Supermassive black holes as giant Bose-Einstein condensates

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Published in:
Europhysics Letters

DOI:
10.1209/0295-5075/83/10008

Citation for published version (APA):
Supermassive black holes as giant Bose-Einstein condensates

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received 14 January 2008; accepted in final form 26 May 2008
published online 24 June 2008

PACS 04.70.Bu - Classical black holes
PACS 04.20.Cv - Fundamental problems and general formalism
PACS 04.20.Jb - Exact solutions

Abstract - The Schwarzschild metric has a divergent energy density at the horizon, which motivates a new approach to black holes. If matter is spread uniformly throughout the interior of a supermassive black hole, with mass $M \sim M_\odot = 2.34 \times 10^6 M_\odot$, it may arise from a Bose-Einstein condensate of densely packed H atoms. Within the relativistic theory of gravitation with a positive cosmological constant, a bosonic quantum field is coupled to the curvature scalar. In the Bose-Einstein condensed ground state an exact, self-consistent solution for the metric is presented. It is regular with a specific shape at the origin. The redshift at the horizon is finite but large, $z \sim 10^{14} M_\odot/M$. The binding energy remains as an additional parameter to characterize the BH; alternatively, the mass observed at infinity can be any fraction of the rest mass of its constituents.

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On the basis of the Schwarzschild, Kerr and Kerr-Newman metrics, it is generally believed that black holes (BHs) are singular objects with all matter localized in the center or, if rotating, on an infinitely thin ring. Recent approaches challenge this unintuitive assumption and consider matter just spread throughout the interior [1–3]. Here we shall follow this line of research. To start, let us just look at some orders of magnitude. For solar-mass neutron stars it is known that the density is about $3 \times 10^{14} \text{ g/cm}^3$. Hence, we have to employ the relativistic theory of gravitation with a positive cosmological constant, a bosonic quantum field is coupled to the curvature scalar, first for a uniform ground-state wavefunction and next for a space-dependent one. Here to this end, we consider a static metric with spherical symmetry,

$$ds^2 = U(r)c^2 dt^2 - V(r)dr^2 - W^2(r)(d\theta^2 + \sin^2 \theta d\varphi^2).$$

The gravitational energy density arises from the Landau-Lifshitz pseudo-tensor [9], generalized to become a tensor

$$\rho_{\text{grav}} = \frac{1}{8\pi G}T_{\mu\nu}U^\mu U^\nu - \frac{1}{2}T^\mu_\nu U^\mu \nabla_\nu U_\rho \nabla^\rho U^\mu.$$
in Minkowski space [8,10]. For (1) it takes the form
\[ t^{00} = \frac{c^4W^2}{8\pi G r^6} \left( -\frac{r^2V'W'}{V} + r^2V' - 5r^2W'^2 + 2r^3V'W' + 3rVW - 2r^2V - 3W^2 \right). \] (2)

Let us start with the general theory of relativity (GTR). The Schwarzschild metric reads in the harmonic gauge
\[ U_S = \frac{1}{V_S} = 1 - \frac{2M}{r - M}, \quad W_S = r + M. \] (3)

(We put \( G = c = \hbar = 1 \).) It is singular at the horizon \( r_h = M \) and involves the gravitational energy density
\[ t^{00} = \frac{1}{4\pi r^2} \frac{d}{dr} \left( M(r + M)^3(2r + M) \right)^{1/2}. \] (4)

Its quadratic divergence at \( r_h \) presents a hitherto overlooked peculiarity, that induces a negative infinite contribution to the total energy. For this reason, we shall switch to the RTG with matter not located at the singularity \( r = 0 \), but just spread out within the horizon.

**Quantum field theory of Bose-Einstein condensed black holes.** – Let our H atoms be described by a bosonic field
\[ \hat{\psi}(r,t) = \sum_i a_i \hat{\psi}_i(r)e^{-iE_it}, \]
where \( i = \{ n, \ell, m \}, [\hat{a}_i, \hat{a}^\dagger_j] = \delta_{ij} \) and eigenfunctions as \( \psi_i(r) = \phi_n(r)Y_{\ell m}(\theta, \phi) \). The rotating wave approximation then leads to the Lagrangian [11]
\[ L_{\text{mat}} = g^{\mu\nu} \partial_\mu \hat{\psi}^\dagger \partial_\nu \hat{\psi} - \frac{m^2}{2} \hat{\psi}^\dagger \hat{\psi} - \lambda \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} - \frac{\lambda}{4} \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi}. \] (5)

For a field in curved space the renormalization group generates a coupling to the Ricci curvature scalar \( R \) [12]. Its strength \( \xi \) is for now a phenomenological parameter. The dimensionless coupling \( \lambda = 8m^2cq/\hbar^4 \) with \( g = 4\pi\hbar^2a_s/m \) models the two-particle interaction by the scattering length. For hydrogen in flat space one has [13]
\[ a_s = 0.32a_0 \quad \text{singlet state}, \quad a_s = 1.34a_0 \quad \text{triplet}. \]

We shall continue with the singlet value \( \lambda = 0.81 \times 10^{-7} \).

With \( \Psi_0 = (2E_0N_0)^{1/2}\psi_0 e^{-iE_0t} \) for \( N_0 \) ground-state atoms, the relativistic Gross-Pitaevskii equation reads
\[ \left( \frac{1}{\sqrt{g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu + m^2 + \xi R + \frac{\lambda\Psi_0^2}{4E_0} \right) \Psi_0 = 0. \] (6)

A homogeneous ground state, \( \Psi_0(r,t) = \Psi_0 e^{-iE_0t} \) occurs when
\[ \frac{E_0^2}{U} + m^2 + \xi R + \frac{\lambda}{4E_0} \|\Psi_0\|^2 = 0. \] (7)

We focus on the RTG, which describes gravitation as a field in Minkowski space [6,7] and possesses the same gravitational energy momentum tensor and thus also the gravitational energy density eq. (2) [8]. It extends the Hilbert-Einstein action with the cosmological term and a bimetric coupling between the Minkowski (γ) and Riemann (g) metrics,
\[ L = -\frac{R}{16\pi} - \rho_\Lambda + \frac{1}{2}\rho_\Lambda^\gamma g_{\mu\nu} g^{\mu\nu} + L_{\text{mat}}. \] (8)

(For \( \rho_\Lambda = 0 \) it is just a field theory for the GTR.) One has
\[ \rho_{\text{tot}} = \rho + \rho_\Lambda + \rho_{\phi} = \frac{\rho_{\phi}}{2U} - \frac{\rho_\Lambda}{2V} - \frac{\rho_{\phi}r^2}{W^2}, \]
\[ p_{\text{tot}} = p_i - \rho_\Lambda - \rho_{\phi} = \frac{\rho_{\phi}}{2V} + \frac{\rho_{\phi}r^2}{W^2} \]
with \( i = r, \theta, \phi \). The value \( \rho_{\phi} = \rho_\Lambda \) is imposed to have a Minkowski metric in the absence of matter. One may fix them to the observed positive cosmological constant [8]. However, historically the opposite choice \( \rho_\Lambda < 0 \) was considered and the cosmological data were described by an additional inflaton field [7]. The new point of the RTG is that \( g_{00} = U \) can be very small. Despite the smallness of \( \rho_{\phi} \), the \( \rho_{\phi}/U \)-terms become relevant near the horizon [6,8], bringing
\[ R = -8\pi T_{00} = 8\pi \left( -\rho + p_r + p_\theta + p_\phi + \frac{\rho_{\phi}}{U} \right). \]

To start, let us consider the stiff equation of state
\[ \rho = \frac{1}{2} \rho_c \left( \frac{U_c}{U} + 1 \right), \quad p_i = p = \frac{1}{2} \rho_c \left( \frac{U_c}{U} - 1 \right), \]
where \( \rho_c = 3/32\pi M^2 \). For \( U_c = 0 \) it is the vacuum equation of state \( p = -\rho = \text{const} \). One gets
\[ R = 8\pi \rho_c \frac{U_c + \rho_{\phi}}{U} - 16\pi \rho_c. \] (10)

Equation (7) has a solution due to the \( \xi R \)-term. The constant and \( 1/U \)-terms imply
\[ \xi = \xi_0 \left( 1 + \frac{\lambda\Psi_0^2}{4E_0m^2} \right), \quad E_0^2 = 8\pi \xi(\rho_c U_c + \rho_{\phi}), \]
respectively. The dimensionless parameter
\[ \xi_0 = \frac{2}{3}m^2M^2 = 1.80 \times 10^{54} \left( \frac{M}{M_*} \right)^2, \] (14)
appears to be large, but since \( R \sim 1/M^2 \) the combination \( \xi R \) is just of order \( m^2 \), making its effect of order unity. The metric can be solved as below; we omit details.

**Self-consistent field theory.** – Rather than imposing an equation of state, the material energy momentum tensor \( T^\mu_\nu_{\text{mat}} \) should be derived from first principles, i.e. from the quantum field theory for the H atoms. Its energy density reads, if we exclude the effect of the \( \xi R \)-term,
\[ \rho_m = \frac{\langle \partial_t \psi^\dagger \partial_t \psi \rangle}{U} + \frac{\langle \partial_\mu \psi^\dagger \partial_\mu \psi \rangle}{V} + \frac{\langle \partial_\mu \psi^\dagger \partial_\mu \psi \rangle}{W^2} + \frac{\langle \partial_\mu \psi^\dagger \partial_\mu \psi \rangle}{W^2 \sin^2 \theta} + m^2 \langle \psi^\dagger \psi \rangle + \frac{\lambda}{4} \langle \psi^\dagger \psi \rangle. \] (15)
The pressures \( p_\theta^m, p_\phi^m, p_\nu^m \) have this shape with signature \((++-), (++-), \) and \((-+--), \) respectively. Spherical symmetry will imply that \( p_\theta^m = p_\phi^m \equiv p_\perp^m. \) For a uniform ground state \( \rho_m \) is isotropic,
\[
(\rho_m, \rho_m) = \frac{1}{2} \left( \frac{E_0}{U} \pm \frac{m^2}{E_0} \right) |\Psi_0^m| \left| \frac{\lambda |\Psi_0^m|}{16E_0^2} \right|^2. \tag{16}
\]
They consist of a vacuum part \( p = -\rho = \text{const} \) and a stiff part \( p = +\rho \sim 1/U, \) the types studied in [3] and above. In the non-relativistic \((E_0 = m) \) and flat-space \((U = 1) \) limit, they reduce for \( \lambda = 0 \) to \( \rho_m = mc^2 |\Psi_0^m|^2 \) and \( \rho_m = 0. \)

Because of the \( \xi R \)-term in (5), the Einstein equations embody a direct backreaction of matter on curvature, \( G^\mu^\nu - 8\pi T^\mu^\nu = 16\pi \xi \langle 4^\psi \rangle G^\mu^\nu. \) To connect to the standard notation, \( G^\mu^\nu = 8\pi T^\mu^\nu \), we define \( T^\mu^\nu \) by
\[
T^\mu^\nu = \frac{T^\mu^\nu + T^\mu^\nu}{1 + B \rho_m} \equiv T^\mu^\nu + T^\mu^\nu, \tag{17}
\]
with direct backreaction strength of matter on the metric
\[
B = 16\pi \xi \langle 4^\psi \rangle = \frac{8\pi \xi |\Psi_0^m|^2}{E_0}. \tag{18}
\]
For \( \lambda = 0 \) the curvature scalar follows from (16) as
\[
R = 8\pi \left( \frac{E_0 |\Psi_0^m|^2 + \rho_m}{U} - \frac{2m^2}{E_0} |\Psi_0^m|^2 \right). \tag{19}
\]
Solving eq. (7), we find two relations and a consequence,
\[
B = 1, \quad E_0^2 = 8\pi \xi \rho_m, \quad |\Psi_0^m|^2 = \frac{E_0}{8\pi \xi} = \frac{\rho_m}{E_0}. \tag{20}
\]
(The GTR situation, reached by taking \( \rho_m \to 0 \) first, would not allow a meaningful solution.) The first identity expresses a 100\% direct backreaction of matter on the metric. This motivates the introduction of the parameters [8]
\[
\mu = \sqrt{16\pi \rho_m} = \sqrt{2} \Lambda, \quad \bar{\mu} = \mu M = 7.90 \times 10^{-15} \frac{M}{M_*}. \tag{21}
\]

Instead of searching for a finite \( U, \) as for boson stars [14], we assume a very small \( U \) with \( U(0) = 0, \) coded by a parameter \( v, \)
\[
U = \frac{1}{2} \mu \xi v^2 W^2. \tag{22}
\]
In terms of the mass function \( M(r), \) defined by
\[
V = \frac{W^2}{1 - 2M/r}, \tag{23}
\]
the 00 and 11 Einstein equations take the form
\[
M' = 4\pi W^2 \rho_{tot}, \quad \frac{W - 2M}{2UW^2} = \frac{M}{W^3} = 4\pi W^2 \rho_{tot}. \tag{24}
\]
The Ansatz (22) solves them and yields, due to (20),
\[
v = 1, \quad M = \frac{W}{4} + \frac{W^3}{16M^2} \frac{2m^2 M^2}{3\xi}. \tag{25}
\]

For the Schwarzschild black hole the horizon occurs when \( M = M \) for \( W = 2M. \) Concerning the outside metric, we will be close to that situation. This implies again that a mass \( M \) corresponds to \( \xi = 2m^2 M^2/3. \)

Let us introduce the “Riemann” variables \( x \) and \( y \) by
\[
x = \frac{W}{2M}, \quad y = \sqrt{1 - x^2}, \tag{26}
\]
so that \( U = 2\bar{\mu} x^2. \) The \( \rho_m \)-terms in (8) violate general coordinate invariance and impose the harmonic gauge,
\[
\frac{U'}{U} + \frac{V'}{V} + \frac{4W'}{W} = \frac{4xV}{W^2}. \tag{27}
\]
With (22), (23) and \( M = \frac{1}{2} M(x + x^2) \) from (25), it brings
\[
2x' = \frac{2x' - 2x''}{1 - x^2} = \frac{2x''}{x} = \frac{8r x^2}{x^2 (1 - x^2)}. \tag{28}
\]

Going to the inverse function \( r(x) \) makes it linear,
\[
x^2 (1 - x^2) r'' + x (3 - 4x^2) r' = 4r. \tag{29}
\]
The solution is then remarkably simple,
\[
r = r_1 \left( 1 + \frac{y}{\sqrt{x}} \right) x^{\sqrt{1 - y} - \sqrt{x}} \tag{30}
\]
(The second independent solution with \( \sqrt{5} \to -\sqrt{5} \) is singular.) This determines the metric function \( V, \)
\[
V = \frac{2W'^2}{y^2} = \frac{5M^2}{2y_1^5} x^{4 - 2\sqrt{5}(1 + y)^2} \sqrt{5}. \tag{31}
\]

Putting these results together, it now follows that
\[
\rho = \frac{3}{64\pi M^2}, \quad p = -\frac{3}{64\pi M^2}, \tag{32}
\]
as the \( 1/U \)-terms cancel due to the relation \( E_0^2 = 8\pi \xi \rho_m. \) So, after all, we reproduce the vacuum equation of state.

To normalize \( |\Psi_0|, \) we need the 3d volume element in the future time direction, \( d\Sigma^m = dr d\phi d\mu \sqrt{-g_3} \equiv d^3 \mu \) \( d\Sigma, \) set by the timelike unit vector \( n^\mu = \delta^\mu_0 / \sqrt{U} \) and \( g_3 = -W^4 \sin^2 \theta. \) This results in \( d\Sigma = dr d\phi \sqrt{U} W^2. \)

The general inner product [12] \( \langle \psi_1, \psi_2 \rangle = -i \int d\Sigma^m \langle \psi_1 | \delta_{\mu \nu} \psi_2 + \psi_2 \delta_{\mu \nu} \psi_1 \rangle \) defines the orthonormality
\[
\langle \psi_1, \psi_2 \rangle = \langle E_1, E_2 \rangle \int d\Sigma \psi_i \psi_j^* = \delta_{ij}. \tag{33}
\]

With \( d\Sigma = dy d\Omega 8M^3/\mu \) it yields
\[
|\Psi_0^m|^2 = 2E_0 N_0 |\Psi_0^m|^2 = \frac{N_0 \bar{\mu}}{32\pi M^3}, \tag{34}
\]

having proper ground-state occupation, \( \int d\Sigma |\Psi_0|^2 = N_0. \) In eq. (13) the correction term is of order
\[
\lambda = \frac{3\xi}{32\pi m^2} = \frac{3\lambda}{64\pi m^3 M^2} \approx 7.58 \times 10^{-12} \frac{M^2}{M_*^2}. \tag{35}
\]

These corrections seem relevant for BHs with masses \( M \sim 2.75 \times 10^{-6} M_* \sim 645 M_0. \) Much less below \( M_0, \) the hydrogen atoms will get ionized, calling for fermionic fields for protons and electrons, which by a BCS pairing can again undergo a BEC transition. This BCS-BEC scenario is beyond the aim of the present paper.
The exterior. – At the horizon $r_h \approx r_1$, $y_h \ll 1$ one has
\begin{equation}
U = 2\bar{\mu}^2, \quad V = \frac{5}{2}, \quad W = 2M, \quad W' = \frac{1}{2\sqrt{5}}y_h.
\end{equation}
We have to connect this to the vacuum solution outside the BH. Well away from matter, the harmonic constraint brings the Schwarzschild shape (3), where $M \equiv M(r_h)$ is the mass, essentially as observed at infinity. The values (35) are far from Schwarzschild’s ones, even when $r$ is near $M$ (e.g., $W_{25} = 1$). The problem, nevertheless, appears to be consistent. Near $r_h$ we need the deformation of the Schwarzschild metric which regularizes its singularity due to the bimetric coupling [6,7]. An elegant scaling form for small $\mu$ was given by the present author, [8],
\begin{equation}
\begin{aligned}
& r = M \left( 1 + \frac{\eta (\xi^2 + \xi + \log \eta + 2)}{1 - \eta (\xi^2 + \xi + \log \eta + 2)} \right), \quad U = \eta e^\xi, \\
& V = \frac{e^\xi}{\eta (1 + e^\xi)^2}, \quad W = \frac{2M}{1 - \eta e^\xi - \bar{\mu}^2 (\xi + w_0)},
\end{aligned}
\end{equation}
Here $\xi$ is the running variable and $\eta$ a small scale. For $\eta e^\xi = O(1)$ it coincides with the Schwarzschild solution. Matching with the interior appears to be possible,
\begin{equation}
\begin{aligned}
& e^{\xi_h} = \sqrt{5}\bar{\mu}, \quad \eta = \frac{2}{\sqrt{5}} \bar{\mu}, \quad W' = e^{\xi_h} + \frac{\bar{\mu}^2}{\eta} = \frac{3}{2\sqrt{5}} \bar{\mu},
\end{aligned}
\end{equation}
implying $y_h = 3\bar{\mu}$. Taken together, the three regimes, interior, horizon and exterior, provide an exact solution of the problem. At the origin it exhibits the singularities
\begin{equation}
U = \bar{U}_1 r^{-\gamma}, \quad V = \frac{1}{2} \gamma^2_{\pi} W^2 r^{-\gamma_{\pi}-2}, \quad W = \bar{W}_1 r^{\frac{1}{2} \gamma_{\pi}},
\end{equation}
where $\gamma_{\pi} = \frac{1}{2}(\sqrt{5} + 1)$ is the golden mean. But if we take $W$ as the coordinate, we have in the interior the shape
\begin{equation}
ds^2 = \frac{1}{2} \mu^2 W^2 dt^2 - \frac{2dW^2}{1 - W^2/4M^2} - W^2 d\Omega^2,
\end{equation}
which is regular at its origin, with the term $2dW^2$ coding the above singularities.
We may rewrite the exterior solution by eliminating $\xi$,
\begin{equation}
\begin{aligned}
r &= M \frac{1 + U + (2\bar{\mu}/\sqrt{5})(\log U + 2)}{1 - U - (2\bar{\mu}/\sqrt{5})(\log U + 2)}, \\
V &= \frac{U}{(U + 2\bar{\mu}/\sqrt{5})^2}, \\
W &= \frac{2M}{1 - U + 2\bar{\mu}^2 - \bar{\mu}^2 \log(U/2\bar{\mu}^2)}.
\end{aligned}
\end{equation}
This describes the free space region $r \gg M$, where $2\bar{\mu}^2 \ll U \ll 1 + O(\bar{\mu})$. At the cosmic scale $r \sim 1/\mu$ Newton’s law picks up the Yukawa-type factor $\cos \mu r$, due to the tachyonic nature of gravitation in the RTG with $\rho_{bi} > 0$ [8]. The interior shape can also be expressed with $U$ as running variable, where it lies in the range $(0, 2\bar{\mu}^2)$. Due to eq. (29) it also holds that
\begin{equation}
W = 2Mx = \frac{1}{\mu} \sqrt{2U}, \quad y = \sqrt{1 - \frac{U}{2\bar{\mu}^2}}
\end{equation}
The locus and the metric function $V$ are given by (29), with $r_1 \approx M$, and (30), respectively.

Contributions to the energy. – With the weight $dV$ given below (32), the standard expression for the energy, $\int d\mathcal{V} \rho$ scales as $1/\bar{\mu}$ and even diverges logarithmically at $r = 0$. However, in the RTG the energy is determined by eq. (2). At the origin it diverges as $r^{\sqrt{5}-5}$, which is integrable. The gravitational energy inside the BH reads
\begin{equation}
U_{\text{grav, int}} = 4\pi \int_0^M dr t^{100} = -842.898 M.
\end{equation}
The total energy density reads $\Theta^{00} = \rho^{00} + VW W^4 \rho_{\text{int}}/r^4$. We can calculate the material and the bimetric energy,
\begin{equation}
\begin{aligned}
U_{\text{mat}} &= 4\pi \int_0^M r^2 dr W^{4} \frac{VW}{r^4} \frac{3}{64\pi M^2} = 169.431 M, \\
U_{\text{bi}} &= 4\pi \int_0^M r^2 dr \frac{VW^4}{r^4} \frac{1}{64\pi M^2 x^2} = 686.466 M.
\end{aligned}
\end{equation}
Together they make up for
\begin{equation}
U_{\text{interior}} = U_{\text{grav, int}} + U_{\text{mat}} + U_{\text{bi}} = 13 M.
\end{equation}
The energy density in the skin layer first has a large positive and then a large negative part, due to the term $3V'$. The integrated effect is obtained easily since, in the formulation of the Einstein equations in Minkowski space, the total energy density is a total derivative, [8]. The region $r > M$ thus yields $U_{\text{exterior}} = -12M$. Together with the interior it confirms the total energy $U = Mc^2$, expected from the decay of the metric, $g_{00} = 1 - 2GM/c^2 r$.

Non-uniform ground state. – Till now we assumed that the ground-state wave function has a constant amplitude, and drops to zero at the horizon. Clearly, this cannot be exact. Considering $\Psi_0 \to \Psi_0(r)$, we first take into account that in deriving the Einstein equations, partial integrations are to be performed. This brings derivatives of $B \sim |\Psi_0|^2$, and induces an extra term $T^{\mu \nu}_B$,
\begin{equation}
(1 + B)G_{\mu \nu} = 8\pi (T^{\mu \nu}_m + T^{\mu \nu}_A + T^{\mu \nu}_\text{bi} + T^{\mu \nu}_b).
\end{equation}
The elements of $(T_B)_\mu^\nu \equiv \text{diag}(\rho_B, -p_B^- - p_B^+ - p_B^-)$ are
\begin{equation}
\begin{aligned}
\rho_B &= \frac{1}{8\pi} \left( B'' + \frac{2r B'}{V} - \frac{2BU'}{V^2} \right), \\
p_B^- &= -\frac{1}{8\pi} \left( \frac{2B'U'}{UV} + \frac{2BW'}{VW} \right), \\
p_B^+ &= -\frac{1}{8\pi} \left( \frac{2B''}{V} + \frac{2r B'}{W} - \frac{B'W'}{VW} \right).
\end{aligned}
\end{equation}
Equation (43) now leads to a total energy momentum tensor
$$T_{\mu\nu}^{\text{tot}} = T_{\mu\nu}^m + T_{\mu\nu}^A + T_{\mu\nu}^b + T_{\mu\nu}^B = \frac{1}{B} T_{\mu\nu}^m + T_{\mu\nu}^A + T_{\mu\nu}^b + T_{\mu\nu}^B.$$  

(45)

$$T_{\mu\nu}^B \equiv \text{diag}(\rho, -p_r, -p_\perp, -p_\perp)$$ has at $\lambda = 0$ the elements
$$\rho = \frac{1}{8\pi(1 + B)} \left( \frac{2B'W'}{V^2} - \frac{B''}{V^2} - \frac{m^2 B + B^2}{2\xi} + \frac{8\xi B^2}{V} \right),$$
$$p_r = \frac{1}{8\pi(1 + B)} \left( \frac{2B'W'}{V^2} - \frac{B''}{V} - \frac{m^2 B + B^2}{2\xi} - \frac{8\xi B^2}{V} \right),$$
$$p_\perp = \frac{1}{8\pi(1 + B)} \left( \frac{B'W'}{V^2} - \frac{2rB'}{W^2} - \frac{m^2 B + B^2}{2\xi} - \frac{8\xi B^2}{V} \right).$$  

(46)

The Gross-Pitaevskii equation (6) reads in terms of $B$
$$-[(6\xi + 1)B + 1] \left( \frac{B''}{2V} + \frac{rB'}{W^2} \right) + \frac{B^2}{4BV} + m^2 B(1 - B) + \lambda m^2 B^2 = \left( E_0^2 - \frac{1}{2}\xi^2 \right) \frac{B}{U}. \quad (47)$$

We can now first verify that the total energy momentum tensor is conserved due to the harmonic condition (27). The singular term $B/U$ also drops out from (47) for
$$E_0^2 = 8\pi\xi \rho_0 = \frac{1}{2}\xi^2. \quad (48)$$

This was already used in (46) to cancel the 1/$U$-terms. With $\xi \equiv 2m^2 M_0^2/\beta$ and $\mu \equiv \mu_0 M_1$ it implies again $E_0 = \mu \mu_0 \sqrt{\beta}$. For very small $B < 1/\xi \sim 10^{-54}$ there will be an exponential falloff, $B \sim \exp(-\sqrt{5m}r_0)$, so the horizon is a fraction of the Compton length thick. In this narrow range $\rho, p_r$ and $p_\perp$ vanish smoothly. In the regime $B \gg 1/\xi$, eq. (47) simplifies and actually reduces to eq. (7),
$$-\frac{B''}{V} - \frac{2rB'}{W^2} = \frac{1 - \lambda}{\lambda}B - B^2 \frac{1}{2M_1^2}. \quad (49)$$

Here we can consider $B$ as vanishing sharply, $B \sim r_h - r$, so that $T_{\mu\nu}^B \neq 0$, keeping the ultimate exponential tail and decay of $\rho$ and the $p_\perp$ in mind. Equation (46) then brings
$$\rho = \frac{1}{8\pi(1 + B)} \left( \frac{B''}{2V^2} \frac{4 + 2B + \lambda(4B + 3B^2)}{8M_1^2} \right),$$
$$p_r = \frac{-1}{8\pi(1 + B)} \left( \frac{2B'W'}{V^2} + \frac{6B + 3\lambda B^2}{8M_1^2} \right),$$
$$p_\perp = \frac{1}{8\pi(1 + B)} \left( \frac{B'W'}{V^2} - \frac{4 + 2B + \lambda(4B + 3B^2)}{8M_1^2} \right).$$  

(50)

Let us first return to the interior where $U = 2\mu^2 x^2$, $V = 8M_1^2 x^2/y^2$ and $W = 2M_1 x$. Equation (49) can be written as
$$\frac{1}{4} x^2 B_{yy} - y B_y + (1 - \lambda)B = 1. \quad (51)$$

To understand the structure of the problem, we again take $\lambda = 0$. Then for any $A$ there is the solution
$$B(x) = 1 + Ay = 1 + A\sqrt{1 - x^2}. \quad (52)$$

Expressing the shapes (50) in $y$, we have
$$\rho = -p_\perp = \frac{1}{64\pi M_1^2(1 + B)}(2B + yB_y + 4),$$
$$p_r = \frac{-3}{64\pi M_1^2(1 + B)}(2B - yB_y). \quad (53)$$

Surprisingly, their $A$-dependence factors out, keeping a vacuum equation of state $\rho = -p = 3/64\pi M_1^2$, so (52) is a non-uniform, exact solution of the same metric. The horizon $B = 0$ is now located at $r_h > r_1$ where $y_h = -1/A$. (Equation (29) continues to negative $y \approx \sqrt{5(r_1 - r)}/4r_1$ for $r > r_1$.) However, a problem shows up with the matching, since $W'(r_h) \sim -1/A$ cannot be of order $\tilde{\mu} \sim 10^{-14}$ anymore. We thus have to deviate from the exact solution, which leads in general to a numerical problem. Analytically, this question can be considered for large $A$, by adding $1/A^2$ corrections to the previous solution. Expanding in $1/A$ at fixed $s \equiv A\sqrt{5(r_1 - 1)/4}$, we arrive at
$$B = 1 - s - \sqrt{5} \frac{s^2}{2A^2} - \frac{7s^3}{6A^2} + \frac{b_1(s)}{A^2},$$
$$U = \frac{2\mu^2}{A^2} - \frac{w_1(s)}{A^2},$$
$$\frac{2}{5} V = 1 - s - \frac{3\mu^2}{A} + \frac{13s^2}{A^2} + \frac{v_1(s)}{A^2},$$
$$W = M_1 = -\frac{1}{2A^2} + \frac{w_1(s)}{A^2}. \quad (54)$$

The additional terms, found to be
$$b_1 = b_{10} + b_{11} s, \quad u_1 = u_{10} + 4w_{11} \ln(2 - s),$$
$$v_1 = v_{10}, \quad w_1 = w_{10} - w_{11} \ln(2 - s), \quad (55)$$
produce an anisotropy, $\rho \neq -p_r \neq -p_\perp = \rho$. The horizon $B = 0$ is located at $s_h = 1 + \sqrt{5}/2A + O(1/A^2)$, where $W' \sim \tilde{\mu}$ from (37) can be attained by tuning $w_{11} = 1 + 3A\tilde{\mu} + O(1/A)$. The maximum of $W$ at $s = 0$ in the absence of the $w_{11}$-term has now been shifted to the horizon, which shows that the horizon has a proper solution, with $A$ remaining a free parameter. $U$ has a maximum at $s = 1 - \sqrt{3}$ . The mass seen at infinity, $M \equiv M(r_h) = \frac{1}{2}W(r_h)$, coincides with $M_1$ to order $1/A^2$. At the horizon we can fix $r_1$ from (36),
$$r_1 = M_1 \left[ 1 - \frac{4}{\sqrt{5}A} + \frac{4\tilde{\mu}}{\sqrt{5}} + 2\ln(2\mu^2) + O \left( \frac{1}{A^2} \right) \right]. \quad (56)$$

**Properties of the solution.** – The ground-state occupation number becomes upon neglecting the $1/A^2$ corrections
$$N_0 = \int dV|\Psi_0|^2 = 2\sqrt{3} \frac{M}{m} \int_{-1/A}^{1} dy (1 + Ay). \quad (57)$$

We may write the two leading orders as
$$M = \nu N_0 m, \quad \nu = \frac{1}{\sqrt{3}(2 + A)} \approx \frac{1}{\sqrt{3}A}. \quad (58)$$
Clearly, the energy $Mc^2$ of the BH can be any fraction of the rest energy $Nmc^2$ of the constituent hydrogen atoms. If $\nu$ starts at a value $\nu_c < 1$, our BH is likely approached in an explosive manner, possibly related to jets of quasars.

We found the sharpness of the horizon to be a fraction of the Compton length of H. On a much larger scale $\ell_{\text{grav}}$ there is near-horizon growth of the metric functions $U/\eta \approx \eta V \approx c^2$, taking place in the millimeter range,

$$\ell_{\text{grav}} = \frac{d\ell}{d\xi} = 2\eta M = \frac{4}{\sqrt{3}} \tilde{\mu} M = 4.88 \times 10^{-3} M^2/L^2 \text{ m.} \quad (59)$$

It is a realistic value, small compared to the size of the BH, and still large compared to the Bohr radius.

The interacting situation. – If the non-linearity $\lambda$ is relevant, a numerical solution is called for. Assuming the same leading-order behaviors near $r = 0$, the above structure survives. Mass and particle number remain is relevant, a numerical solution is called for. Assuming $\xi < 0$, so $B < 0$ due to (18). To avoid a singularity in e.g. (46), a lower bound $B(0) > -1$ will be required. As already indicated by the exact solution (52), $B$ will then go to zero at some point well below $M$. This prevents fitting to the external metric and excludes our BH solution.

We failed to apply our approach to the GTR, technically because it lacks compensation for the singular $1/U$-terms. If no other solution exists for the considered physical situation, the GTR must be abandoned and replaced by another theory, the RTG being the first candidate. In view of its smaller symmetry group, this may have far reaching consequences for singularities in classical gravitation and for quantum approaches to gravitation, while Minkowski space-time needs no quantization.

**References**

The author has benefited from discussions with S. Carlip, K. Skenderis and B. Mehmari.

REFERENCES