



UvA-DARE (Digital Academic Repository)

Pension systems, intergenerational risk sharing and inflation

Beetsma, R.M.W.J.; Bovenberg, A.L.

Publication date

2006

Document Version

Submitted manuscript

[Link to publication](#)

Citation for published version (APA):

Beetsma, R. M. W. J., & Bovenberg, A. L. (2006). *Pension systems, intergenerational risk sharing and inflation*. (European Economy, Economic Papers; No. 257). European Commission. <http://www1.feb.uva.nl/pp/bin/568fulltext.pdf>

General rights

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

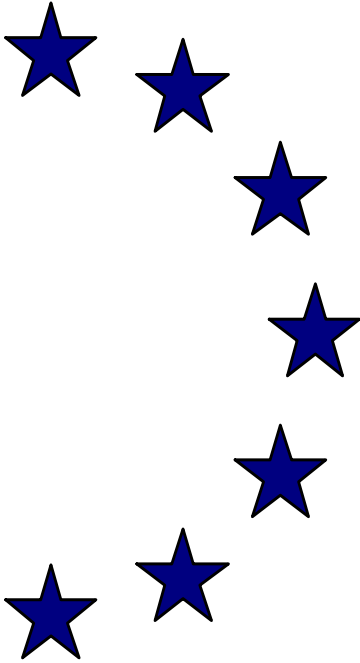
Disclaimer/Complaints regulations

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

EUROPEAN ECONOMY

EUROPEAN COMMISSION
DIRECTORATE-GENERAL FOR ECONOMIC
AND FINANCIAL AFFAIRS

ECONOMIC PAPERS



ISSN 1725-3187

http://europa.eu.int/comm/economy_finance

N° 257

October 2006

**Pension systems, intergenerational
risk sharing and inflation**

by

R. Beetsma (University of Amsterdam) and
A. L. Bovenberg (Tilburg University)

Economic Papers are written by the Staff of the Directorate-General for Economic and Financial Affairs, or by experts working in association with them. The “Papers” are intended to increase awareness of the technical work being done by the staff and to seek comments and suggestions for further analyses. Views expressed represent exclusively the positions of the author and do not necessarily correspond to those of the European Commission. Comments and enquiries should be addressed to the:

European Commission
Directorate-General for Economic and Financial Affairs
Publications
BU1 - -1/13
B - 1049 Brussels, Belgium

ISBN 92-79-01198-7

KC-AI-06-257-EN-C

©European Communities, 2006

Pension systems, intergenerational risk sharing and inflation*

Roel M. W. J. Beetsma**

University of Amsterdam, Tinbergen Institute, Netspar, CEPR and CESifo

A. Lans Bovenberg***

Tilburg University, Netspar, CEPR and CESifo

October 2006

ABSTRACT:

We investigate intergenerational risk sharing in two-pillar pension systems with a pay-as-you-go pillar and a funded pillar. We consider shocks in productivity, depreciation of capital and inflation. The funded pension pillar can be either defined contribution or defined benefit, with benefits defined in real or nominal terms or indexed to wages. Optimal intergenerational risk sharing can be achieved only in the presence of a defined benefit pension system with appropriate restrictions on investment policy of the funded pillar. In this way, both generations have similar exposures to financial and human capital risks.

Keywords: (funded) pensions, fiscal policy, nominal assets, risk-sharing, overlapping generations.

JEL codes: E21, H55, J18.

* Part of this paper was written while R. Beetsma was visiting DG-ECFIN of the European Commission, which he thanks for its hospitality. He also thanks seminar participants at DG-ECFIN for their helpful comments. In particular, he is grateful to Heikki Oksanen for many stimulating discussions.

** Mailing address: Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands; Phone: +31.20.5255280; fax: +31.20.5254254; e-mail: R.M.W.J.Beetsma@uva.nl.

*** Mailing address: CentER, Tilburg University, P.O.Box 90153, 5000 LE Tilburg, The Netherlands; Phone: +31.13.4662912; Fax: +31.13.4663066; E-mail: A.L.Bovenberg@uvt.nl.

1 Introduction

Everywhere in the industrialized world, population aging is putting social security systems under financial strain. As a result, social security systems are being reformed in many countries. In particular, various countries move from pure pay-as-you-go (PAYG) systems to pension systems that include a larger funded component. At the same time, defined-benefit systems in which benefits are guaranteed by public or corporate sponsors are being replaced by defined-contribution systems in which benefits are subject to various risks.¹

This paper explores how different pension systems affect the intergenerational sharing of risks and how they can help to share these risks optimally over generations.² To that end, we formulate a model in which a young and an old generation live under the same government and overlap during one period. The economy is subject to three sources of shocks: productivity, depreciation of capital and inflation. We consider two-pillar pension systems, with a PAYG first pillar and a funded second pillar, which may be of the DC type or various DB types. A number of papers have studied how pension systems affect intergenerational risk sharing but most of them focus on risk sharing within PAYG systems only – see, for example, Hassler and Lindbeck (1997), Thøgersen (1998), Krueger and Kubler (2002), and Wagener (2004). Bohn (2003) investigates intergenerational risk sharing, but he does not study the implications of various types of funded pension systems in this regard.

The key feature of our setup is that financial markets are incomplete for two reasons. The first reason is that generations cannot trade with each other before the shocks hit the economy, because the young generation is born only after these shocks have materialized. The other reason for market incompleteness is that human capital is not traded, so that the old generation cannot acquire a claim on human capital and in this way share in the wage risk faced by the young generation.³ As a result of these two sources of market incompleteness, the portfolio choices of the pension fund and the closure rule for the government budget constraint do have real effects: the two generations cannot fully offset the transactions of the pension fund and the government. While a defined-contribution second pillar does not add anything to the transaction possibilities in financial markets and thus leaves allocations unaffected, defined-benefit pension funds create new opportunities for intergenerational risk sharing that private agents do not offset through transactions in financial markets. The reason for these new risk-sharing possibilities offered by defined-

¹For descriptions of pension reforms in the European Union, see EPC (2006).

²For early work on intergenerational risk sharing and social security, see Enders and Lapan (1982) and Gordon and Varian (1988). More recent contributions include a.o. Demange (2002), De Menil and Sheshinski (2003) and Gottardi and Kubler (2006). For broader recent perspectives, see Shiller, 1999, and Lindbeck and Persson, 2003.

³These two sources of market incompleteness are closely related under an alternative interpretation of the reason why the young generation cannot participate in the financial market. This alternative interpretation is that the young cannot borrow against their human capital to invest in financial capital (see also Constantinides et al., 2002, and Carroll et al., 2005). This short-selling constraint of the young originates also in the lack of tradability of human capital.

benefit pensions is that when funded pension benefits are defined independently from ex-post returns on financial assets, the young generation becomes the residual claimant of the assets of the fund and thus shares in the risks associated with the financial returns on these capital assets (see also Modigliani and Muralidhar, 2004). Moreover, by linking benefits to wages, the old acquire an implicit claim on human capital. In effect, by not matching the risks of its liabilities with the risks in assets, a defined-benefit pension fund allows the young generation to exchange human capital risks and financial risks with the old generation, thereby reducing market incompleteness.

We consider three types of defined-benefit systems. The first is when the benefit paid out to the old is defined in nominal terms, the second is when this benefit is defined in real terms and the final system assumes that the benefit is indexed to wages. These three alternative systems imply that the young sell, respectively, nominal debt, real debt or wage-indexed debt to the old generation and invest the fund's capital for their own risk in assets that the pension fund buys on financial markets.

The key question is whether the pension system allows for optimal intergenerational risk sharing in a decentralized market economy in which incomplete financial markets prevent generations from trading all risks. We find that optimal intergenerational risk sharing and optimal intergenerational redistribution is achieved with a combination of a first pillar PAYG pension system and a second pillar defined-benefit pension fund with restrictions on its portfolio and the way benefits are defined and thus respond to risks. While the first pillar aims at systematic redistribution between generations in accordance with the relative social preference weight given to the old and young generation, the second pillar allows for optimal sharing of financial-market and inflation risk between the generations.⁴ In particular, the pension fund should invest in equity and nominal bonds to implement the optimal exposures of the generations to capital risks and inflation risks. Both pillars can optimally share wage risks by linking pension benefits to wages. If the funded benefits can be linked to wages, the first, PAYG pillar can be targetted exclusively at optimal ex ante redistribution, while the second, funded pillar is responsible for optimal risk sharing.

The optimal pension arrangement ensures that each individual has the same exposure to aggregate depreciation and productivity risks by having the same implicit ownership share of the aggregate capital stock as the same implicit share of aggregate human capital. The actual magnitude of this optimal ownership share depends on the social preference weight and risk aversion. Inflation risk, in contrast, is no aggregate risk and can be completely eliminated. If the relative social preference weight attached to the old and young generation is equal and both generations feature the same relative risk aversion, both generations should have identical effective holdings of physical and human capital and nominal assets. If the social preference weight on the young generation is largest and

⁴The second pillar is fully funded in the sense that, from an ex-ante perspective (i.e., on average), the old generation does not receive any resources through this pillar (see also Oksanen, 2006).

relative risk aversion is uniform across the population, the effective holding of human and physical capital of the young should be larger, so that they are more exposed to productivity and depreciation risks. Since aggregate inflation risk is zero and both generations pay the same tax share, nominal assets holdings should be effectively identical across individuals, thereby eliminating the inflation risk for each individual. The requirement that the exposure to inflation risk be equalized across generations makes defined benefit systems in which benefits are defined in nominal terms unattractive. With such a pension system, the existing long position of the old generation in nominal bonds becomes even longer and their exposure to inflation risk becomes larger. As a result, the fund would need to invest more than its initial value in nominal debt and go short in real debt in order to effectively equalize the nominal exposures of the two generations.

The remainder of this paper is structured as follows. Section 2 presents the model and solves for the social planner's solution, which thus prescribes optimal intergenerational risk sharing and redistribution. Section 3 presents the decentralized economy, while Section 4 loglinearizes the conditions determining the solution for the decentralized economy. Section 5 analyses the laissez-faire economy in the presence of incomplete financial markets and the absence of pension arrangements. Section 6 introduces the various possible pension arrangements. This sets the stage for the normative analysis in Section 7, which investigates under what pension arrangements the market economy is able to replicate the social optimum. Finally, Section 8 concludes the main body of this paper. Appendix A contains some definitions. All technical details are found in the Appendices B - E. These appendices are not for publication, but will be made available via the web or can be obtained directly from the authors.

2 The command economy

2.1 Individuals and preferences

The model represents a closed economy. It incorporates three periods ($t - 1$, t and $t + 1$) and two generations who each live for two periods. In period $t - 1$, a generation of mass $1 - \delta > 0$ is born. This generation lives through periods $t - 1$ and t . We call this generation the "old generation." The representative agent within this generation features the following utility function:

$$U(c_{y,t-1}, c_{ot}) = u(c_{y,t-1}) + \beta \mathbf{E}_{t-1} [u(c_{ot})], \quad (1)$$

where $c_{y,t-1}$ denotes consumption when this agent is young, while c_{ot} represents consumption when the agent is old. In period t , a new generation of mass $\delta > 0$ is born. This generation features the same utility function $U(c_{yt}, c_{o,t+1})$, but now defined over young consumption in period t , c_{yt} , and old consumption in period $t + 1$, $c_{o,t+1}$. This generation is termed the "young generation". At the end of period $t + 1$, the model ends. The lives of the two generations thus overlap in period t . The total population in that period is unity.

2.2 Production

The output levels of the single good in periods $t - 1$ and $t + 1$ are exogenously given at per-capita levels η_{t-1} and η_{t+1} (i.e. measured per person who is alive in the respective period). Production is endogenous only in period t , when the two generations co-exist, and is then given by

$$Y_t = A_t F(K_t, L_t), \quad (2)$$

where K_t represents the aggregate capital stock and L_t aggregate employment. A_t denotes total factor productivity (TFP), which is stochastic. The production function exhibits constant returns to scale. In our closed economy, the capital stock K_t is the result of investment in the previous period $t - 1$. The old generation is retired in period t , while each young individual exogenously supplies an amount of labor \bar{N} in that period. Aggregate employment thus amounts to $\bar{L} \equiv \delta \bar{N}$.

2.3 Public expenditures and resource constraints

Exogenous public spending in periods $t - 1$ and t is given by g_{t-1} and g_t , respectively. We introduce public spending in order to have a role for taxes and public debt in the market economy to be studied below. For convenience, public spending does not enter the utility functions of private agents. With public spending exogenous, this assumption is of no consequence, though.

The resource constraints in periods $t - 1$, t and $t + 1$ are given by, respectively,

$$(1 - \delta) c_{y,t-1} = (1 - \delta) \eta_{t-1} - K_t - g_{t-1}, \quad (3)$$

$$\delta c_{yt} + (1 - \delta) c_{ot} = A_t F(K_t, \delta \bar{N}) + (1 - \zeta_t) K_t - g_t, \quad (4)$$

$$\delta c_{o,t+1} = \delta \eta_{t+1}, \quad (5)$$

where $0 \leq \zeta_t \leq 1$ is the stochastic depreciation rate of the capital shock. Uncertainty in the depreciation rate may arise from (unexpected) changes in relative prices causing changes in the value of capital, disasters affecting the amount of capital that can be transformed back into consumption goods, etcetera. The left-hand sides of these expressions denote aggregate consumption in the economy. The right-hand side of (3) represents total endowment income minus the investment in physical capital and public expenditures. The right-hand side of (4) stands for total production minus total public expenditures but plus what is left over of the capital stock after taking into account depreciation. At the aggregate level, no storage technology is available to transfer resources from period t into period $t + 1$. Hence, consumption outlays in period $t + 1$ are constrained by the total endowment in that period.

2.4 The social planner's solution

The vector of the stochastic shocks in the command economy is $\xi_t^s \equiv \{A_t, \zeta_t\}$. It is unknown in period $t-1$, but becomes known before period t variables are determined. In period $t-1$, the social planner commits to a state-contingent plan. Hence, the consumption levels in period t are functions of the shocks: $c_{ot} = c_{ot}(\xi_t^s)$ and $c_{yt} = c_{yt}(\xi_t^s)$.

The planner weighs all individuals in a generation equally and aims to maximize the sum of the discounted expected utilities of the current and future generations' individuals, where the relative weight of the individuals born in t is given by $\chi^p > 0$. By varying χ^p , we can map out all Pareto optimal solutions.

We can write the planner's problem as (where we have used (5) to eliminate $c_{o,t+1}$) :

$$\begin{aligned} \mathcal{L} = & \int \left[(1-\delta) [u(c_{y,t-1}) + \beta u(c_{ot}(\xi_t^s))] + \chi^p \delta \beta [u(c_{yt}(\xi_t^s)) + \beta u(\eta_{t+1})] \right. \\ & \left. + \beta \lambda_t(\xi_t^s) \left[A_t F(K_t, \delta \bar{N}) + (1-\zeta_t) K_t - \delta c_{yt}(\xi_t^s) \right. \right. \\ & \left. \left. - (1-\delta) c_{ot}(\xi_t^s) - g_t \right] \right] f(\xi_t^s) d\xi_t^s \\ + & \lambda_{t-1} [(1-\delta) \eta_{t-1} - K_t - (1-\delta) c_{y,t-1} - g_{t-1}]. \end{aligned}$$

Here, $f(\xi_t^s)$ stands for the probability density function for the vector of stochastic shocks ξ_t^s . The Lagrange multipliers for the resource constraints in period $t-1$ and t are denoted by λ_{t-1} and $\lambda_t(\xi_t^s)$, respectively. Maximization of the planner's program with respect to $c_{y,t-1}$, K_t , $c_{yt}(\xi_t^s)$, and $c_{ot}(\xi_t^s)$ for all ξ_t^s yields the following first-order conditions:

$$\begin{aligned} u_c(c_{y,t-1}) &= \lambda_{t-1}, \\ \lambda_{t-1} &= \int \beta \lambda_t(\xi_t^s) (1 + r_t^{kn}) f(\xi_t^s) d\xi_t^s, \\ \chi^p u_c(c_{yt}(\xi_t^s)) &= \lambda_t(\xi_t^s), \forall \xi_t^s, \\ u_c(c_{ot}(\xi_t^s)) &= \lambda_t(\xi_t^s), \forall \xi_t^s, \end{aligned}$$

where the first-order derivative of $u(\cdot)$ is denoted by a subscript "c", F_{Kt} stands for the marginal product of capital (suppressing the arguments of the function) and $r_t^{kn} \equiv A_t F_{Kt} - \zeta_t$, which in the sequel we will refer to as the *net-of-depreciation return on capital*. By eliminating the Lagrange multipliers from these first-order conditions, we establish:

$$\chi^p u_c(c_{yt}) = u_c(c_{ot}), \forall \xi_t^s, \quad (6)$$

$$u_c(c_{y,t-1}) = \beta \mathbf{E}_{t-1} [(1 + r_t^{kn}) u_c(c_{ot})]. \quad (7)$$

If a decentralized equilibrium is to replicate the planner's solution, these optimality conditions need to be met in addition to the resource constraints (3) and (4).

2.5 Loglinearization of the social planner's solution

For future use, we loglinearize the planner's system of first-order conditions (6) and (7). We do this in two steps. First, we set up the system when all shocks happen to be at their

expected values (for lack of a better term, we call this the *median system*). The variables in this system are denoted with an upperbar.⁵ Second, we find the log-linearized system of responses to the shocks (we call this the *system in logdeviations*). The stochastic variables in the model exhibit lognormal distributions:

$$\ln A_t = \ln \bar{A} + \omega_{At}, \quad \ln \zeta_t = \ln \bar{\zeta} + \omega_{\zeta t},$$

where the shocks ω_{At} and $\omega_{\zeta t}$ are normally distributed with mean zero and respective variances σ_A^2 and σ_ζ^2 . For convenience, we assume that these shocks are all uncorrelated. The median system is:

$$\chi^p u_c(\bar{c}_{yt}) = u_c(\bar{c}_{ot}), \quad (8)$$

$$\delta \bar{c}_{yt} + (1 - \delta) \bar{c}_{ot} = \bar{A}F(K_t, \bar{L}) + (1 - \bar{\zeta}) K_t - g_t, \quad (9)$$

plus expressions (3) and (7), the latter of which we can write out as (see Appendix B):

$$u_c(c_{y,t-1}) = \beta (1 + \bar{r}_t^{kn}) u_c(\bar{c}_{ot}) \exp [\text{Var}_t(\phi_{pt}) / 2], \quad (10)$$

where ϕ_{pt} , which is defined in Appendix B, has zero mean and is a linear function of the shocks ω_{At} and $\omega_{\zeta t}$. This linear function is derived from the system in logdeviations.

That system, obtained after loglinearizing (4) and (6), is given by:

$$a_{cy}\widehat{c}_{yt} + a_{co}\widehat{c}_{ot} = a_Y\omega_{At} - a_\zeta\omega_{\zeta t}.$$

and

$$\widehat{c}_{yt} = \frac{\sigma^o}{\sigma^y} \widehat{c}_{ot}, \quad (11)$$

where a hat above a variable denotes a logarithmic deviation from the steady state (e.g., $\widehat{c}_{yt} = \ln(c_{yt}/\bar{c}_{yt})$). Furthermore, a_{cy} , a_{co} , a_g , a_Y and a_ζ represent the shares of median young's consumption, old's consumption, government spending, production and depreciation in median total resources in period t (see Appendix A for a formal definition). Finally, $\sigma^o \equiv -\bar{c}_{ot}u''(\bar{c}_{ot})/u'(\bar{c}_{ot})$ and $\sigma^y \equiv -\bar{c}_{yt}u''(\bar{c}_{yt})/u'(\bar{c}_{yt})$ stand for the coefficients of relative risk aversion for the old and the young, respectively (evaluated at the median outcome). The solution of the system in logdeviations is derived in Appendix B.

3 The decentralized economy

This section describes the decentralized market economy in which individuals and firms maximize their objective functions under the relevant constraints. A key question will

⁵To emphasize, the “median” variables are the values of the variables when all shocks happen to be equal to their expected values. Hence, a “median” variable is *not* necessarily equal to its expected value. Further, we note that we do not use upperbars for variables that are predetermined at the start of period t , such as K_t .

be under what circumstances a market economy can replicate the command optimum. We note that we can interpret the optimal risk-sharing condition (6) as the condition for ex-ante trade in risks between the young and the old in complete financial markets. However, in a decentralized economy, the two generations cannot trade risk in financial markets, because the young generation is born only after the shocks have materialized. Indeed, in the absence of pension arrangements, the old bear all the depreciation risk $\omega_{\zeta t}$ and cannot shift this risk toward the young. Both generations are exposed to production risk ω_{A_t} , but it is unlikely that the allocation of risk across generations is optimal. Hence, the generations would like to trade this risk but they cannot do this on financial markets. Other institutions thus have to fill the gap of the missing market for risk sharing between the old and the young generations. We shall explore to what extent the pension system can perform that role.

We allow for inflation in our decentralized economy. Inflation in period t is defined as $\pi_t \equiv \frac{P_t - P_{t-1}}{P_{t-1}}$, where P_t is the price level in period t . We view inflation as a source of risk. This is the relevant view when inflation is beyond the control of the domestic policymakers, for example, when the country is part of a monetary union (like the Euro area) and monetary policy is thus conducted at a supranational level. Another source for inflation risk is a stochastic link between monetary policy and inflation because of velocity shocks. In any case, inflation is assumed to be an exogenous stochastic process, such that

$$\ln(1 + \pi_t) - \ln(1 + \bar{\pi}_t) = \hat{\pi}_t,$$

is normally distributed with mean zero and variance σ_{π}^2 . Hence, the vector of three risk factors in the decentralized market economy is $\xi_t \equiv \{A_t, \zeta_t, \hat{\pi}_t\}$.

The timing of events in the market economy is as follows:

1. The “old generation” (i.e., those who are young in period $t-1$) make their investment decisions, while the government issues debt to cover its period $t-1$ expenditures.
2. The set of shocks ξ_t materializes.
3. The government levies period- t taxes and issues new debt. At the same time, firms take hiring and production decisions, while young individuals decide on their savings.

3.1 Individual budget constraints

The exogenous endowments $\eta_{t-1} > 0$ and $\eta_{t+1} > 0$ are owned by the old and the young, respectively, while labor income in period t accrues to the young. The budget constraints facing the old in periods $t-1$ and t and the young in periods t and $t+1$ are thus given by:

$$c_{y,t-1} = \eta_{t-1} - \left(b_{t-1}^r + b_{t-1}^n + k_t + \theta_{t-1}^f \right), \quad (12)$$

$$c_{yt} = N_t w_t (1 - \theta_t^w) - \theta_t^p - \tau_t + \frac{1-\delta}{\delta} \left(r_{t-1}^a - r_{t-1}^f \right) \theta_{t-1}^f - s_t, \quad (13)$$

$$c_{ot} = \frac{\delta}{1-\delta} (\theta_t^p + N_t w_t \theta_t^w) + \left(1 + r_{t-1}^f \right) \theta_{t-1}^f + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^n + \quad (14)$$

$$(1 + r_{t-1}^r) b_{t-1}^r + [r_t^k + (1 - \zeta_t)] k_t - \tau_t$$

$$c_{o,t+1} = \eta_{t+1} + (1 + r_t^r) s_t - \tau_{t+1}, \quad (15)$$

where b_{t-1}^r denotes real (indexed) public debt directly held by households, b_{t-1}^n is direct nominal (non-indexed) public debt holdings, k_t is the direct claim of households on the capital stock in period t , θ_{t-1}^f is an exogenous mandatory contribution to the pension fund, N_t is the amount of time worked in period t , w_t is the real wage per unit of labor supplied, τ_t is lump-sum tax payments (levied in equal amounts on both generations) in period t , θ_t^p is a lump-sum pay-as-you-go (PAYG) social security contribution made by the period- t young and paid out to the old in the same period, $\theta_t^w N_t w_t$ is a wage-linked PAYG social security contribution made by the period- t young and paid out to the old in the same period (where θ_t^w is thus the premium rate), s_t is savings of the young in period t , r_{t-1}^f is the realized real return to the old on their contributions to the funded pension scheme, r_{t-1}^a is the average net return on the assets held by the pension funds, i_{t-1} is the nominal interest rate on nominal public debt determined at the moment of issuance, r_{t-1}^r is the real return on indexed debt issued in period $t-1$ and is determined at the moment of issuance, r_t^k is the real return per unit of capital in production, r_t^r is the real return on savings in period t and, finally, τ_{t+1} is lump-sum tax payments in period $t+1$ (levied only on the young generation). In deriving (14), we have used that the gross real return in period t on one dollar invested in nominal debt in period $t-1$ amounts to $(1 + i_{t-1}) P_{t-1}/P_t = (1 + i_{t-1}) / (1 + \pi_t)$.

The term $\frac{1-\delta}{\delta} \left(r_{t-1}^a - r_{t-1}^f \right) \theta_{t-1}^f$ arises from the possibility that the ex-post payment to the old generation from the pension fund in period t , $(1 - \delta) \left(1 + r_{t-1}^f \right) \theta_{t-1}^f$, may differ from the ex-post value of the fund, $(1 - \delta) \left(1 + r_{t-1}^a \right) \theta_{t-1}^f$, in which case the young generation has a residual claim on the pension fund. Given that the number of young is δ , each young person receives an amount $\frac{1-\delta}{\delta} \left(r_{t-1}^a - r_{t-1}^f \right) \theta_{t-1}^f$. This term arises if the pension fund faces mismatch risk so that the return on the assets r_{t-1}^a can differ from the return paid on the pension contributions r_{t-1}^f . The deficits or surpluses that may arise as a consequence of the mismatch accrue to the young generations. In other words, the young absorb the mismatch risk. The young are thus in fact the owners of the pension fund, which has certain liabilities (to the old) and assets, such that the liabilities are covered from an ex-ante point of view (as we shall impose later on).

3.2 Individual and firm optimization

As is standard, we solve the model by working backwards through the game. A young person born in period t chooses savings s_t to maximize its utility, (1), forwarded by one period, subject to its budget constraints (13) and (15), so that

$$u_c(c_{yt}) = \beta(1 + r_t^r)u_c(c_{o,t+1}), \quad (16)$$

where all uncertainty is resolved when this saving decision is taken.

In period t , a continuum of perfectly competitive representative firms, with mass normalized to unity, produce according to (2) and maximizes profits

$$A_t F(K_t, L_t) - w_t L_t - r_t^k K_t, \quad (17)$$

over L_t and K_t , taking as given the wage rate and the rental rate of capital. This yields the following first-order conditions:

$$A_t F_{L_t} = w_t, \quad (18)$$

$$A_t F_{K_t} = r_t^k, \quad (19)$$

where $A_t F_{L_t}$ is the marginal product of labor (suppressing the arguments of the function).

In period $t - 1$, the old generation decides on the allocation of its savings over the various assets. They maximize (1) over b_{t-1}^r , b_{t-1}^n and k_t , where $c_{y,t-1}$ and c_{ot} are given by (12) and (14), respectively. The first-order conditions are:

$$\beta(1 + r_{t-1}^r) \mathbf{E}_{t-1} [u_c(c_{ot})] = u_c(c_{y,t-1}), \quad (20)$$

$$\beta(1 + i_{t-1}) \mathbf{E}_{t-1} \left[\left(\frac{1}{1 + \pi_t} \right) u_c(c_{ot}) \right] = u_c(c_{y,t-1}), \quad (21)$$

$$\beta \mathbf{E}_{t-1} [(1 + r_t^k - \zeta_t) u_c(c_{ot})] = u_c(c_{y,t-1}). \quad (22)$$

3.3 The government budget constraint

The public budget constraint in period $t - 1$ reads as:

$$d_{t-1} = g_{t-1}, \quad (23)$$

where d_{t-1} represents the real value of the aggregate public debt issued in period $t - 1$. Without any taxation in that period, it equals the exogenous public expenditures in period $t - 1$. The government in period $t - 1$ issues both nominal debt $d_{t-1}^m \geq 0$ and real (i.e. price-indexed) debt $d_{t-1}^r \geq 0$:

$$d_{t-1} = d_{t-1}^r + d_{t-1}^m. \quad (24)$$

The shares of the two types of debt are exogenous.

The public budget constraint in period t amounts to:

$$\tau_t = (1 + r_{t-1}^r) d_{t-1}^r + \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n + g_t - d_t. \quad (25)$$

The first two terms at the right-hand side of (25) represent the redemption of, respectively, indexed and nominal public debt (including interest payments). The third term on the right-hand side is total real government expenditure. The final right-hand term subtracts newly-issued public debt in period t , d_t ,⁶ to arrive at the tax financing needs of the government. This new public debt is the only asset available to transfer resources between periods t and $t + 1$. The term on the left-hand side represents aggregate lump-sum tax revenue, which adjusts to maintain the government's budget balance.

The period $t + 1$ government budget constraint is given by:

$$\delta \tau_{t+1} = (1 + r_t^r) d_t, \quad (26)$$

where we have used that the (real) interest rate on the public debt issued in period t is r_t^r . Taxes in period $t + 1$, which are levied only on the young generation, should pay for the redemption of the public debt plus the interest on the debt.

3.4 Market equilibrium conditions

The goods market equilibrium conditions in periods $t - 1$, t and $t + 1$ are given by (3), (4) and (5), respectively. Given that the masses of the old and the young generations are $1 - \delta$ and δ , respectively, the factor market equilibria are:

$$K_t = (1 - \delta) (k_t + k_t^f), \quad L_t = \delta N_t = \delta \bar{N} \equiv \bar{L}, \quad (27)$$

where k_t^f denotes the pension fund's investment in physical capital *per old person*. With a total mass of $1 - \delta$ individuals in period $t - 1$, equilibrium in both debt markets in period $t - 1$ requires that:

$$(1 - \delta) (b_{t-1}^r + b_{t-1}^{rf}) = d_{t-1}^r, \quad (1 - \delta) (b_{t-1}^n + b_{t-1}^{nf}) = d_{t-1}^n, \quad (28)$$

where b_{t-1}^{rf} and b_{t-1}^{nf} denote the pension fund's investments in real and nominal debt *per old person*. Finally, since the only way to transfer resources between periods t and $t + 1$ is public debt, equilibrium in the market for debt issued in period t requires that:

$$\delta s_t = d_t. \quad (29)$$

Imposing the bond market equilibrium condition (29) on (16) allows us to solve for the interest rate r_t^r on the debt issued in period t . Of course, equilibrium requires that (5) holds.

⁶In period t , real and nominal debt are equivalent since inflation uncertainty is absent in period $t + 1$.

4 The loglinearized model

For later use, this section log-linearizes the model. The complete system to be loglinearized consists of the old generation's first-order conditions (20), (21) and (22), consumption of the old generation in period $t - 1$:

$$(1 - \delta) c_{y,t-1} = (1 - \delta) \eta_{t-1} - (d_{t-1}^r + d_{t-1}^n + K_t), \quad (30)$$

and consumption of the young and old generations in period t :

$$c_{yt} = \frac{1}{\delta} A_t \bar{L} F_{Lt} - g_t + G_t^y, \quad (31)$$

$$c_{ot} = \frac{1}{1-\delta} (1 + r_t^{kn}) K_t - g_t + G_t^o. \quad (32)$$

where G_t^y and G_t^o are the generational accounts of each young or old person, respectively. These generational accounts depend on the type of pension system, as we shall see below. Since the sum of the generational accounts over all individuals should be zero, we have that:

$$G_t^y = - \left(\frac{1-\delta}{\delta} \right) G_t^o. \quad (33)$$

In the same way as we loglinearized the social planner's solution, we loglinearize the decentralized solution in two steps. First, we set up the *median system* when all shocks happen to be equal to their expected values. Then, we set up the system in logarithmic deviations (as a result of the exogenous shocks) of variables from their corresponding values in the median system.

4.1 The median system

Appendix C shows that median system is given by:

$$u_c(c_{y,t-1}) = \beta (1 + r_{t-1}^r) u_c(\bar{c}_{ot}) \exp [\sigma_{\phi r}^2 / 2], \quad (34)$$

$$u_c(c_{y,t-1}) = \beta \frac{1 + i_{t-1}}{1 + \bar{\pi}_t} u_c(\bar{c}_{ot}) \exp [\sigma_{\phi n}^2 / 2], \quad (35)$$

$$u_c(c_{y,t-1}) = \beta (1 + \bar{r}_t^{kn}) u_c(\bar{c}_{ot}) \exp [\sigma_{\phi K}^2 / 2], \quad (36)$$

$$(1 - \delta) c_{y,t-1} = (1 - \delta) \eta_{t-1} - (g_{t-1} + K_t), \quad (37)$$

$$\bar{c}_{yt} = \frac{1}{\delta} \bar{A} \bar{L} F_{Lt} - g_t + \bar{G}_t^y, \quad (38)$$

$$\bar{c}_{ot} = \frac{1}{1-\delta} (1 + \bar{r}_t^{kn}) K_t - g_t + \bar{G}_t^o, \quad (39)$$

where

$$\phi_{rt} \equiv -\sigma^o \widehat{c}_{ot}, \quad (40)$$

$$\phi_{nt} = -\widehat{\pi}_t - \sigma^o \widehat{c}_{ot}, \quad (41)$$

$$\phi_{Kt} = (a_{Kn}/a_K)\omega_{At} - (a_\zeta/a_K)\omega_{\zeta t} - \sigma^o \widehat{c}_{ot}, \quad (42)$$

$$\sigma_{\phi r}^2 \equiv \text{Var}_t(\phi_{rt}); \quad \sigma_{\phi n}^2 \equiv \text{Var}_t(\phi_{nt}); \quad \sigma_{\phi K}^2 \equiv \text{Var}_t(\phi_{Kt}),$$

where \widehat{c}_{ot} is given below and a_{Kn} , a_K , and a_ζ denote the shares of median capital rentals, overall capital income, and capital depreciation in median resources in period t (see Appendix A for a formal definition). This system would need to be solved for r_{t-1}^r , i_{t-1} , $c_{y,t-1}$, \bar{c}_{yt} , \bar{c}_{ot} and K_t .

4.2 The system in logarithmic deviations

We loglinearize (31) and (32) to arrive at:

$$\widehat{c}_{yt} = \frac{a_L}{a_{cy}}\omega_{At} - \frac{a_{Go}}{a_{cy}}\widehat{G}_t^o, \quad (43)$$

$$\widehat{c}_{ot} = \frac{a_{Kn}}{a_{co}}\omega_{At} - \frac{a_\zeta}{a_{co}}\omega_{\zeta t} + \frac{a_{Go}}{a_{co}}\widehat{G}_t^o, \quad (44)$$

where a_L and a_{Go} stand for the shares of median gross labor income and the median generational account in median resources in period t (see Appendix A for a formal definition). A positive productivity shock ($\omega_{At} > 0$) raises consumption of both the young (through higher wage income) and the old (through higher dividend income). Higher than expected depreciation reduces the amount of capital left over from production and at given generational accounts hurts only the old.

5 Laissez-faire: no pension system

This section investigates the solution to the model in the absence of a pension system (i.e. $\theta_t^p = \theta_t^w = \theta_{t-1}^f = b_{t-1}^{rf} = b_{t-1}^{nf} = k_t^f = 0$), which we call “laissez-faire”. Imposing these assumptions on (14), using (25) to eliminate τ_t , (19) to eliminate r_t^k (and using $r_t^{kn} = A_t F_{Kt} - \zeta_t$), (27) to eliminate k_t (and using $k_t^f = 0$), (28) to eliminate b_{t-1}^r and b_{t-1}^n (and using $b_{t-1}^{rf} = b_{t-1}^{nf} = 0$), and (32) to eliminate c_{ot} , we find that

$$G_t^o = \frac{\delta}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n + d_t, \quad (45)$$

which we can linearize into (see Appendix D):

$$\widehat{G}_t^o = -\frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} \frac{a_{dn}}{a_{Go}} \widehat{\pi}_t + \left(\frac{1-\delta}{\delta}\right) \frac{a_{dr}}{a_{Go}} \widehat{d}_t = \widehat{G}_t^y.$$

We substitute this into (43) and (44) to eliminate \widehat{G}_t^o and obtain the final solutions for \widehat{c}_{yt} and \widehat{c}_{ot} under laissez-faire:

$$\widehat{c}_{yt} = \frac{a_L}{a_{cy}} \omega_{At} + \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} \frac{a_{dn}}{a_{cy}} \widehat{\pi}_t - \left(\frac{1-\delta}{\delta}\right) \frac{a_{dr}}{a_{cy}} \widehat{d}_t, \quad (46)$$

$$\widehat{c}_{ot} = \frac{a_{Kn}}{a_{co}} \omega_{At} - \frac{a_\zeta}{a_{co}} \omega_{\zeta t} - \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} \frac{a_{dn}}{a_{co}} \widehat{\pi}_t + \left(\frac{1-\delta}{\delta}\right) \frac{a_{dr}}{a_{co}} \widehat{d}_t. \quad (47)$$

Both the young and the old generation's consumption increase with positive productivity shocks ($\omega_{At} > 0$). A positive inflation shock benefits (harms) consumption of the young (old) generation because it erodes the real value of the outstanding public debt held by the old, which is in part financed by taxes on the young. Finally, depreciation shocks affect only the old generation. Clearly, only in very special circumstances would \widehat{c}_{yt} and \widehat{c}_{ot} co-move in the socially-optimal way prescribed by (11). The laissez-faire economy thus leaves room for welfare improvements through appropriate risk-sharing mechanisms.

6 The pension system

The pension system consists of two pillars. The first pillar is a PAYG system consisting of a lump-sum part and a wage-indexed part. PAYG financing implies that total contributions by the young, $\delta\theta_t^p + \delta\theta_t^w N_t w_t$, equal the aggregate payments to the old, $(1-\delta) * \left\{ \frac{\delta}{1-\delta} \theta_t^p + \frac{\delta}{1-\delta} \theta_t^w N_t w_t \right\}$. With a fixed parameter $\theta_t^w > 0$, the PAYG system is of the defined-contribution rather than of the defined-benefit type. In this case, pension benefits are exposed to wage risk because they vary with wage income. Through their PAYG pensions, the old generations thus bear some of the wage risk.

The second pillar of the pension system is funded rather than PAYG. This means that the old generation (in expected value) finances its own pension benefits. In particular, in period $t-1$, each old person contributes an amount θ_{t-1}^f into the second pillar. The pension fund can invest these contributions in real debt, nominal debt and physical capital, so that:

$$\theta_{t-1}^f = b_{t-1}^{rf} + b_{t-1}^{nf} + k_t^f. \quad (48)$$

The average net return on the assets held by the pension funds r_{t-1}^a is computed from:

$$(1 + r_{t-1}^a) \theta_{t-1}^f = (1 + r_{t-1}^r) b_{t-1}^{rf} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^{nf} + (1 + r_t^{kn}) k_t^f. \quad (49)$$

Even though a funded pension system does not redistribute between generations ex ante, it may help to share shocks ex post between the generations depending on how the assets and liabilities of the fund respond to shocks. Whereas the funding requirement demands that the assets and liabilities are equal in value ex ante, the values may diverge ex post if the shocks affect the assets and liabilities differently. For the moment, we take the pension fund's contribution policy θ_{t-1}^f and the investment policy b_{t-1}^{rf} , b_{t-1}^{nf} and k_t^f as given, and consider various types of liabilities of the pension funds.

6.1 The various second-pillar systems

6.1.1 Defined contribution

In a *defined-contribution* (DC) type of pension fund, the value of the liabilities always matches that of the assets, not only ex ante but also ex post. Hence, the total pension benefits of the old in period t , $(1 + r_{t-1}^f) \theta_{t-1}^f$, coincide with the actual (gross) returns on the pension contributions in all states of the world. The risks facing the old thus depend directly on the investment policy of the fund.⁷ Indeed, under a DC system, the old are the residual claimants of the fund. A DC system does not involve the young at all. It thus does not provide intergenerational risk-sharing opportunities in addition to those already provided by the capital market.

The generational accounts for the DC system are given by:

$$G_t^o = \frac{\delta}{1-\delta} (\theta_t^p + \theta_t^w \frac{1}{\delta} A_t \bar{L} F_{Lt}) + \frac{\delta}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n + d_t = -\frac{\delta}{1-\delta} G_t^y \quad (50)$$

Hence, both *implicit* public debt, $\theta_t^p + \theta_t^w \frac{1}{\delta} A_t \bar{L} F_{Lt}$, and *explicit* public debt (d_{t-1}^r , d_{t-1}^n and d_t) feature in the generational accounts of the two generations. The benefit to the old of explicit public debt in period $t-1$ depends on the size of the young generation δ . If $\delta = 0$, all explicit public debt issued in period $t-1$ must be paid off by the old generation through their own tax payments. Hence, public debt does not redistribute across generations. If δ is close to one, virtually all explicit debt issued in the previous period is paid off through the tax payments by the young. For debt issued in period t , the relative sizes of the two generations are irrelevant, since d_t is completely paid off by the young.

Suppose that we hold constant the government's debt policies (d_{t-1}^n , d_{t-1}^r and d_t) and the parameters θ_t^p and θ_t^w characterizing the pension system's first pillar. In that case, the introduction of a DC funded system does not affect individual consumption decisions.⁸ Hence, the equilibrium values for r_{t-1}^r , i_{t-1} and K_t (the latter is one-to-one linked to the real return on capital) are the same with or without DC funded pension system; in the absence of short-selling constraints, any investment b_{t-1}^{rf} , b_{t-1}^{nf} or k_t^f implemented by the pension fund is offset by an equal reduction in individual holdings of real debt, nominal debt or capital. This confirms the standard textbook result that a DC pension fund does not affect the equilibrium as long as individuals can freely participate in the capital market (e.g., Blanchard and Fischer, 1989, or Heijdra and Van der Ploeg, 2002), extended to additional instruments. In effect, we arrive at a kind of Modigliani-Miller (or Ricardian equivalence) neutrality result for pension funds: the financing of a DC pension fund does not affect the equilibrium.

⁷The DC system can thus guarantee real benefits ex ante by investing in price-indexed bonds. With the help of the government issuing real bonds, a DC system thus allows for defined benefits.

⁸Computational details and the solution of decentralized economy model with pensions are found in Appendix D. By noting that the expressions (??) and (??) for $c_{y,t-1}$ and c_{ot} are not affected by $\theta_{t-1}^f = 0$, we show formally that the DC system does not impact individual consumption decisions.

The linearized generational accounts are:

$$\widehat{G}_t^o = \theta_t^w \frac{a_L}{a_{Go}} \omega_{At} - \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\bar{\pi}_t} \frac{a_{dn}}{a_{Go}} \widehat{\pi}_t + \left(\frac{1-\delta}{\delta}\right) \frac{a_{dr}}{a_{Go}} \widehat{d}_t = \widehat{G}_t^y. \quad (51)$$

The generational account of the old generation improves when $\omega_{At} > 0$ and $\theta_t^w > 0$, because the young pay more wage-linked PAYG premia. It also improves if $\widehat{d}_t > 0$ allows for a lower tax burden on the old generation. With positive nominal public debt (so that $a_{dn} > 0$), a positive inflation shock erodes the value of the nominal assets held by the old, thereby reducing the generational account of the old and benefiting the young tax payers in period t .

6.1.2 Defined benefit systems

A *defined-benefit* (DB) type of pension fund allows for additional intergenerational risk sharing if the assets of the pension fund do not match its liabilities. The generational accounts then include “correction terms,” which involve the ex-post return differences between the liabilities and the assets of the pension fund.⁹ These correction terms enter the generational accounts in exactly the same way as d_t , since these “debts” are completely paid off by the young only; unlike the explicit public debt issued in period $t - 1$, these debts are not shared between the young and the old depending on the relative sizes of the generations.

Depending on the type of pension liability, we distinguish between various types of defined-benefit systems with, respectively, nominal, real (i.e., price indexed) and wage-indexed liabilities. The neutrality result for the introduction of a second pillar disappears under these defined benefit systems. Intuitively, these pension systems introduce new ways to share risks between the generations that are not offered by financial markets. Hence, agents can not offset the transactions of the pension fund. To ensure equilibrium in the asset markets in period $t - 1$, equilibrium asset returns generally change in response to changes in the funded pension system.

Defined nominal benefits (DNB) If the liabilities of the pension fund are in nominal terms, each old person receives a pre-fixed nominal pension benefit θ_t^{fn} . We refer to this defined-benefit scheme with defined nominal benefits as the *DNB scheme*. Given that the benefit in period t is defined in nominal terms, it should be valued in the same way as a nominal bond. Hence, the relevant discount rate is the nominal interest rate i_{t-1} . The system is fully funded if it does not redistribute between the two generations ex ante. Hence, the old generation should pay in period $t - 1$ for its pension benefit in period t . In other words, the discounted value to the old generation in period $t - 1$ of the pension benefit expected in period t should match the pension contribution in period t , θ_{t-1}^f , so

⁹The value of these return differences is zero in period $t - 1$ (i.e. ex ante) for the old generation since the discount rates of this generation are used to value pension liabilities when imposing the funding requirement imposing ex-ante equality between assets and liabilities.

that $\theta_{t-1}^f = \theta_t^{fn} / (1 + i_{t-1})$. Whereas the nominal return on the pension contribution is thus the nominal interest rate (i.e. $\theta_t^{fn} = (1 + i_{t-1})\theta_{t-1}^f$), the real return $r_{t-1}^f = r_{t-1}^{fn}$ on the contribution is given by:

$$1 + r_{t-1}^{fn} = \frac{1+i_{t-1}}{1+\pi_t}. \quad (52)$$

The funding requirement implies that the old are indifferent about an additional marginal contribution into the fund (and the associated additional benefit) since (21) holds, so that (using (52))

$$1 + r_{t-1}^{fn} = \frac{1}{1 + \pi_t} \frac{1}{\beta E_{t-1}[m_t / (1 + \pi_t)]},$$

where

$$m_t \equiv u_c(c_{ot}) / u_c(c_{y,t-1}),$$

denotes the stochastic discount rate of the old and $m_t = u_c(c_{ot}) / u_c(c_{y,t-1})$.

The surplus of the pension fund, $(1 - \delta)(r_{t-1}^a - r_{t-1}^{fn})\theta_{t-1}^f$, accrues to the young, who thus absorb the mismatch risk. Each young person in period t receives $\frac{1-\delta}{\delta}$ times the value of the fund per old minus the real value of the old's nominal benefit (the second equality below follows from using (48) to eliminate θ_{t-1}^f):

$$\begin{aligned} (r_{t-1}^a - r_{t-1}^r)\theta_{t-1}^f &= (1 + r_{t-1}^r) b_{t-1}^{rf} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^{nrf} + (1 + r_t^{kn}) k_t^f - \frac{1+i_{t-1}}{1+\pi_t} \theta_{t-1}^f \\ &= \left[(1 + r_{t-1}^r) - \frac{1+i_{t-1}}{1+\pi_t} \right] b_{t-1}^{rf} + \left[(1 + r_t^{kn}) - \frac{1+i_{t-1}}{1+\pi_t} \right] k_t^f. \end{aligned}$$

The first line shows that the young effectively issue a nominal bond of size θ_{t-1}^f to the old and invest the resources for their own risk according to the portfolio of the pension fund. The pension system thus allows the young to participate in the capital market before they are born and before they accumulate any assets. The young can in fact go short in nominal assets and long in other assets. In this way, they can share risks with the old generations. In particular, the young share in depreciation risk if the pension fund invests in equity (i.e. $k_t^f > 0$). In this case, the young effectively participate in the stock market through the defined-benefit pension fund. At the same time, the young are short in inflation risk if the pension fund does not invest all its assets in nominal bonds (i.e. $\theta_{t-1}^f - b_{t-1}^{nrf} = b_{t-1}^{rf} + k_t^f > 0$). The second line indicates that the mismatch risk is absent if the pension fund matches its liabilities by investing its entire pension portfolio in nominal debt (so that $b_{t-1}^{rf} = k_t^f = 0$ and $b_{t-1}^{nrf} = \theta_{t-1}^f$). In that case, the DNB system is equivalent to a DC system that invests only in nominal bonds. Hence, in the second pillar, the young are neither short nor long in inflation or share-market risk.

The generational account of the old under the *DNB* system amounts to:

$$\begin{aligned} G_t^o &= \frac{\delta}{1-\delta} (\theta_t^p + \theta_t^w \frac{1}{\delta} A_t \bar{L} F_{Lt}) + \frac{\delta}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n \\ &\quad + d_t - \left[(1 + r_{t-1}^r) - \frac{1+i_{t-1}}{1+\pi_t} \right] b_{t-1}^{rf} - \left[(1 + r_t^{kn}) - \frac{1+i_{t-1}}{1+\pi_t} \right] k_t^f = -\frac{\delta}{1-\delta} G_t^y. \end{aligned} \quad (53)$$

Appendix D loglinearizes (53) and uses the result to arrive at a reduced form for loglinearized consumption.

Defined real benefits (DRB) In a DRB system, the period- t old receive a defined real benefit θ_t^{fr} . The real return on the pension contribution is now denoted r_{t-1}^{fr} . The funding requirement implies that the pension contribution equals the discounted value of the benefit, where the appropriate discount rate in this case is the real return on indexed debt, r_{t-1}^r . Hence,

$$r_{t-1}^{fr} = r_{t-1}^r. \quad (54)$$

The funding requirement implies also that the old are indifferent about an additional marginal contribution into the fund (and the associated additional benefit) since (20) holds, so that (using (54))

$$1 + r_{t-1}^{fr} = 1/(\beta E_{t-1}[m_t]).$$

Using (54), each period- t young receives $\frac{1-\delta}{\delta}$ times

$$\begin{aligned} (r_{t-1}^a - r_{t-1}^r)\theta_{t-1}^f &= (1 + r_{t-1}^r) b_{t-1}^{rf} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^{nf} + (1 + r_t^{kn}) k_t^f - (1 + r_{t-1}^r) \theta_{t-1}^f \\ &= \left[\frac{1+i_{t-1}}{1+\pi_t} - (1 + r_{t-1}^r) \right] b_{t-1}^{nf} + (r_t^{kn} - r_{t-1}^r) k_t^f. \end{aligned}$$

The young in effect issue indexed bonds of size θ_{t-1}^f to the old and invest the resources for their own risk according to the portfolio of the pension fund. This way, the young can in fact go short in real debt and, depending on the equity investments of the pension fund k_t^f , they can share equity market risks with the old. Moreover, if the pension fund invests in nominal assets (i.e. $b_{t-1}^{nf} > 0$), the young share inflation risk. In case the pension fund invests only in price-indexed bonds (i.e. $b_{t-1}^{nf} = k_t^f = 0$ and $b_{t-1}^{rf} = \theta_{t-1}^f$), however, the young do not bear any mismatch risk. In that case, the DC and DRB systems become identical for given initial size θ_{t-1}^f if the DC scheme invests in real public debt only. Mismatch risk is thus necessary for the young to share in the equity market and inflation risk.

The generational account of the old under the *DRB* system is given by:

$$\begin{aligned} G_t^o &= \frac{\delta}{1-\delta} (\theta_t^p + \theta_t^w \frac{1}{\delta} A_t \bar{L} F_{Lt}) + \frac{\delta}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n \\ &\quad + d_t + \left[(1 + r_{t-1}^r) - \frac{1+i_{t-1}}{1+\pi_t} \right] b_{t-1}^{nf} + (r_{t-1}^r - r_t^{kn}) k_t^f = -\frac{\delta}{1-\delta} G_t^y. \end{aligned} \quad (55)$$

Appendix D linearizes (55) and finds a reduced form for loglinearized consumption of the old and the young in period t .

Defined wage-indexed benefit (DWB) With defined wage-indexed benefits, the pension benefit is indexed to the wage rate (the DWB system). In that case, the benefit to the old in period t is:

$$\theta_t^{dw} \frac{1}{\delta} A_t F_{Lt} \bar{L} = \left(1 + r_{t-1}^{fw}\right) \theta_{t-1}^f, \quad (56)$$

where θ_t^{dw} captures the fixed (non-stochastic) factor that links the benefit to the wage and $r_{t-1}^f = r_{t-1}^{fw}$ represents the real return on each euro of contribution in period $t - 1$. The system is fully funded if the old pay at the margin the value they attach to the wage-indexed claim so that $u_c(c_{y,t-1}) = \beta E_{t-1} \left[u_c(c_{ot}) \left(1 + r_{t-1}^{fw}\right) \right]$, which by using (56) to eliminate r_{t-1}^{fw} is equivalent to

$$u_c(c_{y,t-1}) = \frac{\beta}{\delta} \left(\theta_t^{dw} / \theta_{t-1}^f \right) F_{Lt} \bar{L} E_{t-1} [u_c(c_{ot}) A_t]. \quad (57)$$

Substitution of (56) into this expression to eliminate θ_t^{dw} yields:

$$1 + r_{t-1}^{fw} = \frac{A_t}{\beta E_{t-1} [m_t A_t]}.$$

The young receive the residual value of the pension fund. With $r_{t-1}^f = r_{t-1}^{fw}$, each one of them thus gets $\frac{1-\delta}{\delta}$ times

$$\begin{aligned} (r_{t-1}^a - r_{t-1}^{fw}) \theta_{t-1}^f &= (1 + r_{t-1}^r) b_{t-1}^{rf} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^{nf} + (1 + r_t^{kn}) k_t^f - \left(1 + r_{t-1}^{fw}\right) \theta_{t-1}^f \\ &= (r_{t-1}^r - r_{t-1}^{fw}) b_{t-1}^{rf} + \left[\frac{1+i_{t-1}}{1+\pi_t} - (1 + r_{t-1}^{fw}) \right] b_{t-1}^{nf} + \left(r_t^{kn} - r_{t-1}^{fw} \right) k_t^f \end{aligned}$$

Under the DWB system, the young in effect issue a wage-indexed bond to the old and invest the borrowed resources in indexed and nominal bonds and physical capital, conform the portfolio decisions of the pension fund (see (48)). Wage risk is not traded in financial markets¹⁰ so that the DWB pension fund always suffers from mismatch risk. In a way, the pension fund introduces new possibilities for implicitly trading risk factors. In particular, DWB pensions funds allow the young generations not only to participate in the equity market but also to shed wage risk. When pension liabilities depend on non-traded risk factors, the valuation of pension liabilities typically becomes problematic and diverges between various agents. For the valuation of non-traded wage-indexed pension liabilities, see De Jong (2005). By using the stochastic discount factor of the old generation to value liabilities, we in effect employ the valuation of the old generations in imposing the funding requirement.

The generational accounts under the *DWB* system are given by:

$$\begin{aligned} G_t^o &= \frac{\delta}{1-\delta} \left[\theta_t^p + \frac{1}{\delta} A_t \bar{L} F_{Lt} \left(\frac{1-\delta}{\delta} \theta_t^{dw} \right) \right] + \frac{1}{1-\delta} A_t \bar{L} F_{Lt} \theta_t^w + \frac{\delta}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r \\ &\quad - (1 + r_{t-1}^r) b_{t-1}^{rf} + d_t + \frac{\delta}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^m - \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^{nf} - (1 + r_t^{kn}) k_t^f \\ &= -\frac{\delta}{1-\delta} G_t^y. \end{aligned} \quad (58)$$

¹⁰Trade in equity allows only a particular combination of productivity and depreciation risk to be traded.

Appendix D linearizes (58) and uses the result to find a reduced form for loglinearized consumption levels in period t .

7 Optimal pension arrangements

This section studies how the decentralized market economy can replicate the social optimum with an appropriate choice of the pension system. The following proposition gives a necessary and sufficient condition for the replication of the social optimum:

Proposition 1 *When a policy produces $u_c(c_{ot}) = \chi^p u_c(c_{yt})$ for all possible realizations of the shock vector ξ_t and the production function exhibits constant returns to scale, then the competitive equilibrium reproduces the socially-optimal allocation under all types of funded pension systems.*

Proof. Add δ times equation (31) and $(1 - \delta)$ times equation (32). Using $r_t^{kn} = A_t F_{Kt} - \zeta_t$ and (33), the resulting equation can be simplified to (4), which for given K_t coincides with the planner's resource constraint. The combination of (a) the expression $u_c(c_{ot}) = \chi^p u_c(c_{yt})$, which in the proposition holds by assumption, (b) expression (4), (c) equation (22), and (d) equation (30) exactly coincides with the system (3), (4), (6) and (7) to be solved under the planner. Hence, the decentralized economy is solved for the same combination(s) $\{c_{y,t-1}, c_{yt}, c_{ot}, K_t\}$ as in the social planner's problem. ■

For the remainder of the analysis, we assume that the pension system parameters θ_t^p , θ_t^w and θ_t^{dw} are not contingent on the shocks. Although the potential objections to making these parameters dependent on the shocks are not modelled explicitly, frequent changes in the pension parameters inevitably lead to political struggles and introduce additional uncertainty not directly linked to the fundamental economic shocks themselves.

7.1 Optimum with equally-weighted generations ($\chi^p = 1$)

This sub-section assumes that the two generations are weighted equally in social welfare (i.e. $\chi^p = 1$). This particular case is of special interest because it allows for simple and intuitive analytical solutions, without having to resort to loglinearizations.

With $\chi^p = 1$, the necessary and sufficient condition in Proposition 1 for reproducing the social optimum reduces to $c_{yt} = c_{ot}$. Hence, by (31) and (32), the social optimum is reproduced whenever the generational accounts vary such that:

$$\frac{1}{1-\delta} (1 + r_t^{kn}) K_t - \frac{1}{\delta} A_t \bar{L} F_{Lt} = -\frac{1}{\delta} G_t^o = \frac{1}{1-\delta} G_t^y, \quad (59)$$

where we have also used (33). Hence, if individual profit income plus the scrap value of capital, $\frac{1}{1-\delta} (1 + r_t^{kn}) K_t$, exceeds individual wage income, $\frac{1}{\delta} A_t F_{Lt} \bar{L}$, then the old would have more resources for consumption in period t than the young if the generational accounts are zero so that intergenerational transfers are absent. To ensure $c_{yt} = c_{ot}$, the

generational accounts should offset these differences in individual incomes between the generations. To illustrate, an increase in depreciation, which under *laissez-faire* reduces only the resources of the old, requires an decrease (increase) in the generational account of the young (old) so that the young also share in this adverse shock.

Substituting the expressions for the generational accounts under the various pension fund systems, we can show how the policy parameters should be set to reproduce the social optimum. For the DC system, substituting (50) into (59), the requirement for reproducing the social optimum becomes:

$$\begin{aligned} & \frac{1}{1-\delta} \left(\theta_t^p + \theta_t^w \frac{1}{\delta} A_t \bar{L} F_{Lt} \right) + \frac{1}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \frac{1}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n + \frac{1}{\delta} d_t \\ = & \frac{1}{\delta} A_t F_{Lt} \bar{L} - \frac{1}{1-\delta} (1 + r_t^{kn}) K_t. \end{aligned} \quad (60)$$

Replication of the social optimum requires that this expression hold for all possible realizations of the shock vector ξ_t . If at all possible, this imposes certain restrictions on the pension system.

Similarly, for the *DNB* system, substitution of (53) into (59) yields the following requirement for establishing the social optimum:

$$\begin{aligned} & \frac{1}{1-\delta} \left(\theta_t^p + \theta_t^w \frac{1}{\delta} A_t \bar{L} F_{Lt} \right) + \frac{1}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \frac{1}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n + \frac{1}{\delta} d_t \\ & + \frac{1}{\delta} \left[\frac{1+i_{t-1}}{1+\pi_t} - (1 + r_{t-1}^r) \right] b_{t-1}^{rf} + \frac{1}{\delta} \left[\frac{1+i_{t-1}}{1+\pi_t} - (1 + r_t^{kn}) \right] k_t^f \\ = & \frac{1}{\delta} A_t F_{Lt} \bar{L} - \frac{1}{1-\delta} (1 + r_t^{kn}) K_t. \end{aligned} \quad (61)$$

This expression differs from (60) by its second line.

For the *DRB* system, substituting (55) into (59), we find that the social optimum is established if:

$$\begin{aligned} & \frac{1}{1-\delta} \left(\theta_t^p + \theta_t^w \frac{1}{\delta} A_t \bar{L} F_{Lt} \right) + \frac{1}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \frac{1}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n + \frac{1}{\delta} d_t \\ & + \frac{1}{\delta} \left[(1 + r_{t-1}^r) - \frac{1+i_{t-1}}{1+\pi_t} \right] b_{t-1}^{rf} + \frac{1}{\delta} (r_{t-1}^r - r_t^{kn}) k_t^f \\ = & \frac{1}{\delta} A_t F_{Lt} \bar{L} - \frac{1}{1-\delta} (1 + r_t^{kn}) K_t. \end{aligned} \quad (62)$$

Hence, this expression also differs from (60) by its second line.

Finally, under the *DWB* system, the requirement for the social optimum is (substitute (58) into (59)):

$$\begin{aligned} & \frac{1}{1-\delta} \left[\theta_t^p + \frac{1}{\delta} A_t \bar{L} F_{Lt} \left(\frac{1-\delta}{\delta} \theta_t^{dw} + \theta_t^w \right) \right] + \frac{1}{1-\delta} (1 + r_{t-1}^r) d_{t-1}^r + \\ & \frac{1}{1-\delta} \frac{1+i_{t-1}}{1+\pi_t} d_{t-1}^n + \frac{1}{\delta} d_t - \frac{1}{\delta} (1 + r_{t-1}^r) b_{t-1}^{rf} - \frac{1}{\delta} \frac{1+i_{t-1}}{1+\pi_t} b_{t-1}^{nf} - \frac{1}{\delta} (1 + r_t^{kn}) k_t^f \\ = & \frac{1}{\delta} A_t \bar{L} F_{Lt} - \frac{1}{1-\delta} (1 + r_t^{kn}) K_t. \end{aligned} \quad (63)$$

In theory, a shock-contingent debt policy d_t can always produce the social optimum. In reality, however, public debt policies are typically restricted, for example, in view of supranational agreements like under Europe's Stability and Growth Pact (see Beetsma and Uhlig, 1999, and Beetsma and Debrun, 2006). Even in the absence of such restrictions, ex-post fine-tuning of the resources of the two generations through debt policy is difficult from a practical point of view. For the remainder of this section we therefore assume that d_t is not contingent on shocks.

By inspecting (60), we immediately see that:

Proposition 2 *A DC funded system can not replicate the social optimum for arbitrary realizations of the shock vector ξ_t .*

For the DC system, replication of the social optimum fails on several accounts. By its very nature, the pension system does not allow the young to acquire a claim on the net-of-depreciation return on capital. Hence, the young do not share in depreciation risks ζ_t . Furthermore, the young do not hold a claim on nominal debt. Hence, the old can not shed the inflation risk of holding nominal bonds if $d_{t-1}^n > 0$.

The other funded systems can all be designed so as to achieve the social optimum. The following proposition makes this more precise:

Proposition 3 *An appropriate combination of the two pension pillars can replicate the social optimum.*

In particular, with a DRB or DNB fund in the second pillar, the following arrangements achieve this. The first pension pillar implements optimal ex-ante redistribution and optimal ex-post redistribution of wage risk by setting a lump-sum pension premium $\theta_t^p = -(1 + r_{t-1}^r)(K_t + g_{t-1})$ and a wage-linked PAYG pension premium $\theta_t^w = 1 - \delta$. The second pension pillar provides for the optimal ex-post redistribution of productivity, depreciation and inflation risks. A DRB pension fund should have a portfolio with $k_t^f = \frac{\delta}{1-\delta}K_t$ and $b_{t-1}^{nf} = \frac{\delta}{1-\delta}d_{t-1}^n$ while a DNB fund should have $k_t^f = \frac{\delta}{1-\delta}K_t$ and $b_{t-1}^{rf} = -k_t^f - \frac{\delta}{1-\delta}d_{t-1}^n$ so that $b_{t-1}^{nf} = \theta_{t-1}^f + \frac{\delta}{1-\delta}d_{t-1}^n$.

With a DWB fund, the first pension pillar provides for the appropriate ex-ante redistribution by setting a lump-sum pension premium $\theta_t^p = -(1 + r_{t-1}^r) \left[g_{t-1} + K_t - \frac{1-\delta}{\delta} \theta_{t-1}^f \right]$ and $\theta_t^w = 0$. The second pension pillar provides for the optimal ex-post sharing of productivity, depreciation and inflation risks with $\theta_t^{dw} = \delta$ and a portfolio with $k_t^f = \frac{\delta}{1-\delta}K_t$ and $b_{t-1}^{nf} = \frac{\delta}{1-\delta}d_{t-1}^n$.

Proof. Follows directly by substitution of the proposed arrangements into (62), (61) and (63), respectively, and making use of (23), (24) and (48). ■

As regards to the funded part of the pension system, all arrangements require $k_t^f = \frac{\delta}{1-\delta}K_t$. By having a defined-benefit pension fund to invest part of its assets in equity capital, the young share in the depreciation risk that would otherwise be solely borne by the old. The share of the pension portfolio invested in equity capital is determined by

the requirement that each agent bears the same depreciation risk. In particular, if the young generation is relatively larger (δ is larger), the proportion of total capital held by the fund rises as the depreciation risk has to be shifted towards a larger young generation and away from a smaller old generation. In fact, using (27) (i.e. $K_t = (1 - \delta)(k_t + k_t^f)$), we can write $k_t^f = \frac{\delta}{1-\delta}K_t$ as $k_t/(k_t^f + k) = (1 - \delta)$. In other words, the share of capital held by the old is equal to its population share, so that each individual effectively (directly or indirectly as a residual claimant to the pension fund) holds exactly the same amount of capital. The condition $k_t/k_t^f = (1 - \delta)/\delta$ in effect determines the optimal share of DC pensions relative to DB pensions.¹¹ We see that in a more aged society (i.e., δ is becoming small), the DC part of the second pension pillar should be larger. Intuitively, with a relatively small young generation, the old can shed less capital risk. This result may help to explain the trend away from DB to DC pension systems in aging societies.

The pension arrangements also share wage risks. By setting $\theta_t^w = 1 - \delta$ under the DNB and DRB funds and $\theta_t^{dw} = \delta$ under the DWB scheme, the human capital claims effectively held by the old versus the young coincide with their population shares. Hence, each individual has exactly the same exposure to not only depreciation risk but also human capital risk. The pension system thus allows the old to get rid of depreciation risk and the young to shed wage risk.

As regards inflation risks, the DRB and DWB systems require the pension fund to invest in nominal debt such that $b_{t-1}^{nf} = \frac{\delta}{1-\delta}d_{t-1}^n$. Recall that b_{t-1}^{nf} is the amount of nominal bonds in the pension fund per old person. Hence, $(1 - \delta)b_{t-1}^{nf} = \delta d_{t-1}^n$ is the total amount of nominal bonds in the pension fund, which is effectively held by the young generation as a whole. The young generation thus effectively holds a share δ of the overall stock of nominal assets d_{t-1}^n , which coincides with its population share δ . Each person thus has the same exposure to inflation risk through nominal asset holdings. Inflation then does not redistribute across generations because each person pays the same amount of taxes and thus benefits in the same way from the erosion of the real value of debt service as a result of inflation. An unexpected increase in inflation leaves all agents unaffected: the decline in the real value of their nominal asset holdings is exactly offset by lower tax payments on account of a decline in public debt service.

By defining its pension liabilities to the old generation in nominal terms, the DNB system increases the exposure of the old generation to inflation risks. Investments in nominal assets by the DNB fund then have to ensure that the young not only share the inflation risk implied by nominal public debt d_{t-1}^n but also take over the inflation risk implied by the nominal nature of the funded pension benefits. A DNB fund thus must invest more than its overall liabilities in nominal debt: $b_{t-1}^{nf} = \theta_{t-1}^f + \delta d_{t-1}^n / (1 - \delta)$. The large long position of the fund in nominal debt and the positive investment in equity, which ensures the correct exposure of the young to depreciation risk, imply that the fund has to

¹¹Recall that members of the old generation are indifferent between a physical capital investment k_t via a defined contribution pension fund or just directly investing themselves in physical capital.

go short in real debt. Indeed, the short position in real debt $b_{t-1}^{rf} = -\frac{\delta}{1-\delta}(K_t + d_{t-1}^n)$ is the counterpart of the required exposure of the young to inflation risk (which is associated with nominal public debt d_{t-1}^n) and depreciation risk. Hence, a DNB pension fund can replicate the social optimum only if short-selling constraints are absent.

Let us now turn to optimal intergenerational ex-ante redistribution established by the first pension pillar. With DNB and DRB systems, optimal intergenerational redistribution requires $\theta_t^p = -(1 + r_{t-1}^r)(K_t + g_{t-1})$ and $\theta_t^w = 1 - \delta$. θ_t^w ensures that the old get the appropriate implicit claim on human capital, whereas θ_t^p ensures that the young obtain the correct implicit claim on economy-wide saving in period $t - 1$. With a fully funded second pillar, the young on average do not have a claim on saving in period $t - 1$. To establish equal consumption of all individuals in period t , the PAYG system thus has to ensure that the two generations share not only human capital but also the claims on capital and public resources.

With a DWB system, the parameter θ_t^p has to establish both the correct average share of the young on saving as the correct average share of the old on human capital. The DWB system does not require systematic redistribution to the old through the wage-indexed component of the PAYG system. As a result, the lump-sum component of the PAYG system should include an extra systematic transfer from the young to the old. By including a component $-(g_{t-1} + K_t)$ in $\theta_t^p / (1 + r_{t-1}^r)$, from an ex-ante perspective, gross investment income is evenly spread over all individuals in period t . Similarly, by including the component $\frac{1-\delta}{\delta}\theta_{t-1}^f$ in $\theta_t^p / (1 + r_{t-1}^r)$, wage income in period t is spread equally over all individuals. Hence, consumption levels are equalized from an ex ante perspective.

We can conclude that an optimal funding arrangement ensures that all individuals in society, young or old, hold the same claims on human capital, physical capital and nominal assets. Under DRB and DNB systems, the first pillar of the pension system provides for both ex-ante redistribution and some risk sharing. With a DWB system, in contrast, these two roles of the pension system can be completely separated. The first, PAYG pillar can be targetted exclusively at optimal ex ante redistribution, while the second, funded pillar is responsible for optimal risk sharing.

7.2 Optimum with diverging generational weights ($\chi^p \neq 1$)

With diverging generational weights (i.e. $\chi^p \neq 1$), replication of the social optimum in a decentralized economy is established in two steps. First, the median solution of the decentralized economy needs to coincide with that under the social planner. Given the values of the potential other parameters of the pension arrangement, one can set the lump-sum component of the first pillar of the pension system, θ_t^p , such that the system (3), (8), (9) and (10) is fulfilled. In this way, the first pillar establishes the appropriate amount of intergenerational redistribution. Second, the responses of the pension policies need to be such that (11) holds. Under $\chi^p = 1$, we have that $\bar{c}_{yt} = \bar{c}_{ot}$ and, hence, $\sigma^o = \sigma^y$, so that $\hat{c}_{yt} = \hat{c}_{ot}$. In the more general case where χ^p differs from 1 and thus \bar{c}_{yt} and \bar{c}_{ot} differ from

each other, we still have that $\widehat{c}_{yt} = \widehat{c}_{ot}$ if the felicity function features constant relative risk aversion (CRRA) so that $\sigma^o = \sigma^y$ is constant. Without constant relative risk aversion, σ^o and σ^y may deviate from each other so that the optimal logdeviations of consumptions of the old and young differ from each other according to $\widehat{c}_{yt} = (\sigma^o/\sigma^y) \widehat{c}_{ot}$.

Substitution of (43) and (44) into this latter requirement for optimal risk sharing yields:

$$a_{Go} \widehat{G}_t^o = \frac{\tilde{a}_{co} a_L - \tilde{a}_{cy} a_{Kn}}{\tilde{a}_{co} + \tilde{a}_{cy}} \omega_{At} + \frac{\tilde{a}_{cy}}{\tilde{a}_{co} + \tilde{a}_{cy}} a_\zeta \omega_{\zeta t}, \quad (64)$$

where $\tilde{a}_{cy} \equiv a_{cy}/\sigma^y$ and $\tilde{a}_{co} \equiv a_{co}/\sigma^o$. Under each of our pension systems, \widehat{G}_t^o behaves in a different way. We explore only the three types of DB systems because we already established that the DC pension system was not able to replicate the social optimum with $\chi^p = 1$.

Proposition 4 (1) Consider a two-pillar pension system with a DRB second pillar. By setting $\theta_t^w = 1/(1 + \tilde{a}_{cy}/\tilde{a}_{co})$ and $k_t^f = (\tilde{a}_{cy}/\tilde{a}_{co}) K_t / [(1 - \delta)(1 + \tilde{a}_{cy}/\tilde{a}_{co})]$, we simultaneously achieve optimal risk-sharing of ω_{At} and $\omega_{\zeta t}$ shocks. In addition, inflation risks are optimally shared if $b_{t-1}^{nf} = \delta d_{t-1}^n / (1 - \delta)$.

(2) Consider a two-pillar pension system with a DNB second pillar. By setting $b_{t-1}^{nf} = \theta_{t-1}^f + \delta d_{t-1}^n / (1 - \delta)$, $b_{t-1}^{rf} = -k_t^f - \frac{\delta}{1-\delta} d_{t-1}^n$, and otherwise keeping the arrangement the same as under (1), the social optimum is replicated.

(3) Consider a two-pillar pension system with a DWB second pillar. By setting $\theta_t^w + \frac{1-\delta}{\delta} \theta_t^{dw} = 1/(1 + \tilde{a}_{cy}/\tilde{a}_{co})$ and otherwise keeping the arrangement the same as under (1), the social optimum is replicated.

Proof. See Appendix E. ■

This proposition generalizes Proposition 3 to the case $\chi^p \neq 1$. If $\chi^p = 1$, $\bar{c}_{yt} = \bar{c}_{ot}$ and thus $\sigma^y = \sigma^o$ and $\tilde{a}_{cy}/\tilde{a}_{co} = \delta/(1 - \delta)$. In that case, arrangement (3) in Proposition 4 reduces to $\theta_t^w + \frac{1-\delta}{\delta} \theta_t^{dw} = 1 - \delta$, $k_t^f = \delta K_t / (1 - \delta)$ and $b_{t-1}^{nf} = \delta d_{t-1}^n / (1 - \delta)$, which is exactly the corresponding arrangement proposed in Proposition 3 when $\theta_t^w = 0$. Similarly, we can see that arrangements (1) and (2) in Proposition 4 reduce to the corresponding arrangements in Proposition 3.

To obtain some more intuition for arrangement (3) with $\chi^p \neq 1$,¹² we write:

$$\begin{aligned} \theta_t^w + \frac{1-\delta}{\delta} \theta_t^{dw} &= \frac{1}{1 + \tilde{a}_{cy}/\tilde{a}_{co}} = \frac{(1-\delta)(\bar{c}_{ot}/\sigma^o)}{\delta(\bar{c}_{yt}/\sigma^y) + (1-\delta)(\bar{c}_{ot}/\sigma^o)}; \\ k_t^f &= \frac{1}{1-\delta} \frac{\delta(\bar{c}_{yt}/\sigma^y)}{\delta(\bar{c}_{yt}/\sigma^y) + (1-\delta)(\bar{c}_{ot}/\sigma^o)} K_t = \frac{1-\theta_t^w}{1-\delta} K_t. \end{aligned}$$

For the moment, suppose that $\sigma^y = \sigma^o$. If the social welfare weight on the young is relatively large (i.e. $\chi^p > 1$), we have $\bar{c}_{yt} > \bar{c}_{ot}$ and $\theta_t^w < 1 - \delta$ so that $1 - \theta_t^w - \frac{1-\delta}{\delta} \theta_t^{dw} > \delta$

¹²The intuitions are very similar for arrangements (1) and (2) and are, therefore, not explicitly described.

and $(1 - \delta)k_t^f > \delta K_t$. The young thus obtain a larger claim on human capital and financial capital than their population share δ and thus bear a larger share of the risks associated with human and financial capital. Intuitively, the young are richer and thus absorb a larger share of all risks.¹³ To make the young richer on average, not only θ_t^w but also θ_t^p has to be adjusted accordingly. If in addition the old are more risk averse than the young ($\sigma^o > \sigma^y$), the young obtain an even larger claim on human and financial capital (i.e. $1 - \theta_t^w - \frac{1-\delta}{\delta}\theta_t^{dw}$ and k_t^f) so that they bear a larger share of the aggregate risks. To prevent the old becoming poorer on average, the government has to adjust θ_t^p so that the old collect relatively more safe income.

A wage-linked first PAYG pillar is required (i.e. $\theta_t^w \neq 0$) unless we have a DWB second pillar. The reason is that depreciation and productivity risks imply independent wage and capital risks. Accordingly, all generations must have an implicit claim on both human capital and equity capital to optimally share these risks. Without a wage-linked first and second pillar (i.e. $\theta_t^w = \theta_t^{dw} = 0$), the old do not have a claim on human capital and thus do not share in wage risks. The overall optimal exposure of the old to wage risk in the first and second pillars $\frac{\delta}{1-\delta} \theta_t^w + \theta_t^{dw}$ is given by $\frac{\delta(\bar{c}_{ot}/\sigma^o)}{\delta(\bar{c}_{yt}/\sigma^y) + (1-\delta)(\bar{c}_{ot}/\sigma^o)}$.

In contrast to productivity and depreciation risks, inflation risk is not an aggregate risk. Hence, it can be completely eliminated with the social welfare weight χ^p and differences in risk aversion not affecting the optimal allocation. Through the pension fund holding nominal bonds, each young person holds the same amount of nominal bonds as an old person.¹⁴ Since each person pays the same amount of taxes and thus benefits in the same way from lower debt service as a result of unexpected inflation eroding the real value of nominal bonds, nobody is exposed to inflation risk. With unanticipated inflation, lower tax payments exactly compensate for a lower real value of (effective) nominal bond holdings.

8 Conclusions

This paper investigated how different pension arrangements affect intergenerational risk sharing in a three-period overlapping generations model. The two generations in the model overlap only during the middle period. As a result, risks cannot be directly traded between the two generations, thereby creating a rationale for institutions that take over this role. In particular, the combination of a first PAYG pension pillar aimed at intergenerational redistribution and a second, funded defined-benefit pension pillar aimed at sharing financial market, wage and inflation risks can achieve the social optimum.

This paper has focussed on the role of the pension system in conducting intergenera-

¹³Our finding of $\bar{c}_{yt} = \bar{c}_{ot}$ combined with $\bar{c}_{yt} > \bar{c}_{ot}$ implies that the fluctuations, as a result of shocks, in the young's consumption exceed the fluctuations in the old's consumption.

¹⁴ $b_{t-1}^{nf} = \delta d_{t-1}^n / (1 - \delta)$ implies $(1 - \delta) b_{t-1}^{nf} = \delta d_{t-1}^n$. Hence, the aggregate nominal bonds holdings of the pension fund $(1 - \delta) b_{t-1}^{nf}$ equal a share δ of the total amount of nominal bonds in the economy. Hence, the young hold an aggregate share of nominal bonds that corresponds to their population share.

tional risk sharing. An alternative instrument for intergenerational risk sharing is fiscal policy. In our model, the scope for optimal risk sharing through fiscal policy is rather limited. With a richer menu of taxes, fiscal policy can play a larger role in optimal risk sharing. To illustrate, taxes on labor and capital income may help to optimally share productivity and depreciation risks across generations. At the same time, however, fiscal discretion may give rise to other, political risks. This may in fact further increase the role of the pension system in optimal intergenerational risk sharing.

Our paper allows for a large number of further extensions. First, the menu of shocks could be extended to include, for example demographic shocks (such as shocks to fertility and longevity – see, for example, Auerbach and Hassett, 2002, and Andersen, 2005) and health shocks. A second extension would be to incorporate discretionary monetary policy endogenously determining inflation. Indeed, monetary policy provides another potential instrument for optimal risk sharing. However, the potential beneficial role for discretionary monetary policy in conducting risk sharing has to be weighed against potential political risks in conducting monetary policy. As a third extension, we can allow for intragenerational heterogeneity in risk preferences. In that case, mandatory pension funds with uniform investments and liabilities are not able to tailor to individual preferences. If young individuals do not have access to financial markets to construct their own tailor-made portfolio, mandatory pension funds may give rise to welfare losses compared to a first-best world in which all individuals can buy their own tailor-made portfolio. These losses have to be traded off against the potential benefits of pension funds in allowing young generations to share in financial-market risks.

We have assumed that the young cannot participate in capital markets at all to share financial-market risks. One interpretation is that human capital is not tradable and that the young therefore cannot borrow at all against their human capital to invest in financial capital. In practice, however, the young may be able to participate in equity-market risk that materializes during their working career, either by borrowing or by investing all their saving in the risk-bearing capital. Indeed, capital markets allow in principle for risk-sharing between overlapping generations, especially if the young can borrow. In this regard, our calculations thus are likely to overstate the potential risk-sharing benefits from defined-benefit pension plans. At the same time, however, by modelling only two generations, we have underestimated the potential gains of pension funds from risksharing between non-overlapping generations. Indeed, with many non-overlapping generations, old generations can benefit from sharing risks with not only the young generations that they overlap with but also future generations that are not yet born when they are alive (see Van Hemert, 2006). In other words, it may pay to allow not only the young generation but also other future generations to trade with the old through the pension system. We would like to explore how sensitive our results are with respect to alternative assumptions about the extent to which the young and the future generations can participate in capital markets and pension institutions. Whereas more scope for the young to participate in

capital markets reduces the value-added of pension funds, including more generations in pension arrangements increases the potential of pension funds to create value by opening up new ways to conduct intergenerational risk sharing.

References

- [1] Andersen, T.M., 2005, Social Security and Longevity, *CEifo Working Paper*, No. 1577.
- [2] Auerbach, A.J. and K.A. Hassett, 2002, Optimal Long-Run Fiscal Policy: Constraints, Preferences and the Resolution of Uncertainty, *mimeo*, University of California at Berkeley and American Enterprise Institute.
- [3] Beetsma, R. and X. Debrun, 2006, The New Stability and Growth Pact: A First Assessment, *European Economic Review*, forthcoming.
- [4] Beetsma, R. and H. Uhlig, 1999, An Analysis of the Stability and Growth Pact, *Economic Journal* 109, 546-571.
- [5] Bohn, H., 2003, Intergenerational Risk Sharing and Fiscal Policy, *mimeo*, University of California at Santa Barbara.
- [6] Bovenberg, A.L. and H. Uhlig, 2005, Pension Systems and the Allocation of Macroeconomic Risk, *in preparation*, Tilburg University/Humboldt University.
- [7] Carroll, D.R., Porapokkarm, P and E. Young, 2005, Social Security and Intergenerational Risk Sharing, *mimeo*, University of Virginia.
- [8] Constantinides, G.M., Donaldson, J.B. and R. Mehra, 2002, Junior Can't Borrow: A New Perspective on the Risk Premium Puzzle, *Quarterly Journal of Economics* 117, 269-296.
- [9] De Jong, F., 2005, Valuation of Pension Liabilities in Incomplete Markets, *mimeo*, Tilburg University.
- [10] Demange, G., 2002, On Optimality in Intergenerational Risk Sharing, *Economic Theory* 20, 1-27.
- [11] De Menil, G. and E. Sheshinski, 2003, The Optimal Balance of Intergenerational Transfers and Funded Pensions in the Presence of Risk, *mimeo*, EHESS/Hebrew University of Jerusalem.
- [12] Enders, W. and H.E. Lapan, 1982, Social Security Taxation and Intergenerational Risk Sharing, *International Economic Review* 23, pp.647-658.

- [13] EPC, 2006, Working Group on Ageing Population, http://europa.eu.int/comm/economy_finance/epc/epc_country_fiches_en.htm.
- [14] Gordon, R.H. and H.R. Varian, 1988, Intergenerational Risk Sharing, *Journal of Public Economics* 37, 185-202.
- [15] Gottardi, P. and F. Kubler, 2006, Social Security and Risk Sharing, *mimeo*, Universities of Venice/Mannheim.
- [16] Fehr, H. and C. Habermann, 2005, Risk Sharing and Efficiency Implications of Progressive Pension Arrangements, *CESifo Working Paper*, No. 1568.
- [17] Hassler, J. and A. Lindbeck, 1997, Intergenerational Risk Sharing, Stability and Optimality of Alternative Pension Systems, *mimeo*, Institute for International Economic Studies, Stockholm.
- [18] Heijdra, B. and F. van der Ploeg, 2002, *Foundations of Modern Macroeconomics*, Oxford University Press, Oxford.
- [19] Krueger, D. and F. Kubler, 2002, Intergenerational Risk-Sharing via Social Security when Financial Markets are Incomplete, *American Economic Review, Papers and Proceedings* 92, 2, 407-410.
- [20] Lindbeck, A. and M. Persson, 2003, The Gains from Pension Reform, *Journal of Economic Literature* 41, 74-112.
- [21] Modigliani, F., and A. Muralidhar, 2004, *Rethinking Pension Reform*, Cambridge University Press.
- [22] Oksanen, H., 2006, Actuarial Neutrality across Generations Applied to Public Pensions under Population Ageing: Effects on Government Finances and National Saving", in *The Balance Sheet of Social Security Pensions II*, Proceedings No. 7, Institute of Economic Research at Hitotsubashi University, Tokyo.
- [23] Shiller, R.J., 1999, Social Security and Institutions for Intergenerational, Intragenerational, and International Risk Sharing, *Carnegie-Rochester Series on Public Policy* 50, 165-204.
- [24] Thøgersen, O., 1998, A Note on Intergenerational Risk Sharing and the Design of Pay-As-You-Go Pension Programs, *Journal of Population Economics* 11, 373-378.
- [25] Van Hemert, O., 2006, Dynamic Investor policies, *Ph.D thesis*, Tilburg University.
- [26] Wagener, 2004, On Intergenerational Risk Sharing within Social Security Schemes, *European Journal of Political Economy* 20, 181-206.

Appendix:

A Definitions

We first introduce a number of definitions. $\bar{T}_t \equiv \bar{A}F(K_t, \bar{L}) + (1 - \bar{\zeta})K_t$ represents median (or average) total resources in period t . The shares of median young's consumption, old's consumption and government spending in total median resources in that period are defined as, respectively:

$$a_{cy} \equiv \frac{\delta \bar{c}_{yt}}{\bar{T}_t}; \quad a_{co} \equiv \frac{(1 - \delta) \bar{c}_{ot}}{\bar{T}_t}; \quad a_g \equiv \frac{g_t}{\bar{T}_t}.$$

Notice that $a_{cy} + a_{co} + a_g = 1$. We define the share of output (a_Y), capital providers (a_K), the net share of capital providers (a_{Kn}), the depreciation share (a_ζ) and the share of labor providers (a_L) in total resources as, respectively (note that $a_K + a_L = 1$):

$$a_Y \equiv \frac{\bar{A}F_t}{\bar{T}_t}; \quad a_K \equiv \frac{[\bar{A}F_{Kt} + (1 - \bar{\zeta})] K_t}{\bar{T}_t}; \quad a_{Kn} \equiv \frac{\bar{A}F_{Kt}K_t}{\bar{T}_t}; \quad a_\zeta \equiv \frac{\bar{\zeta}K_t}{\bar{T}_t}; \quad a_L \equiv \frac{\bar{A}F_{Lt}\bar{L}}{\bar{T}_t}.$$

where $F_t \equiv F(K_t, \bar{L})$. Finally, we define the following ratios:

$$\begin{aligned} a_{KL} &\equiv \frac{\bar{A}K_t\bar{L}F_{KLt}}{\bar{T}_t}; & a_{LL} &\equiv \frac{\bar{A}\bar{L}^2F_{LLt}}{\bar{T}_t}; & a_{KLf} &\equiv \frac{\bar{A}\bar{L}F_{KLt}(1 - \delta)k_t^f}{\bar{T}_t}; \\ a_{Gy} &\equiv \frac{\delta \bar{G}_t^y}{\bar{T}_t}; & a_{Go} &\equiv \frac{(1 - \delta) \bar{G}_t^o}{\bar{T}_t} = -a_{Gy}; & a_{dr} &\equiv \frac{\delta \bar{d}_t}{\bar{T}_t}; \\ a_{dn} &\equiv \frac{(1 - \delta) d_{t-1}^n}{\bar{T}_t}; & a_{brf} &\equiv \frac{(1 - \delta) b_{t-1}^{rf}}{\bar{T}_t}; & a_{bnf} &\equiv \frac{(1 - \delta) b_{t-1}^{nf}}{\bar{T}_t}; \\ a_{kf} &\equiv \frac{\bar{A}F_{Kt}(1 - \delta)k_t^f}{\bar{T}_t}; & a_{\zeta kf} &\equiv \frac{\bar{\zeta}(1 - \delta)k_t^f}{\bar{T}_t}, \end{aligned}$$

where $F_{KLt} \equiv F_{KL}(K_t, \bar{L})$, $F_{LLt} \equiv F_{LL}(K_t, \bar{L})$, etc.