UvA-DARE (Digital Academic Repository)

Reconstruction of a collaborative mathematical learning process

Pijls, M.H.J.; Dekker, R.; van Hout-Wolters, B.H.A.M.

DOI

[10.1007/s10649-006-9051-3](https://doi.org/10.1007/s10649-006-9051-3)

Publication date

2007

Document Version

Final published version

Published in

Educational Studies in Mathematics

[Link to publication](#)

Citation for published version (APA):

Pijls, M. H. J., Dekker, R., & van Hout-Wolters, B. H. A. M. (2007). Reconstruction of a collaborative mathematical learning process. *Educational Studies in Mathematics*, 65(3), 309-329. <https://doi.org/10.1007/s10649-006-9051-3>

General rights

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

RECONSTRUCTION OF A COLLABORATIVE MATHEMATICAL LEARNING PROCESS

ABSTRACT. The study focused on the interaction between two secondary school students while they were working on computerized mathematical investigation tasks related to probability theory. The aim was to establish how such interaction helped the students to learn from one another, and how it may have hindered their learning process. The assumption was that interaction is beneficial for students if they can perform certain key activities, namely showing, explaining, justifying, and reconstructing their work. Both students attained mathematical level raising. However, the student who explained frequently and criticized himself attained more mathematical level raising than the student who did not explain her work frequently or criticize herself.

KEY WORDS: collaborative learning, process model, mathematical level raising, computer simulation

1. INTRODUCTION

One of the difficulties for teachers is to observe the learning process of students who are working collaboratively. The first sight of a classroom with pairs of students talking vividly to one another may be satisfying, but does not tell us if they are learning mathematics. What do they really discuss with one another? How does the interaction develop? Do they support or hinder one another's learning?

The importance of interactions with peers for students' mathematical learning has become evident in recent decades (Teasley, 1995; Webb, 1991; Yackel et al., 2000; Wood, 1999). Some studies (Roschelle, 1992; Trognon, 1993) analyzed in detail the interaction between two students in relation to the content they learned; Roschelle did so in the field of physics, and Trognon with respect to the "four cards" problem.

The study reported here aimed at analyzing the collaborative learning process of two students in a situation that was in a way very close to the usual situation in their classroom, i.e. students were working in pairs with a computer and learning materials, and the teacher was playing a minimal role. The learning materials, however, were very experimental and developed especially for these students. Earlier research made it clear that, in this type of education, the students are not always motivated to perform investigations with the help of a computer (PRINT, 1998; Prent, 1999).

We ascribed this to the fact that many computer simulations on probability theory were rather abstract. That is why we tried to develop simulations in a way that would be very accessible for students, offering them many opportunities to explore the subject on a concrete level. We developed investigation tasks in order to make them reflect on their experiences with the computer games. The tasks were intended for the students to experience the *grid structure* by playing several games. The tasks were presented to the students before they started to work on a chapter in their regular textbook. We expected that the investigative tasks would allow students to experience the properties of a grid and profit from this experience while working on their textbook chapter. The underlying structure of all the different games in the computer simulation was a grid, in several shapes, probabilities and appearance. The idea was to allow students to develop a feel for the grid structure, by providing several examples.

We took the learning results of both students as a starting point for the analysis of their utterances during the lessons. Our focus was on events indicating whether or not certain concepts in probability theory were being learned. Our goal was to provide a detailed illustration of the learning process of two students, and to formulate hypotheses about interaction and mathematical learning.

2. THEORETICAL BACKGROUND

2.1. *Learning mathematics*

In lessons dealing with the transition from a concrete or practical approach to a problem toward a more abstract approach, it makes sense to provide students with meaningful problems (Freudenthal, 1973) to be collaboratively solved. Inspired by Van Hiele's level theory (Van Hiele, 1986), we defined the transition from the perceptual to the conceptual level as "mathematical level raising". In the different stages of their learning process, students approach problems in different ways. Before they learn a concept, they approach it at a perceptual level; that is, they see it "as it is" and do not recognize properties of a mathematical concept in it. After they have acquired a certain mathematical concept, students will recognize aspects of that concept in a realistic problem and can use these aspects to solve the problem. They thus treat the problem at another level. Level raising has taken place, namely from the perceptual to the conceptual level. An example from our experimental learning materials is given below.

Figure 1 shows the game board of the gambling game TIC-TAC. The task related to this game is:

Figure 1. The game TIC-TAC.

One can play a little ball by clicking with the mouse on its starting point at the bottom left. There is a 50% probability that a ball will go up, and a 50% probability that it will go to the right. The question is: “What is the probability of ending up in the box with 100 points?” Students’ answers were:

1. “There are nine boxes, so the probability is $1/9$.”
2. “The probability of ending up in the middle is greater than the probability of ending up in an end point.”
3. “I enumerated all possible routes toward the box with 100 points; the probability is 28.”
4. “I enumerated routes in the grid and the probability is 28 divided by the total number of routes; $28/256$.”

The first two answers occurred as student answers in the study described in this article. With respect to the concept of calculating probabilities in a grid, we regard the first answer as an answer at the perceptual level, which is actually not correct but shows the prior knowledge of the students. The fourth answer is an answer at the conceptual level. In an earlier study (Pijls, et al., 2003), we categorized two stages in the learning process between the perceptual and conceptual level, namely “beginning level raising”¹ and “semi level raising”. We defined the former as the beginning of concept building with the help of one’s own constructions (as shown in the second answer), while the latter occurs when students seem to master certain concepts, but do not really understand them, and apply them incorrectly (as in the third answer). We assume that in this case their knowledge is not rooted in own constructions. These two stages, though not necessarily both, may occur in a student’s learning process. In both stages, students have not yet mastered a concept; however, we value beginning level raising higher than semi level raising, since in the former case students understand what they know at that moment, while this is not the case with semi level raising.

A and B are working on the same mathematical problem. Their work is different.

A is working		B is working
<i>A asks B to show his work</i>	<i>What are you doing? What have you got?</i>	<i>B asks A to show her work</i>
A becomes aware of her own work		B becomes aware of his own work
A shows her own work	I am doing this... I have got this...	B shows his own work
A becomes aware of B's work		B becomes aware of A's work
<i>A asks B to explain his work</i>	<i>Why are you doing that? How did you get that?</i>	<i>B asks A to explain her work</i>
A thinks about her own work		B thinks about his own work
A explains her own work	I'm doing this, because... I have got this, because...	B explains his own work
A thinks about B's work		B thinks about A's work
<i>A criticizes B's work</i>	<i>But that's wrong, because...</i>	<i>B criticizes A's work</i>
A thinks about B's criticism		B thinks about A's criticism
A justifies her own work	I thought it was right, because...	B justifies his own work
A thinks about her justification		B thinks about his justification
<i>A criticizes her own work</i>	<i>Oh no, it isn't right, because...</i>	<i>B criticizes his own work</i>
A reconstructs her work	I'll better do it like this...	B reconstructs his own work
bold: key activities		
standard: mental activities		
<i>italic: regulating activities</i>		

Figure 2. Process model of Dekker and Elshout-Mohr (1998).

2.2. Collaborative learning

When students are trying to learn something difficult, they benefit from sharing their ideas with their peers, especially when their peers have a different point of view. Several studies mentioned differences between students (Kieran and Dreyfus, 1998; Sfard, 2003; Webb, 1991), in particular differences in thinking, as a prerequisite for a discussion in which students can learn. Webb (1991) showed a correlation between learning and giving explanations. Dekker and Elshout-Mohr (1998) developed a process model for the analysis of interaction between two students working on a mathematical task. The point of departure is that the students' mathematical work is different (see Figure 2).

The assumption is that the performance of four key activities leads to mathematical level raising. These are: showing one's work, explaining one's work, justifying one's work, and reconstructing one's work. These activities are evoked by peers who perform such regulating activities as

asking one to show one's work, to explain one's work and to criticize one's work. The key activities are assumed to be related to reflection, either because they give rise to reflection (showing one's work) or because they are the result of it (justifying and reconstructing). As the middle column of the model shows, the key activities can be observed when students are at work. Confusion may arise between explaining and justifying; one should keep in mind that justification is a reaction to criticism ("My answer is right, because. . ."). The other two key activities (i.e., to show or to reconstruct one's work) can also cause confusion. If there are indications that a student's answer has evolved from another answer, we call it a reconstruction. Mental activities – such as thinking about and criticizing one's own work – accompany the key activities, but are less observable. Although the process model describes learning, it is also an instrument to analyze those parts of students' dialogues that give rise to mathematical level raising. The idea behind the model is that it describes the activities students can learn from. In principle, one person could perform both regulating and key activities.

Apart from differences between students, what other prerequisites are there for collaborative learning? What about the learning materials, and the role of the computer? To make it possible for the students to profit from one another, the instructional materials must provoke an interesting discussion. Dekker (1994) formulated four criteria for learning materials for mathematical level raising in collaborative learning. Problems must be: *real or meaningful*, in order to motivate and stimulate the students; *complex*, so that the students need to work together; *to construct something*, in order to reveal the students' thoughts and to promote discussion; and of course *aimed at level raising*. The use of the computer may support these four criteria.

However, fulfilling these prerequisites does not guarantee that an interaction will be fruitful. Sfard and Kieran (2001) analyzed the discursive interactions of pairs of 13-year-old students on two levels: the level of immediate mathematical content and the level of meta-messages and engagement in the communication. They found the students' communication to be unhelpful because of the ineffectiveness of their communication. Kieran (2001) found different patterns of interaction between pairs of students, and showed that adolescents can experience some difficulty in following their own thinking in novel problem situations.

2.3. Aim of the study

The aim of the study was to investigate whether we could explain the students' mathematical level raising attained in the post-test by the expected occurrence of key activities according to the process model. For example, if we knew from pretest and post-test that a student had mastered a certain

concept, we expected to find in the transcript an event in which the student showed – and, possibly, explained, justified and reconstructed – his or her work related to this concept.

3. METHOD

The students took part in a field experiment on students' collaborative investigations using a computer (see Pijls et al., 2003). The experiment took place in a Montessori school in Amsterdam; the subjects were 16-year-old students who were mainly studying applied mathematics. The students were used to working on their own, while the teacher played the role of a coach. The experiment extended over 10 lessons of 45 minutes each.

3.1. *The theme of routes and probabilities*

The mathematical concept to be learned is a model to calculate binomially distributed probabilities, namely a *grid structure*. By calculating the number of routes in a grid ending in a node, one creates Pascal's triangle. The aim of the lessons was, firstly, to learn an enumeration of routes procedure (adding the number of routes towards two 'preceding' points, the so-called 'angular point method'), and then to calculate probabilities in a grid and to use the grid as a model for solving word problems. The difficulty for students was that the enumeration procedure might become a meaningless 'trick'. This is what we regard as semi level raising, which may result in counting mistakes and not recognizing occasions of applying the grid structure.

3.2. *Learning materials*

In order to make the enumeration of routes in a grid meaningful for students, we developed investigation tasks in a computer simulation. The simulation (Pijls, 2001) comprised several computer games with the grid as an underlying model. The idea was that students could explore the probabilities in a grid in an informal way by playing the games. The investigation tasks were intended to encourage students to reflect on their experiences with the computer games and to develop the mathematical concept of enumerating routes in a grid. The investigation tasks were followed by the tasks in their regular textbook (Boer et al., 1998). This textbook is a modern book, in the tradition of realistic mathematics education, teaching new mathematical concepts according to the principles of guided reinvention. In the book, students worked on small, relatively closed tasks in order to reinvent

mathematical concepts, which are then summarized in “theory blocks”. We assumed that, by providing the students with investigation tasks before they worked with the textbook, they could explore the domain, which, in turn, would help them to attain level raising. During the investigation tasks, the students worked together using one set of activity sheets as well as their own exercise book. The investigation tasks consisted of five tasks that concern the analysis and playing of a game. The aim of the first task was to raise the question of counting routes. The aim of the second task was to make clear that this game had the same underlying structure; it was made smaller so that students have the opportunity to develop an enumeration strategy. The third task let students explore so-called ‘grids with a hole’. The fourth task let them experiment with probabilities. The idea was that in order to answer the question students should calculate all probabilities. The aim of the fifth task was to let students experience an asymmetric distribution.

The chapter on Routes and Probabilities in their mathematics textbook started with exercises that made students enumerate routes in a grid by systematically focusing on the number of paths towards a certain angular point. Secondly, students were shown how to use the grid as a model and the relation between Pascal’s triangle and enumerating routes in a grid. Finally, they were taught to calculate probabilities in a grid and how to use this to solve word problems. Although the didactical rationale behind the textbook exercises was to make students reinvent the concepts they learned, many of the tasks were closed and fragmented, leaving little room for own students’ creativity. Our intention was to create opportunities for students’ own inventions with the investigative tasks and the computer simulation.

3.3. Role of the teacher

In our experiment, the learning materials were developed for independent learning. The usual role of the teacher at the Montessori school in Amsterdam is to coach the students and to give them help when they ask for it. Since the materials were still in a developmental stage and the researcher was the most familiar with them, we decided to work together as a team, that is, the teacher – who knew the students well – to coach the students, and the researcher to provide help with the mathematics when the students asked for it. The students were used to working self-reliantly in pairs. At the beginning of the lessons, the students were told to execute all tasks together and in what time sequence they had to finish their work. At the end of the lessons, they were to check their work against correction sheets. Since we wanted to stay very close to their normal school routine, the students were not given any special training on interaction.

3.4. *Data collection*

We audio-taped the students during all their lessons. We collected written student products and log files to clarify any inaudible remarks on the audiotapes. For the present study, we selected one pair of students by taking samples of the audiotapes of all pairs. The occurrence of an interesting discussion concerning the relevant theme during their first lesson was our rationale for selecting Susan and Peter. They formed a high-average pair: Susan scored high and Peter scored average on prior knowledge. Furthermore, in her pretest on the theme of Routes and Probabilities, Susan made extensive elaborations by drawing diagrams, and Peter answered a difficult question correctly. So we chose the students in such a way that we expected them to show their work, criticize one another and attain mathematical level raising.

3.5. *Analysis*

We read the transcript several times and this led to suppositions about the learning processes of Susan and of Peter (cf. Dekker et al., 2001). Then we asked ourselves this question: Which passages in the verbatim protocol account for the learning results in their post-test? With the agreement of two coders, we selected four episodes comprising events in which a concept was learned and an event in which this did not happen. We hoped that the latter would help us to understand why some concepts had not been learned. We analyzed the mathematical level of the utterances and the interaction with help of the process model of Dekker and Elshout-Mohr (1998), which we described in Section 2.2.

4. RESULTS

4.1. *Pretest and post-test*

The pre- and posttest each consisted of 11 open problems on the theme of Routes and Probabilities. It concerned enumerating routes in a grid, calculating probabilities in a grid, enumeration problems to solve with a grid and probability problems to solve with a grid. The problems could be answered at a perceptual level and at a conceptual level. As an example in a task on enumerating routes in a grid, students were given a part of a map with the question how many shortest paths one could take from point A to point B. When students enumerated routes by drawing possible paths and counted them, we regarded this as an answer at the perceptual level, whereas systematically counting routes by the 'angular point method' or with Pascal's triangle was determined as an answer at the conceptual level.

In her pretest, Susan approached the questions at the perceptual level. Thus, she could solve easy problems, but not those that were more complex. She exhibited a concrete approach to the tasks, by trying to use known procedures and by enumerating concrete routes. In her post-test, Susan showed that she had learned to enumerate routes according to the procedure taught in their textbook, the angular point method. She had also learned to use the grid as a model to solve enumeration problems, although she was not clear about when this method can be used and when not. She knew how to calculate probabilities in a concrete grid by enumerating routes and multiplying by probabilities. Susan had attained the conceptual level for the enumeration of routes and calculating probabilities in a grid, but she had not fully learned to use the grid as a model. She attained semi level raising for this concept, since she showed not to know exactly for what kind of word problems the grid could or could not be used.

In his pretest, Peter's answers demonstrated a preference for formulas. He tackled the enumeration problems and the complex route enumeration problem by applying a formula (either his own or one he had been taught) that was not correct. It is striking that he answered a question about calculating probabilities in a grid correctly. In his post-test, Peter showed that he had learned to count routes and to use the grid as a model to solve enumeration problems. He knew when to use it and when not. He had refined his method in order to solve probability problems in a grid. Peter had attained the conceptual level for all concepts.

To summarize, Susan and Peter both started at the perceptual level. However, Peter showed a tendency to approach problems at a conceptual level, while Susan explored the perceptual level. At the end of the experiment, Susan had attained the conceptual level for two of the concepts, and Peter for all concepts.

4.2. Reconstruction of the learning process

In order to reconstruct the learning process, we selected four episodes we considered to be central with respect to mathematical level raising. These were:

1. Susan developed the notion that in order to calculate probabilities in a grid, she and Peter had to enumerate routes. Peter subsequently constructed a formula to enumerate routes.
2. Peter wanted to refine his formula to enumerate routes; Susan did not really respond to this.
3. Susan and Peter reconstructed Peter's formula to enumerate routes.
4. Peter explained to Susan the enumeration formula given in their mathematical textbook.

Analysis showed us that episodes 1, 3, and 4 contain utterances indicating mathematical level raising, and that during episodes 2 mathematical level raising could have taken place but did not. For each episode, we wanted to establish whether there was a correspondence between learning and the interaction as described in the process model. First, we tried to determine the mathematical level of the utterances (analogous to our previous study (Pijls et al., 2003)) and then coded the utterances with the help of the process model. Both codings were performed with the agreement of two coders.

4.2.1. *First episode: The notion of enumerating routes*

Susan and Peter were working on the task shown Figure 1. They have been playing the game TIC-TAC on the computer and realized that the little ball hits the boxes in the middle more often. Below we show a part of Susan and Peter's discussion on task (1b); (. . .) indicates that a part of the dialogue has been deleted for the sake of clarity.

- 1 S: There are nine possibilities.
- 2 P: (. . .) But the probability that it will come here is greater than that it will come here.
- 3 S: Oh, why?
- 4 P: Oh, no, that is, yes, of course, because the probability that it will come here is greater than that it will come here, because the probability is greater that it will go like this, than that it will go up all the time.
- 5 S: Okay, but the computer is also doing something. . . okay.
- 6 P: Comput., yes, but it's about the theories.
- 7 S: Yes, that's true. . .
- 8 P: I mean, there are. . .
- 9 S: But how can you calculate that?
- 10 P: For example, that one can go like this or like this, or like that, all kind of routes, but that one has only one route.
- 11 S: Yes, but look, with that one you then have. . .
- 12 P: Oh no, because that one can't, well that's very difficult (silence) (. . .)
- 13 S: Different possibilities, then you have to. . . all those routes, you know... (. . .)
- 14 S: You see, it goes as often to the 5 as to the 1.
- 15 P: No, I mean: You can see that the ball is going more often to the middle than here and here.
- 16 S: Yes...
- 17 P: I think that the probability is . . . I think, er. . . the greatest probability is that it will go to the 6, because that's in the middle. The probability increases as you go toward the middle. (. . .) I think the probability is 5% here, and then it doubles, and then it doubles all the time, or something like that. . . What's the probability. . . ? (. . .)
- 18 S: So then you have to take percents. 1/9, what's that?
- 19 P: 1/9? ... something like 9 percent, 9.333 percent (. . .) But we can also. . . we keep it apart.

Susan showed Peter her answer (line 1), which is in accordance with their prior knowledge of probability theory, namely that in order to calculate a probability, one must first determine all the possibilities. Peter criticized this by showing his answer: The probability that a ball will end up in the middle is greater than the probability that it will end up at an endpoint. Susan asked Peter to explain this, and he explains (line 4) that the probability that a ball will zigzag is greater than that the probability that it will go upward all the time. Susan criticized this (line 5) by stating that there is no structure: possibly the computer influences the game. In addition, Peter criticized himself (line 6) by saying that there must be some theory behind it. For him, mathematics is about theories. Susan agreed and asked for an explanation. Peter explained his idea by refining it (line 10): A box in the middle can be reached by all kind of routes, while a box at an endpoint can be reached in only one way.

They were both puzzled (lines 11 and 12); this was difficult for them. Then Susan reconstructed her answer at the conceptual level (line 13). She realized that they had to count routes in order to calculate probabilities. After this statement, there was a long silence: They did not know how to enumerate routes. Susan then showed (line 14): The probability is symmetric around the middle. Peter showed that the probability is greater in the middle, and (line 18) that he wanted to calculate the probability. He estimated that it was 5% at an endpoint and then doubled each time it moved closer to the middle. When he started to calculate, so did Susan, and she returned to her original answer of $1/9$. Susan proposed writing down this answer and proceeding with the next question. Peter, however, was not satisfied: He wanted to think about it and would return to the question later. They went on with the second task.

Interpretation of the first episode

Susan formulated an important conclusion: In order to calculate the probability, one has to count routes. She gathered a lot of information about grids and probabilities. However, she did not manage to use this information to answer the question and she relied on her original answer at the perceptual level. Peter was clearly searching for a theory. At the end of the episodes, Susan did not ask for explanations nor did Peter ask Susan why she had decided the answer was $1/9$. He criticized himself. In the first part of this episode, mathematical level raising went hand in hand with a discussion in which both Susan and Peter showed their own work and criticized one another's work. In the last part of the episode, they returned to their original answer ($1/9$), but they no longer criticized one another and they stayed at the same mathematical level.

4.2.2. *Second episode: Peter reconstructs his enumeration procedure*

After the first episode Susan and Peter continued with the next investigation tasks, which concern other type of grids. They made calculations to determine optimal chances to win. However, there was no link in their reasoning to the previous task and they did not mention any more the need to enumerate possible routes in order to calculate the probabilities in a grid. At the beginning of the next lesson, Peter returned to the question from the first episode. He has been thinking about this problem outside the lesson and he has reconstructed his enumeration procedure. He now shows his work to Susan. Their dialogue is given in below.

- 1   P: I assumed that every time that, every time that the ball moves toward the middle, that the probability that it, er, one upward and then this way and that way, so here the probability is 1, 2, 3, 4 and 5, and then I've added those chances, that's 25, and then this is three 25ths, so twelve 100ths
- 2   S: Yes.
- 3   P: Is 12 percent.
- 4   S: Yes.
- 5   P: I think, but I'm not sure, so it would be, yes, 20% and that 4%. Well, shall we (. . .)
- 6   P: You may come up with another theory, but anyway I think it's better than 1/9.
- 7   S: Okay.
- 8   P: At least, for me. I don't know what you think.
- 9   S: Yes, it's okay, add it to our stuff... And do you also have something for this one? Er, have a look.
- 10   P: Wow. Erm, let me see, er...
- 11   S: Shall we proceed? Add it. . . How do you write it?
- 12   P: Well, I'll write.

Peter had worked out his idea that the probability increases toward the middle. He showed his new theory to Susan (line 1): If I name the probability at one endpoint 1, and in the next box 2, and so on, then it makes 25 altogether, so the probability that the ball will come land in the 100 box is 3/25, or about 12%. Susan (lines 2 and 4) did not ask for an explanation, nor did she criticize him; instead, she agreed. Peter then worked it out (line 5): The probability will be 20% in the middle and 4% at an endpoint. Peter seemed to justify his answer (line 6), though not with a content-related argument. Susan agreed with him (line 7), but did not explicitly agree with the content of his work. Peter was not sure of his answer and asked Susan what she thought (line 8). Susan agreed with his idea in a practical way (line 9), and asked whether he could solve another question, too, as she wanted to proceed with the next one. Peter then replaced their original answer (1/9) by 12%, and they went on with the next question.

Task 4b

In the Galton game, one can distribute a total of 100 points over the nine boxes. Of course, the purpose is to do this in such a way that you win as many points as possible. Determine for each box on the bottom row the probability that a ball will end up in it.

Figure 3. Task 4b.

Interpretation of the second episode

Peter came up with an idea he was not sure of. He started to defend it, but since he received no feedback from Susan regarding the content, he could not really explain, justify, or reconstruct it. There was nobody to indicate the weak points of his theory. Since he had offered his idea at the beginning of the lesson, they did not have the actual computer game at hand to test his idea. Susan accepted his theory without criticism. It seems that she was not interested in *why* his theory would work; instead, she wanted to go on with the next task. In fact, Peter criticized Susan's idea of $1/9$.

Although Peter's theory was a good try, it is not correct. He rejected the $1/9$ probability and wanted to make a theory based on the conclusion that the probability increases toward the middle and that it is symmetric around the midpoint in an enumeration method. In terms of mathematical level raising, he might have benefited from someone asking him for an explanation. No level raising took place during this interaction.

4.2.3. Third episode: Reconstructing the enumeration procedure

Susan and Peter returned to the question from the first episode during the second lesson, when they were working on the fourth task on a Galton-board-like game (Figure 3). The students had the opportunity to drop 100 balls at the same time, and so gain insight on the distribution of the balls.

The aim of this task was to let the students realize the relation between the distribution of the balls and the probability to reach a certain box. The episode started when Peter mentioned that this question made him think of task 1b, which we have discussed before.

- 1 S: Here? But do you mean the same as...?
- 2 P: Yes, erm, I don't know. . . I think that in fact that theory is a bit vague.
- 3 S: And here will come. . . Here comes 2, here comes 0? Yes, that sounds. . .
The ball comes here the most.
- 4 P: Yes, I know (. . .)
- 5 S: And then here, less. . . Here it comes less often.
- 6 P: Yes, but I did 'plus one' all the time, 1, 2, 3, 4, 5, but it's also possible
that it doubles all the time.
- 7 S: Yes.
- 8 P: Wait a minute (noise of paper)
(silence)
- 9 P: Like this. 16/46, . . .
(. . .)
- 10 P: This is more likely than this. . . So I think that the probability
- 11 S: Yes, okay. doubles all the time.
- 12 P: What did you have? I mean. . .
- 13 S: Yes, erm. . . I really don't have anything more, but I think your answer
is reasonable. (. . .)
- 14 S: Erm, oh. I only calculated it for the box in the middle.
- 15 P: Yes, as I said, 16/46, 8/46, 4/46. The number is divided by two.

Peter wanted to use the same theory as in task 1b (first episode). Susan asked for an explanation (line 1), and Peter said that he was still not sure of his theory. Susan then repeated the conclusion that the probability of ending up in the middle was greater than the probability of ending up in one of the endpoints (lines 3 and 5). It might be on the basis of experiments with dropping 100 balls at a time that Peter returned to his idea that the probability "doubles all the time". He concluded from the distribution of the balls in the Galton-board, that the probability to reach a certain box increased at a stronger rate than 'adding one'. He looked back at task 1b and now worked it out in such a way that the probability doubled for each box toward the middle. Then the probability of ending up in the middle was 16/46 ($\approx 30\%$). He reconstructed his answer, and then (line 10) showed his reconstructed work. In line 12 he may have meant that the probability of ending up in the middle was more likely to be 30% than 20%. Peter asked Susan to show him her work (line 12). Susan simply agreed with Peter's work (line 13); she did not really react to the content of what he had said, nor did she show him her work. They then wanted to express the probability in percentages, and here Susan took the initiative in the calculations (line 14). When she had finished the calculations, she realized that she had worked out the probability for only one box. Peter explained that with that probability, one could easily calculate the other probabilities. This was their answer to question 4b. They also reconstructed their answer to question 1b (first episode) in the same way.

Interpretation of the third episode

The game made Peter think about task 1b and it gave him the opportunity to reflect on his ideas and to reconstruct his answer. Susan's calculating skills helped him to work out the next version of his theory. Therefore, they reconstructed their answer to question 1b (see Figure 1). Although this was a beautiful construction in terms of probabilities, the theory is not correct. Susan and Peter had made a good start with regard to mathematical level raising, and we believe that these explorations gave them a sense of the distribution of paths and probabilities in a grid. As in the previous episode, Peter asked Susan to criticize his work and to show him her work, but she did not react to the content; thus, Peter did not develop his theory further.

4.2.4. Fourth episode: Enumerating routes in their mathematical textbook

After the third episode, Susan and Peter finished the investigation tasks, and started to work on the textbook's "Routes and Probabilities" section. After the first task, in which they had to mention possible paths in a grid, they start with the task shown in Figure 4.

- 1 S: Er, 1-3.
- 2 P: Of course not. . . Yes, yes. 3-1 I say. 3-1, 2-3, or 2-2, only to come here, you must go along these two sides.
- 3 S: Huh?
- 4 P: Don't you understand?
- 5 S: No, I don't understand this question at all!
- 6 P: Look, the final score is this.
- 7 S: Yes.
- 8 P: To come here, first you have to be here.
- 9 S: Yes.
- 10 P: If it'd been here, it would've been 3-1. . . (...)(...)(...)
- 11 P: But I want to find a way to calculate it. Now I count. . . It's just 2 to the power of 2. . . No, er, yes. 2 times 2 times 2, 2 to the power of 3 is it. . . (...)(...)(...)
- 12 P: But how do you calculate it, why does it take so long?
- 13 S: Now, erm, three upward. . .
- 14 P: But you just have to add that and that?
- 15 S: Huh?
- 16 P: That one and that one, and then you have that M.
- 17 S: Okay. And this is 10? (...)
- 18 S: Very easy indeed. . . (...)
- 19 A: Susan and Peter, how did you calculate that?
- 20 P: We just made a thing. . . a grid.
- 21 S: . . . grid.
- 22 A: Is that possible for question 11a?
- 23 S: That's possible for all questions.
- 24 P: Yes, that's possible for all questions.

2. The result of 2-3 must be preceded by 2-2 or by 1-3.

- a) Indicate these predecessors in the system of coordinates.
- b) How many routes are there from 0-0 to 2-2?
- c) How many routes are there from 0-0 to 1-3?
- d) How can you obtain the number of routes from 0-0 to 2-3 from the answers to tasks b) and c)?

Figure 4. Task 2 from the students' mathematics textbook.

The aim of this task was for the students to reinvent the procedure for enumerating routes. We expected them to refer to the investigation task in which they realized that they had to enumerate routes in order to calculate probabilities, but this did not happen.

The episode below starts while they were working on task 2d.

Susan showed her answer (line 1). The answer is correct, but Peter seemed to mix up the 3 and the 1, so he criticized her and started to show Susan his idea. Susan asked him to explain (lines 3 and 5). Peter did so (lines 6, 8, and 10), but still made the error of saying 3-1 instead of 1-3; however, he explained and understood the idea of the angular point method correctly. The task led them to the conclusion that in order to know the routes toward 2-3, one must add the routes to 1-3 and 2-2.

Peter made it clear (line 11) that he was not satisfied with the fact that the angular point method for enumerating routes that was suggested in the book is still recursive: In order to calculate the routes to a certain point, one first has to know the routes to its predecessors. He wanted to find a direct formula for enumerating routes, and he made a guess that it would be 2 to the power of 2. Then they read an explanation of the angular point method in their textbook, and Peter picked up the idea quickly. Susan, however, still enumerated routes.

After some time, Peter realized (line 12) that they were not working on the same task. He asked Susan to show him where she was. Susan showed that she was still enumerating routes (line 13). Peter criticized her method and she asked him for an explanation. Peter explained that you have to add the number of routes toward two endpoints. Susan asked for further explanation. Between lines 17 and 18, there is a long episode (which we have omitted, because it is repetitive) in which Susan asks Peter for an explanation and he explains the angular point method. Susan got the idea and checked her work with Peter. Then they worked it out together. Susan had picked up the angular point method and adopted it very well.

A peer ("A") then asked them to show him their work (line 19) and whether task 11a) could be solved with a grid. This task was one of a series

of problems that had to be solved with a grid. Susan and Peter were working comfortably with the angular point method of enumerating routes in a grid and in the context of the textbook; every task could be solved with a grid. This may have led them to the idea that any enumeration problem could be solved with it. Especially the fact that Peter showed Susan the technique but did not explain it, might have perpetuated the misunderstanding and reinforced Susan's misconception.

Interpretation of the fourth episode

Peter had understood the idea of the angular point method very quickly. However, he was not careful about the details. Susan did not pick up this concept from the textbook exercises and continued to count routes by hand, which took a lot of time. Peter noticed this and explained how to count routes. She understood the routine and applied it. When they were asked by another student whether the routine could be used for all enumeration problems, they said it could, and this possibly strengthened Susan's idea that enumerating routes could be used to solve enumeration problems, as she showed in her post-test. Especially since no explanation was asked for and no criticism was given, she remained at the stage of semi level raising for the concept of solving enumeration problems with the help of a grid.

5. SUMMARY

5.1. Key activities and mathematical level raising

The questions here are: Can we account for the concepts Peter and Susan learned through their dialogue? Did they support or hinder one another? With the results of their post-test in mind, we selected four episodes in which we found indications that they had or had not mastered a certain concept. These episodes were analyzed on the occurrence of activities of the process model. Table I presents a summary of the activities that occurred in the episodes, organized according to the concepts to which they are related. With 'attained level' we indicate the level that the students attained in the post-test.

As can be seen, both students attained the conceptual level for the concept of routes and probabilities in a grid. Both showed their work and both reconstructed it; both asked one another to show their work and they criticized one another. However, Susan did not explain her work, while Peter did. Actually, Susan asked Peter to explain his work several times, while Peter did not ask her. Another difference is that Peter criticized his own work while Susan did not. So, in general, Peter showed more reflection on his own work than Susan did. This may have been an advantage in attaining the conceptual level for "routes and probabilities as a model."

TABLE I
Key activities in the selected episodes

		Routes	Probabilities	Routes as a model
Susan	Process	Shows 3 <i>Asks to show 1</i> <i>Asks to explain 3</i>	Shows 6 Reconstructs 3 <i>Asks to show 1</i> <i>Asks to explain 2</i>	Shows 2
	Attained level	Conceptual	Conceptual	Semi level raising
Peter	Process	Shows 2 Explains 5 <i>Criticizes 2</i>	Shows 10 Explains 4 Justifies 1 Reconstructs 3 <i>Asks to show 3</i> <i>Criticizes 2</i> Self-criticizes 5	Shows 2
	Attained level	Conceptual	Conceptual	Conceptual

As we know from the results of the post-test both students attained mathematical level raising for the subject of routes and probabilities. The one who both explained and generated self-criticism attained level raising with the subsequent concept, too. The one who showed her work and re-constructed it, without giving explanations, attained semi level raising for the subsequent concept. These observations lead to three hypotheses about key activities and mathematical level raising:

- Giving explanations about a concept may facilitate further level raising;
- Criticizing oneself is crucial for further level raising;
- Reconstruction which is not preceded by explanation may lead to semi level raising in the continuation of the learning process.

As mentioned under “Learning Materials,” the investigation task on probabilities was an important one. Table I shows that a lot of activities took place for this concept. Peter did well in this crucial part of the learning materials, and it helped him later in the learning process.

6. DISCUSSION

This experiment concerned the learning of a part of probability theory. The question is whether we can generalize the results for other domains of mathematics. Since the interactions between Susan and Peter deal with the transition from a concrete toward an abstract approach of a grid, we believe that conclusions about their learning process would hold for any other topic in mathematics where level raising takes place.

One of the students clearly profited more than the other. What happened? What was the role of the learning materials and the composition of the pairs? The learning materials, especially the investigation tasks, indeed generated differences between the students. These differences gave rise to discussion, but Susan did not explain her work. Was it because Peter did not ask her to? Possibly, but when he asked her to show her work, she did not always react. She seemed to think that Peter's answer was more worthwhile than her own. Susan scored higher on prior knowledge than Peter. Nevertheless, she seems to have the feeling that he knows better. She gives the impression of a student who works hard. Her drive to proceed with the tasks now affects her adversely, because for investigation tasks it is important to take as much time as you need to find the answer. What does Susan need in order to give explanations? A partner who asks her to explain her work and gives her the feeling that her answer makes sense, and in the background a teacher who makes it clear that they can take their time to find the right answer.

Although the students were not explicitly asked to make sense of one another's contributions, the format of the lessons made it clear that they had to solve the problems together. However, nothing was said about how they should do this. We expected them to learn from performing key activities, without telling them that they had to do this, or rewarding this. We did this to evoke key activities by means of the tasks and the composition of the pairs. Since one of the students performed less key activities than the other one, it might be a point of attention in our next experiment to make it explicitly clear to students that they have to make sense of one another's work. This is one of the "social norms" (Wood, 1999) that may induce meaningful interaction.

In the experimental setting, the role of the teacher was to provide help when the students requested it. This help was mainly focused on the mathematical content of the tasks because of the developmental stage of the learning materials. As shown in Dekker and Elshout-Mohr (2004), it might be useful to have the teacher encourage the performance of key activities. In our next experiment, we will compare the effects of teacher help that is focused only on the mathematical content of the tasks with the effects of a teacher who only ensures that the key activities are performed.

ACKNOWLEDGEMENTS

This research was funded by the NWO (Netherlands Organization for Scientific Research), under grant no. 575-36-003 A. The opinions expressed do not necessarily reflect the views of the NWO. We are grateful to Terry

Wood for sharing our first analysis and for her helpful comments on earlier drafts of this paper.

NOTE

1. Beginning level raising was indicated as 'start level raising' in Pijls et al. (2003).

REFERENCES

- Boer, W., Bouwman, J., Van Dijk, B., Van der Eijk, E., Van de Giessen, C., De Goede, W. et al.: 1998, *Moderne Wiskunde 7e editie havo bovenbouw wiskunde A1 en B1-deel 1* [Modern Mathematics], Wolters-Noordhoff bv, Groningen.
- Dekker, R.: 1994, 'Graphs, small groups and the process of level raising', in A. Antibbi (ed.), *Représentations graphique et symbolique de la maternelle à l'université, Tome 1*, Université Paul Sabatier, Toulouse, pp. 184–189.
- Dekker, R. and Elshout-Mohr, M.: 1998, 'A process model for interaction and mathematical level raising', *Educational Studies in Mathematics* 35(3), 303–314.
- Dekker, R. and Elshout-Mohr, M.: 2004, 'Teacher interventions aimed at mathematical level raising during collaborative learning', *Educational Studies in Mathematics* 56(1), 39–65.
- Dekker, R., Elshout-Mohr, M. and Wood, T.: 2001, 'Working together on assignments: Multiple analysis of learning events', in J.V.D. Linden and P. Renshaw (eds.), *Dialogic Learning: Shifting Perspectives to Learning, Instruction, and Teaching*, Kluwer Academic Publishers, Dordrecht, pp. 145–170.
- Freudenthal, H.: 1973, *Mathematics as an Educational Task*, Reidel, Dordrecht.
- Kieran, C.: 2001, 'The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges', *Educational Studies in Mathematics* 46, 187–228.
- Kieran, C. and Dreyfus, T.: 1998, 'Collaborative versus individual problem solving: Entering another's universe of thought', in A. Olivier and K. Newstead (eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, PME, Stellenbosch, pp. 112–119.
- Pijls, M.H.J.: 2001, *Whoopy Trainer* (Version 1.0) [computer software], Antropi V.O.F., Almere.
- Pijls, M.H.J., Dekker, R. and Van Hout-Wolters, B.H.A.M.: 2003, 'Mathematical level raising through collaborative investigations with the computer', *The International Journal of Computers for Mathematical Learning* 8(2), 191–213.
- PRINT: 1998, Reports of meetings with teachers that participated in the project 'Project Invoering Nieuwe Technologieën'.
- Prent: 1999, Reports of meetings with teachers that participated in the project 'Praktische Opdrachten en Nieuwe Technologieën'.
- Roschelle, J.: 1992, 'Learning by collaborating: Convergent conceptual change', *The Journal of the Learning Sciences* 2(3), 235–276.
- Sfard, A.: 2003, 'Communicational conflict and learning agreement: What turns obstacles to mathematical communication into effective triggers for learning?', *Proceedings of the 10th European Conference for Research on Learning and Instruction*, Cooperativa Libreria Editrice Università di Padova, Padova, pp. 85–86.
- Sfard, A. and Kieran, C.: 2001, 'Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of student' mathematical interactions', *Mind, Culture, and Activity* 8(1), 42–76.

- Teasley, S.D.: 1995, 'The role of talk in children's peer collaborations', *Developmental Psychology* 31(2), 207–220.
- Trogonon, A.: 1993, 'How does the process of interaction work when two interlocutors try to resolve a logical problem?', *Cognition and Instruction* 11(3–4), 325–345.
- Van Hiele, P.M.: 1986, *Structure and Insight*, Academic Press, Orlando.
- Webb, N.M.: 1991, 'Task-related verbal interaction and mathematics learning in small groups', *Journal of Research in Mathematics Education* 22(5), 360–389.
- Wood, T.: 1999, 'Creating a context for argument in mathematics class', *Journal for Research in Mathematics Education* 30(2), 171–191.
- Yackel, E., Rasmussen, C. and King, K.: 2000, 'Social and sociomathematical norms in an advanced undergraduate mathematics course', *Journal of Mathematical Behavior* 19, 275–287.

*Universiteit van Amsterdam
Graduate School of Teaching and Learning
Wibautstraat 2-4, 1091 GM Amsterdam
The Netherlands
E-mail: m.h.j.pijls@uva.nl*