this job $(i, j^\circ) \notin \mathcal{A}^*$ for all $i \neq i^\circ$.

One can generate the list by sorting the jobs by nondecreasing h -values. As long as the decision maker considers these values sufficiently small, the related jobs are suitable alternatives for this person. An option is to choose a list length a priori; then the h -values give an indication of the loss of optimality. In the given example (Figure 1 shows H) for, say, $i^\circ = 2$, we find a list with the jobs 1 and 4, without loss of optimality. For, say, $i^\circ = 1$ and a maximal optimality loss of 1, we find a list with the jobs 3 and 1.

Holder (2005) introduced a restriction on the number of different c_{ij} -values used as input into his model, namely, $c_{ij} \in \{0, 1, 2, 3, 4\}$, to promote the existence of alternative solutions with equal criterion values. The advantage of our alternative procedure is that such a restriction can be omitted. If it is appropriate to use a wider variety of cost coefficients, then we expect to find more different hard dual values and the size of the list can be better controlled. There is no guarantee, however, that all the hard dual values are unique.

In this alternative procedure, the use of hard dual values is essential. The dual values as produced by many linear assignment algorithms are in general lower bounds for the hard dual values (see matrices C_{red} and H in Figure 1) because the LAP can be seen as a degenerated case of the linear programming problem.

4. Remarks and Conclusion

We make two final remarks:

- Concerning the practical significance of the parametric results, we note that the time for the parametric analysis is dominated by the computation time for solving the

Figure 1. An example with cost matrix C (an optimal solution in bold), reduced cost matrix C_{red} (the elements of $\mathcal{A}^*$ are starred), and H the matrix of the hard dual values.

$$C = \begin{pmatrix} 4 & 0 & \mathbf{3} & 9 \\ \mathbf{3} & 8 & 8 & 5 \\ 5 & \mathbf{1} & 6 & 8 \\ 4 & 2 & 4 & \mathbf{6} \end{pmatrix} \quad C_{red} = \begin{pmatrix} 1 & 1 & 0^* & 4 \\ 0^* & 9 & 5 & 0^* \\ 0 & 0^* & 1 & 1 \\ 0^* & 2 & 0 & 0^* \end{pmatrix} \quad H = \begin{pmatrix} 1 & 2 & 0^* & 4 \\ 0^* & 9 & 6 & 0^* \\ 1 & 0^* & 2 & 2 \\ 0^* & 2 & 1 & 0^* \end{pmatrix}$$

related linear assignment problem. Dell'Amico and Toth (2000) show that linear assignment problems can be solved in short times (seconds) for sizes at least up to $n = 1,000$. Thus, the proposed analysis is suitable in the context of the motivating application of Holder (2005), where about 1,000 persons (sailors) are involved in the job-assignment process.

- We can also give a similar parametric analysis for the *bottleneck* (or *minmax*) version of ASP_θ . Because reoptimization can also be done in $O(n^2)$ time, according to Derigs and Zimmerman (1978), the efficiency is similar to the sum version, with the bottleneck linear assignment problem solvable in $O(n^{2.5})$ time, according to Punnen and Nair (1994).

We have described an efficient procedure for doing a parametric analysis of linear assignment problems; it may be useful for personnel-assignment problems as well as other applications of linear assignment. The procedure has a complexity of $O(n^3)$, in contrast to a known interior-point-based method that has a complexity of $O(n^8 / \ln n)$.

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