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Nonlocal stochastic mixing-length theory and the velocity profile in the turbulent boundary layer

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Abstract

Turbulence mixing by finite size eddies will be treated by means of a novel formulation of nonlocal K -theory, involving sample paths and a stochastic closure hypothesis, which implies a well defined recipe for the calculation of sampling and transition rates. The connection with the general theory of stochastic processes will be established. The relation with other nonlocal turbulence models (e.g. transilience and spectral diffusivity theory) is also discussed. Using an analytical sampling rate model (satisfying exchange) the theory is applied to the boundary layer (using a scaling hypothesis), which maps boundary layer turbulence mixing of scalar densities onto a *nondiffusive* (Kubo–Anderson or kangaroo) type stochastic process. The resulting transport equation for longitudinal momentum $P_x \equiv \rho \bar{U}$ is solved for a unified description of both the inertial and the viscous sublayer including the crossover. With a scaling exponent $\varepsilon \approx 0.58$ (while local turbulence would amount to $\varepsilon \rightarrow \infty$) the velocity profile $\bar{U}_+ = f(y_+)$ is found to be in excellent agreement with the experimental data. *Inter alia* (i) the significance of ε as a turbulence Cantor set dimension, (ii) the value of the integration constant in the logarithmic region (i.e. if $y_+ \rightarrow \infty$), (iii) linear timescaling, and (iv) finite Reynolds number effects will be investigated. The (analytical) predictions of the theory for near-wall behaviour (i.e. if $y_+ \rightarrow 0$) of fluctuating quantities also perfectly agree with recent direct numerical simulations.

1. Introduction

1.1. The velocity profile

Dimensional analysis of simple shear flow along an infinitely extended smooth surface immediately yields that the nondimensional mean velocity $\bar{U}_+ \equiv \bar{U}/u_*$ (in the x -direction) must be a function of $y_+ \equiv u_* y/\nu$, where u_* is a reference (friction) velocity, y is the distance from the surface, and ν is the kinematic viscosity. The friction

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velocity is defined by means of the total stress $\tau \equiv \rho u_*^2$ in the boundary layer (ρ being the fluid mass density). In fully developed turbulent flows all fluctuating velocities scale with u_* (for recent reflections on this inner scaling, see e.g. [1, 2]).

In the viscous sublayer (adjacent to the surface) one has $v d\bar{U}/dy = \tau/\rho$ (i.e. $d\bar{U}_+/dy_+ = 1$) so that

$$\bar{U} = u_*^2 y / \nu \quad (1)$$

if $y \rightarrow 0$. On the other hand, in the outer layer of the flow one expects the existence of an inertial subrange *à la* Kolmogorov for which stress due to molecular viscosity should be unimportant. This will obviously be possible only if $\kappa d\bar{U}_+/dy_+ = 1/y_+$, which implicitly defines the Von Kármán constant κ (and which amounts to so-called complete similarity in terms of the local Reynolds number $Re_+ = y_+$). In the inertial sublayer one thus finds that $\bar{U} = (u_*/\kappa) \ln(y/y_0)$ if $y \rightarrow \infty$, where y_0 arises as an integration constant. For rough surfaces y_0 is known as the geometrical roughness length, but for a smooth surface (i.e. if $u_* y_0/\nu \ll 1$) it is a measure of the thickness of the viscous sublayer. Letting $y_0 \equiv (\nu/u_*)e^{-\gamma}$ one obtains

$$\bar{U} = \frac{u_*}{\kappa} \left[\ln \left(\frac{u_* y}{\nu} \right) + \gamma \right]. \quad (2)$$

The logarithmic velocity profile is one of the most famous results in the study of turbulent flows. It was first given by Von Kármán [3] and, independently, by Prandtl [4]. The above derivation is due to Millikan [5] (and to Landau [6]). It clearly shows the generality (or model independence) of the result. However, the asymptotic nature of the logarithmic profile (valid only for $y \rightarrow \infty$) prohibits a smooth connection to the linear profile in the viscous sublayer (for $y \rightarrow 0$). In the connection (or crossover) region the purely dimensional argument breaks down and one must resort to more specific modeling.

Physical modeling of the turbulent boundary layer should *ipso facto* imply a calculation of the integration constant γ , thereby explaining its origins. So far, however, its numerical value has been determined mainly by fitting experimentally measured profiles to (2); see e.g. [7, 8]. This yields $\gamma/\kappa \approx 4-6$, along with $\kappa \approx 0.39 \pm 0.02$. Both Landau and Lifshitz [6] and Monin and Yaglom [9] quote $\kappa \approx 0.4$ and $\gamma/\kappa \approx 5.0$ as an established result. So, any nontrivial boundary layer model should – by connecting (in the crossover region) the viscous and inertial sublayers in a smooth and unified manner – predict a value of $\gamma \approx 2$.

The proper description of the crossover region is crucial for the calculation of the integration constant γ in the asymptotic profile (2). Such modeling should provide a realistic description of scalar transport, not only of longitudinal momentum $P_x = \rho \bar{U}$ but also e.g. of particles, temperature, humidity. The crossover region is of considerable practical importance, especially if the surface is a source (or sink) of tracer material, e.g. aerosol particles [10]. Whether such particles are entrained in the flow or are re-adsorbed by the surface is effectively determined by the physics in this

extended viscous sublayer. For atmospheric flow ($\nu = 0.15 \text{ cm}^2/\text{s}$) with $u_* \approx 0.1 \text{ m/s}$ the relevant distances are of the order of a centimeter.

We consider “fully developed turbulent flow” along an infinitely extended smooth surface. Let the mean flow with velocity $\bar{U}(y)$ be in the x -direction, with y being the coordinate normal to the surface. Under zero pressure gradient conditions the standard Reynolds–Navier–Stokes equation for $\bar{U}(y)$ then reads (e.g. [9])

$$\frac{\partial \bar{U}}{\partial t} = \frac{\partial}{\partial y} (\tau_R / \rho) + \nu \frac{\partial^2 \bar{U}}{\partial y^2}, \quad (3)$$

where τ_R is the Reynolds stress and $\partial \bar{U} / \partial t$ has been kept in order to emphasize the nature of (3) as a transport equation (which is useful for modeling within the framework of stochastic processes). With $U = \bar{U} + u$ and $V = \bar{V} + v$, where u and v are the fluctuating velocity components in the x and y -directions respectively (while $\bar{V} = 0$), one has

$$\tau_R / \rho = -\overline{uv}, \quad (4)$$

where the overbar denotes (time) averaging as in $\bar{U}$ and $\bar{V}$, and which reveals the closure problem.

A widely used closure method models the Reynolds stress according to the ideas of molecular viscosity, i.e. as $\tau_R / \rho \equiv \nu_R \partial \bar{U} / \partial y$ (e.g. [11]). In fact, a great variety of local gradient models has been proposed (e.g. [12]). E.g., a widely used model is known as local K -theory (where $\nu_R = \ell u_*$, with a Pandtl-type mixing length $\ell = \kappa y$). All models yield the logarithmic velocity profile (2) as a consequence of (3) in the steady state ($\partial \bar{U} / \partial t = 0$) outside the viscous sublayer, i.e. if τ_R is constant. Deep inside the viscous sublayer, Eq. (3) always implies constant viscous stress which yields the linear profile (1).

Both (1) and (2) are shown as solid lines in Fig. 1, along with a reference set of experimental data (see Appendix E). Despite the fact that data for very small y -values are difficult to obtain (in particular at sufficiently high Reynolds numbers) and that the logarithmic profile *per se* only applies in a somewhat limited y -range (due to the finite boundary layer thickness δ with typically $u_* \delta / \nu \lesssim 10^5$), the separate validity of (1) and (2) is obvious and allows for an empirical determination of κ and γ with an accuracy of the order of ten percent. The straight line in Fig. 1 – used as a reference throughout the present work – amounts to $\kappa = 0.39$ and $\gamma = 1.8$.

1.2. Nonlocal transport theory

Local gradient closure models only account for turbulence mixing by infinitesimally small eddies. In contrast with molecular diffusion, however, such a local description (on the hydrodynamical scale) of turbulence transport is well-known to be inadequate (see e.g. [13]). For example, such models fail for the description of near-source plume

Fig. 1. Reference data set for the mean velocity profile $\bar{U}_+(y_+)$ (at $Re = \infty$; see Appendix E). The solid lines represent the linear profile (according to Eq. (1)) and the logarithmic profile (according to Eq. (2) with $\kappa = 0.39$, and $\gamma/\kappa = 4.6$).

dispersion. In the present article it will be shown that nonlocal effects (i.e. finite size eddies) are crucial for an appropriate modeling of the turbulent boundary layer.

Nonlocal effects were investigated for the first time by Richardson [14], who studied a two-particle diffusion problem. Further developments were due to e.g. Roberts [15], Schönfeld [16], Spiegel [17], and Fiedler [18]. Schönfeld's concept of integral diffusivity was later taken up by Berkowicz and Prahm [19] in their theory of spectral turbulent diffusivity (see also Berkowicz [20]). A discretized version of Fiedler's model was, independently, propounded by Stull [11] as "transient turbulence" theory. Most models are concerned with isotropic homogeneous turbulence rather than boundary layer turbulence. In particular, none of them has been applied to the problem of a unified description of the mean velocity profile. Finally, improvements on local K -theory are of up-to-date interest for subgrid closure modeling in large eddy simulations (e.g. [21]), in particular for inhomogeneous turbulence.

The present formulation – in Section 2 – of a nonlocal closure model connects the description of turbulence transport (e.g. of tracer particles) directly to experimentally accessible data on fluctuating velocity fields. The theory will be developed by means of a novel analysis of turbulence sample paths and a stochastic closure hypothesis. The ensuing model will be seen to be intimately related to the general theory of continuous (but *not necessarily diffusive*) Markov processes (e.g. [22, 23]). The analysis gives rise

to an integro-differential transport equation of the Chapman–Kolmogorov (or master) type, which describes the migration of an average density (e.g. $P_x = \rho \bar{U}$). The transition rates in this master equation can be computed on the basis of (experimental) sampling rates. If they satisfy certain symmetry conditions, turbulence is mixing by “exchange”. The model involves both an eddy-viscosity and a mixing-length.

In Section 3 the theory of Section 2 will be generalized to the boundary layer situation. This involves a scaling hypothesis for the sampling (and, hence, for the transition) rates. The scaling also defines a statistical mapping of trajectories in a fictitious space (attached to each point in real space) onto physical space *per se* (i.e. of Eulerian onto Lagrangian sample paths). Explicit results are obtained for the case of exponential Eulerian sampling rates, which transform into algebraic rates for Lagrangian trajectories. The associated transition rates generate a nondiffusive stochastic process of the *kangaroo* (or strong collision) type, which underscores the inadequacy of gradient (i.e. diffusive, or Fokker–Planck) type turbulence modeling. A suggestion will also be made as to the significance of the model’s scaling exponent ε as a fractal dimension of a turbulence Cantor set.

In Section 4 boundary layer scaling is extended to the viscous sublayer on the basis of a selfconsistent analysis of the fully three-dimensional fluctuating velocity field (see Appendix C). The theory involves only two intrinsic parameters, *viz.* the novel exponent ε (local transport would amount to $\varepsilon \rightarrow \infty$) and the viscous sublayer length a_+ . Fortunately, the latter is fairly accurately known from data concerning the normal velocity fluctuations near the wall, both from experiments (e.g. [24]) and direct numerical simulations (e.g. [25]). In addition, it will be related to a (experimentally) well-studied longitudinal correlation function. The perfect consistency between the data and the (analytical) predictions of the present theory for near-wall fluctuations will be noteworthy.

1.3. Results

While our model applies to the mixing of any scalar tracer (with density $P(y, t)$), the solution of the transport problem in the case of $P_x = \rho \bar{U}$ allows for an easy comparison of the theory with a large set of existing data on the mean velocity profile (in particular for pipe and channel flow) thereby yielding a value for ε . Therefore, in Section 5 the integro-differential master equation for the mean velocity (or rather: its gradient) is solved (partly analytically) in the steady state. Integrating the gradient from the surface (at $y = 0$) through the crossover region, the asymptotic logarithmic behaviour (as $y \rightarrow \infty$) will be found including the constant $\gamma(\varepsilon)$. This result describes the velocity profile in the boundary layer in a unified manner, i.e. in both the inertial and the viscous sublayer. With $\varepsilon \approx 0.58$ (for $a_+ \approx 16$ and $\kappa \approx 0.39$, corresponding to $\gamma \approx 1.8$) the resulting mean velocity profile is in excellent agreement with the experimental data. Some final remarks are made in Section 6.

2. Nonlocal stochastic model

2.1. Sample paths

Consider the Reynolds stress (4). Using the definition of time-averaging it may be written as

$$\tau_R/\rho = -\frac{1}{T} \int_0^T u(y, t + \tau) v(y, t + \tau) d\tau, \quad (5)$$

where T is understood to be sufficiently large. On the other hand, the averaging period should be less than typical mean response times. For instance, for Brownian motion it should be very large compared to the velocity autocorrelation time (i.e. on the Rayleigh scale). However, once this condition has been satisfied, T should be as small as possible (i.e. on the Smoluchovski scale). For turbulence there is usually no problem in judiciously choosing T [9, 11, 26]. Let us cast (5) in the form of a spatial rather than a time integral. A stochastic trajectory (or sample path) $\eta(y, t)$ can be defined on the basis of the fluctuation velocity $v(y, t)$ through $v(t + \tau) = d\eta(\tau)/d\tau$. See Fig. 2 for an illustrative example of both $v(t)$ and the trajectory

$$\eta(y, \tau) = \int_0^\tau v(y, t + s) ds, \quad (6)$$

where we have let $\eta(y, 0) \equiv 0$. The resulting Reynolds stress reads

$$\tau_R/\rho = -\frac{1}{T} \int_{R(y, T)} u[y, t + \tau(\eta)] d\eta, \quad (7)$$

where $R(y, T)$ indicates the range of values taken by $\eta(y, \tau)$ during $\tau \in (0, T)$ and where $u[y, t + \tau(\eta)]$ has the significance of the x -component of the fluid velocity fluctuation along the trajectory. Note that $\tau(\eta)$ is a multivalued function. Therefore, we rewrite the formal expression (7) as

$$\tau_R/\rho = -\frac{1}{T} \int_{-\infty}^{\infty} \sum_{n=1}^{N(\eta, y, T)} u[y, t + \tau_n(\eta)] d\eta, \quad (8)$$

where the range $R(y, T)$ is now accounted for by the number $N(\eta, y, T)$ of crossings (or visits) of the sample path $\eta(y, \tau)$ with a fixed value of the coordinate η during $\tau \in (0, T)$.

It will be useful to consider (8) as an integral over a monotonously increasing coordinate. Therefore, one should distinguish between η -crossings with positive or negative velocity $v_n \equiv v[t + \tau_n(\eta)]$. Let $N_+(\eta, y, T)$ be the number of times the sample path visits the value η with $v_n > 0$. Similarly, let $N_-(\eta, y, T)$ be the number of times the path visits the value η with $v_n < 0$. Notice that N_+ and N_- differ at most by one. The

Fig. 2. Sketch of a typical trace of the fluctuating velocity $v(t)$ at a fixed point in space (upper figure) and the ensuing sample path $\eta(t)$ with its indicated “visits” at a particular value of the coordinate η (lower figure).

Reynolds stress now becomes

$$\tau_R/\rho = -\frac{1}{T} \int_{-\infty}^{\infty} \left(\sum_{n=1}^{N_+(\eta, y, T)} u[y, t + \tau_n(\eta|+)] - \sum_{n=1}^{N_-(\eta, y, T)} u[y, t + \tau_n(\eta|-)] \right) d\eta, \quad (9)$$

where the conditional functions $\tau_n(\eta|\pm)$ have been defined as the n th $\tau(\eta)$ for which $v > 0$ (upper sign) or $v < 0$ (lower sign). Note that in the notation of (9), for instance,

the mean velocity $\bar{v}(y)$ reads

$$\bar{v}(y) = \frac{1}{T} \int_{-\infty}^{\infty} (N_+(y, \eta, T) - N_-(y, \eta, T)) d\eta, \quad (10)$$

which is easily shown to tend to zero for all trajectories $\eta(y, T)$ which expand slower than T^α with $\alpha < 1$. For Brownian motion e.g. one has $\alpha = 1/2$.

Let us now define the mean visiting rates

$$\lambda_{\pm}(y, \eta) = \frac{1}{T} N_{\pm}(y, \eta, T). \quad (11)$$

Eq. (11) is easily rewritten as a time average and supplemented with an ensemble averaging. In fact, this combination is common practice experimentally (see further Appendix A). Note that for a periodic velocity field with amplitude v_0 and frequency ω the deterministic rate $\lambda_{\pm}(\eta) = \omega/2\pi$ is nonzero only if $|\eta| < v_0/\omega$. Hence, if v_0 were e.g. exponentially distributed this rate would be exponential in η . On the other hand, in an ensemble where v_0 is Gaussian distributed it would be a complementary errorfunction. In fact, for Brownian motion one finds $\lambda_+(\eta > 0) = (2\nu/\ell^2) \text{erfc}(\eta/\ell)$ with $\ell \equiv 2(\nu T)^{1/2} \rightarrow 0$ on the Smoluchovski scale. In that case $\lambda_+(\eta < 0) = 0$, while $\lambda_-(\eta) = \lambda_+(-\eta)$ are related by mirror symmetry.

Let us also introduce the mean conditional velocities

$$\bar{u}(y, \eta, t | \pm) = \frac{1}{N_{\pm}(y, \eta, T)} \sum_{n=1}^{N_{\pm}(y, \eta, T)} u[y, t + \tau_n(\eta | \pm)]. \quad (12)$$

With (11) and (12), Eq. (9) becomes

$$\tau_R/\rho = - \int_{-\infty}^{\infty} (\lambda_+(y, \eta) \bar{u}(y, \eta, t | +) - \lambda_-(y, \eta) \bar{u}(y, \eta, t | -)) d\eta. \quad (13)$$

It is worth noticing that (13) is merely a reorganized version of the original formula (5). As yet the formulae are still exactly equivalent.

2.2. Closure hypothesis

Eq. (13) is of a form which allows for the implementation of a closure hypothesis. That is, from here on a turbulence model will be used. On the other hand, the analysis will be such that it also applies to molecular diffusion. Therefore, let the coordinate η defined by the sample path (6) be mapped onto the real space coordinate y' . For the purpose of the present section, let us consider a homogeneous medium so that this mapping can be done without further ado. Boundary layer scaling will be considered in Section 3.

Because $u = U - \bar{U}$ in (5), one has $\bar{u}(y, \eta, t | \pm) = \bar{U}(y, \eta, t | \pm) - \bar{U}(y, t)$ in (13). Note that (3) with (13) for τ_R/ρ describes the transport of longitudinal momentum (as

a tracer density) in the y -direction. Now let the transport over a distance $\Delta y = |\eta|$ be effectively instantaneous (i.e. within the averaging-time window) and consider how it contributes to τ_R/ρ at the observation point y . There are four different $(\eta, \pm)$ -combinations. *First*, a fluid element with $v(y) > 0$ either (i) arrives at y from below (if $\eta > 0$), or (ii) leaves y (if $\eta < 0$). Therefore, let $\bar{U}(y, \eta > 0, t|+) = \bar{U}(y - \eta, t)$ and $\bar{U}(y, \eta < 0, t|+) = \bar{U}(y, t)$. *Second*, if $v(y) < 0$ the fluid element either (iii) leaves y (if $\eta > 0$), or (iv) arrives at y from above (if $\eta < 0$). Hence, $\bar{U}(y, \eta > 0, t|-) = \bar{U}(y, t)$ and $\bar{U}(y, \eta < 0, t|-) = \bar{U}(y - \eta, t)$. Notice that the model is defined compactly by $\bar{U}(y, \eta, t|\pm) = \bar{U}[y - \eta\theta(\pm\eta), t]$, $\theta(\eta)$ being the unit step function. That is, setting

$$\begin{aligned}\bar{U}(y, \eta, t|+) &= \bar{U}(y - \eta, t) & \text{if } \eta > 0, \\ \bar{U}(y, \eta, t|-) &= \bar{U}(y - \eta, t) & \text{if } \eta < 0,\end{aligned}\quad (14)$$

and $\bar{U}(y, \eta, t|\pm) = \bar{U}(y, t)$ in all other cases, Eq. (13) may be written as

$$\tau_R/\rho = \int_{-\infty}^{\infty} A(y, \eta)(\bar{U}(y + \eta, t) - \bar{U}(y, t))d\eta, \quad (15)$$

where we have defined $A(y, \eta < 0) \equiv -\lambda_+(y, -\eta)$ and $A(y, \eta > 0) \equiv \lambda_-(y, -\eta)$. This result is in agreement with a recent Lagrangian analysis of the Reynolds stress by Bernard and Handler ([13], esp. Section 9).

If the visiting (mixing) rates obey mirror symmetry (so that $A = \pm\lambda_+(|\eta|)$ is odd in η), (15) reduces to

$$\tau_R/\rho = \int_0^{\infty} \lambda(y, \eta)(\bar{U}(y + \eta, t) - \bar{U}(y - \eta, t))d\eta, \quad (16)$$

where $\lambda(y, \eta) \equiv |A(y, \eta)|$. Differentiating with respect to y , (16) can be shown to coincide with the result from transilient turbulence theory [11] (in its continuous form [18]). If the sampling rates $\lambda(y, \eta)$ only depend on η , (16) further agrees with spectral diffusivity theory [19, 20]. In the latter case the rates have been modeled as $\lambda(\eta) = \int_{\eta}^{\infty} p(\ell)\omega(\ell)d\ell$, where $p(\ell)$ is the probability density of occurrence of an eddy of length ℓ and with a typical transport frequency $\omega(\ell)$. Indeed, mixing over a distance $\Delta y = \eta$ can only be due to eddies of size $\ell \geq \eta$.

2.3. Transport equation and transition rates

Reinsert the Reynolds stress τ_R/ρ into the (momentum) transport equation (3). Using (15) – for the time being neglecting molecular diffusion – gives

$$\frac{\partial \bar{U}}{\partial t} = \frac{\partial}{\partial y} \int_{-\infty}^{\infty} A(y, \eta)(\bar{U}(y + \eta, t) - \bar{U}(y, t))d\eta. \quad (17)$$

Differentiating the first term in the integrand with respect to y and partially integrating it, leads to

$$\frac{\partial \bar{U}}{\partial t} + \frac{\partial}{\partial y} (A(y) \bar{U}(y, t)) = \int_{-\infty}^{\infty} \mathbf{W}(y, \eta) \bar{U}(y + \eta, t) d\eta, \quad (18)$$

where $A(y) = \int_{-\infty}^{\infty} A(y, \eta) d\eta$ and

$$\mathbf{W}(y, \eta) = \frac{\partial A(y, \eta)}{\partial y} - \frac{\partial A(y, \eta)}{\partial \eta}. \quad (19)$$

Since $A(y, \eta)$ is discontinuous at $\eta = 0$ with $A(y, +0) - A(y, -0) = 2\lambda(y, 0)$, the transport kernel $\mathbf{W}(y, \eta)$ defines a transition rate $W(y, \eta)$ according to (e.g. [23, 27])

$$\mathbf{W}(y, \eta) = W(y, \eta) - 2\lambda(y, 0)\delta(\eta), \quad (20)$$

so that (with $\lambda \equiv |A(y, \eta)|$)

$$W(y, \eta) = \pm \left(\frac{\partial \lambda(y, \eta)}{\partial y} - \frac{\partial \lambda(y, \eta)}{\partial \eta} \right), \quad (21)$$

where the upper (lower) sign applies if $\eta > 0$ (< 0). Note from the definition of A in Eq. (15) that $\lambda(\eta < 0) \equiv \lambda_+(-\eta)$ and $\lambda(\eta > 0) \equiv \lambda_-(-\eta)$. The inverse of (21) is easily found by considering it as an ordinary first order differential equation for $\lambda(y, \eta)$. Integrating it, yields

$$\begin{aligned} \lambda(y, \eta \geq 0) &= \int_{\eta}^{\infty} W(y + \eta - \eta', \eta') d\eta', \\ \lambda(y, \eta \leq 0) &= \int_{-\infty}^{\eta} W(y + \eta - \eta', \eta') d\eta'. \end{aligned} \quad (22)$$

Taking $\eta = 0$ and adding the two expressions for $\lambda(y, 0)$, one obtains

$$2\lambda(y, 0) = \int_{-\infty}^{\infty} W(y - \eta, \eta) d\eta. \quad (23)$$

Using (23) and (20), Eq. (18) may be written as

$$\frac{\partial \bar{U}}{\partial t} + \frac{\partial}{\partial y} (A(y) \bar{U}(y, t)) = \int_{-\infty}^{\infty} (W(y, \eta) \bar{U}(y + \eta, t) - W(y - \eta, \eta) \bar{U}(y, t)) d\eta. \quad (24)$$

The right hand side of (24) describes a stochastic process with transition rates $W(y, \eta)$, i.e. rates of jumping (mixing) over $\Delta y = |\eta|$ (up or down) towards the point y . Putting $\eta = y' - y$ one may define $W(y|y') \equiv W(y, y' - y)$. Similarly, $W(y - \eta, \eta)$ is the rate of mixing over $\Delta y = |\eta|$ away from y , so that $W(y'|y) \equiv W(y', y - y')$. The

corresponding transport kernel $\mathbf{W}(y|y')$ satisfies $\int_{-\infty}^{\infty} \mathbf{W}(y|y') dy = 0$, as it should. Of course, $\mathbf{W}(y|y')$ has significance as a statistical matrix only if its offdiagonal elements are nonnegative, i.e. if $W(y|y') \geq 0$.

2.4. Exchange

Let us consider the subclass of turbulence processes (24) for which the transition rates $W(y, \eta)$ have the property of “strong exchange”. Such processes represent transport in flow systems with a “strong eddies” structure as they are defined by $W(y, \eta) = W(y + \eta, -\eta)$, which is equivalent to

$$W(y|y') = W(y'|y). \quad (25)$$

Exchange expresses a strong correlation between the mixing from y to y' and back to y (for any pair of points). Note that (25) is quite different from detailed balancing in thermal equilibrium (e.g. [23]).

By (21) exchange is related to a symmetry property for (the spatial rate of change of) the visiting rates. Note that if on the left hand side in (25) one has $W(y|y' > y) \equiv W(y, \eta > 0)$, one must have $W(y'|y < y') \equiv W(y + \eta, -\eta)$ on the right hand side (and vice versa). With $\lambda(y|y') \equiv \lambda(y, \eta)$, Eq. (25) implies

$$\frac{\partial}{\partial y} \lambda(y|y') = -\frac{\partial}{\partial y'} \lambda(y'|y), \quad (26)$$

where $\partial/\partial y$ and $\partial/\partial y'$ now denote partial differentiation for constant y' and y , respectively. E.g. assuming $y' > y$, increasing y – as in $\lambda(y|y')$ – decreases $\eta = y' - y$ and, hence, typically increases the rate. On the other hand, in that case increasing y' – as in $\lambda(y'|y)$ – one instead increases η , leading to a corresponding lowering of the rate. The condition (26) shows how these changes for upward and downward visiting rates are related under exchange.

If $\lambda(y, \eta)$ only depends on η , (26) simplifies to $\partial\lambda(\eta > 0)/\partial\eta = -\partial\lambda(\eta < 0)/\partial\eta$ so that (assuming $\lambda(\eta)$ to be continuous at $\eta = 0$) the visiting rate should then be even in η . This property is implicit in the derivation of the spectral diffusivity equation for homogeneous turbulence in Ref. [19]. In that case the explicit flow term in the transport equation vanishes because $A(y) = 0$. This feature is more generally related to exchange. Namely, using (26) one easily shows that $dA/dy = 0$ if $\lambda(\pm\infty|y) = 0$. As in that case also $A(\pm\infty) = 0$, one has $A(y) = 0$.

3. The boundary layer

3.1. Scaling hypothesis

In the preceding it has been assumed that the statistical nature of the turbulence allowed for a trivial mapping of η onto y' . While this may be correct for statistically homogeneous turbulence, it is certainly not valid in the boundary layer.

Let $\eta(y, t)$ be the sample path attached to the point y , and let $v(y, t)$ be the fluctuating velocity at y (like in Section 2), so that

$$\frac{d\eta}{d\tau} = v(y, t + \tau). \quad (27)$$

Further, let $\eta_*(y, t)$ denote the actual trajectory of a (co-moving) fluid particle with fluctuation velocity $v(y_*, t)$, where $y_* = y + \eta_*$. That is,

$$\frac{d\eta_*}{d\tau} = v(y + \eta_*, t + \tau). \quad (28)$$

The corresponding trajectory is given by

$$\eta_*(y, \tau) = \int_0^\tau v(y + \eta_*(s), t + s) ds. \quad (29)$$

Boundary layer scaling then amounts to the hypothesis that there exists an invariant characteristic time scale τ_0 (defined at some y_0), such that $d\tau/d\tau_0 = y/y_0$ scales with the distance y from the surface. In other words, for each value of y there exists a characteristic frequency scale Ω for the fluctuations, with $\Omega \propto 1/y$. For the time being it will further be assumed that the normal velocity itself is nonscaling (i.e., does not depend on y). This latter condition (which holds only in the inertial sublayer) will be relaxed in the discussion of viscous sublayer turbulence (i.e. if $y \rightarrow 0$) in Section 4. The hypothesis asserts that apart from (global) time scaling, inertial sublayer turbulence is (locally) homogeneous. Therefore, both η and η_* (i.e. the size of characteristic eddies) scale with y , which reflects the statistical self-similarity of the flow (see e.g. [3, 26, 28]). Hence, one may write

$$d\eta = y d\varphi_0, \quad (30)$$

and

$$d\eta_* = y_* d\varphi_0, \quad (31)$$

where $\varphi_0(\tau_0)$ is a nondimensional invariant function. The Jacobian $J(\eta, \eta_*) = |d\eta/d\eta_*|$ of the mapping $\eta(\eta_*)$ reads

$$J(\eta, \eta_*) = y/y_*. \quad (32)$$

The inverse $\eta_*(\eta)$ is explicitly given by

$$\eta_* = y(e^{\eta/y} - 1). \quad (33)$$

Eq. (33) maps the fictitious path space $\eta \in (-\infty, \infty)$ onto the real path space $\eta_* \in (-y, \infty)$, as it should be for the boundary layer.

3.2. Sampling rates model

The differential form of (33) implies

$$d\eta = \left(\frac{1}{1 + \eta_*/y} \right) d\eta_*. \quad (34)$$

By means of (34) one may rewrite the Reynolds stress (13) as

$$\tau_R/\rho = - \int_{-y}^{\infty} (\lambda_*^+(y, \eta_*) \bar{u}(y, \eta_*, t|+) - \lambda_*^-(y, \eta_*) \bar{u}(y, \eta_*, t|-)) d\eta_*, \quad (35)$$

where

$$\lambda_*^\pm(y, \eta_*) = \left(\frac{1}{1 + \eta_*/y} \right) \lambda_\pm(y, \eta) \quad (36)$$

represent the visiting rates at η_* . The closure model (14) now implies the linear mapping of η_* onto the y -axis. That is, the boundary layer transport equation becomes

$$\frac{\partial \bar{U}}{\partial t} = \int_{-y}^{\infty} (W(y, \eta_*) \bar{U}(y + \eta_*, t) - W(y - \eta_*, \eta_*) \bar{U}(y, t)) d\eta_*, \quad (37)$$

where we have already set $A(y) = 0$ because the visiting rates model presented below satisfies exchange (Section 2.4).

By (11) the visiting rates are defined as the rates at which the path $\eta(y, t)$ samples the value η . This path is defined by (6), i.e. it is constructed on the basis of the fluctuating velocity field data at a single point y . Hence, recalling the definition of $\lambda(y, \eta)$ in terms of the $\lambda_\pm$ [from Eq. (21)], one expects $\lambda = \lambda(y, |\eta|)$ by symmetry. Moreover, due to scaling (Eq. (30)) one should have $\lambda = \lambda(y, |\eta|/y)$. We thus propose to model the rates by

$$\lambda(y, \eta) = \frac{D}{y} e^{-\varepsilon|\eta|/y}, \quad (38)$$

as is suggested by the form of the mapping (33). The parameters $D > 0$ and $\varepsilon > 0$ will be determined further on. Substituting (38) on the right hand side of (36) and using (33) to express η in terms of η_* , one obtains

$$\lambda_*(y, \eta_*) = \frac{D}{y} \left(\frac{1}{1 + \eta_*/y} \right)^{1 \pm \varepsilon}, \quad (39)$$

where the upper (lower) sign applies if $\eta_* > 0$ (< 0). Of course, now always $y \geq 0$ and $\lambda_*(y, \eta_*) \equiv 0$ if $\eta_* < -y$. With $\eta_* = y' - y$, (39) reads

$$\lambda_*(y|y') = D y^{\pm \varepsilon} y'^{-(1 \pm \varepsilon)}. \quad (40)$$

By (26) one easily verifies that (40) satisfies exchange. The transition rates $W(y, \eta_*)$ belonging to (39) follow from (21) with η being replaced by η_* . This yields

$$W(y, \eta_*) = \frac{\varepsilon D}{y^2} \left(\frac{1}{1 + \eta_*/y} \right)^{1 \pm \varepsilon}. \quad (41)$$

Since Eq. (41) shows that $W(y, \eta_*) = (\varepsilon/y)\lambda_*(y, \eta_*)$ is always non-negative, it indeed generates a stochastic process. Using (40) one has

$$W(y|y') = \varepsilon D y^{-(1 \mp \varepsilon)} y'^{-(1 \pm \varepsilon)}. \quad (42)$$

It is easily checked that $W(y|y') = W(y'|y)$, i.e. that these rates satisfy exchange in the form of (25).

3.3. Comments

A stochastic process with factorizing transition rate matrix is known as a “kangaroo process” [23, 29]. In general for such processes one has $W(y|y') = \phi(y)q(y')$. A kangaroo process (in particular, with $\phi(y) = 1$ or $q(y) = 1$) is also known as a “Kubo–Anderson process”. It has *inter alia* been used to describe motional narrowing in spin systems [27, 30, 31]. An application to Mössbauer spectroscopy is given in Refs. [32–34]. In atomic collision theory [35–38] and in laser linewidth calculations [29, 39, 40] the model has been applied in the “strong collision” limit. This limit is the opposite of diffusive motion as described by local gradient models. Its application to (boundary layer) turbulence suggests that “strong eddies” play a key role in turbulence mixing (see also Section 2.4).

The kangaroo process (42) has fractal features, due to scaling. Let

$$Q(y') = \int_0^\infty W(y|y') dy \quad (43)$$

be the intensity function and let $\mathbb{T}_\tau(y|y')$ denote the transition (or transport) probability over a short period of time $\tau \rightarrow 0$, at least short with respect to typical mean response times (e.g. $\tau \approx T$, the integration time in the Reynolds stress). $\mathbb{T}_\tau(y|y')$ is also known as short time propagator or Green’s function. Then

$$\mathbb{T}_\tau(y|y') = (1 - \tau Q(y))\delta(y - y') + \tau W(y|y'). \quad (44)$$

Notice that $\mathbb{W}(y|y') = \partial \mathbb{T}_\tau / \partial \tau$ equals the transport kernel (20). Now let $\Pr(Y > y)$ denote the probability that during τ the system (e.g. a tracer particle) jumps from y_0 to beyond y . $\Pr(Y > y)$ may also be called the gap distribution [28, 41]. In terms of (44) one has

$$\Pr(Y > y) = \int_y^\infty \mathbb{T}_\tau(y''|y_0) dy''. \quad (45)$$

Using $W(y|y')$ from (42), (45) yields

$$\Pr(Y > y) = C_0 y^{-\varepsilon}, \quad (46)$$

where the constant is given by $C_0 = \tau D / y_0^{1-\varepsilon}$. Upon comparing the Paretian distribution (46) with the number of gaps $\text{Nr}(L > \ell) = N_0 \ell^{-d}$ of length $L > \ell$ in a Cantor set $\mathcal{C}_\infty$ with fractal dimension $0 < d < 1$ (e.g. [28, 42]), one has $d = \varepsilon$.

The above interpretation of boundary layer turbulence in terms of a Cantor set with fractal dimension $d = \varepsilon$ hinges on the new scaling exponent ε being sufficiently small ($\varepsilon < 1$). This is tantamount to the turbulence being sufficiently nonlocal. It will be shown that for fully developed boundary layer turbulence $\varepsilon \approx 0.58$. Note that according to (37) locality requires $\varepsilon = \infty$. However, a gradient expansion in terms of y is not even rigorously possible for any finite value of ε (due to the Paretian tails in the rates), i.e. the limit $\varepsilon \rightarrow \infty$ is highly nonuniform. This finding confirms an earlier suggestion of Bernard and Handler [13].

3.4. The velocity profile

Consider (37) – for the scalar momentum density $P_x = \rho \bar{U}(y)$ – with the rates (39)–(42), and reinsert the diffusion term as in (3). Any sampling rate with a spatial range ℓ (of the order of a few molecular mean free paths) which is effectively zero on the hydrodynamic scale will do. For example, take $\lambda_v(\eta) = (v/2\ell^2) e^{-|\eta|\ell/\ell^*}$ with $\ell \rightarrow 0$. In the notation of (18) one then has

$$\frac{\partial \bar{U}}{\partial t} = v \frac{\partial^2 \bar{U}}{\partial y^2} + \int_0^\infty \mathbb{W}(y|y') \bar{U}(y', t) dy', \quad (47)$$

because now $A(y) = 0$. In the steady state $\partial \bar{U} / \partial t = 0$, and using (19) one obtains

$$v \frac{\partial \bar{U}}{\partial y} + \int_0^\infty A(y|y') \bar{U}(y') dy' = u_*^2, \quad (48)$$

which defines the friction velocity u_* as usual through the total stress $\tau/\rho = u_*^2$. With $A(y|y')$ based on (40), the result reads

$$v \frac{\partial \bar{U}}{\partial y} + D \left(y^\varepsilon \int_y^\infty y'^{-(1+\varepsilon)} \bar{U}(y') dy' - y^{-\varepsilon} \int_0^y y'^{-(1-\varepsilon)} \bar{U}(y') dy' \right) = u_*^2. \quad (49)$$

A partial integration in both integrals reduces the integro-differential equation (49) to an integral equation for the velocity gradient. One finds

$$v \frac{\partial \bar{U}}{\partial y} + \frac{D}{\varepsilon} \left[\int_y^\infty \left(\frac{y}{y'} \right)^\varepsilon \frac{\partial \bar{U}}{\partial y'} dy' + \int_0^y \left(\frac{y}{y'} \right)^{-\varepsilon} \frac{\partial \bar{U}}{\partial y'} dy' \right] = u_*^2. \quad (50)$$

Let us write (50) in terms of the variable $z = \ln(y/y_0)$, with y_0 again being an arbitrary length scale (without loss of generality, we take $y_0 = 1$). This yields

$$v e^{-z} \frac{\partial \bar{U}}{\partial z} + \frac{D}{\varepsilon} \int_{-\infty}^{\infty} e^{-\varepsilon|z-z'|} \frac{\partial \bar{U}}{\partial z'} dz' = u_*^2. \quad (51)$$

If $z \rightarrow \infty$ (i.e. $y \rightarrow \infty$) the viscous term obviously vanishes. Since according to (2) in that limit $\bar{U} \rightarrow u_* z/\kappa$ (so that $\partial \bar{U}/\partial z \rightarrow u_*/\kappa$), (51) shows that $D = \frac{1}{2}\varepsilon^2 \kappa u_*$.

The Fredholm equation (51) can be solved analytically. This feature is due to the nature of the integral kernel $K(z) = e^{-\varepsilon|z|}$. Namely, considering $\mathcal{J} \equiv \int_{-\infty}^{\infty} dz' K(z-z')$ as a linear operator [43,44] which maps the function $\varphi = \partial \bar{U}/\partial z$ onto another function, say $\psi = \mathcal{J}\varphi$, one obtains $\mathcal{J}^{-1} = \frac{1}{2}\varepsilon[1 - \varepsilon^{-2}(\partial/\partial z)^2]$ for the inverse operator (i.e. $\mathcal{J}^{-1}\mathcal{J} = I$). Therefore, operating with $\mathcal{J}^{-1}$ from the left on (51) reduces it to the second order differential equation $\mathcal{J}^{-1}(2v e^{-z}\varphi/\varepsilon) + \kappa u_* \varphi = u_*^2$.

Rewriting this equation for φ in terms of the variable $\xi = 2\varepsilon(\kappa u_* y/v)^{1/2}$ (i.e. $\xi \sim e^{z/2}$) it becomes a Lommel equation for $\Phi = \partial \bar{U}/\partial y$, which reads

$$\xi^2 \Phi'' + \xi \Phi' - (\xi^2 + \sigma^2)\Phi = k, \quad (52)$$

where a prime denotes differentiation with respect to ξ , the Bessel index $\sigma = 2\varepsilon$, and Lommel's constant $k = -(\sigma u_*)^2/v$. As a result the velocity gradient can be expressed as the sum of a Lommel- and a Bessel function (of imaginary argument), viz.

$$\Phi = k s_{-1,\sigma}(i\xi) + B I_{\sigma}(\xi), \quad (53)$$

with $B = (k\pi/2\varepsilon)\text{cosec}(\varepsilon\pi)$.

Since $s_{-1,\sigma}(i\xi) \rightarrow -\sigma^{-2}$ if $\xi \rightarrow 0$, according to (53) one has $\Phi(y) \rightarrow u_*^2/v$ if $y \rightarrow 0$ for the velocity gradient, in agreement with (1) for the viscous sublayer. On the other hand, in the inertial sublayer Eq. (53) has the asymptotic form $\Phi(y) \rightarrow k s_{-1,\sigma}(i\xi)$ as $y \rightarrow \infty$, with $s_{-1,\sigma}(i\xi) \rightarrow -\xi^{-2}$ being the second Lommel function. So, in that case $\Phi(y) \rightarrow u_*/\kappa y$. This leads to the logarithmic profile (2) with

$$\gamma(\varepsilon) = \frac{1}{\varepsilon} - 2(\psi_E(1+\varepsilon) - \ln \varepsilon) + \ln \kappa, \quad (54)$$

where $\psi_E(x) = d \ln \Gamma/dx$ is Euler's psi (or digamma) function (see Appendix B).

The universal (i.e. κ -independent) function $\gamma_{\varepsilon} = \gamma(\varepsilon) - \ln \kappa$ according to (54) is shown in Fig. 3. If $\varepsilon \rightarrow \infty$ one has $\psi_E(1+\varepsilon) \approx (\ln \varepsilon) + 1/(2\varepsilon) - B_2/(2\varepsilon^2) + \dots$, with Bernoulli number $B_2 = \frac{1}{6}$. So, in that case $\gamma_{\varepsilon} \approx B_2/\varepsilon^2$ tends to zero, as it should in the local transport limit. Indeed, $\gamma(\varepsilon \rightarrow \infty) = \ln \kappa$ is the result from diffusive K -theory. On the other hand, in the limit of extremely *nonlocal* transport (i.e. $\varepsilon \rightarrow 0$) one has $\psi_E(1+\varepsilon) \rightarrow -\gamma_E$ with $\gamma_E = 0.5772\dots$. Therefore, in that case $\gamma_{\varepsilon} \approx 1/\varepsilon$ tends to infinity as shown in Fig. 3. We have calculated γ_{ε} both from (54) and by means of direct numerical integration of (52). The resulting values for γ_{ε} were found to be equal within any pre-set accuracy. Fig. 3 clearly demonstrates the influence nonlocality in turbulence mixing has on the additive constant in the logarithmic velocity profile.

Fig. 3. Plot of the universal (i.e. κ -independent) logarithmic profile constant $\gamma(\varepsilon) - \ln \kappa$, according to Eq. (54), as a function of the nonlocality parameter (scaling exponent) ε . Local K -theory corresponds to $\varepsilon = \infty$.

4. Generalized scaling

4.1. Scaling function

Very close to the surface ($yu_*/\nu \ll 1$) time scaling alone is insufficient. Namely, as $y \rightarrow 0$, viscous friction reduces the velocity fluctuations and, thereby, the effective size of the eddies. This feature influences the mean velocity profile $\bar{U}(y)$ in the viscous sublayer and the crossover region, and thus has its consequences for the value of γ . In what follows we will show that the theory of Section 3 can be generalized to arbitrary scaling and point out the existence of a simple (inverse) scaling function (based on fundamental properties arising from the fully three-dimensional Navier–Stokes equations for the fluctuating velocities; see Appendix C).

Let $\eta(y, \tau)$ again (like in Section 2) be the sample path attached to the point y , and let $v(y, t)$ be the fluctuation velocity at y , i.e. $d\eta = v(y) d\tau$. Similarly, let $\eta_*(y, t)$ again (like in Section 3) denote the actual trajectory of a fluid particle with velocity $v(y_*, t)$, where $y_* = y + \eta_*$. So, $d\eta_* = v(y_*) d\tau$. Now let there exist space-time scaling such that for the coordinate η one has $d\eta = \varphi(y) d\varphi_0$. This defines the general scaling function $\varphi(y)$.

Similarly, $d\eta_* = \varphi(y + \eta_*)d\varphi_0$. Hence, one has

$$d\eta = \frac{\varphi(y)}{\varphi(y + \eta_*)} d\eta_*, \quad (55)$$

which again defines the Jacobian of the mapping $\eta(\eta_*)$. This is the generalized version of (34). Integrating (55), one obtains

$$\frac{\eta}{\varphi(y)} = \int_0^{\eta_*} \frac{1}{\varphi(y + \eta'_*)} d\eta'_*. \quad (56)$$

The model (38) for the sampling rates $\lambda(y, \eta)$ generalizes to

$$\lambda(y, \eta) = \frac{D}{\varphi(y)} e^{-\varepsilon|\eta|/\varphi(y)}, \quad (57)$$

while the rates at η_* are again given by $\lambda_*(y, \eta_*) = \lambda(y, \eta)|\partial\eta/\partial\eta_*|$. Consequently, one has

$$\lambda_*(y, \eta_*) = \frac{D}{\varphi(y + \eta_*)} \exp\left(\pm \int_0^{\eta_*} \frac{\varepsilon}{\varphi(y + \eta'_*)} d\eta'_*\right). \quad (58)$$

The upper (lower) sign in the exponent applies if $\eta_* < 0$ (> 0). This may also be written as

$$\lambda_*(y, \eta_*) = \pm \frac{D}{\varepsilon} \frac{\partial}{\partial \eta_*} \exp\left(\pm \int_0^{\eta_*} \frac{\varepsilon}{\varphi(y + \eta'_*)} d\eta'_*\right). \quad (59)$$

Rewriting $\lambda_*(y, \eta_*)$ in terms of $y' = y + \eta_*$ and (like in Section 3) defining $\lambda_*(y|y') \equiv \lambda_*(y, \eta_*)$, one finds

$$\lambda_*(y|y') = \frac{D}{\varphi(y')} \left(\frac{\kappa(y')}{\kappa(y)} \right)^{\pm \varepsilon}, \quad (60)$$

or

$$\lambda_*(y|y') = \pm \frac{D}{\varepsilon} \frac{\partial}{\partial y'} \left(\frac{\kappa(y')}{\kappa(y)} \right)^{\pm \varepsilon}, \quad (61)$$

with

$$\kappa(y) = \kappa_0 \exp\left(\int_{y_0}^y \frac{dy'}{\varphi(y')}\right), \quad (62)$$

where κ_0 and y_0 are constants (of integration of $\partial\kappa/\partial y = \kappa/\varphi$) which are immaterial for the rates $\lambda_*(y|y')$. In what follows we take (without loss of generality) $\kappa_0 = \kappa(y_0)$ such that $\kappa(y) \rightarrow y$ if $y \rightarrow \infty$. The generalized rates (60)–(61) still satisfy exchange, as is seen by means of (26).

As in Section 2, the transition rates are given by $W(y|y') = \pm \partial \lambda_*(y|y')/\partial y$. Hence (recalling $\partial \ell/\partial y = \ell/\varphi$), the visiting rates (60)–(61) imply that

$$W(y|y') = \frac{\varepsilon}{\varphi(y)} \lambda_*(y|y'), \quad (63)$$

which is always nonnegative and, therefore, indeed generates a stochastic process. By virtue of (61) one may also write

$$W(y|y') = \frac{\varepsilon D}{\varphi(y) \varphi(y')} \left(\frac{\ell(y)}{\ell(y')} \right)^{\pm \varepsilon}. \quad (64)$$

One recovers (42) in the inertial sublayer, i.e. if $\ell(y) \approx \varphi(y) \rightarrow y$ (and similarly for y'). As $W(y|y') = \ell(y)\varphi(y')$ factorizes for arbitrary scaling, it always represents a kangaroo process (Section 3.3). In Section 4.2 it will be argued that the physics of boundary layer turbulence allows for a simple third order polynomial expression, *viz.*

$$\frac{1}{\varphi(y)} = \frac{1}{y} \left[1 + \left(\frac{a}{y} \right)^2 \right] \quad (65)$$

for the inverse scaling function in terms of the inverse distance to the surface.

4.2. Velocity fluctuations

The scaling function $\varphi(y)$ can be considered as the product of two functions, *viz.* $v(y)$ for the normal velocity component and $\ell(y)$ ($= d\tau/d\tau_0$) for the scaling of time, such that $\varphi = v\ell$. As pointed out in Section 3, the basic scaling hypothesis for fully developed boundary layer turbulence amounts to $\ell(y) = y$. It implies that $v(y) = 1$ in the inertial sublayer. On its turn $v(y) = 1$ should imply that $(\overline{v^2})^{1/2} \sim v(y)$ does not depend on y in that region, a feature for which indeed exists ample experimental evidence (e.g. [9, 45–48]). Since the flow in the viscous sublayer is also “fully turbulent” (despite the fact that it is sometimes misleadingly denoted as the laminar sublayer), linear time scaling is expected to hold down to molecular distances from the surface (e.g. [9, 49]). In Appendix C it is shown that $\ell(y) = y$ is consistent with a systematic calculation of the properties of the (fluctuating) flow adjacent to the surface. In fact, nonlinear time scaling is ruled out.

Of course, $v(y) = 1$ can not hold down to the surface. Near $y = 0$ the normal velocity goes to zero according to $v(y) = (y/a)^2$ by virtue of continuity [6, 9, 48]. The possibility of $a = \infty$ has been considered (either ex- or implicitly via the Reynolds stress τ_R/ρ) in e.g. Refs. [50, 51]. In that case one would have $v(y) \sim y^3$ (if $y \rightarrow 0$) and mixing length scaling $\varphi(y) \sim y^4$. However, the latter is ruled out in Appendix C since it will be shown that $\varphi(y) \sim y^n$ (with $n > 1$) implies $\tau_R/\rho \sim y^n$ (if $y \rightarrow 0$) for the Reynolds stress. On the other hand, it is also shown that in the power series expansion of τ_R/ρ the term with $n = 4$ is always absent. Therefore, we take $v(y) = (y/a)^2$ if $y \rightarrow 0$.

Appendix C also yields an expression for the length scale a , viz.

$$a = \left(\frac{2v}{\kappa u_*} \int_0^\infty R_{22}(x, 0, 0) dx \right)^{1/2}, \quad (66)$$

where $R_{22}(x, 0, 0)$ represents the longitudinal correlation of the normal velocity component close to the surface, with $R_{22}(0, 0, 0) = 1$ (e.g. [48, 52–54]). The integral in (71) defines a correlation length $\bar{x}$, such that $a_+ = (2\bar{x}_+/\kappa)^{1/2}$ (where $a_+ = u_* a/v$ and $\bar{x}_+ = u_* \bar{x}/v$). The experimental data indicate that $\bar{x}_+ \approx 30$ –60 (with an average $\bar{x}_+ \approx 45$), which implies $a_+ \approx 12$ –18 (with an average $a_+ \approx 15$). This value for a_+ is perfectly consistent with data from measurements concerning the root-mean-square normal velocity fluctuations $(\bar{v}^2)^{1/2}$ (see below).

Let us now consider $\varphi^{-1} = 1/\varphi(y)$ as a function of y^{-1} . It has the property $\varphi^{-1} \rightarrow y^{-1}$ if $y^{-1} \rightarrow 0$, while $\varphi^{-1} \rightarrow y^{-n}$ if $y^{-1} \rightarrow \infty$. In addition, it should be a continuous and differentiable function on $y^{-1} \in (0, \infty)$. Hence, it may have the polynomial representation $\varphi^{-1} = \sum_{k=1}^n c_k y^{-k}$. In view of the preceding, $n = 3$ while $c_1 = 1$ and $c_3 = a^2$. Since throughout $\ell(y) = y$, the inverse $v^{-1} = 1/v(y)$ therefore reads $v^{-1} = 1 + c_2 y^{-1} + (y/a)^{-2}$. Consequently, one has $v(y) = (y/a)^2 [1 - c_2 y/a^2 + \dots]$ if $y \rightarrow 0$. However, in Appendix C it is shown that in the Taylor expansion of $(\bar{v}^2)^{1/2} \sim v(y)$ the cubic term $\sim y^3$ is always absent. Therefore, $c_2 = 0$ so that

$$v(y) = \frac{(y/a)^2}{1 + (y/a)^2}. \quad (67)$$

The ensuing result for $\varphi(y)$ has been given in Eq. (65). Comparison of $(\bar{v}^2)^{1/2} \approx u_* v(y)$ with the available data from both experiments and numerical simulations (e.g. [1, 2, 24, 25, 46–48, 55–64]) indeed yields $c_2 \approx 0$ and confirms that $a_+ \approx 10$ –20 with an average $a_+ \approx 15$.

5. The velocity profile

Substituting (59) into (48) – recalling that $\Lambda(y|y') = \pm \lambda_*(y|y')$ depending on $y' > y$ (upper sign) or $y' < y$ (lower sign) – and performing a partial integration like in Section 3.4, one obtains

$$v \frac{\partial \bar{U}}{\partial y} + \frac{D}{\varepsilon} \left[\int_y^\infty \left(\frac{\ell(y)}{\ell(y')} \right)^\varepsilon \frac{\partial \bar{U}}{\partial y'} dy' + \int_0^y \left(\frac{\ell(y)}{\ell(y')} \right)^{-\varepsilon} \frac{\partial \bar{U}}{\partial y'} dy' \right] = u_*^2. \quad (68)$$

This result reduces to (50) if $\ell(y) = y$, i.e. in the inertial sublayer. It is again possible to cast the transport integral in exponential form, now by defining the variable

$z = \ln[\ell(y)/\ell_0]$. This yields

$$v \frac{\partial \bar{U}}{\partial y} + \frac{D}{\varepsilon} \int_{-\infty}^{\infty} e^{-\varepsilon|z-z'|} \frac{\partial \bar{U}}{\partial z'} dz' = u_*^2, \quad (69)$$

with $\partial/\partial z = \varphi(y) \partial/\partial y$. Like in Section 3.4, the operator $\mathcal{J}^{-1} = \frac{1}{2}\varepsilon[1 - \varepsilon^{-2}(\partial/\partial z)^2]$ reduces (69) to the second order differential equation

$$\mathcal{J}^{-1}(2v\Phi/\varepsilon) + \kappa u_* \varphi(y)\Phi = u_*^2, \quad (70)$$

where $\Phi = \partial \bar{U}/\partial y$. Substituting (65) into (62) yields

$$\ell(y) = y \exp \left[-\frac{1}{2} \left(\frac{a}{y} \right)^2 \right], \quad (71)$$

so that (68) may be written as

$$v \frac{\partial \bar{U}}{\partial y} + \frac{D}{\varepsilon} \int_0^{\infty} \left(\frac{y}{y'} \right)^{\varepsilon'} \exp \left[-\frac{\varepsilon}{2} \left| \left(\frac{a}{y} \right)^2 - \left(\frac{a}{y'} \right)^2 \right| \right] \frac{\partial \bar{U}}{\partial y'} dy' = u_*^2, \quad (72)$$

where we have defined $\varepsilon' = \varepsilon (-\varepsilon)$ if $y' > y$ ($y' < y$). Eq. (72) with $a \neq 0$ has so far eluded an analytical solution (see Section 3 if $a = 0$). However, its numerical solution is easily obtained by virtue of the reduction of (69) to (70). Rewriting the latter in terms of $y \equiv \kappa y_+$ and defining $\bar{\mathcal{U}} \equiv \kappa \bar{U}_+$, it becomes a Sturm–Liouville equation for the gradient function $f(y) = \bar{\mathcal{U}}'$, viz.

$$-\varepsilon^{-2} \varphi(y) [\varphi(y) f']' + [1 + \varphi(y)] f(y) = 1, \quad (73)$$

where a prime denotes differentiation with respect to y , where $\varphi(y) = y^3/(a^2 + y^2)$ with $a = \kappa a_+$. For fixed a one is left with a one-parameter (ε) problem.

The value of the new exponent ε can in principle be determined in two ways. On the “input side” of the model it may be obtained by processing fluctuating normal velocity data as described in Section 2 and comparing the resulting sampling rates $\lambda(y, \eta)$ with (57). Alternatively, on the “output side” of the model the value of ε may be found by comparing mean velocity distribution data with the solution of (73). As a lot of data on the velocity profile has been reported in the literature, we presently adopt the output method. A rough estimate for ε can be made using the $\varepsilon = \infty$ value $\gamma(a) \approx 1.5$ for the logarithmic profile constant if $a \approx 6$ from Appendix D. Since the experimental data indicate that $\gamma \approx \gamma_e + \gamma(a)$ has the value $\gamma \approx 2$, it follows that $\gamma_e \approx 0.5$. Hence, Fig. 3 yields $\varepsilon \approx 0.5$. A better estimate can be obtained by a best-fit of the solution of (73) to the entire measured profile (rather than merely to its logarithmic portion).

Eq. (73) has been integrated by means of a simple (matrix inversion) routine, starting at $y = 0$ (the surface) with $f(0) = 1$. A final integration then yields the mean velocity profile $\bar{\mathcal{U}}(y)$, which is universal in the sense that it is independent of the Von Kármán constant κ (which merely relates the theoretical variables to the experimental ones by the simple rescaling $\bar{\mathcal{U}} = \kappa \bar{U}_+$ and $y = \kappa y_+$). A sample of the results is

Fig. 4. Theoretical mean velocity profiles $\bar{u}(y)$ according to Eq. (72) (or Eq. (73)), for a few values of the exponent ε (with fixed viscous length $a = 0$ in (a)) and for a few values of a (with fixed $\varepsilon = \infty$ in (b)). The universal variables are $\bar{u} = \kappa \bar{U}_+$ and $y = \kappa y_+$.

depicted in Fig. 4a and b, for fixed a (for a few ε values) and for fixed ε (for a few a values), respectively. The logarithmic profile constant γ increases upon decreasing ε or increasing a .

Fig. 5. Plot of the best unconditional least-squares fit of the theoretical profile to the experimental reference data set (see Fig. 1 and Appendix E), corresponding to $\kappa = 0.39$, $a_+ = 16$, and $\varepsilon = 0.58$. This amounts to $\gamma = 1.8$ in Eq. (2).

Fig. 5 shows our best-fit to the experimental data for $\bar{U}_+(y_+)$ for $1 < y_+ < 10^4$. It corresponds to $\kappa \approx 0.39$, $a_+ \approx 16$ and yields $\varepsilon \approx 0.58$. These optimal values have been obtained by means of a least-squares computation [65] *without* a priori constraints on the triple $(\kappa, a_+, \varepsilon)$. Assuming a constant relative data error over the entire y_+ -range, the theoretical curve fits the experimental data with an accuracy of approximately two percent. The data (in particular in the crossover region $10 < y_+ < 100$) have been processed for their residual Reynolds number dependence. The final reference data set is given in Table 1 in Appendix E. Statistical tests on the optimum triple have been performed as well. The ensuing scatter plots (Fig. 7 in Appendix E) show that the 95% confidence intervals amount to $0.37 \lesssim \kappa \lesssim 0.42$, $14 \lesssim a_+ \lesssim 18$, and $0.4 \lesssim \varepsilon \lesssim 0.8$.

6. Final remarks

In this article a novel theory has been formulated for *nonlocal* turbulence transport of scalar quantities. Emphasis has been on the mixing of longitudinal momentum in the turbulent boundary layer over a flat surface. The model is based on rewriting the Reynolds stress (4) in terms of sample paths (Section 2.1) and imposing a stochastic closure hypothesis (Section 2.2). The sampling rates $\lambda(y|y')$ define transition rates

$W(y|y')$ in a Chapman–Kolmogorov (master) transport equation for the density $P_x = \rho \bar{U}(y)$.

The sample paths (attached to a fixed point y in space) are mapped onto tracer trajectories in the boundary layer by means of a scaling hypothesis (Section 3.1 and 4.1). The sampling rates model (57) satisfies exchange (Section 2.4), which reflects the “strong eddies” structure of the flow. The ensuing transition rates define a *nondiffusive* (kangaroo or Kubo–Anderson) stochastic process (Section 3.3). The theory involves both an eddy viscosity (defining the Von Kármán constant $\kappa \approx 0.39$) and a novel mixing exponent ε . The latter can be used to define a mixing length ($\ell \approx y/\varepsilon$ if $\varepsilon \rightarrow \infty$; $\ell \approx y e^{1/\varepsilon}$ if $\varepsilon \rightarrow 0$). For $\varepsilon \rightarrow \infty$ the model reduces to local (diffusive) K -theory. In addition, it involves a correlation length ($a_+ \approx 16$) for the viscous sublayer (Section 4.2).

In the steady state the transport integral equation for the mean momentum density $P_x = \rho \bar{U}(y)$ is easily solved (analytically for $a_+ = 0$ in Section 3.4 and for $\varepsilon = \infty$ in Appendix D; numerically in Section 5) since it can be transformed into the differential equation (73). The resulting profile $\bar{U}(y)$ is compared with experimental data for $\bar{U}_+(y_+)$ in Fig. 5. This leads to $\varepsilon \approx 0.58$, which underscores the nonlocality of turbulent transport and points to the significance of ε as a fractal dimension of a turbulence Cantor set (Section 3.3). It is gratifying that this value for ε has been determined within the triple $(\kappa, a_+, \varepsilon)$ with $\kappa \approx 0.39$ – confirming its by-now accepted value – and $a_+ \approx 16$. This result for a_+ also agrees with its value estimated on the basis of both the root-mean-square $v(y)$ of the normal velocity fluctuations – according to (67) – and of their longitudinal correlation function $R_{22}(x)$ as occurring in (66). The latter involves a correlation length $\bar{x}_+ = \kappa a^2/2$, for which the data indeed indicate $\bar{x}_+ \approx 45$.

While the actual solution of the transport problem in the specific case of $P_x = \rho \bar{U}$ allowed us to estimate ε , the present analysis of the stress applies to the mixing of any scalar tracer quantity. E.g., in Appendix A it is *inter alia* shown how the visiting rates $\lambda(\eta)$ can be calculated for a Brownian particle. For turbulence, Appendix C provides a discussion of the velocity fluctuations underlying – together with linear time-scaling $\ell(y) = y$ – the eddy scaling function $\mathcal{J}(y)$ in the rates (57). In fact, nonlinear time scaling (e.g. $\ell(y) \sim y^2$) is ruled out. This *en passant* yields the results $v(y) \approx (y/a)^2$ and $\tau_R/\rho \approx \kappa u_*^3 y^3 / \nu a^2$ in the viscous sublayer, which are in perfect agreement with recent experiments and direct numerical simulations. In addition, in the inertial sublayer – where $v(y) \approx 1$ – linear time-scaling implies the logarithmic profile Eq. (2), which is considered to be a well-established result.

Appendix A. Sampling rates: Markov processes and Brownian motion

Let us write the mean sampling rates (11) as a time-average, *viz.*

$$\lambda_{\pm}(y, \eta) = \frac{1}{T} \int_0^T \sum_n \delta(\tau - \tau_n(\eta | \pm)) d\tau, \quad (\text{A1})$$

where the conditional functions $\tau_n(\eta | \pm)$ have been defined in (9). Eq. (A1) is identical to

$$\lambda_{\pm}(y, \eta) = \frac{1}{T} \int_0^T |v(\tau)| \delta[\eta - \eta(\tau)] \theta[\pm v(\tau)] d\tau, \quad (\text{A2})$$

where $v(\tau) = d\eta/d\tau$ is the velocity at y and $\eta(\tau) = y_{\xi}(\tau | y', 0)$ is the trajectory with $y = y'$ at $\tau = 0$ for the sample ξ . Using $f[v(\tau)] = \int_{-\infty}^{\infty} f(v) \delta[v - v(\tau)] dv$ with $f(v) = |v| \theta(\pm v)$ and recalling the definition of $\Lambda(y, \eta)$ in Eq. (15), one obtains

$$\Lambda(y, y' - y) = \pm \frac{1}{T} \int_0^T d\tau \int_0^{\infty} dv v P_{\xi}(y, \mp v, \tau | y', 0), \quad (\text{A3})$$

with

$$P_{\xi}(y, \mp v, \tau | y', 0) \equiv \delta(y - y_{\xi}(\tau | y', 0)) \delta(v \pm v_{\xi}(\tau)), \quad (\text{A4})$$

where the upper (lower) sign applies if $y' > y$ ($y' < y$). The rate Λ in (A3) will be a stochastic function if $\{y_{\xi}(\tau), v_{\xi}(\tau)\}$ is a stochastic process, which may or may not be ergodic (e.g. [9, 66]). In fact, application of ergodic theorems to (A3) is hampered by the finite duration of the sample time T . Fortunately, the experimental data are typically gathered by means of averaging over a set of finite T samples, i.e. by combined time- and ensemble averaging.

By imposing the initial condition y' at time t (i.e. $\tau = 0$) on the sample functions of a stochastic process one selects a subensemble (e.g. [23]). Let the subensemble be such that during the time-lapse $T \rightarrow 0$ (on the relevant time scale) arrivals at y at time τ with $v > 0$ (< 0) almost surely come from the region $y' < y$ ($> y$). In that case it is allowed to extend the velocity integration in (A3) to $v = -\infty$, which yields

$$\Lambda(y, y' - y) = \lim_{T \rightarrow 0} \frac{1}{T} \int_0^T d\tau \langle -v_{\xi}(\tau) \delta(y - y_{\xi}(\tau | y', 0)) \rangle, \quad (\text{A5})$$

where the brackets indicate the average with respect to the ensemble $\{\xi\}$. The integrand in (A5) is the time-derivative of the gap distribution (45), so that the result may be written as

$$\Lambda(y, y' - y) = - \int_y^{\infty} \frac{\mathbb{T}_{\tau}(y'' | y') - \delta(y'' - y')}{\tau} dy'', \quad (\text{A6})$$

where the short time propagator ($\tau \rightarrow 0$) is given by

$$\mathbb{T}_{\tau}(y | y') = \langle \delta(y - y_{\xi}(\tau | y', 0)) \rangle. \quad (\text{A7})$$

For a stationary Markov process the subensemble defined by $\mathbb{T}_\tau$ will be time-homogeneous and imply a time-independent transition rate $\mathbb{W} = \partial \mathbb{T}_\tau / \partial \tau$. In that case – e.g. using (44) – Eq. (A6) readily yields

$$\mathbb{W}(y, y' - y) = \frac{\partial \Lambda(y, y' - y)}{\partial y}, \quad (\text{A8})$$

which is identical to (19).

Brownian motion is also time-homogeneous, but its Green's function does not satisfy (44). However, (A6)–(A7) do apply since Brownian motion has the properties implied in (A5). Namely, consider (A3)–(A4) and average over $\{\xi\}$. This gives

$$\Lambda(y, y' - y) = \pm \frac{1}{T} \int_0^T d\tau \int_0^\infty dv v P(y, \mp v, \tau | y', 0). \quad (\text{A9})$$

For Brownian motion, $P(y, v, \tau | y', 0) = \langle P_\xi(y, v, \tau | y', 0) \rangle$ obeys Kramers' Fokker–Planck equation for the Rayleigh particle, corresponding to the (Lagrangian) Langevin equations

$$dx/dt = v, \quad dv/dt = -\gamma v + \xi(t), \quad (\text{A10})$$

with

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t + \tau) \xi(t) \rangle = 2\gamma \vartheta \delta(\tau), \quad (\text{A11})$$

where $\vartheta = k_B \Theta / m$ (Θ being temperature, m the particle mass). In this case the averaging period in (A9) should be very large on the Rayleigh scale $1/\gamma$. In this limit (i.e. $\gamma T \rightarrow \infty$) the particle loses all memory of its initial velocity and the solution for its propagator reduces to (e.g. [67])

$$P(y, v, \tau | y', 0) = N \exp \left[-\frac{1}{2\vartheta} \left(v - \frac{y - y'}{2\tau} \right)^2 - \frac{\gamma}{4\vartheta\tau} (y - y')^2 \right], \quad (\text{A12})$$

where $N = (\gamma/2\tau)^{1/2}/2\pi\vartheta$. Substitution of (A12) into (A9), and taking the limit $T \rightarrow 0$ (on the Smoluchovski scale), leads to

$$\Lambda(y, y' - y) = \pm (2v/\ell^2) \operatorname{erfc}(|y - y'|/\ell), \quad (\text{A13})$$

with $\ell \equiv 2(vT)^{1/2} \rightarrow 0$ and $v \equiv \vartheta/\gamma$ being Einstein's viscosity. Using (A13) in for example Eq. (17) indeed yields the diffusion equation $\partial \bar{U}/\partial t = v \partial^2 \bar{U}/\partial y^2$.

Finally, let us use (A13) to obtain an independent estimate of (an upper bound) for ε . Namely, upon comparing (A13) and (57) for $y \approx y'$ one finds the relation $\varepsilon/y \lesssim (\pi v_R T)^{-1/2}$, where $v_R = \kappa u_* y$ and T is a so-called integral time scale. Taking for the latter $T \gtrsim y/u_*$ (e.g. [68]), one has $\varepsilon \lesssim (\pi\kappa)^{-1/2}$. Hence, $\varepsilon \lesssim 0.9$ in close agreement with the result $\varepsilon \approx 0.58$ (Section 5).

Appendix B. The nonlocal logarithmic profile constant

It will be useful to rewrite (53) in the form (e.g. [69])

$$\Phi = -k \left(K_\sigma(\xi) \int_0^\xi I_\sigma(\xi') \frac{d\xi'}{\xi'} + I_\sigma(\xi) \int_\xi^\infty K_\sigma(\xi') \frac{d\xi'}{\xi'} \right), \quad (\text{B1})$$

where $I_\sigma(\xi)$ and $K_\sigma(\xi)$ are Bessel functions. The velocity $\bar{\mathcal{U}} = \kappa \bar{U}_+$ is given by

$$\bar{\mathcal{U}} = -k' \int_0^{\xi_\kappa} \Phi(\xi') \xi' d\xi', \quad (\text{B2})$$

with $k' \equiv 2/k$. Using Eq. (52) for the integrand in (B1) one obtains

$$\bar{\mathcal{U}} = -k' \left(\xi \Phi'(\xi) - \sigma^2 \int_0^\xi [\Phi(\xi') - u_\star^2/v] \frac{d\xi'}{\xi'} \right), \quad (\text{B3})$$

with $\Phi' = \partial\Phi/\partial\xi$. By means of the asymptotic properties of modified Bessel functions, it is easy to show that $\Phi \rightarrow -k/\xi^2$ as $\xi \rightarrow \infty$. As it should, by virtue of (B2) this implies the logarithmic profile $\bar{\mathcal{U}} \rightarrow 2 \ln \xi$ in that limit. It also implies that the first contribution on the right hand side of (B3) vanishes asymptotically, while the logarithmic profile results from the second term in the integral. Taking the logarithmic contribution apart, one thus has

$$\bar{\mathcal{U}} = \lim_{a \rightarrow 0} \left(2 \ln(\xi/a) + \sigma^2 k' \int_a^\infty \Phi(\xi') \frac{d\xi'}{\xi'} \right), \quad (\text{B4})$$

where terms that vanish if $\xi \rightarrow \infty$ have been disregarded. Upon comparing this with Eq. (2) and substituting (B1) for $\Phi(\xi)$, this leads to

$$\gamma(\varepsilon) = \lim_{a \rightarrow 0} (\ln[\kappa(\sigma/a)^2] - 2\sigma^2 A(\sigma)), \quad (\text{B5})$$

where

$$A(\sigma) = \int_a^\infty K_\sigma(\xi) \frac{d\xi}{\xi} \int_0^\xi I_\sigma(\xi') \frac{d\xi'}{\xi'} + \int_a^\infty I_\sigma(\xi) \frac{d\xi}{\xi} \int_\xi^\infty K_\sigma(\xi') \frac{d\xi'}{\xi'}. \quad (\text{B6})$$

Interchanging integrations in the second contribution, (B6) may be written as $A = A_0 + A_1$ with

$$A_0 = 2 \int_a^\infty K_\sigma(\xi) \frac{d\xi}{\xi} \int_0^\xi I_\sigma(\xi') \frac{d\xi'}{\xi'}, \quad (\text{B7})$$

and

$$A_1 = - \int_a^\infty K_\sigma(\xi) \frac{d\xi}{\xi} \int_0^a I_\sigma(\xi') \frac{d\xi'}{\xi'}. \quad (\text{B8})$$

First consider A_0 . Invoking the series expansion for $I_\nu(\xi')$ and integrating termwise over ξ' , the subsequent integration over ξ can be done for $a = 0$ except in the leading term. Using Eq. 6.561.16 from Ref. [70], one finds

$$A_0 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n(\sigma+n)(\sigma+2n)} + \frac{2}{\sigma\Gamma(\sigma+1)} \int_a^\infty \left(\frac{1}{2}\xi\right)^\sigma K_\sigma(\xi) \frac{d\xi}{\xi}. \quad (\text{B9})$$

The sum in (B9) is easily related to two Euler psi-functions (using Eq. 6.3.16 from Ref. [71]), which yields

$$\sum_{n=1}^{\infty} \frac{1}{n(\sigma+n)(\sigma+2n)} = \left(\frac{1}{\sigma}\right)^2 (2\psi_E(1 + \tfrac{1}{2}\sigma) - \psi_E(1 + \sigma) + \gamma_E), \quad (\text{B10})$$

where $\gamma_E = 0.5772 \dots$ is Euler's constant. The integral in (B9) may be evaluated using the cosine integral (Eq. 9.6.25 from Ref. [71]) for $\xi^\sigma K_\sigma(\xi)$. In the ensuing double integral the integrations may be interchanged, and invoking the properties of $\text{Ci}(x) = -\int_x^\infty t^{-1} \cos(t) dt$ one obtains

$$A_0 = \left(\frac{1}{\sigma}\right)^2 \left(\psi_E(1 + \tfrac{1}{2}\sigma) - \tfrac{1}{2}\psi_E(1 + \sigma) - \tfrac{1}{2}\gamma_E \right. \\ \left. - \frac{2}{B(\sigma, 1/2)} \int_0^\infty (1+t^2)^{-\sigma-1/2} \ln t \, dt - \ln a \right), \quad (\text{B11})$$

where $B(x, y)$ is the beta function. Terms that vanish if $a \rightarrow 0$ have been disregarded in (B11) under the assumption that $\sigma \in (0, 1)$. In the end this constraint can be removed by analytic continuation.

The integral in (B11) is evaluated by noting that $\ln t = \partial t^\mu / \partial \mu$ at $\mu = 0$. One thus arrives at a partial derivative of a beta function, which leads to

$$\int_0^\infty (1+t^2)^{-\sigma-1/2} \ln(t) \, dt = \tfrac{1}{4} B(\tfrac{1}{2}, \sigma) (\psi_E(\tfrac{1}{2}) - \psi_E(\sigma)). \quad (\text{B12})$$

Since $\psi_E(\frac{1}{2}) = -\gamma_E - 2 \ln 2$, substitution of (B12) into (B11) and use of the recurrence relation $\psi_E(1 + \sigma) = \psi_E(\sigma) + 1/\sigma$, yields

$$A_0 = \left(\frac{1}{\sigma}\right)^2 \left(\psi_E(1 + \tfrac{1}{2}\sigma) - \frac{1}{2\sigma} + \ln(2/a) \right). \quad (\text{B13})$$

For A_1 from (B8) one only needs $I_\sigma(\xi)$ and $K_\sigma(\xi)$ for $\xi \rightarrow 0$. The result reads

$$A_1 = -\frac{1}{2} \left(\frac{1}{\sigma} \right)^3. \quad (\text{B14})$$

For A one thus finds

$$A(\sigma) = \left(\frac{1}{\sigma} \right)^2 \left(\psi_E(1 + \tfrac{1}{2}\sigma) - \frac{1}{\sigma} + \ln(2/a) \right). \quad (\text{B15})$$

With this upshot for $A(\sigma)$ the limit $a \rightarrow 0$ in (B5) for the integration constant $\gamma(\varepsilon)$ can be taken and yields Eq. (54).

Appendix C. Time-scaling analysis of near-wall fluctuations

The scaling function $\mathcal{s}(y)$ (Eq. (65)) is connected with the normal velocity-scaling function $v(y)$ [Eq. (67)] via $\ell(y) = y$, according to $\mathcal{s} = v\ell$. Since $v(y) \sim (\overline{v^2})^{1/2}$, we will investigate the velocity fluctuations near the wall under linear time-scaling.

The “fully developed turbulence flat boundary layer” is defined by the feature that *all* mean quantities are stationary and translation invariant (in the (x, z) -plane parallel to the surface). Without loss of generality we take the mean flow to be in the x -direction, i.e. $\bar{U}_1 \equiv \bar{U}$ and $\bar{U}_3 \equiv 0$. Substituting the Reynolds decompositions ($U_i \equiv \bar{U}_i + u_i$ and $P \equiv \bar{P} + p$) into the Navier–Stokes equations (the incompressibility condition $\partial U_i / \partial x_i = 0$ yields $\bar{U}_2 = 0$) leads to

$$\frac{\partial}{\partial y} \overline{uv} = v \frac{\partial^2 \bar{U}}{\partial y^2}, \quad \bar{P} = -\rho \overline{v^2}, \quad (\text{C1})$$

subject to the nontrivial constraint

$$\overline{vw} = 0. \quad (\text{C2})$$

Of course, since $\tau_R/\rho = -\overline{uv}$, (C1) is the steady state version of Eq. (3). For the fluctuating velocities (u, v, w) one obtains

$$\begin{aligned} \frac{\partial u}{\partial t} + v \frac{\partial \bar{U}}{\partial y} + \bar{U} \frac{\partial u}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + v \Delta u - u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - w \frac{\partial u}{\partial z} + \frac{\partial}{\partial y} \overline{uw}, \\ \frac{\partial v}{\partial t} + \bar{U} \frac{\partial v}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + v \Delta v - u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - w \frac{\partial v}{\partial z} + \frac{\partial}{\partial y} \overline{v^2}, \\ \frac{\partial w}{\partial t} + \bar{U} \frac{\partial w}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + v \Delta w - u \frac{\partial w}{\partial x} - v \frac{\partial w}{\partial y} - w \frac{\partial w}{\partial z}, \end{aligned} \quad (\text{C3})$$

where Δ is the three-dimensional Cartesian Laplacian. In addition it will be useful to explicit the Poisson equation determining the pressure fluctuations (e.g. [48]). Taking

the divergence of (C3) and using $\partial u_i / \partial x_i = 0$, one has

$$\begin{aligned} \frac{1}{\rho} \Delta p = & -2 \frac{\partial v}{\partial x} \frac{\partial \bar{U}}{\partial y} - 2 \frac{\partial^2 uv}{\partial x \partial y} - 2 \frac{\partial^2 uw}{\partial x \partial z} - 2 \frac{\partial^2 vw}{\partial y \partial z} - \frac{\partial^2}{\partial x^2} (u^2) \\ & - \frac{\partial^2}{\partial y^2} (v^2 - \bar{v}^2) - \frac{\partial^2}{\partial z^2} (w^2). \end{aligned} \quad (\text{C4})$$

From the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (\text{C5})$$

and (under “stick” boundary conditions) $u \sim w \sim y$ if $y \rightarrow 0$, one has $v \sim y^2$. Hence, by (C4) $\Delta p = \mathcal{O}(y^2)$. By Green’s theorem for the Laplace equation, up to this order in y the pressure fluctuations are therefore fixed by two independent fields (*viz.* the pressure and its normal gradient) at the surface.

Let us now implement the time-scaling hypothesis by means of a parameter σ , as follows. Let both $t \rightarrow \sigma t$ and $y \rightarrow \sigma y$. In addition, let $u \equiv \sigma u$, $v \equiv \sigma^2 v$, $w = \sigma \omega$, and $\bar{U} \equiv \sigma \bar{U}$. For the purpose of the analysis, σ is considered as a small parameter. In the end one resets $\sigma = 1$. Introducing the scaling into (C3)–(C5), and expanding all fields in powers of σ (e.g. $u \equiv u^{(0)} + \sigma u^{(1)} + \sigma^2 u^{(2)} + \dots$), one collects all terms of equal power in σ . The leading contributions from (C3) are of order σ^{-1} , and yield

$$\frac{\partial^2 u^{(0)}}{\partial y^2} = 0, \quad \frac{\partial \phi^{(0)}}{\partial y} = 0, \quad \frac{\partial^2 \omega^{(0)}}{\partial y^2} = 0, \quad (\text{C6})$$

so that

$$u^{(0)} = y f(x, z, t), \quad \phi^{(0)} = \phi(x, z, t), \quad \omega^{(0)} = y g(x, z, t), \quad (\text{C7})$$

where f , ϕ and g are as yet arbitrary (and random) functions. From Eq. (C4) one obtains $\partial^2 \phi^{(0)} / \partial y^2 = 0$ in order σ^{-2} , which is already satisfied by (C7). The next to leading term from (C4) reads $\partial^2 \phi^{(1)} / \partial y^2 = 0$, so that

$$\phi^{(1)} = y \psi(x, z, t), \quad (\text{C8})$$

with ψ being another arbitrary function (thereby confirming the remark below (C5)). Using (C8) the terms of order σ^0 from (C3) yield

$$\begin{aligned} u^{(1)} &= \frac{1}{2! \mu} y^2 \frac{\partial \phi}{\partial x} + \frac{1}{3! \nu} y^3 \frac{\partial f}{\partial t}, \quad v^{(0)} = \frac{1}{2! \mu} y^2 \psi, \\ \omega^{(1)} &= \frac{1}{2! \mu} y^2 \frac{\partial \phi}{\partial z} + \frac{1}{3! \nu} y^3 \frac{\partial g}{\partial t}. \end{aligned} \quad (\text{C9})$$

In its leading order σ , Eq. (C5) now leads to

$$\frac{\partial f}{\partial x} + \frac{1}{\mu} \psi + \frac{\partial g}{\partial z} = 0. \quad (\text{C10})$$

Hence, apart from the pressure fields ϕ and ψ , either f or g will be an independent source function. Consider then the constraint (C2). In leading order ($\sim \sigma^3$) it implies that $\overline{\psi g} = 0$, which fixes the (statistically) independent triple (ϕ, ψ, g) .

The second to leading terms ($\sim \sigma^0$) from (C4) connect $\rho^{(2)}$ with $\rho^{(0)}$ and, upon using (C7) for the latter, yield

$$\rho^{(2)} = -\frac{1}{2!} y^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right). \quad (\text{C11})$$

With (C8) and (C11) for the pressure fluctuations, and (C7) and (C9) for the velocities, the contributions of order σ from (C3) imply that

$$\begin{aligned} u^{(2)} &= -\frac{1}{3!} y^3 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} \right) + \frac{1}{3! \mu} y^3 \frac{\partial \psi}{\partial x} + \frac{1}{4! \mu v} y^4 \frac{\partial^2 \phi}{\partial x \partial t} + \frac{1}{5! v^2} y^5 \frac{\partial^2 f}{\partial t^2}, \\ v^{(1)} &= -\frac{1}{3! \mu} y^3 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + \frac{1}{4! \mu v} y^4 \frac{\partial \psi}{\partial t}, \\ \omega^{(2)} &= -\frac{1}{3!} y^3 \left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial z^2} \right) + \frac{1}{3! \mu} y^3 \frac{\partial \psi}{\partial z} + \frac{1}{4! \mu v} y^4 \frac{\partial^2 \phi}{\partial z \partial t} + \frac{1}{5! v^2} y^5 \frac{\partial^2 g}{\partial t^2}. \end{aligned} \quad (\text{C12})$$

Using $v^{(1)}$ from (C12) one verifies that (C5) is satisfied in order σ^2 , by virtue of (C10). This feature propagates to all higher orders since (D3) and (D4) are compatible with a diffusive scalar transport equation for $Q \equiv \partial u_i / \partial x_i$, assuming Q to be nonzero for the sake of the argument, which reads $\partial Q / \partial t + \bar{U} \partial Q / \partial x = v \Delta Q$. In the absence of volume sources this implies (e.g. [44]) the identical vanishing of its solution if both Q and its normal derivative $\partial Q / \partial n$ are zero at a boundary. By (C10) this is obviously the case here.

The constraint (C2) is satisfied through order σ^4 by virtue of the mutual independence of (ϕ, ψ, g) , and using (C8) for $\rho^{(1)}$ Eq. (C4) yields

$$\rho^{(3)} = -\frac{1}{3!} y^3 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right). \quad (\text{C13})$$

From (C3) it then follows that

$$v^{(2)} = -\frac{2}{4! \mu} y^4 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right) - \frac{1}{5! \mu v} y^5 \frac{\partial}{\partial t} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + \frac{1}{6! \mu v^2} y^6 \frac{\partial^2 \psi}{\partial t^2}. \quad (\text{C14})$$

Continuity is of course satisfied for $u^{(2)}$, $v^{(2)}$, and $\omega^{(2)}$, by (C10). In addition, Eq. (C2) is satisfied through order σ^5 because of statistical homogeneity (in x , z , and t), which *inter alia* implies

$$\overline{\psi(x, z, t) \psi(x', z', t')} = \mathcal{K}_{\psi\psi}(|\Delta x|, |\Delta z|, |\Delta t|), \quad (\text{C15})$$

where $\Delta x = x - x'$, etc., so that e.g. $\overline{\psi \partial \psi / \partial z} = 0$.

Let us now turn to the root-mean-square $(\overline{v^2})^{1/2}$ near the wall, where $v = v_1 y + \frac{1}{2!} v_2 y^2 + \frac{1}{3!} v_3 y^3 + \dots$ (e.g. [25, 57]) with

$$\begin{aligned} v_1 &= 0, & v_2 &= \frac{1}{\mu} \psi, & v_3 &= -\frac{1}{\mu} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right), \\ v_4 &= \frac{1}{\mu v} \frac{\partial \psi}{\partial t} - \frac{2}{\mu} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right), \end{aligned} \quad (\text{C16})$$

which at once implies $\overline{v_2 v_3} = 0$. Hence $(\overline{v^2}) = \frac{1}{4} (\overline{v_2^2}) y^4 + \mathcal{O}(y^6)$ so that in the power series expansion of $(\overline{v^2})^{1/2} \approx u_*(y)$ the term of order y^3 is always absent (the case $\psi = 0$ is considered further on). In Section 4.2 this feature is used to arrive at (67) for $v(y)$, which (with $\ell(y) = y$) implies (65) for $s(y)$. Summarizing, one may write

$$\overline{v^2} = (y^4/4\mu^2) \mathcal{K}_{\psi\psi}(0, 0, 0) + \mathcal{O}(y^6), \quad (\text{C17})$$

where we have used (C15) for the correlation function.

Consider now $\tau_R/\rho = -\overline{uv}$ near the wall. Setting $u = u_1 y + \frac{1}{2!} u_2 y^2 + \frac{1}{3!} u_3 y^3 + \dots$, one has

$$u_1 = f, \quad u_2 = \frac{1}{\mu} \frac{\partial \phi}{\partial x}, \quad u_3 = \frac{1}{v} \frac{\partial f}{\partial t} + \frac{1}{\mu} \frac{\partial \psi}{\partial x} - \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial z^2} \right). \quad (\text{C18})$$

Using (C16) and (C18) gives

$$\tau_R/\rho = -\frac{1}{2} (\overline{u_1 v_2}) y^3 + \left(\frac{1}{24} \overline{u_1 v_4} + \frac{1}{12} \overline{u_3 v_2} \right) y^5 + \dots, \quad (\text{C19})$$

where it is used that $\overline{u_1 v_3} = \overline{u_2 v_2} = 0$ so that terms of order y^4 are always absent. One also has $\overline{u_2 v_3} = 0$. Integration of (C10) over $x' \in (-\infty, x)$, and use of the independence of ψ and ϕ , then yields

$$\tau_R/\rho = (y^3/2\mu^2) \int_0^\infty \mathcal{K}_{\psi\psi}(x, 0, 0) dx + \mathcal{O}(y^5), \quad (\text{C20})$$

the correlation function being defined in (C15).

The absence of a term of order y^4 in τ_R/ρ is worth noticing. Eg., Reichardt [72] advocated $n = 3$, but in a footnote remarked that $n = 4$ might apply if certain correlations (*viz.* $f \partial g / \partial z$) were absent. However, above we have shown that (f, g) can not be a pair of independent random functions and that $f \partial g / \partial z = 0$ only if $\psi = 0$. Hama [50] found that $n = 4$, but reviewing the author's analysis (noticing the asymptotic regime where Hama's $\ell^* \sim y$) shows that his viscous sublayer data much better fit $\ell^* \sim y^{3/2}$, which implies $n = 3$. Van Driest [51] proposed a viscous damping factor near the wall which entailed $n = 4$ (see Ref. [73] for empirical critique). Neither Hinze [26] nor Spalding [74] decides between $n = 3$ and 4. However, using data of Refs. [45, 46] Coantic [75] suggested that "most likely" $n = 3$. This value is also favoured by Landau and Lifshitz [6], Monin and Yaglom [9], and Townsend [48].

Finally, in a recent comparison [57] of experimental data and results from direct numerical simulations (e.g. [25,62]) it is concluded that indeed $n = 3$, in agreement with Eq. (C20).

Let us now consider the exceptional case $\psi = 0$. By (C16) then $v_2 = v_4 = 0$, so that in (C19) both terms of order y^3 and y^5 would vanish. On the other hand, even if $\psi = 0$, still $v_3 \neq 0$. Hence, in that case one would have $v(y) \sim y^3$ while $\tau_R/\rho \sim y^6$. These functions, however, are not in accord with the data (e.g. [57]). Moreover, they can be shown to be mutually incompatible. Namely, consider Eq. (73). Let $v(y) = \mathcal{A}y^k[1 + \mathcal{O}(y)]$ so that $\mathcal{J}(y) = \mathcal{A}y^{k+1}[1 + \mathcal{O}(y)]$ with $k \geq 2$, and let $f(y) = 1 + c_1y + c_2y^2 + \dots$. One finds that $f(y) = 1 - \mathcal{A}y^{k+1}[1 + \mathcal{O}(y)]$, and noticing that $\tau_R/\rho u_*^2 = 1 - f(y)$ one concludes that $\tau_R/\rho u_*^2 = \mathcal{A}y^{k+1}[1 + \mathcal{O}(y)]$. So, if (the normal velocity fluctuations) $v(y) \sim y^k$ for $y \rightarrow 0$, then (the eddy-viscosity) $\tau_R/\rho \sim y^n$ with $n = k + 1$. Clearly, this fluctuation-dissipation relation is incompatible with the case $\psi = 0$ (i.e. $k = 3$, $n = 6$). Hence, $\psi \neq 0$ and $k = 2$, $n = 3$.

Finally, note that the case $\psi \neq 0$ only admits linear time-scaling. Namely, with $t \sim y^m$ the model would imply $n - k = m$ (instead of one). On the other hand, it is not difficult to see that the result $k = 2$ and $n = 3$ from the fluctuation analysis is compatible with any value of m . Internal consistency then requires $m = 1$ (which rules out e.g. $t(y) \equiv y^2$).

The above analysis with $k = 2$ assumes $a = \kappa a_+$ to be nonzero. With $\mathcal{A} = 1/a^2$ (see below (73)), and using $(v^2)^{1/2} \approx u_* v(y)$ with $v(y) \approx (y/a)^2$ along with (C17), one obtains

$$\mathcal{K}_{\psi\psi}(0,0,0) = 4\mu^2 u_*^2/a^4. \quad (\text{C21})$$

On the other hand, using $\tau_R/\rho \approx u_*^2 y^3/a^2$ (i.e. $\tau_R/\rho \approx \kappa u_*^3 y^3/va^2$) along with (C20), one finds

$$\int_0^\infty \mathcal{K}_{\psi\psi}(x,0,0) dx = 2\kappa\mu^2 u_*^3/va^2. \quad (\text{C22})$$

Combining (C21)–(C22) and defining $R_{22}(x,0,0) = \mathcal{K}_{\psi\psi}(x,0,0)/\mathcal{K}_{\psi\psi}(0,0,0)$ yields Eq. (66).

Appendix D. Local generalized scaling solution

Consider Eq. (73) with $\varepsilon = \infty$, so that the velocity profile is given by

$$\bar{u}(y) = \int_0^y \left(1 + \frac{t^3}{a^2 + t^2}\right)^{-1} dt. \quad (\text{D1})$$

It is convenient to introduce three coefficients A , B and C with $A - B \equiv 1$, $AB \equiv C$ and $AC \equiv a^2$, so that A is the real root of the cubic equation $A^3 - A^2 - a^2 = 0$. The

solution for A may be written as $A = (\alpha_1 + \alpha_2) + \frac{1}{3}$, where

$$\alpha_{1,2} = \left[\frac{1}{27} + \frac{1}{2} a^2 \pm a \left(\frac{1}{27} + \frac{1}{4} a^2 \right)^{1/2} \right]^{1/3}. \quad (\text{D2})$$

Let $N_0 \equiv A^2/(1 + 3B)$, $N_1 \equiv 1 - N_0$, $N_2 \equiv 1 - (1 - 3B)N_0/A$ and $\Delta \equiv (3 + 4/B)^{1/2}$. Then (D1) yields

$$\begin{aligned} \bar{u} = N_0 \ln \left(1 + \frac{y}{A} \right) + \frac{1}{2} N_1 \ln \left(1 - \frac{y}{A} + \frac{y^2}{C} \right) \\ + \frac{1}{\Delta} N_2 \left[\arctan \left(\frac{1}{\Delta} \right) - \arctan \left(\frac{1 - 2y/B}{\Delta} \right) \right]. \end{aligned} \quad (\text{D3})$$

If $y \rightarrow \infty$, one obtains the logarithmic profile as $\bar{u} \rightarrow \ln(y) + \gamma_a$. Comparison with Eq. (2) yields $\gamma = \gamma(a)$ with $\gamma(a) = \gamma_a + \ln \kappa$. For γ_a one finds

$$\gamma_a = -N_0 \ln(A) - \frac{1}{2} N_1 \ln(C) + \frac{1}{\Delta} N_2 \operatorname{arccot} \left(-\frac{1}{\Delta} \right). \quad (\text{D4})$$

Since the experimental data indicate that $a \approx 6$ (see Section 4.2), consider (D4) for $a \gg 1$. Using the above formulae one obtains $A = a^{2/3} [1 + \frac{1}{3} a^{-2/3} + \dots]$, $B = a^{2/3} [1 - \frac{2}{3} a^{-2/3} + \dots]$, $C = a^{4/3} [1 - \frac{1}{3} a^{-2/3} + \dots]$, $N_0 = \frac{1}{3} a^{2/3} [1 + a^{-2/3} + \dots]$, $N_1 = -\frac{1}{3} a^{2/3} [1 - 2a^{-2/3} + \dots]$, $N_2 = a^{2/3} [1 + \frac{2}{3} a^{-2/3} + \dots]$, and $\Delta = 3^{1/2} [1 + \frac{2}{3} a^{-2/3} + \dots]$. Substitution into (D4) leads to

$$\gamma_a \approx (2\pi/3^{3/2}) a^{2/3} - \ln(a^{2/3}) - \frac{1}{3}, \quad (\text{D5})$$

where terms that vanish if $a \rightarrow \infty$ (which are of order $a^{-2/3} \ln a$) have been disregarded. For instance, $\gamma_a \approx 2.5$ if $a \approx 6$. Hence, with $\kappa \approx 0.4$ one would then have $\gamma(a) \approx 1.5$.

Appendix E. Reynolds number effects

Quite some effort has been spent to measure the mean velocity profile in a turbulent layer over a smooth surface (e.g. [7, 9]). In the range where the profile is logarithmic (typically $y_+ > 10^2$), the data from various authors show little spread ($< 5\%$), although the accuracy is not always clearly stated. In the crossover region ($1 \lesssim y_+ \lesssim 10^2$) one clearly observes larger scatter (e.g. [76] p. 198).

The notion of a “law of the wall” $\bar{u} = f(y)$ hinges on the existence of a Reynolds number independent limit in terms of the so-called “inner variables”. However, in the crossover layer the data are necessarily obtained at relatively small Reynolds numbers (typically of the order of $\text{Re} \approx 3\text{--}40 \times 10^3$ based on bulk velocity and layer- or channel width). For such numbers there may still be a residual Re-dependence in boundary layer quantities near the wall. In fact, there is a justified revival in interest in these features [1, 2, 60, 64].

Fig. 6. Examples of Reynolds number dependence of the mean velocity data in pipe flow (at $y_+ = 28.5$; upper figure) and in channel flow (at $y_+ = 32$; lower figure). $R = 1$ corresponds to $Re = 1725$ and $Re = 1830$, respectively. The data in this figure are due to Ref. [79]. The solid line is a fit according to Eq. (E1).

Table 1
Data file boundary layer velocity profile

y_+	$\bar{U}_+$	y_+	$\bar{U}_+$
4.7	4.6	78.0	15.2
10.0	7.4	100	16.0
11.0	7.9	145	16.9
14.0	9.4	200	17.9
17.5	10.4	450	19.7
20.0	10.8	600	20.5
22.5	11.4	2000	23.6
28.5	12.2	4000	25.7
32.0	12.7	8000	27.5
39.0	13.1	10000	28.0
50.0	13.7		

$\bar{U}_+$ as a function of y_+ for $Re = \infty$. Typical data error: $\Delta\bar{U}_+/\bar{U}_+ \approx (\pm)3\%$. This reference data set has been obtained by processing existing experimental data (such as shown in Fig. 6) according to Eq. (E1).

Residual Re-dependence of the velocity profile in the crossover region has been discussed in Refs. [77, 78] using data from *inter alia* Nikuradse [8] and Laufer [46]. We have re-examined these data and found that they are consistent with

$$\bar{U}_+(\text{Re}) \approx \bar{U}_+(\infty) + c_0 \text{Re}^{-1} + \mathcal{O}(\text{Re}^{-2}), \quad (\text{E1})$$

valid for $y_+ \gtrsim 5$. For pipe flow $c_0 \approx (7 \pm 3) \times 10^3$, for channel flow $c_0 \approx (4 \pm 2) \times 10^3$ (for Reynolds numbers based on bulk velocity and pipe- or channel full-width). The coefficient for the second order term in (E1) is typically found to be nonnegative.

Fig. 7. Scatter plots showing the statistical uncertainty in the optimal parameter values for $(\kappa, a_+, \varepsilon)$. The figures have been generated by Gaussian randomizing the reference data set (with the standard deviation of the best fit). (a) (resp. (b)) shows a projection on the (κ, a_+) -plane [resp. (ε, κ) -plane]. The solid lines indicate the 95% confidence levels.

Fig. 7. Continued.

Fig. 6 shows two examples from the work of Patel and Head [79]. Their conclusion that (for pipe flow, in particular) the data of Ref. [8] do not accord with the generic trends unless $Re > 10^4$ is presently confirmed. Only with that restriction can Nikuradse's [8] data be fitted to Eq. (E1), with $c_0 \approx 7 \times 10^3$ (for $Re \leq 10^4$ his data would yield a negative coefficient in the second order term). The recent data of Ref. [2] also comply with (E1). According to these authors the residual Re -dependence of the mean profile data is likely to be the effective result of much stronger Re -dependences in quantities which are characteristic for the fluctuations (such as our a_+ and ϵ). In the present article we therefore limit ourselves to estimating values as $Re \rightarrow \infty$. The resulting reference data set for the mean velocity profile is given in Table 1.

The reference data has been shown in Fig. 5, along with the best-fit of the present theory. It has been determined by means of an unconstrained least-squares optimizing algorithm (e.g. [65]) – taking a fixed relative error for weighting the data points – and corresponds to $\kappa \approx 0.39$, $a_+ \approx 16$ and $\varepsilon \approx 0.58$. The average mismatch per data point amounts to approximately 2%. The uncertainty in these optimal values has been investigated both by plotting contours of constant relative error and by making scatter plots (by Gaussian randomizing the data). Typical results for the latter are shown in Fig. 7. Notice that although the uncertainty in ε is quite large, it is almost surely less than one (see also Section 3.3 and Appendix A). On the 95% confidence level lines the average mismatch per data point has increased to about 2.5 percent.

An unconditional fit to the raw (i.e. mixed Re-numbers) data of Ref. [8] (Table 3), yields – apart from $\varepsilon \approx 2.5$ – unlikely high values for $\kappa (\approx 0.43)$ and $a_+ (\approx 22)$. The average mismatch per data point also increases (to $> 3\%$). On the other hand, the possibility of a moderate increase of the Von Kármán constant with decreasing Re-number has been suggested before (e.g. [48, 68, 80]) while the more pronounced variation in a_+ and ε seems to be in line with the findings of Refs. [1, 2, 64].

Note added in proof

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