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Research Note

Legendre expansion of the neutrino-electron scattering kernel

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Abstract. The expansion of the neutrino-electron scattering rate in a Legendre series of the scattering angle is extended to include quadratic terms. This extension provides a considerable improvement of the 'fit' to the scattering rate. On the other hand, the effect of the quadratic terms on the neutrino transport during the infall phase of a Type II supernova is found to be negligible. This is partly due to the specific state of the matter background and the shape of the neutrino spectra which suppress the phase space where the quadratic Legendre approximant could be significant. Furthermore, the intrinsic structure of the Boltzmann equation causes suppression of (nearly) coherent scattering in the forward direction where the scattering rate deviates most from a linear approximation.

Key words: dense matter – scattering – supernovae

1. Introduction

Neutrino-electron scattering (NES) plays an important role in the infall phase of a Type II supernova explosion (Bowers & Wilson 1982; Bruenn 1985; Myra et al. 1987; Bruenn 1988; Mezzacappa & Bruenn 1993a). The transport of neutrinos is governed by the Boltzmann-equation in which NES enters through the collision kernel (we adopt the convention $\hbar = c = 1$):

$$B_{\rm NES}[\mathscr{F}_\nu] = \int \frac{d^3p'}{(2\pi)^3}$$

$$\left[R^{in}(p,p')\,\mathscr{F}_\nu'\,(1-\mathscr{F}_\nu) - R^{out}(p,p')\,\mathscr{F}_\nu\,(1-\mathscr{F}_\nu') \right] \quad (1)$$

In this equation $\mathscr{F}_\nu(x,p)$ is the neutrino distribution function, which depends on spacetime coordinates $x=(t,\mathbf{x})$ and the neutrino four-momentum $p=(\omega,\omega\,\mathbf{\Omega})$. The primed distribution function has the following meaning: $\mathscr{F}_\nu' = \mathscr{F}_\nu(x,p')$. The scattering rates $R^{in/out}$ also depend on the matter temperature T and electron degeneracy ξ_e, but this has not been explicitly indicated in Eq. (1).

In most practical situations where neutrino transport is computed in a supernova setting, the Boltzmann equation is not solved directly, but some approximate method is used. In another paper (Smit et al.1995; S95 from hereon) the role of NES was investigated in the multigroup flux-limited diffusion (MGFLD) approach. In MGFLD angular moments of the Boltzmann equation are taken and then solved for the angular moments of the distribution function; for a description of MGFLD theory, see for example Cernohorsky & Van Weert (1992). This approach requires that the scattering rates are expanded in a power series of the cosine of the scattering angle, $\cos\theta = \mathbf{\Omega}\cdot\mathbf{\Omega}'$, in order to write the collision kernel in terms of the angular moments of the distribution function.

For isoenergetic scattering processes in which $\omega' = \omega$, a Legendre expansion up to first order is sufficient in the moment approach because higher order terms cancel identically. For NES the higher order terms do not cancel. In S95 a linear Legendre expansion of NES was used. To see whether higher order Legendre coefficients are important or not in MGFLD, we have computed the second order Legendre coefficient and investigated its contribution in a typical Type II supernova setting.

2. Legendre expansion of NES

The Nth order Legendre approximation of the NES scattering rate R^{out} is written as follows:

$$R^{(N)\,out}(\omega,\omega',\cos\theta) = \frac{1}{2}\sum_{l=0}^{N}(2l+1)\,\Phi_l^{out}(\omega,\omega')\,P_l(\cos\theta) \quad (2)$$

where the $P_l(\cos\theta)$ are Legendre polynomials. A similar approximation can be written for R^{in}. Expressions for the exact scattering rate R^{out} and the linear approximant $R^{(1)\,out}$ were derived by Yueh & Buchler (1977). To go one step beyond this

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we computed the quadratic term Φ_2^{out}; an expression is given in Appendix A. The expression contains an integral (cf. Eq. (A2)) over electron energy which has to be done numerically when the electrons are arbitrarily degenerate. In this work we used a 30-point Gauss-Legendre integration combined with a trailing 10-point Gauss-Laguerre integration.

In this section we compare the angular dependence of R^{out} with the two Legendre approximants $R^{(1)\,out}$ and $R^{(2)\,out}$. In computing these rates, we have set $\hbar = c = 1$ and also set the Fermi-constant $G = 1$. In the figures below the units of the rates are therefore arbitrary. Fig. 1 shows the scattering rate $R^{out}(\omega, \omega', \cos\theta)$ as a function of the scattering angle at a fixed incoming neutrino energy ω for several scattered energies ω'. The electron temperature and degeneracy, $T = 1.6$ MeV and $\xi_e = 17$ respectively, were chosen to represent the matter at infall (before bounce) in a typical supernova collapse. Down-scattering is the dominant process and therefore we depict only $\omega' < \omega$. The incoming energy $\omega = 30$ MeV is taken slightly above the Fermi-energy of the electrons. One sees from Fig. 1 that the functional form of the scattering rate varies considerably: $R^{out}(\cos\theta)$ is an almost linear function of $\cos\theta$ for $\omega' = 5$ MeV, whereas it becomes very nonlinear when ω' approaches ω. Figures 2 and 3 each show a selected curve from Fig. 1 and their Legendre approximants $R^{(1)\,out}$ (dashed curves) and $R^{(2)\,out}$ (dotted curves). In Fig. 2 the quadratic approximation evidently provides a better 'fit' to the scattering rate than the linear approximation. The fit becomes better as the transferred energy $\omega - \omega'$ increases. In Fig. 3 $R^{out}(\cos\theta)$ is more complicated, and in terms of a good fit, neither of the two approximants seems to be the better one. The peak of R^{out} near $\cos\theta = 1$ in Fig. 3 becomes more pronounced as ω' approaches ω and eventually (not shown in our figure) becomes a delta-peak in the limit of coherent scattering, $\omega' = \omega$. The picture sketched above is found for a wide range of the neutrino energies ω and ω', matter temperature and electron degeneracy.

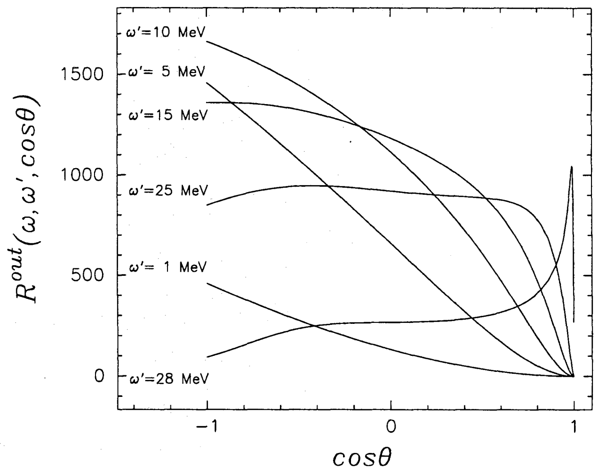

Fig. 1. The scattering rate $R^{out}(\omega, \omega', \cos\theta)$ (in arbitrary units) at fixed neutrino energy $\omega = 30$ MeV and six different ω' energies: 1, 5, 10, 15, 25 and 28 MeV. The neutrino energies ω' are indicated in the figure. The electrons are taken at a temperature $T = 1.6$ MeV and degeneracy $\xi_e = 17$ (dimensionless).

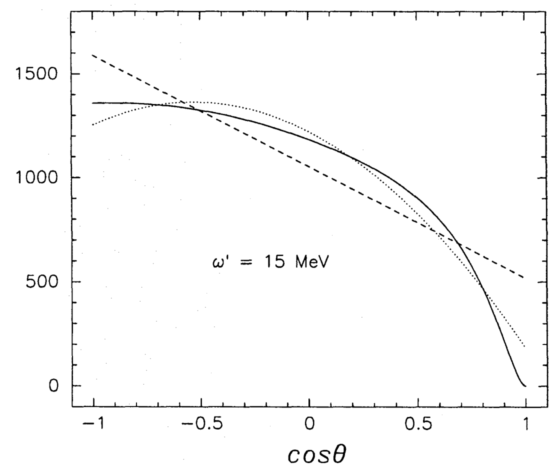

Fig. 2. Selected curve from Fig. 1 with $\omega' = 15$ MeV. Shown are R^{out} (solid line), the linear Legendre approximant (dashed line) and the quadratic Legendre approximant (dotted line).

3. Neutrino transport with quadratic NES

If and when the quadratic approximation leads to significantly improved results will depend on what part of the ω, ω' space is relevant in a neutrino transfer problem. This in turn depends on the distribution function $\mathscr{F}_\nu$ itself because the NES collision kernel is a quadratic functional of $\mathscr{F}_\nu$. Therefore, no definite prediction can be made beforehand and a full transport calculation has to be performed.

Cernohorsky (1994) presented a convenient formalism handling NES. Appendix B lists the extension of his formulae required for the quadratic approximation $R^{(2)\,out}$. We performed numerical transport computations on the same stationary background model that we used in S95. This model is a 1.17 $M_\odot$ iron core during infall, with a central density $\rho_c = 4.1 \times 10^{12}$ g cm^{-3}, temperature $T_c = 2.3$ MeV and electron degeneracy $\xi_{e,c} = 24$. The maximum infall velocity is $v = -1.34 \times 10^4$ km s^{-1} at masspoint $M = 0.91\ M_\odot$ ($\rho = 2.0 \times 10^{10}$ g cm^{-3}). The stationary state solutions of neutrino transport, in which NES was computed without the quadratic Legendre term, were presented in S95. Computing stationary state neutrino transport with the quadratic term included we find no significant change in any relevant quantity (e.g. neutrino fraction, flux, or transfer rates of lepton number, energy or momentum to the stellar matter). The NES coefficients κ^p are all smaller than κ^f by a factor of order 10^6, and in combination with f, p and q (see Appendix B for an explanation of the symbols) their contribution to the moment equations is even less.

4. Discussion

Mezzacappa & Bruenn (1993b) were able to solve the Boltzmann equation, even with NES included (Mezzacappa & Bruenn

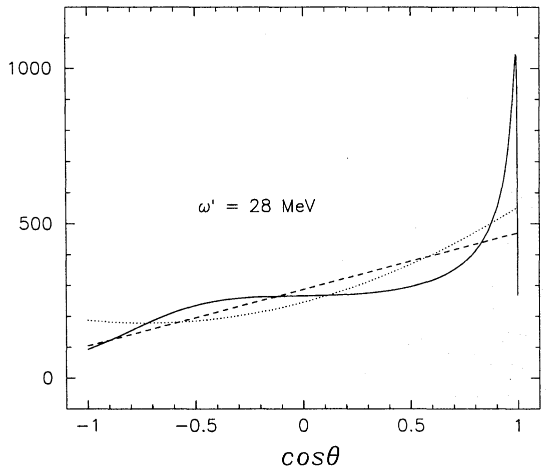

Fig. 3. Selected curve from Fig. 1 with $\omega' = 28$ MeV. The line styles are explained in the caption of Fig. 2.

1993a). In their treatment the NES rates need no expansion into a Legendre series and are retained exactly. For comparison they also computed the results with a MGFLD approximation which involved a linear expansion of NES. They found that differences between the Boltzmann and the MGFLD transport were somewhat larger for several relevant neutrino quantities (such as the neutrino density and luminosity) when NES was included in the transport. They concluded however that those effects did not result from truncating the neutrino-electron scattering kernels at $R^{(1)\,in/out}$ but rather from the intrinsic difference between the two transport methods. Because the distribution function $\mathscr{F}_\nu$ was found to be fairly isotropic at matter densities above $\rho \approx 5 \times 10^{11}$ g cm^{-3}, they expected higher moments of the NES kernel not to contribute much during infall. This was also suspected by Yueh & Buchler (1977), but based on transport calculations using a matter background late in the infall phase, a central density $\rho_c = 10^{14}$ g cm^{-3}.

By deriving an analytic expression for the quadratic Legendre term we were able to show explicitly that in MGFLD neutrino transport it is sufficient to retain only the first two terms of the NES Legendre expansion, at least at temperatures and electron degeneracy that typify the supernova setting during infall. This result must be due to the fact that the Boltzmann equation rules out the parts of the (ω, ω') phase space where the scattering rates $R^{in/out}$ are not well described by the Legendre approximants $R^{(1)\,in/out}$. For coherent scattering ($\omega' = \omega$), the scattering rates $R^{in/out}$ become singular when the scattering angle $\theta \longrightarrow 0$, but the Boltzmann equation *must* suppress this type of "collision" because it does not change the distribution $\mathscr{F}_\nu$. The reduction of the quadratic terms at other parts of the phase space is due to the specific state of the neutrino fluid in this matter setting. The second eddington factor $p(\omega)$ is close to $\frac{1}{3}$ (see eq. B7) in a stellar region that is actually somewhat larger than the diffusive region (where $f \propto \partial e/\partial r$). In the outer atmospheric regions $p \longrightarrow 1.0$, and the quadratic terms are not suppressed, but there NES scattering rate is too low to establish any change in the neutrino fluid.

It remains to be investigated if the quadratic extension is relevant in a different matter setting and/or for other neutrino flavors.

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Appendix A: quadratic expansion of NES rate

In order to retain consistency in notation with our previous work, we use the notation of Cernohorsky (1994; C94 for short). We copy his Eqs. (3.2) and (3.3):

$$\Phi_l^{out}(\omega, \omega') = \alpha_I\, A_l^I(\omega, \omega') + \alpha_{II}\, A_l^{II}(\omega, \omega') \tag{A1}$$

$$A_l^k(\omega, \omega') = \frac{\mathscr{C}}{\omega^2 \omega'^2} \int_{max(0,\omega'-\omega)}^{\infty}$$

$$dE\, F_e(E)\, \left[1 - F_e(E + \omega - \omega')\right] H_l^k(\omega, \omega', E) \quad k = \text{I,II} \tag{A2}$$

In the second equation $F_e(E)$ is the distribution function of electrons with energy E. The constants α_i , i = I, II are combinations of the coupling constants (see C94). The functions H_l^I and H_l^{II} for $l = 0$ and $l = 1$ are as given in Yueh & Buchler (1977; YB77), with the corrections below equation C.50 in Bruenn (1985). The dimensional constant $\mathscr{C}$ is given in C94.

The quadratic corrections $H_2^k(\omega, \omega', E)$ are found by evaluating the integral in Eq. A.3 of YB77 and we obtain:

$$\begin{aligned} &\omega^2 \omega'^2 H_2^I \\ &= a_2^I(\omega, \omega') + b_2^I(\omega, \omega')\, E + c_2^I(\omega, \omega')\, E^2 \quad E \geq \omega' \\ &= \Upsilon_2^I(\omega, \omega', E) + \Theta(\omega' - \omega)\, \Gamma_2^I(\omega, \omega', E) \quad E < \omega' \end{aligned} \tag{A3}$$

$$\begin{aligned} &\omega^2 \omega'^2 H_2^{II} \\ &= a_2^{II}(\omega, \omega') + b_2^{II}(\omega, \omega')\, E + c_2^{II}(\omega, \omega')\, E^2 \quad E \geq \omega' \\ &= \Upsilon_2^{II}(\omega, \omega', E) + \Theta(\omega' - \omega)\, \Gamma_2^{II}(\omega, \omega', E) \quad E < \omega' \end{aligned} \tag{A4}$$

with

$$\begin{aligned} a_2^I(\omega, \omega') \;=\; & (\tfrac{16}{35}\omega^9 - \tfrac{36}{35}\omega^8\omega' + \tfrac{64}{105}\omega^7\omega'^2)\,\Theta(\omega' - \omega) \\ & + (\tfrac{4}{15}\omega^4\omega'^5 - \tfrac{8}{5}\omega^3\omega'^6 + \tfrac{128}{35}\omega^2\omega'^7 \\ & - \tfrac{24}{7}\omega\omega'^8 + \tfrac{8}{7}\omega'^9)\,\Theta(\omega - \omega') \end{aligned} \tag{A5}$$

$$a_2^{II}(\omega, \omega') \;=\; a_2^I(\omega', \omega) \tag{A6}$$

$$\begin{aligned} b_2^I(\omega, \omega') \;=\; & (\tfrac{72}{35}\omega^8 - \tfrac{24}{7}\omega^7\omega' + \tfrac{4}{3}\omega^6\omega'^2)\Theta(\omega' - \omega) \\ & + (-\tfrac{24}{7}\omega'^8 + \tfrac{48}{7}\omega\omega'^7 - 4\omega^2\omega'^6 \\ & + \tfrac{8}{15}\omega^3\omega'^5)\,\Theta(\omega - \omega') \end{aligned} \tag{A7}$$

$$b_2^{II}(\omega,\omega') = -b_2^{I}(\omega',\omega) \tag{A8}$$

$$c_2^{I}(\omega,\omega') = (\tfrac{96}{35}\omega^7 - \tfrac{12}{5}\omega^6\omega' + \tfrac{4}{15}\omega^5\omega'^2)\,\Theta(\omega'-\omega) + (\tfrac{96}{35}\omega'^7 - \tfrac{12}{5}\omega\omega'^6 + \tfrac{4}{15}\omega^2\omega'^5)\,\Theta(\omega-\omega') \tag{A9}$$

$$c_2^{II}(\omega,\omega') = c_2^{I}(\omega',\omega) \tag{A10}$$

and

$$\begin{aligned}\Upsilon_2^{I}(\omega,\omega',E) = {} & \tfrac{8}{7}E^9 + \tfrac{24}{7}E^8(2\omega-\omega') \\ & + \tfrac{48}{35}E^7(12\omega^2 - 13\omega\omega' + 2\omega'^2) \\ & + E^6(\tfrac{96}{5}\omega^3 - 36\omega^2\omega' + 12\omega\omega'^2) \\ & + \tfrac{4}{5}E^5(12\omega^4 - 42\omega^3\omega' + \tfrac{73}{3}\omega^2\omega'^2) \\ & + \tfrac{4}{3}E^4(-9\omega^4\omega' + 10\omega^3\omega'^2) \\ & + \tfrac{8}{3}E^3\omega^4\omega'^2 \end{aligned} \tag{A11}$$

$$\Upsilon_2^{II}(\omega,\omega',E) = \Upsilon_2^{I}(-\omega',-\omega,E) \tag{A12}$$

$$\begin{aligned}\Gamma_2^{I}(\omega,\omega',E) = {} & -\tfrac{4}{105}E^2(\omega'-\omega)(72\omega^6 + 9\omega^5\omega' + 16\omega^4\omega'^2 \\ & + 16\omega^3\omega'^3 + 16\omega^2\omega'^4 + 9\omega\omega'^5 + 72\omega'^6) \\ & + \tfrac{4}{105}E(\omega'-\omega)^2(54\omega^6 + 18\omega^5\omega' + 17\omega^4\omega'^2 \\ & + 16\omega^3\omega'^3 + 15\omega^2\omega'^4 + 90\omega'^6) \\ & - \tfrac{4}{105}(\omega'-\omega)^3(12\omega^6 + 9\omega^5\omega' \\ & + 7\omega^4\omega'^2 + 6\omega^3\omega'^3 + 6\omega^2\omega'^4 + 30\omega'^6) \end{aligned} \tag{A13}$$

$$\Gamma_2^{II}(\omega,\omega',E) = \Gamma_2^{I}(-\omega',-\omega,E) \tag{A14}$$

In the equations above Θ is the unit step function.

Appendix B: quadratic NES in neutrino transport

B.1. Boltzmann equation

We adopt here the formalism of C94 for NES in neutrino transport theory. The quadratic term in the Legendre series merely extends the formulae without changing the basic theory. Therefore we present only the necessary modifications and refer the reader to C94 for symbols that remain unexplained here. When the Legendre expansion eq. (2) is substituted in eq. (1), the scattering kernel can be written as:

$$B_{\text{NES}} = \kappa^0(\omega) - \mathscr{F}_\nu\kappa^e(\omega) + \sum_i \Omega_i\tilde{\kappa}_i(\omega) - \mathscr{F}_\nu \sum_i \Omega_i\kappa_i^f(\omega)$$

$$+\sum_{i,j}\Omega_i\Omega_j\,\tilde{\kappa}_{ij}^p(\omega) - \mathscr{F}_\nu\sum_{i,j}\Omega_i\Omega_j\,\kappa_{ij}^p(\omega) \tag{B1}$$

which is Eq. A.1 in C94 with additional terms involving $\tilde{\kappa}^p$ and κ^p:

$$\tilde{\kappa}_{ij}^p = (5/2)\int d\omega'\omega'^2 e(\omega')\frac{1}{2}\left[3p_{ij}(\omega') - \delta_{ij}\right]\,R_2^{in}(\omega,\omega') \tag{B2}$$

$$\kappa_{ij}^p = (5/2)\int d\omega'\omega'^2 e(\omega')\frac{1}{2}\left[3p_{ij}(\omega') - \delta_{ij}\right]\,\left[R_2^{in}(\omega,\omega') - R_2^{out}(\omega,\omega')\right] \tag{B3}$$

B.2. Moments of the Boltzmann equation

The NES contribution to the first two angular moment equations of the Boltzmann equation becomes (Eqs. A.7 and A.8 of C94):

$$S_{\text{NES}}^e = \int_{4\pi}\frac{d\boldsymbol{\Omega}}{4\pi}B_{\text{NES}} = \kappa^0 - e\,\kappa^e - e\sum_i f_i\,\kappa_i^f - e\sum_{ij}p_{ij}\kappa_{ij}^p \tag{B4}$$

$$S_{\text{NES},i}^f = \int_{4\pi}\frac{d\boldsymbol{\Omega}}{4\pi}\,\Omega_i\,B_{\text{NES}} = -e\,f_i\kappa^e + 1/3\,\tilde{\kappa}_i^f - e\sum_j p_{ij}\,\kappa_j^f - e\sum_{j,k}q_{ijk}\,\kappa_{jk}^p \tag{B5}$$

The third angular moment of $\mathscr{F}_\nu$ is defined as

$$q_{ijk} = \frac{1}{4\pi}\int_{4\pi}d\boldsymbol{\Omega}\,\Omega_i\Omega_j\Omega_k\,\mathscr{F}_\nu \tag{B6}$$

Note that $\tilde{\kappa}^p$ is absent in both equations, due to the fact that Trace($\tilde{\kappa}_{ij}^p$)=0 and that the odd angular integrations of $\tilde{\kappa}^p$ vanish.

In spherical symmetry the moment equations reduce to

$$S_{\text{NES}}^e = \kappa^0 - e\,\kappa^e - ef\,\kappa^f - \frac{1}{2}e(3p-1)\,\kappa^p \tag{B7}$$

$$S_{\text{NES}}^f = \frac{1}{3}\tilde{\kappa} - ef\,\kappa^e - ep\,\kappa^f - \frac{1}{2}e(3q-f)\,\kappa^p \tag{B8}$$

in which

$$f \equiv f_r\,,\qquad p \equiv p_{rr}\,,\qquad q \equiv q_{rrr}\quad\text{and}$$
$$\tilde{\kappa} \equiv \tilde{\kappa}_r\,,\qquad \kappa^f \equiv {\kappa^f}_r\,,\qquad \kappa^p \equiv {\kappa^p}_{rr} \tag{B9}$$

where r is the radial component.

B.3. Closure

The set of moment equations must be closed by prescriptions $p(e,f)$ and $q(e,f)$ (the latter only when NES with the quadratic term is included). We used the Levermore & Pomraning (1981) closure in our transport computations:

$$f = \coth R - 1/R \tag{B10}$$

$$p = \coth R\,(\coth R - 1/R) \tag{B11}$$

from which $p = p(f)$. The third moment q becomes

$$q = \coth^2 R(\coth R - 1/R) - 1/(3R) \tag{B12}$$

or, equivalently (Van Thor et al.1995),

$$q = \frac{p}{3f}(3p-1) + f/3 \tag{B13}$$

Finally we give an expression for the quantity R (see also Eq. A.9 in C94).

$$R = \frac{f}{(p-f^2)} = -\frac{\partial_r\ln e + \kappa^f + \frac{1}{2}\kappa^p\,(3p-1)/f}{\kappa_{tot} + \kappa^0/e - \tilde{\kappa}/(3ef)} \tag{B14}$$

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