Orthogonal rational functions and modified approximants

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Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence in the open unit disk in the complex plane and let
\[
B_0 = 1 \quad \text{and} \quad B_n(z) = \prod_{k=0}^{n} \frac{1}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z}, \quad n = 1, 2, \ldots,
\]
\((\bar{a}_k/|a_k| = -1 \text{ when } a_k = 0)\). Let \( \mu \) be a positive Borel measure on the unit circle, and let \( \{\phi_n\}_{n=0}^{\infty} \) be the orthonormal sequence obtained by orthonormalization of the sequence \( \{B_n\}_{n=0}^{\infty} \) with respect to \( \mu \). Let \( \{\psi_n\}_{n=0}^{\infty} \) be the sequence of associated rational functions. Using the functions \( \phi_n, \psi_n \) and certain conjugates of them, we obtain modified Padé-type approximants to the function
\[
F_\mu(z) = \int_{-\pi}^{\pi} t + z d\mu(\theta), \quad (t = e^{i\theta}).
\]

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1. Introduction

The purpose of this paper is to give certain modified rational approximants to the function
\[
F_\mu(z) = \int_{-\pi}^{\pi} \frac{t + z}{t - z} d\mu(\theta), \quad (t = e^{i\theta})
\]
where $\mu$ is a positive Borel measure on the unit circle in the complex plane. Let

$$T = \{z \in \mathbb{C} : |z| = 1\}, \quad D = \{z \in \mathbb{C} : |z| < 1\}, \quad E = \{z \in \mathbb{C} : |z| > 1\}$$

and let $\alpha_n, n = 0, 1, 2, \ldots$ be given points in $D$ with $\alpha_0 = 0$. The Blaschke factors $\zeta_n$ are given by

$$\zeta_n(z) = \frac{\alpha_n - z}{|\alpha_n| \cdot 1 - \overline{\alpha_n}z}, \quad n = 0, 1, 2, \ldots,$$

where by convention

$$\frac{\overline{\alpha_n}}{|\alpha_n|} = -1 \quad \text{when} \quad \alpha_n = 0.$$

The (finite) Blaschke products are

$$B_n(z) = \prod_{k=1}^{n} \zeta_k(z), \quad n = 1, 2, \ldots \quad \text{and} \quad B_0(z) = 1.$$

We define the linear spaces $\mathcal{L}_n, n = 0, 1, 2, \ldots$ and $\mathcal{L}$ by

$$\mathcal{L}_n = \text{span}\{B_m : m = 0, 1, \ldots, n\} \quad \text{and} \quad \mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n.$$

Clearly $\mathcal{L}_n$ consists of the functions that may be written as

$$\frac{p_n(z)}{\pi_n(z)},$$

where

$$\pi_n(z) = \prod_{k=1}^{n} (1 - \overline{\alpha_k}z), \quad n = 1, 2, \ldots \quad \text{and} \quad \pi_0(z) = 1$$

and $p_n$ belongs to $\Pi_n$, the set of polynomials of degree at most $n$. The star conjugate $f_*$ of a function $f$ is defined as

$$f_*(z) = \overline{f(1/z)}.$$

For $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ the superstar conjugate $f^*$ will be

$$f^*(z) = B_n(z)f_*(z).$$

If $f \in \mathcal{L}_0$, then $f^* = f_*$.

The linear spaces $\mathcal{L}_{n*}, n = 0, 1, 2, \ldots$, and $\mathcal{L}_*$ are defined as

$$\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\} \quad \text{and} \quad \mathcal{L}_* = \{f_* : f \in \mathcal{L}\}.$$
where
\[ \omega_m(z) = \prod_{k=1}^{m}(z - \alpha_k), \quad \text{and} \quad \omega_0(z) = 1. \]

As in [1] we also put
\[ \mathcal{L}_n(\alpha_n) = \{ f \in \mathcal{L}_n : f(\alpha_n) = 0 \}, \quad n = 1, 2, \ldots \]
and similarly
\[ \mathcal{L}_n(1/\alpha_n) = \{ f \in \mathcal{L}_n : f(1/\alpha_n) = 0 \}, \quad n = 1, 2, \ldots. \]

Furthermore, we assume that \( M \) is a linear functional on \( \mathcal{L} + \mathcal{L}_* \) such that for \( f \in \mathcal{L} \) we have
\[ M(f_\ast) = \overline{M(f)}, \quad \text{and} \quad M(ff_\ast) > 0 \quad \text{if} \quad f \neq 0. \]

Then this also holds for \( f \in \mathcal{L} + \mathcal{L}_* \). The functional \( M \) induces an inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{L} \times \mathcal{L} \) by
\[ \langle f, g \rangle = M(fg_\ast), \quad f, g \in \mathcal{L}. \]

Note that \( \mathcal{L} \mathcal{L}_* = \mathcal{L} + \mathcal{L}_* \), as can be seen by partial fraction decomposition. Also for \( f, g \in \mathcal{L}_* \) we may define \( \langle f, g \rangle = M(fg_\ast) \). Then we get
\[ \langle f, g \rangle = \langle g_\ast f_\ast \rangle \quad \text{for} \quad f, g \in \mathcal{L}. \]

As \( \langle g, f \rangle = \overline{M(gf_\ast)} = M(fg_\ast) = \langle f, g \rangle \) for \( f, g \in \mathcal{L} \) and \( \langle f, f \rangle = M(ff_\ast) > 0 \) for \( f \in \mathcal{L}, f \neq 0 \), the inner product is Hermitian and positive-definite on \( \mathcal{L} \times \mathcal{L} \).

In this paper we assume that \( \mu \) is a solution to the following “moment” problem:

Given the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{L} \times \mathcal{L} \) (or the linear functional \( M \) on \( \mathcal{L} + \mathcal{L}_* \)), find a non-decreasing function \( \mu \) on \( [-\pi, \pi] \) (or a positive Borel measure \( \mu \) on \( (-\pi, \pi] \)) such that
\[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(e^{i\theta})g(e^{i\theta})d\mu(\theta) \quad \text{for} \quad f, g \in \mathcal{L} \]
(or \( M(f) = \int_{-\pi}^{\pi} f(e^{i\theta})d\mu(\theta) \) for \( f \in \mathcal{L} + \mathcal{L}_* \)).

This moment problem always has a solution. Two non-decreasing functions which are solutions of the moment problem such that their difference is a constant at all the points at which it is continuous, are considered to be the same solution of the moment problem. We will give modified rational approximants to the function \( F_\mu \) in terms of orthogonal rational functions and their associates. Besides, we obtain some results about related quadrature formulas.

2. Orthogonal rational functions

In our approach orthogonal rational functions will play an important rôle. Let
the sequence \( \{ \phi_n \}_{n=0}^{\infty} \) in \( \mathcal{L} \) be obtained by orthonormalization of the sequence \( \{ \mathbb{B}_n \}_{n=0}^{\infty} \) with respect to the inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{L} \times \mathcal{L} \), i.e.

\[
\phi_n \in \mathcal{L}_n \quad \text{and} \quad \langle \phi_n, \phi_n \rangle = 1, \quad n = 0, 1, 2, \ldots
\]

and

\[
\langle f, \phi_n \rangle = 0 \quad \text{for} \quad f \in \mathcal{L}_{n-1}, \quad n = 1, 2, \ldots.
\]

It follows easily that

\[
\langle f, \phi_n^* \rangle = 0 \quad \text{for} \quad f \in \mathcal{L}_n(\alpha_n), \quad n = 1, 2, \ldots,
\]

because \( \mathbb{B}_n f \in \mathcal{L}_{n-1} \) for such \( f \). Each \( \phi_n \) can be written as

\[
\phi_n(z) = \sum_{k=0}^{n} b^{(n)}_k \mathbb{B}_k(z).
\]

Here the non-zero number \( b^{(n)}_n \) is called the leading coefficient of \( \phi_n \). We assume that the \( \phi_n \) are chosen such that \( b^{(n)}_n > 0 \) and we write \( \kappa_n = b^{(n)}_n \). It is easily shown that

\[
\kappa_n = \frac{\phi_n^*(\alpha_n)}{\phi_n^*(\alpha_n)} = \phi_n(\alpha_n).
\]

Using the uniqueness of the reproducing kernel

\[
\sum_{k=0}^{n} \phi_k(z) \overline{\phi_k(w)}
\]

for the inner product space \( \mathcal{L}_n \) one can show (see for instance [1]) that the following Christoffel-Darboux formula holds

\[
\sum_{k=0}^{n-1} \phi_k(z) \overline{\phi_k(w)} = \frac{\phi_n^*(z) \overline{\phi_n^*(w)} - \phi_n(z) \overline{\phi_n(w)}}{1 - \zeta_n(z) \overline{\zeta_n(w)}},
\]

(2.1)

and equivalently

\[
\sum_{k=0}^{n} \phi_k(z) \overline{\phi_k(w)} = \frac{\phi_n^*(z) \overline{\phi_n^*(w)} - \zeta_n(z) \overline{\zeta_n(w)} \phi_n(z) \overline{\phi_n(w)}}{1 - \zeta_n(z) \overline{\zeta_n(w)}}.
\]

(2.2)

The \( \phi_n \) and \( \phi_n^* \) satisfy the recurrence relations

\[
\phi_n(z) = \epsilon_n \frac{z - \alpha_{n-1}}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}(z) + \delta_n \frac{1 - \alpha_{n-1} z}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \phi_n^*(z), \quad n = 1, 2, \ldots
\]

(2.3)

and (superstar conjugation)

\[
\phi_n^*(z) = -\frac{\overline{\alpha_n}}{|\alpha_n|} \delta_n \frac{z - \alpha_{n-1}}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \phi_{n-1}(z) - \frac{\overline{\alpha_n}}{|\alpha_n|} \epsilon_n \frac{1 - \alpha_{n-1} z}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \phi_n^*(z),
\]

(2.4)

\( n = 1, 2, \ldots \)
with $\phi_0 = \phi^*_0 = \kappa_0$. Here

\begin{equation}
\epsilon_n = - \frac{\alpha_n}{(\alpha_n)} \frac{1}{\alpha_{n-1} \alpha_n} \frac{\phi_n^*(\alpha_{n-1})}{\kappa_n}, \tag{2.5}
\end{equation}

\begin{equation}
\delta_n = \frac{1 - \alpha_{n-1} \alpha_n}{1 - |\alpha_{n-1}|^2} \frac{\phi_n(\alpha_{n-1})}{\kappa_n}. \tag{2.6}
\end{equation}

It follows from the Christoffel-Darboux formula (2.1) with $z = w = \alpha_{n-1}$ that $\epsilon_n \neq 0$. A proof of (2.3) and (2.4) can be found in [1] or in [2], but (2.3) and (2.4) may also be derived from the superstar conjugates with respect to $w$ and with respect to $z$ and $w$ of the Christoffel-Darboux formula. We mention another consequence of the Christoffel-Darboux formula. Taking the superstar conjugate of (2.1) with respect to $z$ and $w$ and writing

$$
\psi_k = B_k, \quad k = 0, 1, \ldots, n; \quad n = 0, 1, \ldots,
$$

we obtain

\begin{equation}
\epsilon_n(z) \psi_k(w) - \phi_n(z) \phi_k(w) \frac{1 - \zeta_n(z) \zeta_n(w)}{1 - \zeta_n(z) \zeta_n(w)} = \sum_{k=0}^{n-1} B_{n-1 \setminus k}(z) B_{n-1 \setminus k}(w) \phi_n(z) \phi_k(w). \tag{2.7}
\end{equation}

For $z = w = \alpha_{n-1}$ this gives

$$
|\phi_n(\alpha_{n-1})|^2 - |\phi_n(\alpha_{n-1})|^2 = |\phi_n^*(\alpha_{n-1})|^2 \left[ 1 - |\zeta_n(\alpha_{n-1})|^2 \right] = \kappa_{n-1}^2 \frac{(1 - |\alpha_n|^2)(1 - |\alpha_{n-1}|^2)}{|1 - \alpha_n \alpha_{n-1}|^2}.
$$

Together with (2.5) and (2.6) this leads to

\begin{equation}
|\epsilon_n|^2 - |\delta_n|^2 = \frac{\kappa_{n-1}^2}{\kappa_n^2} \frac{1 - |\alpha_n|^2}{1 - |\alpha_{n-1}|^2}. \tag{2.8}
\end{equation}

In particular this implies

\begin{equation}
|\epsilon_n| > |\delta_n|. \tag{2.9}
\end{equation}

A different proof of (2.8) can be found in [4].

### 3. Associated functions

Next to the orthogonal functions $\phi_n$ we consider the associated functions $\psi_n$ defined by

$$
\psi_0(z) = - \frac{1}{\kappa_0} - M(\phi_0), \quad (\psi_0(z) = - M(\phi_0)),
$$

and

$$
\psi_n(z) = M(D(t, z) [\phi_n(z) - \phi_n(t)]), \quad n = 1, 2, \ldots
$$
Here \( M \) is acting on \( t \) and

\[
D(t, z) = \frac{t + z}{t - z}.
\]

Obviously \( \psi_n \in \mathcal{L}_n \) for \( n = 0, 1, 2, \ldots \). For \( f \in \mathcal{L}_{(n-1)^*} \), we may write

\[
f(t) = \frac{a(t)}{\omega_{n-1}(t)}\]

with \( a \in \Pi_{n-1} \), so, if \( f \neq 0 \), then

\[
D(t, z) \left[ 1 - \frac{f(t)}{f(z)} \right] = \frac{t + z}{t - z} \left[ 1 - \frac{a(t)}{a(z)} \frac{\omega_{n-1}(z)}{\omega_{n-1}(t)} \right] = \frac{(t + z)[a(z)\omega_{n-1}(t) - a(t)\omega_{n-1}(z)]}{(t - z)a(z)} \frac{1}{\omega_{n-1}(t)}
\]

is in \( \mathcal{L}_{(n-1)^*} \). Hence

\[
M \left( D(t, z) \left[ 1 - \frac{f(t)}{f(z)} \right] \phi_n(t) \right) = 0 \quad \text{for} \quad f \in \mathcal{L}_{(n-1)^*}, \quad f \neq 0.
\]

This gives immediately

\[
\psi_n(z) = M \left( D(t, z) \left[ \phi_n(z) - \frac{f(t)}{f(z)} \phi_n(t) \right] \right) \quad \text{for} \quad f \in \mathcal{L}_{(n-1)^*}, \quad f \neq 0, \quad n = 1, 2, \ldots
\]

(3.1)

For the superstar conjugates of the \( \psi_n \) we have

\[
\psi_0^*(z) = -\frac{1}{\kappa_0}
\]

and, since \( \psi_n \in \mathcal{L}_n \),

\[
\psi_n^*(z) = \mathbb{B}_n(z) M(D(t, 1/\bar{z})[\phi_n(1/\bar{z}) - \phi_n(t)])
\]

\[
= \mathbb{B}_n(z) M(D(1/t, 1/z)[\phi_n(1/\bar{z}) - \phi(1/t)])
\]

\[
= -\mathbb{B}_n(z) M(D(t, z)[\phi_n^*(z) - \phi_n(t)])
\]

\[
= M \left( D(t, z) \left[ \mathbb{B}_n(z) \phi_n^*(z) - \phi_n^*(t) \right] \right), \quad n = 1, 2, \ldots,
\]

so

\[
\psi_n^*(z) = M \left( D(t, z) \left[ \frac{\mathbb{B}_n(z)}{\mathbb{B}_n(t)} \phi_n^*(t) - \phi_n^*(z) \right] \right), \quad n = 1, 2, \ldots
\]

(3.2)

If \( f \in \mathcal{L}_{n^*}(1/\bar{\alpha}_n) \), we may write

\[
f(t) = \frac{(1 - \alpha_n t)b(t)}{\omega_n(t)} \quad \text{with} \quad b \in \Pi_{n-1}.
\]
So, for \( f \neq 0 \),

\[
D(t, z) \left[ \frac{B_n(z) - f(t)}{B_n(t) - f(z)} \right] = \frac{\omega_n(z)}{\pi_n(z)(1 - \alpha_n z)b(z)} \frac{t + z}{t - z} \left[ \pi_n(t)(1 - \alpha_n z)b(z) - \pi_n(z)(1 - \alpha_n t)b(t) \right] \frac{1}{\omega_n(t)}
\]

belongs to \( \mathcal{L}_{n+1}(1/\alpha_n) \), and it follows that

\[
M \left( D(t, z) \left[ \frac{B_n(z) - f(t)}{B_n(t) - f(z)} \right] \phi_n^*(t) \right) = 0.
\]

This gives

\[
\psi_n^*(z) = M \left( D(t, z) \left[ \frac{f(t)}{f(z)} \phi_n(t) - \phi_n^*(z) \right] \right) \text{ for } f \in \mathcal{L}_{n+1}(1/\alpha_n), \ f \neq 0, \ n = 1, 2, \ldots.
\]

The functions \( \psi_n \) and \( \psi_n^* \) satisfy the recurrences

\[
\psi_n(z) = \epsilon_n \frac{z - \alpha_{n-1}}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z) - \delta_n \frac{1 - \alpha_{n-1} z}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}^*(z), \ n = 1, 2, \ldots
\]

and (superstar conjugation)

\[
\psi_n^*(z) = \frac{\alpha_n}{|\alpha_n|} \frac{z - \alpha_{n-1}}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z) - \frac{\alpha_n}{|\alpha_n|} \frac{1 - \alpha_{n-1} z}{1 - \alpha_n z} \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}^*(z), \ n = 1, 2, \ldots
\]

A proof of these recurrence formulas is given in [1], but they also follow easily from the above results. Indeed, writing

\[
A_n(z) = \frac{z - \alpha_{n-1}}{1 - \alpha_n z} \quad \text{and} \quad B_n(z) = \frac{1 - \alpha_{n-1} z}{1 - \alpha_n z},
\]

for \( n \geq 2 \) we have

\[
\psi_n(z) - \epsilon_n A_n(z) \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z)
\]

\[
= M \left( D(t, z) \left[ \phi_n(z) - \frac{f(t)}{f(z)} \phi_n(t) \right] \right) - \epsilon_n A_n(z) \frac{\kappa_n}{\kappa_{n-1}} \ M(D(t, z)[\phi_{n-1}(z) - \phi_{n-1}(t)]),
\]

where \( f \in \mathcal{L}_{n+1}, \ f \neq 0 \) such that \( f(1/\alpha_n) = 0 \), so

\[
f(t) = \frac{(1 - \alpha_n t)p(t)}{\omega_{n-1}(t)} \quad \text{with} \quad p \in \Pi_{n-2}.
\]
Elementary calculations using (2.3) and (2.4) give

\[ \psi_n(z) - \epsilon_n A_n(z) \frac{\kappa_n}{\kappa_{n-1}} \psi_{n-1}(z) = I_1 + I_2 \]

with

\[ I_1 = \delta_n \frac{\kappa_n}{\kappa_{n-1}} B_n(z) M \left( D(t, z) \left[ \phi^*_n(z) - \frac{f(t)}{f(z)} \frac{B_n(t)}{B_n(z)} \phi^*_n(t) \right] \right) \]

\[ = -\delta_n \frac{\kappa_n}{\kappa_{n-1}} B_n(z) \psi^*_{n-1}(z) \]

since \( f(t)B_n(t) \in L_{(n-1)*}(1/\alpha_{n-1}) \), and

\[ I_2 = \epsilon_n \frac{\kappa_n}{\kappa_{n-1}} A_n(z) M \left( D(t, z) \left[ 1 - \frac{f(t)}{f(z)} \frac{A_n(t)}{A_n(z)} \right] \phi_{n-1}(t) \right) = 0 \]

since

\[ f(t)A_n(t) = \frac{(1 - \alpha_n) p(t)}{\omega_{n-1}(t)} \frac{t - \alpha_{n-1}}{1 - \alpha_n t} = \frac{p(t)}{\omega_{n-2}(t)} \in L_{(n-2)*}. \]

Formula (3.5) follows by superstar conjugation. The case \( n = 1 \) is easily verified. Thus the pair \( (\psi_n, -\psi^*_n) \) satisfies the same recurrence as the pair \( (\phi_n, \phi^*_n) \). The initial values are \( (\phi_0, \phi^*_0) = \kappa_0(1, 1) \) and \( (\psi_0, -\psi^*_0) = (-1/\kappa_0)(1, -1) \).

4. Para-orthogonal functions, quadrature formulas and modified approximants

It follows easily from the Christoffel-Darboux formula (2.1) that the zeros of \( \phi_n \) are in \( D \) and that the zeros of \( \phi^*_n \) are in \( E \). Moreover, we have \( |\phi_n(z)| < |\phi^*_n(z)| \) for \( z \in D \) and \( |\phi_n(z)| > |\phi^*_n(z)| \) for \( z \in E \). As we intend to give quadrature formulas with nodes in \( T \) we consider the functions

\[ Q_n(z, w) = \phi_n(z) + w\phi^*_n(z), \quad n = 0, 1, 2, \ldots \]  

(4.1)

with \( w \in T \) arbitrary. Clearly the zeros \( z_1, \ldots, z_n \) of \( Q_n(z, w) \) are all in \( T \) and it is easy to show that they are simple. See [1]. Of course the zeros \( z_j \) depend on \( n \) and \( w \). Since

\[ Q_n(z, w) \perp \mathcal{L}_{n-1} \cap \mathcal{L}_n(\alpha_n), \quad n = 1, 2, \ldots \]

and

\[ \langle Q_n(z, w), 1 \rangle \neq 0 \quad \text{and} \quad \langle Q_n(z, w), \mathbb{B}_n(z) \rangle \neq 0, \quad n = 1, 2, \ldots, \]

where the inner product acts on \( z \), the sequence is called para-orthogonal. As

\[ Q^*_n(z, w) = \overline{w} Q_n(z, w), \]

superstar conjugation with respect to \( z \), the \( Q_n \) are called \( \overline{w} \)-invariant. Notice that the above orthogonality remains valid if for each \( n \) we take for \( w \) a fixed \( w_n \).
in $T$. If

$$\Lambda_{n,i}(z) = \frac{1 - \bar{\alpha}_n z}{1 - \bar{\alpha}_n z_i} \frac{Q_n(z,w)}{(z - z_i)Q_n(z_i,w)}, \quad i = 1, \ldots, n,$$

(4.2)

where the prime means differentiation with respect to $z$, then $\Lambda_{n,i} \in \mathcal{L}_{n-1}$ and we have the quadrature formula (see [1])

$$M(R) = \sum_{j=1}^{n} \lambda_{n,j} R(z_j) \quad \text{for} \quad R \in \mathcal{L}_{(n-1)*} + \mathcal{L}_{n-1},$$

(4.3)

with $\lambda_{n,j} = M(\Lambda_{n,j}) > 0$ for $j = 1, \ldots, n$.

Let us assume now that $z_j = e^{i\theta_j}$, $j = 1, 2, \ldots, n$, with

$$-\pi \leq \theta_1 < \theta_2 < \ldots < \theta_n < \pi.$$

Then, using the functions $\mu_n$ given by

$$\mu_n(\theta) = \begin{cases} 0 & \text{if} \quad -\pi \leq \theta \leq \theta_1, \\
\sum_{j=1}^{k} \lambda_{n,j} & \text{if} \quad \theta_k < \theta \leq \theta_{k+1}, \quad k = 1, \ldots, n-1, \\
M(1) & \text{if} \quad \theta_n < \theta \leq \pi
\end{cases}$$

(or using the measures $\mu_n = \sum_{j=1}^{n} \lambda_{n,j} \delta_{\theta_j}$, where $\delta_{\theta_j}$ is the translated Dirac measure), it follows from Helly's theorems (or from the weak* compactness of the closed unit ball in the dual space of the Banach space $C(T)$), that the moment problem has a solution, say $\mu$. So there is a non-decreasing function (or a positive Borel measure) $\mu$ such that

$$M(R) = \int_{-\pi}^{\pi} R(e^{i\theta}) d\mu(\theta) \quad \text{for} \quad R \in \mathcal{L}_* + \mathcal{L}.$$

(4.4)

It follows from the fact that the inner product is positive definite that the solutions $\mu$ must have infinitely many points of increase (or must be measures with infinite support).

Now let

$$F_{\mu}(z) = \int_{-\pi}^{\pi} \frac{t + z}{t - z} d\mu(\theta), \quad (t = e^{i\theta})$$

(4.5)

and

$$R_n(z,w) = \int_{-\pi}^{\pi} \frac{t + z}{t - z} d\mu_n(\theta) = \sum_{j=1}^{n} \lambda_{n,j} \frac{z_j + z}{z_j - z}.$$

Then $R_n(z,w)$ can be written as

$$R_n(z,w) = \frac{P_n(z,w)}{Q_n(z,w)} \quad \text{with} \quad P_n(z,w) \in \mathcal{L}_n.$$
Indeed, for \( n \geq 2 \) we have by the results of section 3

\[
\psi_n(z) - w\psi_n^*(z) = M\left(D(t,z) \left[ \phi_n(z) - \frac{f(t)}{f(z)} \phi_n(t) \right]\right) + wM\left(D(t,z) \left[ \phi_n^*(z) - \frac{f(t)}{f(z)} \phi_n^*(t) \right]\right)
\]

\[
= M\left(D(t,z) \left[ \phi_n(z) + w\phi_n^*(z) - \frac{f(t)}{f(z)} (\phi_n(t) + w\phi_n^*(t)) \right]\right)
\]

\[
= M\left(D(t,z) \left[ Q_n(z,w) - \frac{f(t)}{f(z)} Q_n(t,w) \right]\right)
\]

for \( f \in \mathcal{L}_{(n-1)*} \cap \mathcal{L}_{ns}(1/\alpha_n), f \neq 0 \). As

\[
D(t,z) \left[ Q_n(z,w) - \frac{f(t)}{f(z)} Q_n(t,w) \right] \in \mathcal{L}_{(n-1)*} + \mathcal{L}_{n-1}
\]

for such \( f \), we have

\[
\psi_n(z) - w\psi_n^*(z) = \sum_{j=1}^{n} \lambda_{nj} \frac{z_j + z}{z_j - z} Q_n(z,w) = P_n(z,w).
\]

The case \( n = 1 \) follows by direct verification, using \( \lambda_{1,1} = M(1) = 1/\kappa_0^2 \).

In [3] a formula like (4.6) could only be obtained in the "cyclic" situation, i.e. in
the case of a finite number of points \( \alpha_n \) repeated in cyclic order.

From the partial fraction decomposition

\[
R_n(z,w) = \sum_{j=1}^{n} \lambda_{nj} \frac{z_j + z}{z_j - z}
\]

it follows that

\[
(z - z_k)R_n(z,w) = \sum_{\substack{j=1 \atop j \neq k}}^{n} \lambda_{nj} \frac{(z_j + z)(z - z_k)}{z_j - z} - \lambda_{nk}(z_k + z),
\]

so the limit \( z \to z_k \) gives

\[
\frac{P_n(z_k,w)}{Q_n(z_k,w)} = -2z_k\lambda_{nk}, \quad k = 1, \ldots, n.
\]

Hence

\[
\lambda_{nk} = -\frac{1}{2z_k} \frac{P_n(z_k,w)}{Q_n'(z_k,w)}, \quad k = 1, \ldots, n.
\]

(4.7)

From the fact that

\[
\int_{-\pi}^{\pi} g(e^{i\theta}) d(\mu - \mu_n)(\theta) = 0 \quad \text{for} \quad g \in \mathcal{L}_{(n-1)*} + \mathcal{L}_{n-1}
\]
if $\mu$ is a solution of the moment problem and $\mu_n$ is a solution of the “truncated” moment problem as above, we get the general form of some results which were obtained in [3], (p. 63, formula (3.43)), for the cyclic situation.

If $g \in L_{(n-1)*} + L_{n-1}$, then $g$ is of the form

$$g(t) = \frac{p(t)}{\omega_{n-1}(t)\pi_{n-1}(t)} \quad \text{with} \quad p \in \Pi_{2n-2},$$

so

$$D(t, z) [g(t) - g(z)] = \frac{(t + z)[\omega_{n-1}(z)\pi_{n-1}(z)p(t) - \omega_{n-1}(t)\pi_{n-1}(t)p(z)]}{(t - z)\omega_{n-1}(t)\pi_{n-1}(t)}$$

$$\frac{1}{\omega_{n-1}(t)\pi_{n-1}(t)}$$

is in $L_{(n-1)*} + L_{n-1}$, and

$$\int_{-\pi}^{\pi} D(t, z)[g(t) - g(z)]d(\mu - \mu_n)(\theta) = 0, \quad (t = e^{i\theta}).$$

As

$$F_\mu(z) - R_n(z, w) = \int_{-\pi}^{\pi} D(t, z)d(\mu - \mu_n)(\theta),$$

it follows now that

$$h(z) = \int_{-\pi}^{\pi} D(t, z)g(t)d(\mu - \mu_n)(\theta)$$

$$= g(z) \int_{-\pi}^{\pi} D(t, z)d(\mu - \mu_n)(\theta) = g(z) (F_\mu(z) - R_n(z, w)).$$

Clearly, $h$ is analytic in $D$ and in $E$ and $h(0) = 0$ and $h(\infty) = \lim_{z \to \infty} h(z) = 0$. For $g(z) = \mathbb{B}_{n-1}(z)$ we get

$$F_\mu(z) - R_n(z, w) = \frac{1}{\mathbb{B}_{n-1}(z)} h_\infty(z),$$

where $h_\infty$ is analytic in $D \cup E$ and $h_\infty(\infty) = 0$ and for $g(z) = 1/\mathbb{B}_{n-1}(z)$ we obtain

$$F_\mu(z) - R_n(z, w) = \mathbb{B}_{n-1}(z) h_0(z),$$

where $h_0$ is analytic in $D \cup E$ and $h_0(0) = 0$. Thus $R_n(z, w)$ is a “modified” Padé-type approximant to $F_\mu$. Notice that the functions $h, h_0, h_\infty$ depend on the parameter $w$. Because the approximants $R_n(z, w)$ have the same structure relative to the orthogonal sequence $\{\phi_n\}_{n=0}^{\infty}$ and the sequence of the associated functions $\{\psi_n\}_{n=0}^{\infty}$ as in the case of so called modified approximants to HPC (= Hermitian Perron-Carathéodory) continued fractions (see [6]), the $R_n(z, w)$ are also called modified approximants in the present situation.
Since
\[ P_n(z, w) = M \left( D(t, z) \left[ Q_n(z, w) - \frac{f(t)}{f(z)} Q_n(t, w) \right] \right) \]
for \( f \in \mathcal{L}_{n-1} \cap \mathcal{L}_{n-1}^*(1/\alpha_n), f \neq 0 \), it follows that for the error \( F_\mu(z) - R_n(z, w) \) we also have (with \( t = e^{i\theta} \))
\[ F_\mu(z) - R_n(z, w) = \int_{-\pi}^{\pi} D(t, z) d\mu(\theta) \]
\[ - \frac{1}{Q_n(z, w)} \int_{-\pi}^{\pi} D(t, z) \left[ Q_n(z, w) - \frac{f(t)}{f(z)} Q_n(t, w) \right] d\mu(\theta) \]
\[ = \frac{1}{f(z)Q_n(z, w)} \int_{-\pi}^{\pi} D(t, z) f(t) Q_n(t, w) d\mu(\theta). \]
See [5].

We conclude the paper with a remark on the quadrature weights. Recall that \( z_1, \ldots, z_n \) are the zeros of \( Q_n \) and that \( |z_j| = 1, j = 1, \ldots, n \). For \( z = z_j \) we have \( \phi_n(z_j) + w\phi_n^*(z_j) = 0 \) and \( \xi_n(z_j) = 1/\xi_n(z_j) \), so by the Christoffel-Darboux formula
\[ \frac{\zeta_n(z_j)}{\phi_n(z_j)} \to \frac{Q_n(z_j, w) - Q_n(t, w)}{z_j - t} = \frac{\zeta_n(z_j)}{z_j - t} \sum_{k=0}^{n-1} \phi_k(z_j) \phi_k(t). \]
Using
\[ \frac{\zeta_n'(z_j)}{\zeta_n(z_j)} = \frac{1 - |\alpha_n|^2}{(z_j - \alpha_n)(1 - \alpha_n z_j)}, \]
the limit \( t \to z_j \) yields
\[ Q_n'(z_j, w) = \frac{1}{\phi_n(z_j)} \frac{1 - |\alpha_n|^2}{(z_j - \alpha_n)(1 - \alpha_n z_j)} \sum_{k=0}^{n-1} |\phi_k(z_j)|^2. \]
Using (2.8) and the recurrence relations (2.3), (2.4), (3.4), (3.5) it follows that the functions \( \phi_n \) and \( \psi_n \) and their superstar conjugates satisfy a determinant formula
\[ \phi_n^*(z) \psi_n(z) + \phi_n(z) \psi_n^*(z) = \frac{1 - |\alpha_n|^2}{1 - \alpha_n z} \frac{-2zB_n(z)}{z - \alpha_n}. \]
As \( \phi_n(z_j) + w\phi_n^*(z_j) = 0 \), this gives
\[ P_n(z_j, w) = \frac{1}{\phi_n^*(z_j)} \frac{-2zB_n(z_j)(1 - |\alpha_n|^2)}{(z_j - \alpha_n)(1 - \alpha_n z_j)}. \]
Using (4.7), we get
\[ \lambda_n = -\frac{1}{2z_j} \frac{P_n(z_j, w)}{Q_n^*(z_j, w)} = \frac{\phi_n^*(z_j)}{\phi_n(z_j)} \frac{B_n(z_j)}{\sum_{k=0}^{n-1} |\phi_k(z_j)|^2}. \]
Since $|z_j| = 1$, we have
\[ \phi_n^*(z_j) = \bar{B}_n(z_j) \overline{\phi_n(z_j)} \]
and we obtain
\[ \lambda_{n,j} = \frac{1}{\sum_{k=0}^{n-1} |\phi_k(z_j)|^2}, \quad j = 1, \ldots, n; \quad n \in \mathbb{N}. \]

References