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Remarks on quantiles and distortion risk measures

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Abstract

Distorted expectations can be expressed as weighted averages of quantiles. In this note, we show that this statement is true, but that one has to be careful with the correct formulation of it. Furthermore, the proofs of the additivity property for distorted expectations of a comonotonic sum that appear in the literature often do not cover the case of a general distortion function. We present a straightforward proof for the general case, making use of the appropriate expressions for distorted expectations in terms of quantiles.

Keywords: comonotonicity, distorted expectation, distortion risk measure, TVaR, quantile.

1 Introduction

It is well-known that a distorted expectation of a random variable (r.v.) can be expressed as a weighted average of its corresponding quantiles; see e.g. Wang (1996) or Denuit et al. (2005). Although this statement is true, one has to be careful to formulate it in an appropriate and correct way. In this short note, we explore this statement and the conditions under which it holds.

A second goal of this note is to present a complete proof for the additivity property which holds for distorted expectations of a comonotonic sum. The proofs of this theorem that are presented in the literature are often incomplete, in the sense that they only hold for a particular type of distortion functions, such as the class of concave distortion functions. We present a straightforward proof for the general case, making use of the appropriate expressions for distorted expectations as weighted averages of quantiles.

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2 Distortion risk measures as mixtures of quantiles

In this section, we investigate the representation of a distorted expectation of a r.v. as a mixture of its quantiles. All r.v.'s that we consider are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The cumulative distribution function (cdf) and the decumulative distribution function (ddf) of a r.v. X are denoted by F_X and $\overline{F}_X$, respectively.

2.1 Distorted expectations

For a given r.v. X, we define its càglàd (continue à gauche, limitée à droite) inverse cdf F_X^{-1}, as well as its càdlàg (continue à droite, limitée à gauche) inverse cdf F_X^{-1+} as follows.

Definition 1 (The inverse cdf's F_X^{-1} and F_X^{-1+}) *For any $p \in [0,1]$, the inverse cdf $F_X^{-1}(p)$ is defined by*

$$F_X^{-1}(p) = \inf \{x \mid F_X(x) \geq p\}, \tag{1}$$

whereas the inverse cdf $F_X^{-1+}(p)$ is defined by

$$F_X^{-1+}(p) = \sup \{x \mid F_X(x) \leq p\}. \tag{2}$$

In these expressions, $\inf \varnothing = +\infty$ *and* $\sup \varnothing = -\infty$ *by convention.*

We recall the following equivalence relations:

$$p \leq F_X(x) \Leftrightarrow F_X^{-1}(p) \leq x, \qquad x \in \mathbb{R} \text{ and } p \in [0,1], \tag{3}$$

and

$$\mathbb{P}[X < x] \leq p \Leftrightarrow x \leq F_X^{-1+}(p), \qquad x \in \mathbb{R} \text{ and } p \in [0,1], \tag{4}$$

which will be used in the derivations hereafter.

In order to define the distorted expectation of a r.v., we have to introduce the notion of distortion function.

Definition 2 (Distortion function) *A distortion function is a non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.*

Any distortion function g can be represented as the following convex combination of distortion functions:

$$g(q) = p_1 g^{(c)}(q) + p_2 g^{(d)}(q) + p_3 g^{(s)}(q), \qquad q \in [0,1], \tag{5}$$

where $p_i \geq 0$ for $i = 1, 2, 3$ and $p_1 + p_2 + p_3 = 1$. In this expression, $g^{(c)}$ is absolutely continuous, $g^{(d)}$ is discrete and $g^{(s)}$ is singular continuous.

Provided the distortion function g has no singular continuous part and is right continuous (r.c.) on $[0,1)$, it can be expressed as

$$g(q)=\int_0^q g'(p)\mathrm{d}p+\sum_{p\in(0,q]}\left[g\left(p\right)-g(p-)\right], \qquad q\in[0,1], \tag{6}$$

where g' has to be understood as an arbitrary function which coïncides with the derivative of g whenever this derivative exists. Furthermore, the sum is taken over all jumps of g in the interval $(0,q]$. Finally, $g(p-)=\lim_{\varepsilon\downarrow 0} g(p-\varepsilon)$, while $[g\left(p\right)-g(p-)]$ is the height of the jump of g at level p.

Wang (1996) introduced a class of risk measures in the actuarial literature, the elements of which are known as distortion risk measures.

Definition 3 (Distorted expectation) *Consider a distortion function g. The distorted expectation of the r.v. X, notation $\rho_g[X]$, is defined as*

$$\rho_g\left[X\right]=-\int_{-\infty}^0\left[1-g\left(\overline{F}_X(x)\right)\right]\mathrm{d}x+\int_0^{+\infty}g\left(\overline{F}_X(x)\right)\mathrm{d}x, \tag{7}$$

provided at least one of the two integrals in (7) is finite.

The functional ρ_g is called the distortion risk measure with distortion function g. Both integrals in (7) are well-defined and take a value in $[0,+\infty]$. Provided at least one of the two integrals is finite, the distorted expectation $\rho_g[X]$ is well-defined and takes a value in $[-\infty,+\infty]$. Hereafter, when using a distorted expectation $\rho_g[X]$, we silently assume that both integrals in the definition (7) are finite, or equivalently, that $\rho_g[X]\in\mathbb{R}$, unless explicitely stated otherwise.

Consider a distortion function g which can be expressed as a strictly convex combination of two distortion functions g_1 and g_2, i.e.

$$g=c_1g_1+c_2g_2 \tag{8}$$

with weights $0<c_i<1$, $i=1,2$, and $c_1+c_2=1$. Assuming that $\rho_g[X]\in\mathbb{R}$ is then equivalent with assuming that $\rho_{g_i}[X]\in\mathbb{R}$, $i=1,2$. Under any of these assumptions, we have that $\rho_g[X]$ is additive with respect to g, in the sense that

$$\rho_g\left[X\right]=c_1\rho_{g_1}\left[X\right]+c_2\rho_{g_2}\left[X\right]. \tag{9}$$

The proofs of the equivalence of the stated assumptions and of (9) follow from the observation that the additivity property (with respect to g) holds for both integrals in (7). Notice that the statements above remain to hold in case $c_i=0$ for $i=1$ or $i=2$, provided g_i is chosen such that $\rho_{g_i}[X]$ is finite.

Hereafter, we will often consider distortion functions that are left continuous (l.c.) on $(0,1]$ or right continuous (r.c.) on $[0,1)$.

The inverse F_X^{-1} defined above belongs to the class of distortion risk measures. Indeed, for $p \in (0,1)$, consider the l.c. distortion function g defined by

$$g(q) = \mathbb{I}\left(q > 1-p\right), \qquad 0 \leq q \leq 1, \tag{10}$$

where we use the notation $\mathbb{I}\left(A\right)$ to denote the indicator function, which equals 1 when A holds true and 0 otherwise. From definition (7) and equivalence relation (3), we find that the corresponding distorted expectation is equal to the p-quantile of X:

$$\rho_g\left[X\right] = F_X^{-1}\left(p\right). \tag{11}$$

2.2 Distorted expectations and r.c. distortion functions

In the following theorem, it is shown that any distorted expectation $\rho_g\left[X\right]$ with r.c. distortion function g can be expressed as a weighted average of the quantiles $F_X^{-1+}\left(q\right)$ of X.

Theorem 4 *When g is a r.c. distortion function, the distorted expectation $\rho_g\left[X\right]$ has the following Lebesgue-Stieltjes integral representation:*

$$\rho_g\left[X\right] = \int_{[0,1]} F_X^{-1+}\left(1-q\right) \mathrm{d}g\left(q\right). \tag{12}$$

Proof. Taking into account that F_X has at most countably many jumps, we have that $\overline{F}_X\left(x\right) = \mathbb{P}\left[X \geq x\right]$ a.e., and we can rewrite the expression (7) for $\rho_g\left[X\right]$ as follows:

$$\rho_g\left[X\right] = -\int_{-\infty}^{0} \left[1 - g\left(\mathbb{P}\left[X \geq x\right]\right)\right] \mathrm{d}x + \int_{0}^{+\infty} g\left(\mathbb{P}\left[X \geq x\right]\right) \mathrm{d}x. \tag{13}$$

As the distortion function g is r.c., we find that $g\left(\mathbb{P}\left[X \geq x\right]\right)$ can be expressed as $\int_{[0,\mathbb{P}[X \geq x]]} \mathrm{d}g(q)$, which has to be understood as a Lebesgue-Stieltjes integral. Applying Fubini's theorem to change the order of integration and noticing (4), the second integral in (13) can be transformed into

$$\begin{aligned} \int_{0}^{+\infty} g\left(\mathbb{P}\left[X \geq x\right]\right) \mathrm{d}x &= \int_{[0,\mathbb{P}[X \geq 0]]} \mathrm{d}g\left(q\right) \int_{0}^{F_X^{-1+}(1-q)} \mathrm{d}x \\ &= \int_{[0,\mathbb{P}[X \geq 0]]} F_X^{-1+}\left(1-q\right) \mathrm{d}g\left(q\right). \end{aligned} \tag{14}$$

Similarly, taking into account that $1 - g\left(\mathbb{P}\left[X \geq x\right]\right)$ can be expressed as $\int_{(\mathbb{P}[X \geq x],1]} \mathrm{d}g(q)$, the first integral in (13) can be transformed into

$$\int_{-\infty}^{0} \left[1 - g\left(\mathbb{P}\left[X \geq x\right]\right)\right] \mathrm{d}x = -\int_{(\mathbb{P}[X \geq 0],1]} F_X^{-1+}(1-q) \mathrm{d}g\left(q\right). \tag{15}$$

Inserting the expressions (14) and (15) into (13) leads to (12). ∎

Theorem 4 can be strengthened in the following sense: if either the distorted expectation $\rho_g[X]$ or the Lebesgue-Stieltjes integral $\int_{[0,1]} F_X^{-1+}(1-q)\mathrm{d}g(q)$ is finite, then also the other quantity is finite and both are equal. Indeed, the case where one starts from a finite $\rho_g[X]$ is considered in the proof of the theorem. On the other hand, in case the integral in (12) is finite, it can be written as the sum of the finite integrals $\int_{[0,\mathbb{P}[X\geq 0]]} F_X^{-1+}(1-q)\mathrm{d}g(q)$ and $\int_{(\mathbb{P}[X\geq 0],1]} F_X^{-1+}(1-q)\mathrm{d}g(q)$. Applying Fubini's theorem leads to the relations (14) and (15), which proves that relation (12) holds.

Using integration by parts, Theorem 4 can be considered as a consequence of Corollary 2.1 in Gzyland and Mayoral (2006). The proof presented above is different and is based on Fubini's theorem.

Suppose that g is r.c. and has no singular continuous part. In this case, g can be expressed as (6) and we can rewrite (12) as follows:

$$\rho_g[X] = \int_0^1 F_X^{-1+}(1-q)g'(q)\mathrm{d}q + \sum_{q\in(0,1]} F_X^{-1+}(1-q)\left[g(q) - g(q-)\right], \tag{16}$$

where the notations are as before, while the sum is taken over all values of q in $(0,1]$ where g jumps.

As there are at most countably many values of $q \in [0,1]$ where the inverses $F_X^{-1}(1-q)$ and $F_X^{-1+}(1-q)$ differ, we can replace F_X^{-1+} by F_X^{-1} in the integral on the right hand side in (16) without changing the value of the integral. On the other hand, in case $F_X^{-1}(1-q)$ and $g(q)$ jump at the same value of q, we have that $F_X^{-1+}(1-q) \neq F_X^{-1}(1-q)$ and in the corresponding term in the sum of (16), we cannot replace $F_X^{-1+}(1-q)$ by $F_X^{-1}(1-q)$.

In order to prove that the càdlàg inverse F_X^{-1+} also belongs to the class of distortion risk measures, let $p \in (0,1)$ and consider the r.c. discrete distortion function g defined by

$$g(q) = \mathbb{I}(q \geq 1-p), \qquad 0 \leq q \leq 1. \tag{17}$$

Taking into account expression (12) for $\rho_g[X]$, we find that

$$\rho_g[X] = F_X^{-1+}(p). \tag{18}$$

The assumption that g is r.c. is essential for (12) to hold. If we assume e.g. that g is l.c., expression (12) for $\rho_g[X]$ above is not valid anymore. This can be illustrated by the l.c. distortion function g that we defined in (10) and for which $\rho_g[X] = F_X^{-1}(p)$. Suppose for a moment that expression (12) is valid for l.c. distortion functions. Applying this formula to the distortion function defined in (10), we find that $\rho_g[X] = F_X^{-1+}(p)$. As $F_X^{-1}(p)$ and $F_X^{-1+}(p)$ are in general not equal, we can indeed conclude that (12) is in general not valid for a l.c. distortion function. The situation where the distortion function g is left continuous will be considered in Theorem 6.

2.3 Distorted expectations and l.c. distortion functions

In order to present a left continuous version of Theorem 4, we introduce the notion of a dual distortion function. Therefore, consider a distortion function g and define the related function $\overline{g} : [0,1] \to [0,1]$ by

$$\overline{g}\,(q) = 1 - g(1-q), \qquad 0 \le q \le 1. \tag{19}$$

Obviously, $\overline{g}$ is also a distortion function, called the dual distortion function of g.

Lemma 5 *For any r.v. X and distortion function g, we have*

$$\rho_{\overline{g}}\,[X] = -\rho_g\,[-X] \tag{20}$$

and

$$\rho_g\,[X] = -\rho_{\overline{g}}\,[-X]\,. \tag{21}$$

Proof. Relation (20) can be proven from definition (7) of a distorted expectation. Indeed, we have

$$\begin{aligned}
\rho_g\,[-X] &= -\int_{-\infty}^{0} \left[1 - g\left(\overline{F}_{-X}(x)\right)\right] \mathrm{d}x + \int_{0}^{+\infty} g\left(\overline{F}_{-X}(x)\right) \mathrm{d}x \\
&= -\int_{-\infty}^{0} \overline{g}\,(F_{-X}\,(x))\,\mathrm{d}x + \int_{0}^{+\infty} \left[1 - \overline{g}\,(F_{-X}\,(x))\right] \mathrm{d}x.
\end{aligned}$$

Substituting $s = -x$ leads to

$$\begin{aligned}
\rho_g\,[-X] &= -\int_{0}^{+\infty} \overline{g}\,(F_{-X}\,(-s))\,\mathrm{d}s + \int_{-\infty}^{0} \left[1 - \overline{g}\,(F_{-X}\,(-s))\right] \mathrm{d}s \\
&= \int_{-\infty}^{0} \left[1 - \overline{g}\,(\Pr\,(X \ge s))\right] \mathrm{d}s - \int_{0}^{+\infty} \overline{g}\,(\Pr\,(X \ge s))\,\mathrm{d}s \\
&= \int_{-\infty}^{0} \left[1 - \overline{g}\left(\overline{F}_X\,(s)\right)\right] \mathrm{d}s - \int_{0}^{+\infty} \overline{g}\left(\overline{F}_X\,(s)\right) \mathrm{d}s,
\end{aligned}$$

where in the last step we used the fact that the Lebesgue measure of the set of all discontinuities of a monotone function is 0. This proves (20).

Relation (21) follows immediately from (20) by noting that $\overline{\overline{g}} \equiv g$. ■

The following theorem can be considered as an adapted version of Theorem 4 for l.c. distortion functions. Notice that for a l.c. distortion function g, we have

$$\int_{[0,1]} F_X^{-1}\,(1-q)\,\mathrm{d}g\,(q) = \int_{[0,1]} F_X^{-1}\,(q)\,\mathrm{d}\overline{g}\,(q)\,, \tag{22}$$

by the definition of Lebesgue-Stieltjes integration for l.c. distortion functions.

Theorem 6 *When g is a l.c. distortion function, the distorted expectation $\rho_g\,[X]$ has the following Lebesgue-Stieltjes integral representation:*

$$\rho_g\,[X] = \int_{[0,1]} F_X^{-1}\,(1-q)\,\mathrm{d}g\,(q)\,. \tag{23}$$

Proof. Let g be a l.c. distortion function. The dual distortion function $\overline{g}$ of g is r.c. Applying (12) and (21) leads to

$$\rho_g[X] = -\rho_{\overline{g}}[-X] = -\int_{[0,1]} F_{-X}^{-1+}(1-q)\mathrm{d}\overline{g}(q).$$

Taking into account the expression

$$F_{-X}^{-1+}(1-q) = -F_X^{-1}(q),$$

as well as the equality (22), we find (23). ■

An alternate proof of Theorem 6 follows from first rewriting $g(\mathbb{P}[X \geq x])$ as $\int_{[0,\mathbb{P}[X\geq x])} \mathrm{d}g(q)$ and $1 - g(\mathbb{P}[X \geq x])$ as $\int_{[\mathbb{P}[X\geq x],1]} \mathrm{d}g(q)$, respectively, and then proceeding as in the proof of Theorem 4.

Theorem 6 can be strengthened in the following sense: if either the distorted expectation $\rho_g[X]$ or the Lebesgue-Stieltjes integral $\int_{[0,1]} F_X^{-1}(1-q)dg(q)$ is finite, then also the other quantity is finite and both are equal.

Suppose that g is a l.c. distortion function without singular continuous part. In this case $\overline{g}$ can be expressed as (6) with g replaced by $\overline{g}$, and we can rewrite the expression (23) for $\rho_g[X]$ as follows:

$$\rho_g[X] = \int_0^1 F_X^{-1}(1-q)g'(q)\mathrm{d}q + \sum_{q\in[0,1)} F_X^{-1}(1-q)\left[g(q+) - g(q)\right], \tag{24}$$

where $g(q+) = \lim_{\varepsilon\downarrow 0} g(q+\varepsilon)$, while the sum is taken over all values of $q \in [0,1)$ where the function g jumps.

The distortion function g defined in (10) is an example of a l.c. discrete distortion function. Its dual distortion function $\overline{g}$ is given by

$$\overline{g}(q) = \mathbb{I}(q \geq p), \qquad 0 \leq q \leq 1.$$

From Theorem 6 it follows that $\rho_g[X]$ is given by $F_X^{-1}(p)$, as we found before.

There are at most countably many values of $q \in [0,1]$ where the inverses $F_X^{-1}(q)$ and $F_X^{-1+}(q)$ differ. This implies that in case g is continuous on $[0,1]$, we can replace F_X^{-1+} by F_X^{-1} in (12) without changing the value of the integral. This observation leads to the following implication:

$$g \text{ is continuous } \Longrightarrow \rho_g[X] = \int_{[0,1]} F_X^{-1}(1-q)\,\mathrm{d}g(q). \tag{25}$$

Notice that this implication follows also directly from (23). Furthermore, when g is absolutely continuous, we can replace $\mathrm{d}g(q)$ by $g'(q)\mathrm{d}q$ in (25), and we find that

$$g \text{ is absolutely continuous } \Longrightarrow \rho_g[X] = \mathbb{E}\left[F_X^{-1}(1-U)\,g'(U)\right], \tag{26}$$

where U is a r.v. uniformly distributed on the unit interval $[0,1]$.

In the literature, much attention is paid to the class of concave (resp. convex) distortion functions. A concave distortion function is continuous on $(0, 1]$ and can only jump at 0, while a convex distortion function is continuous on $[0, 1)$ and can only jump at 1. Concave (resp. convex) distortion functions without jumps in the endpoints of the unit interval are absolutely continuous, which implies that the expressions for $\rho_g[X]$ in (25) and (26) hold in particular for these functions.

Consider a concave distortion function g without a jump at 0. Taking into account (26), one can rewrite the corresponding distorted expectation $\rho_g[X]$ as

$$\rho_g[X] = -\int_0^1 F_X^{-1}(q)\,\phi(q)\,\mathrm{d}q, \tag{27}$$

with

$$\phi(q) = -g'(1-q). \tag{28}$$

Notice that $\phi(q)$ may not exist on a set of Lebesgue measure 0, but this observation does not hurt the validity of (27). A risk measure of the form (27) is called a spectral risk measure with risk spectrum $\phi(q)$; see e.g. Gzyland and Mayoral (2006).

As an example of a concave distortion function, for $p \in [0, 1)$, consider

$$g(q) = \min\left(\frac{q}{1-p}, 1\right), \qquad 0 \le q \le 1. \tag{29}$$

The corresponding distorted expectation $\rho_g[X]$ is denoted by $\mathrm{TVaR}_p[X]$. From (26) we find that $\mathrm{TVaR}_p[X]$ is given by

$$\mathrm{TVaR}_p[X] = \frac{1}{1-p}\int_p^1 F_X^{-1}(q)\mathrm{d}q. \tag{30}$$

2.4 Distorted expectations and general distortion functions

In Theorems 4 and 6, we derived expressions for distortion risk measures $\rho_g[X]$ related to r.c. and l.c. distortion functions g, in terms of the quantile functions F_X^{-1+} and F_X^{-1}, respectively. In general, distortion functions may be neither r.c. nor l.c. However, as will be proven in the following theorem, a general distortion function can always be represented by a convex combination of a r.c. and a l.c. distortion function.

Theorem 7 *Any distortion function g can be represented by a convex combination*

$$g = c_r g_r + c_l g_l, \tag{31}$$

where g_r and g_l are a r.c. and a l.c. distortion function, respectively, and the non-negative weights c_r and c_l sum to 1.
When $c_r \in (0, 1)$, the distorted expectation $\rho_g[X]$ can be expressed as

$$\rho_g[X] = c_r \rho_{g_r}[X] + c_l \rho_{g_l}[X]. \tag{32}$$

Proof. Consider a general distortion function g. For any $p \in (0,1]$, we define

$$D(p) = \sum_{q\in[0,p)} \left[g\left(q+\right) - g\left(q\right)\right],$$

where the sum is taken over the finite or countable set of all values of q in $[0,p)$ where the distortion function is right discontinuous. Furthermore, we set $D(0) = 0$.

In case $D\left(1\right) = 0$, we have that g is r.c., while in case $D(1) = 1$, we find that g is l.c., and in both cases (31) and (32) are obvious.

Let us now assume that $0 < D\left(1\right) < 1$. Define

$$g_l(p) = \frac{D(p)}{D(1)}, \qquad 0 \leq p \leq 1,$$

and

$$g_r(p) = \frac{g(p) - D\left(1\right) g_l(p)}{1 - D\left(1\right)}, \qquad 0 \leq p \leq 1.$$

It is easy to check that g_l and g_r are a l.c. and a r.c. distortion function, respectively. Moreover,

$$g = (1 - D\left(1\right))g_r + D\left(1\right) g_l,$$

so that (31) holds. From (9) and the discussion of that result, we can conclude that under the implicit assumption that $\rho_g\left[X\right] \in \mathbb{R}$, or equivalenty, that $\rho_{g_r}\left[X\right]$ and $\rho_{g_l}\left[X\right]$ are real-valued, relation (32) holds. ■

The expression (32) remains to hold in case $c_r = 0$, provided g_r is chosen such that $\rho_{g_r}\left[X\right]$ is finite, while it also holds in case $c_r = 1$, provided g_l is chosen such that $\rho_{g_l}\left[X\right]$ is finite. Notice that it is always possible to choose such a distortion function, and hereafter, we will make this appropriate choice when $c_r = 0$ or $c_r = 1$.

The intuitive idea behind the proof of the theorem above is that we form a piecewise constant l.c. distortion function g_l by succesively adding all jumps corresponding to right-side discontinuities of g. The rescaled difference $\left(g - D(1)g_l\right)/\left(1 - D(1)\right)$ is a distortion function that is obtained from g by pulling down its graph at its right-side discontinuities, making it a r.c. distortion function. The reader is referred to Dudley and Norvaiša (2011) for related discussions on Young type integrals where the integrand and the integrator may have any kind of discontinuities.

As an illustration of Theorem 7, consider the distortion function g defined by

$$g\left(q\right) = \frac{1}{2}\mathbb{I}\left(\frac{1}{3} < q < \frac{2}{3}\right) + \mathbb{I}\left(\frac{2}{3} \leq q \leq 1\right), \qquad 0 \leq q \leq 1. \tag{33}$$

This distortion function is neither r.c. nor l.c., but it can be represented as follows:

$$g\left(q\right) = \frac{1}{2}\left(g_r\left(q\right) + g_l\left(q\right)\right), \qquad 0 \leq q \leq 1,$$

with

$$g_r\left(q\right) = \mathbb{I}\left(\frac{2}{3} \leq q \leq 1\right) \text{ and } g_l\left(q\right) = \mathbb{I}\left(\frac{1}{3} < q \leq 1\right),$$

where $g_r(q)$ and $g_l(q)$ are a r.c. and a l.c. distortion function, respectively. Taking into account (32), we find that

$$\rho_g\left[X\right] = \frac{1}{2}\left(\rho_{g_r}\left[X\right] + \rho_{g_l}\left[X\right]\right).$$

Consider now a general (not necessarily r.c. or l.c.) distortion function g without singular continuous component. Then $\rho_g\left[X\right]$ can be expressed as (32), where both ρ_{g_r} and ρ_{g_l} have no singular continuous part. Applying (16) and (24) to ρ_{g_r} and ρ_{g_l}, respectively, we find that

$$\begin{aligned}\rho_g\left[X\right] &= \int_0^1 F_X^{-1}\left(1-q\right) g'(q)\mathrm{d}q \\ &+ \sum_{q\in(0,1]} F_X^{-1+}\left(1-q\right)\left[g\left(q\right) - g\left(q-\right)\right] + \sum_{q\in[0,1)} F_X^{-1}\left(1-q\right)\left[g\left(q+\right) - g\left(q\right)\right], \qquad (34)\end{aligned}$$

where we took into account that $g'(q) = c_r g_r'(q) + c_l g_l'(q)$ and that $g_r\left(q+\right) = g_r\left(q\right)$, while $g_l\left(q-\right) = g_l\left(q\right)$.

3 Distortion risk measures and comonotonic sums

A random vector $\underline{X} = (X_1, \ldots, X_n)$ is said to be comonotonic if

$$\underline{X} \overset{\mathrm{d}}{=} \left(F_{X_1}^{-1}\left(U\right), \ldots, F_{X_n}^{-1}\left(U\right)\right), \qquad (35)$$

where U is a uniform $(0,1)$ r.v. and $\overset{\mathrm{d}}{=}$ stands for equality in distribution.

For a general random vector $\underline{X} = (X_1, \ldots, X_n)$, we call $\left(F_{X_1}^{-1}\left(U\right), \ldots, F_{X_n}^{-1}\left(U\right)\right)$ the comonotonic modification of $\underline{X}$, corresponding to the uniform r.v. U. Furthermore, the sum of the components of the comonotonic modification is denoted by S^c:

$$S^c = F_{X_1}^{-1}\left(U\right) + F_{X_2}^{-1}\left(U\right) + \cdots + F_{X_n}^{-1}\left(U\right). \qquad (36)$$

For an overview of the theory of comonotonicity and its applications in actuarial science and finance, we refer to Dhaene et al. (2002a). Financial and actuarial applications are described in Dhaene et al. (2002b). An updated overview of applications of comonotonicity can be found in Deelstra et al. (2010).

The following theorem states that distorted expectations related to general distortion functions are additive for comonotonic sums.

Theorem 8 (Additivity of ρ_g for comonotonic r.v.'s) *Consider a random vector $\underline{X} = (X_1, \ldots, X_n)$, a distortion function g and the distorted expectations $\rho_g\left[X_i\right], i = 1, 2, \ldots, n$. The distorted expectation of the comonotonic sum S^c is then given by*

$$\rho_g\left[S^c\right] = \sum_{i=1}^{n} \rho_g\left[X_i\right]. \qquad (37)$$

Proof. Applying the decomposition (32) in the first and the last steps of the following derivation, while taking into account Theorems 4 and 6 in the second and the fourth steps and, finally, applying the additivity property of the càglàd and càdlàg inverses F^{-1} and F^{-1+} for comonotonic r.v.'s in the third step, we find that

$$\begin{aligned}
\sum_{i=1}^{n} \rho_g[X_i] &= \sum_{i=1}^{n} \left(c_r \rho_{g_r}[X_i] + c_l \rho_{g_l}[X_i]\right) \\
&= c_r \int_{[0,1]} \sum_{i=1}^{n} F_{X_i}^{-1+}(1-q) \mathrm{d}g_r(q) + c_l \int_{[0,1]} \sum_{i=1}^{n} F_{X_i}^{-1}(1-q) \mathrm{d}g_l(q) \\
&= c_r \int_{[0,1]} F_{S^c}^{-1+}(1-q) \mathrm{d}g_r(q) + c_l \int_{[0,1]} F_{S^c}^{-1}(1-q) \mathrm{d}g_l(q) \\
&= c_r \rho_{g_r}[S^c] + c_l \rho_{g_l}[S^c] \\
&= \rho_g[S^c].
\end{aligned}$$

Given that $\rho_g[X_i]$, $i = 1, 2, \ldots, n$, is finite by assumption, we have that $\rho_{g_r}[X_i]$ and $\rho_{g_l}[X_i]$ are finite too, so that all steps in the derivation above are allowed. We can conclude that $\rho_g\left[S^c\right]$ is finite and given by (37). ■

The additivity property of distorted expectations for comonotonic sums presented in Theorem 8 is well-known. However, most proofs that appear in the literature only consider the case where the distortion function is continuous or concave. The proof that we presented here is simple and considers the general case.

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