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ON MODAL LOGICS ARISING FROM SCATTERED LOCALLY COMPACT HAUSDORFF SPACES

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Abstract. For a topological space X, let $\mathsf{L}(X)$ be the modal logic of X where $\Box$ is interpreted as interior (and hence $\Diamond$ as closure) in X. It was shown in [6] that the modal logics $\mathsf{S4}$, $\mathsf{S4.1}$, $\mathsf{S4.2}$, $\mathsf{S4.1.2}$, $\mathsf{S4.Grz}$, $\mathsf{S4.Grz}_n$ ($n \geq 1$), and their intersections arise as $\mathsf{L}(X)$ for some Stone space X. We give an example of a scattered Stone space whose logic is not such an intersection. This gives an affirmative answer to [6, Question 6.2]. On the other hand, we show that a scattered Stone space that is in addition hereditarily paracompact does not give rise to a new logic; namely we show that the logic of such a space is either $\mathsf{S4.Grz}$ or $\mathsf{S4.Grz}_n$ for some $n \geq 1$. In fact, we prove this result for any scattered locally compact open hereditarily collectionwise normal and open hereditarily strongly zero-dimensional space.

1. Introduction

Topological semantics for intuitionistic logic was first developed by Stone [24] and Tarski [25], and for modal logic by Tsao-Chen [26], McKinsey [15], and McKinsey and Tarski [16, 17, 18]. In topological semantics for intuitionistic logic formulas are interpreted as open sets, and in topological semantics for modal logic modal box is interpreted as topological interior, and hence modal diamond as topological closure. For a topological space X, let $\mathsf{L}(X)$ be the set of formulas in the basic modal language that are valid in X. It is well known that $\mathsf{L}(X)$ is a normal extension of Lewis' modal system $\mathsf{S4}$. Much effort has been put into axiomatizing $\mathsf{L}(X)$ for a given topological space X with good separation properties. To name a few results in this direction:

- McKinsey and Tarski [16] developed an algebraic treatment of topological spaces via closure algebras. Their key result establishes that the variety of all closure algebras is generated by the closure algebra of any dense-in-itself separable metrizable space. Since closure algebras serve as algebraic models of $\mathsf{S4}$, their result is often phrased as $\mathsf{S4} = \mathsf{L}(X)$ for any dense-in-itself separable metrizable space X.
- Rasiowa and Sikorski [21, Sections III.7 and III.8] showed that separability can be dropped from the McKinsey-Tarski theorem, and that $\mathsf{L}(X) = \mathsf{S4}$ for any dense-in-itself metrizable space X.
- These results were utilized in [5] to axiomatize $\mathsf{L}(X)$ for every metrizable space X. Let $\mathsf{Iso}(X)$ be the set of isolated points of X. If $\mathsf{Iso}(X)$ is not dense in X, then $\mathsf{L}(X) = \mathsf{S4}$; if $\mathsf{Iso}(X)$ is dense in X and X is not scattered, then $\mathsf{L}(X) = \mathsf{S4.1}$; and if X is scattered, then $\mathsf{L}(X)$ is $\mathsf{S4.Grz}$ or $\mathsf{S4.Grz}_n$ for some $n \geq 1$ depending on the Cantor-Bendixson rank of X.

One of the most studied classes of topological spaces is that of compact Hausdorff spaces. A natural but quite complicated question is to axiomatize $\mathsf{L}(X)$ for an arbitrary compact Hausdorff space X. This question was taken up in [6] in the setting of zero-dimensional

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compact Hausdorff spaces, also known as Stone spaces. It was shown in [6] that each of the logics $\mathsf{S4}$, $\mathsf{S4.1}$, $\mathsf{S4.2}$, $\mathsf{S4.1.2}$, $\mathsf{S4.Grz}$, $\mathsf{S4.Grz}_n$ $(n \geq 1)$, and their intersections can be realized as $\mathsf{L}(X)$ where X is a metrizable Stone space or an extremally disconnected Stone space. We note that for the extremally disconnected setting, these results utilize a set-theoretic assumption beyond ZFC. Thus, upon leaving the setting of metrizable spaces, whether one works within ZFC or an extension of it matters, revealing interesting ties with set theory.

In [6, Question 6.2] it was posed as an open question whether there is a Stone space whose logic is not one of the previously mentioned logics. The goal of the present paper is to answer this question in the affirmative by proving that the Čech-Stone compactification of a space studied by Mrowka [19, 20] is a scattered Stone space whose logic differs from the above logics.

On the other hand, we prove that if X is a scattered Stone space that in addition is hereditarily paracompact, then $\mathsf{L}(X)$ is $\mathsf{S4.Grz}$ or $\mathsf{S4.Grz}_n$ for some $n \geq 1$ depending on the Cantor-Bendixson rank of X. In fact, we prove a stronger result that if X is a scattered locally compact open hereditarily collectionwise normal and open hereditarily strongly zero-dimensional space, then $\mathsf{L}(X)$ is either $\mathsf{S4.Grz}$ or $\mathsf{S4.Grz}_n$ for some $n \geq 1$ depending on the Cantor-Bendixson rank of X. Our results are proved within ZFC, with key technical tool being the notion of modal Krull dimension introduced in [3].

The axiomatization of $\mathsf{L}(X)$ for X a Stone space, or more generally a compact Hausdorff space, remains a challenging open problem, already in the restricted setting of scattered spaces. Indeed, the logic $\mathsf{L}(X)$ of the space X alluded to above which answers [6, Question 6.2] is difficult to axiomatize due to combinatorial complexity of the frames for $\mathsf{L}(X)$. It is likely that there will be different solutions of the problem based on set-theoretic assumptions beyond ZFC.

The paper is organized as follows. In Section 2 we provide the necessary background for the paper. Section 3 presents some basic results about modal Krull dimension for compact Hausdorff spaces, and Section 4 generalizes some of these results to locally compact Hausdorff spaces. In Section 5 we answer [6, Question 6.2] affirmatively by utilizing the work of Mrowka. In particular, we exhibit a scattered Stone space whose logic is not one of $\mathsf{S4.Grz}$ or $\mathsf{S4.Grz}_n$ for $n \geq 1$. The rest of the paper answers negatively the question obtained from modifying [6, Question 6.2] by replacing Stone space with scattered locally compact hereditarily paracompact space. Section 6 contains necessary technical background for Section 7, where a classification of the logics arising as $\mathsf{L}(X)$ for a scattered locally compact hereditarily paracompact space X is given. In fact, we prove the same classification by weakening the hereditarily paracompact condition to the open hereditarily collectionwise normal and open hereditarily strongly zero-dimensional conditions. The final section of the paper closes with a list of open problems.

2. Background

In this section we briefly recall the modal logics of interest, as well as their relational and topological semantics. We also recall the modal Krull dimension of a topological space, which will be one of our key tools in what follows.

2.1. **Modal logics.** The modal logic $\mathsf{S4}$ is the least set of formulas in the basic modal language containing the classical tautologies, the formulas

- $\Box(p \to q) \to (\Box p \to \Box q)$,
- $\Box p \to p$,
- $\Box p \to \Box\Box p$,

and closed under the inference rules of modus ponens $\frac{\varphi,\ \varphi\to\psi}{\psi}$, substitution $\frac{\varphi(p_1,\dots,p_n)}{\varphi(\psi_1,\dots,\psi_n)}$, and necessitation $\frac{\varphi}{\Box\varphi}$. A *normal extension* of $\mathsf{S4}$ is a set of formulas that contains $\mathsf{S4}$ and is closed under modus ponens, substitution, and necessitation.

As is customary, we use the abbreviation $\Diamond\varphi := \neg\Box\neg\varphi$. We will consider the following well-known normal extensions of $\mathsf{S4}$:

- $\mathsf{S4.1} = \mathsf{S4} + \Box\Diamond p \to \Diamond\Box p$,
- $\mathsf{S4.2} = \mathsf{S4} + \Diamond\Box p \to \Box\Diamond p$,
- $\mathsf{S4.1.2} = \mathsf{S4} + \Box\Diamond p \leftrightarrow \Diamond\Box p$,
- $\mathsf{S4.Grz} = \mathsf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$,
- $\mathsf{S4.Grz}_n = \mathsf{S4.Grz} + \mathsf{bd}_n$ for $n \geq 1$,

where $\mathsf{bd}_1 = \Diamond\Box p_1 \to p_1$ and $\mathsf{bd}_{n+1} = \Diamond(\Box p_{n+1} \wedge \neg\mathsf{bd}_n) \to p_{n+1}$.

2.2. **Relational semantics.** An $\mathsf{S4}$-*frame* is a pair $\mathfrak{F} = (W, R)$ where W is a nonempty set and R is a reflexive and transitive relation on W. The modal language is interpreted in $\mathfrak{F}$ as usual (see, e.g., [9] or [8]). We only point out that

$$w \Vdash \Box\varphi \text{ iff } (\forall v)(wRv \Rightarrow v \Vdash \varphi),$$

and hence

$$w \Vdash \Diamond\varphi \text{ iff } (\exists v)(wRv \text{ and } v \Vdash \varphi).$$

We say that φ is *valid* in $\mathfrak{F}$, and write $\mathfrak{F} \Vdash \varphi$, if for each valuation and each $w \in W$ we have $w \Vdash \varphi$. It is well known that $\mathsf{S4} \vdash \varphi$ iff $\mathfrak{F} \Vdash \varphi$ for every $\mathsf{S4}$-frame $\mathfrak{F}$.

For an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ we have that $\sim_R := \{(w, v) \mid wRv \text{ and } vRw\}$ is an equivalence relation on W, whose equivalence classes are called *clusters*. A singleton cluster is called *simple*. The *skeleton* of $\mathfrak{F}$ is the partially ordered set of clusters of $\mathfrak{F}$, see Figure 1.

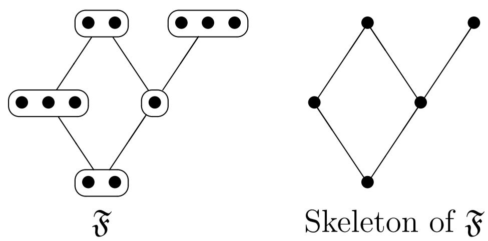

FIGURE 1. An $\mathsf{S4}$-frame $\mathfrak{F}$ and its skeleton.

For $w, v \in W$, we write $w\vec{R}v$ if wRv and $\neg(vRw)$. The *depth* of $\mathfrak{F}$ is n, denoted $\operatorname{depth}(\mathfrak{F}) = n$, if there is a sequence $w_1, \dots, w_n$ in W such that $w_i\vec{R}w_{i+1}$ for $1 \leq i < n$ and no longer sequence has this property.

A *root* of $\mathfrak{F}$ is a point $r \in W$ such that rRw for all $w \in W$. We say that $\mathfrak{F}$ is *rooted* if $\mathfrak{F}$ has a root. We call $\mathfrak{F}$ a *tree* provided that $\mathfrak{F}$ is a rooted partially ordered set such that for all $w, v, u \in W$, if vRw and uRw, then vRu or uRv. We say that $\mathfrak{F}$ is a *quasi-tree* provided its skeleton is a tree.

A *quasi-maximal* point of $\mathfrak{F}$ is $w \in W$ such that for any $v \in W$, if wRv, then vRw. By $\max(\mathfrak{F})$ we denote the set of quasi-maximal points of $\mathfrak{F}$. The following result is well-known (see, e.g., [6, Proposition 2.5] for references and details):

Proposition 2.1.
(1) $\mathsf{S4}$ *is the logic of the class of all finite quasi-trees.*
(2) $\mathsf{S4.1}$ *is the logic of the class of all finite quasi-trees such that the cluster of each quasi-maximal point is simple.*

(3) $\mathsf{S4.2}$ *is the logic of the class of all finite* $\mathsf{S4}$*-frames* $\mathfrak{F}$ *such that* $\max(\mathfrak{F})$ *is a single cluster and the subframe* $W \setminus \max(\mathfrak{F})$ *is a quasi-tree.*
(4) $\mathsf{S4.1.2}$ *is the logic of the class of all finite* $\mathsf{S4}$*-frames* $\mathfrak{F}$ *such that* $\max(\mathfrak{F})$ *is a singleton and the subframe* $W \setminus \max(\mathfrak{F})$ *is a quasi-tree.*
(5) $\mathsf{S4.Grz}$ *is the logic of the class of all finite trees.*
(6) $\mathsf{S4.Grz}_n$ *is the logic of the class of all finite trees of depth* $\leq n$*.*

2.3. **Topological semantics.** For a topological space X, we denote the interior, closure, and derivative operators of X by i_X, c_X, and d_X. We briefly recall that for each $A \subseteq X$ and $x \in X$, we have:

$$
\begin{aligned}
x \in \mathsf{i}_X A \quad &\text{iff} \quad \text{there is an open neighborhood } U \text{ of } x, \text{ such that } U \subseteq A,\\
x \in \mathsf{c}_X A \quad &\text{iff} \quad \text{for each open neighborhood } U \text{ of } x, \text{ we have } U \cap A \neq \varnothing,\\
x \in \mathsf{d}_X A \quad &\text{iff} \quad \text{for each open neighborhood } U \text{ of } x, \text{ we have } (U \setminus \{x\}) \cap A \neq \varnothing.
\end{aligned}
$$

We often omit the subscript when the context is clear. The modal language is interpreted in X by assigning to each modal formula a subset of X, interpreting the classical connectives as the Boolean operations, $\Box$ as interior, and hence $\Diamond$ as closure. Thus, under a given valuation of the propositional variables, we have:

$$
\begin{aligned}
x \Vdash \Box\varphi \quad &\text{iff} \quad \text{for some open neighborhood } U \text{ of } x, \text{ each } y \in U \text{ satisfies } y \Vdash \varphi,\\
x \Vdash \Diamond\varphi \quad &\text{iff} \quad \text{for each open neighborhood } U \text{ of } x, \text{ there is } y \in U \text{ such that } y \Vdash \varphi.
\end{aligned}
$$

We say that a formula φ is *valid* in X, written $X \Vdash \varphi$, if for each valuation and each $x \in X$ we have $x \Vdash \varphi$. It is well known that the set $\mathsf{L}(X) := \{\varphi \mid X \Vdash \varphi\}$ is a normal extension of $\mathsf{S4}$.

Topological semantics generalizes relational semantics for $\mathsf{S4}$. Given an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$, call $U \subseteq W$ and *R-upset* if $w \in U$ and wRv imply $v \in U$. Then the collection τ_R of R-upsets is a topology on W such that $\mathfrak{F} \Vdash \varphi$ iff $(W, \tau_R) \Vdash \varphi$. Such spaces are called *Alexandroff spaces*, and they have the additional property that an arbitrary intersection of open sets is open; equivalently, each point w has a least open neighborhood, namely ${\uparrow}w := \{v \mid wRv\}$. Consequently, the closure of A in an Alexandroff space is ${\downarrow}A := \{v \mid vRw \text{ for some } w \in A\}$. We write ${\downarrow}w$ for ${\downarrow}\{w\}$.

For a topological space X, a subset $A \subseteq X$ is *dense* if $\mathsf{c}A = X$ and it is *nowhere dense* if $\mathsf{ic}A = \varnothing$. A point $x \in X$ is an *isolated point* if $\{x\}$ is open in X. Let $\mathsf{Iso}(X)$ be the set of isolated points of X. Then X is *dense-in-itself* if $\mathsf{Iso}(X) = \varnothing$, X is *weakly scattered* if $\mathsf{Iso}(X)$ is dense, and X is *scattered* if every nonempty subspace Y of X has an isolated point (relative to Y). We say that X is *extremally disconnected* if the closure of each open set is open.

Definition 2.2 ([3])**.** The *modal Krull dimension* of a topological space X, denoted $\operatorname{mdim}(X)$, is defined recursively as follows:

$$
\begin{array}{lll}
\operatorname{mdim}(X) = -1 & \text{if} & X = \varnothing,\\
\operatorname{mdim}(X) \leq n & \text{if} & \operatorname{mdim}(Y) \leq n-1 \text{ for each nowhere dense subspace } Y \subseteq X,\\
\operatorname{mdim}(X) = n & \text{if} & \operatorname{mdim}(X) \leq n \text{ and } \operatorname{mdim}(X) \not\leq n-1,\\
\operatorname{mdim}(X) = \infty & \text{if} & \operatorname{mdim}(X) \not\leq n \text{ for all } n = -1, 0, 1, 2, \ldots
\end{array}
$$

We point out two characterizations of finite modal Krull dimension for nonempty spaces (see [3, Theorem 3.6] for a larger list of equivalent conditions).

Proposition 2.3 ([3])**.** *Let X be a nonempty space X and $n \geq 1$. The following are equivalent:*

(1) $X \Vdash \mathsf{bd}_n$.

(2) $\mathrm{mdim}(X) \leq n-1$.
(3) *There does not exist a sequence* $F_0, \dots, F_n$ *of nonempty closed subsets of* X *such that* $F_0 = X$ *and* F_{i+1} *is nowhere dense in* F_i *for each* $0 \leq i < n$.

For an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$, we have $\mathrm{depth}(\mathfrak{F}) = \mathrm{mdim}(W, \tau_R) + 1$. As pointed out in [3, Section 2], the difference of 1 arises because depth counts the elements in a longest chain of $\mathfrak{F}$ while modal Krull dimension counts the links between elements of such a chain, which is 1 less. It follows that the modal Krull dimension is the topological analogue of the depth of an $\mathsf{S4}$-frame. The following result is well known (see, e.g., [6, Section 2] for references and details):

Proposition 2.4.
(1) $\mathsf{S4}$ *is the logic of the class of all (finite) spaces.*
(2) $\mathsf{S4.1}$ *is the logic of the class of all (finite) weakly scattered spaces.*
(3) $\mathsf{S4.2}$ *is the logic of the class of all (finite) extremally disconnected spaces.*
(4) $\mathsf{S4.1.2}$ *is the logic of the class of all (finite) extremally disconnected weakly scattered spaces.*
(5) $\mathsf{S4.Grz}$ *is the logic of the class of all (finite) scattered spaces.*
(6) $\mathsf{S4.Grz}_n$ *is the logic of the class of all (finite) scattered spaces of modal Krull dimension* $\leq n-1$.

3. Compact Hausdorff spaces of finite modal Krull dimension

In this section we present some results about modal Krull dimension for compact Hausdorff spaces that will be utilized later. In particular, we show that continuous surjections between compact Hausdorff spaces do not increase modal Krull dimension, and we prove that if X is a compact Hausdorff space of finite modal Krull dimension, then X is scattered. This result was also obtained in [4, Remark 6.12] using the machinery of point-free topology. For the benefit of the reader, we give a direct topological proof of this result that requires no knowledge of point-free topology.

We recall that a map $f : X \to Y$ between topological spaces is *continuous* if $f^{-1}[V]$ is open in X whenever V is open in Y, that f is *open* if $f[U]$ is open in Y whenever U is open in X, and that f is *closed* if $f[F]$ is closed in Y whenever F is closed in X. It is well known that a continuous mapping between compact Hausdorff spaces is closed. We call f *irreducible* provided f is a continuous closed surjection such that $f[A]$ is a proper subset of Y whenever A is a proper closed subset of X.

Lemma 3.1. *If* $f : X \to Y$ *is irreducible and* Z *is nowhere dense in* Y, *then* $f^{-1}[Z]$ *is nowhere dense in* X.

Proof. Let $N = f^{-1}[Z]$ and $A = X \setminus N$. Since f is continuous and onto, we have:

$$\begin{aligned} f[\mathsf{c}\mathsf{i}A] &\supseteq f[\mathsf{i}A] = f[\mathsf{i}(X \setminus N)] \\ &= f[\mathsf{i}f^{-1}[Y \setminus Z]] \supseteq f[f^{-1}[\mathsf{i}(Y \setminus Z)]] = \mathsf{i}(Y \setminus Z). \end{aligned}$$

As f is closed, $f[\mathsf{c}\mathsf{i}A]$ is a closed set containing $\mathsf{i}(Y \setminus Z)$. Because Z is nowhere dense in Y, we have that $\mathsf{i}(Y \setminus Z)$ is dense in Y, so $f[\mathsf{c}\mathsf{i}A] = Y$. Since f is irreducible, $\mathsf{c}\mathsf{i}A = X$. Thus, $\mathsf{i}\mathsf{c}N = \varnothing$, and hence $f^{-1}[Z] = N$ is nowhere dense in X. □

Lemma 3.2. *Suppose* X, Y *are compact Hausdorff,* $f : X \to Y$ *is a continuous surjection, and for* $n \geq 1$, $Z_0, \dots, Z_n$ *are nonempty closed subsets of* Y *such that* $Z_0 = Y$ *and* Z_{i+1} *is nowhere dense in* Z_i *for* $0 \leq i < n$. *Then there are nonempty closed subsets* $N_0, \dots, N_n$ *of* X *such that* $N_0 = X$ *and* N_{i+1} *is nowhere dense in* N_i *for* $0 \leq i < n$.

Proof. We construct recursively $N_0, \dots, N_n$ that satisfy the conditions of the lemma.

Basis step: Set $N_0 = X$. Since Z_0 is nonempty and f is onto, we have that N_0 is nonempty. Clearly N_0 is closed in X. It is well known (see, e.g., [14, page 102]) that there is a closed subspace X_0 of X such that $f_0 : X_0 \to Z_0$ restricting f is irreducible.

Recursive step: Let $0 \le i < n$. Then nonempty N_i closed in X, X_i closed in N_i, and an irreducible surjection $f_i : X_i \to Z_i$ restricting f are given. Set $N_{i+1} = f_i^{-1}[Z_{i+1}]$. Since f_i is onto and $Z_{i+1} \neq \varnothing$, we have that $N_{i+1} \neq \varnothing$. Clearly N_{i+1} is closed in X. By Lemma 3.1, N_{i+1} is nowhere dense in X_i. Therefore, N_{i+1} is nowhere dense in N_i. The map $g_{i+1} : N_{i+1} \to Z_{i+1}$ restricting f_i is a continuous surjection. Since both N_{i+1} and Z_{i+1} are compact Hausdorff spaces, there is a closed subspace X_{i+1} of N_{i+1} such that $f_{i+1} : X_{i+1} \to Z_{i+1}$ restricting g_{i+1} is irreducible. The result follows. ☐

Lemma 3.3. *If X, Y are compact Hausdorff and $f : X \to Y$ is a continuous surjection, then* $\mathrm{mdim}(X) \ge \mathrm{mdim}(Y)$.

Proof. First suppose that Y has finite modal Krull dimension, say $\mathrm{mdim}(Y) = n$. If $n = -1$, then $Y = \varnothing$, so $X = \varnothing$, giving $\mathrm{mdim}(X) = -1 = \mathrm{mdim}(Y)$. Suppose $n \ge 0$. Then $Y \neq \varnothing$. Since f is onto, $X \neq \varnothing$, and so $\mathrm{mdim}(X) \ge 0$. Clearly if $n = 0$, then $\mathrm{mdim}(X) \ge \mathrm{mdim}(Y)$. Suppose that $n \ge 1$. Then Proposition 2.3 is applicable, and so there are nonempty closed subsets $Z_0, \dots, Z_n$ of Y such that $Z_0 = Y$ and Z_{i+1} is nowhere dense in Z_i for $0 \le i < n$. By Lemma 3.2, there are nonempty closed subsets $N_0, \dots, N_n$ of X such that $N_0 = X$ and N_{i+1} is nowhere dense in N_i for $0 \le i < n$. Therefore, $\mathrm{mdim}(X) \ge n$ by Proposition 2.3. Thus, $\mathrm{mdim}(X) \ge \mathrm{mdim}(Y)$.

Next suppose that $\mathrm{mdim}(Y) = \infty$. Then $\mathrm{mdim}(Y) \ge n$ for all $n \ge 1$. By Proposition 2.3, for each $n \ge 1$, there are nonempty closed subsets $Z_0, \dots, Z_n$ of Y such that $Z_0 = Y$ and Z_{i+1} is nowhere dense in Z_i for $0 \le i < n$. By Lemma 3.2, there are nonempty closed subsets $N_0, \dots, N_n$ of X such that $N_0 = X$ and N_{i+1} is nowhere dense in N_i for $0 \le i < n$. Applying Proposition 2.3 again yields $\mathrm{mdim}(X) \ge n$ for each $n \ge 1$. Thus, $\mathrm{mdim}(X) = \infty = \mathrm{mdim}(Y)$. ☐

Remark 3.4. By [3, Lemma 3.3], taking subspaces does not increase modal Krull dimension. Lemma 3.3 shows the same for taking continuous images in the class of compact Hausdorff spaces. As the following examples demonstrate, neither compact nor Hausdorff can be dropped from the hypothesis of Lemma 3.3.

For the first example, let X be the real line $\mathbb{R}$ with the discrete topology, Y be $\mathbb{R}$ with the usual topology, and $f : X \to Y$ be the identity map. Then f is a continuous surjection, but $\mathrm{mdim}(X) = 0$ (since the only nowhere dense subset of any nonempty discrete space is $\varnothing$) whereas $\mathrm{mdim}(Y) = \infty$ by [3, Example 3.7.1].

For the second example, let X and Y be the Alexandroff spaces and $f : X \to Y$ the map between them depicted in Figure 2. Then f is a continuous surjection, but $\mathrm{mdim}(X) = \mathrm{depth}(X) - 1 = 2 - 1 = 1$ while $\mathrm{mdim}(Y) = \mathrm{depth}(Y) - 1 = 3 - 1 = 2$.

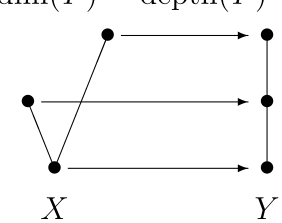

FIGURE 2. Depiction of $f : X \to Y$.

Theorem 3.5. *If X is a compact Hausdorff space of finite modal Krull dimension, then X is scattered.*

Proof. If X is not scattered, then there is a continuous surjection $f : X \to [0,1]$ (see, e.g., [22, Theorem 8.5.4]). Since $\mathrm{mdim}([0,1]) = \infty$, by Lemma 3.3, $\mathrm{mdim}(X) = \infty$, a contradiction. Thus, X is scattered. $\square$

4. Locally compact Hausdorff spaces of finite modal Krull dimension

This section generalizes Theorem 3.5 by replacing the assumption of compactness with local compactness. We also point out a connection between finite modal Krull dimension and the Cantor-Bendixson rank.

For a noncompact locally compact Hausdorff space X, let $\alpha X = X \cup \{\infty\}$ be the one-point compactification of X (see, e.g., [10, Theorem 3.5.11]). The following lemma is useful in relating $\mathrm{mdim}(X)$ and $\mathrm{mdim}(\alpha X)$.

Lemma 4.1.

(1) *Let X be a topological space and $A, B, C \subseteq X$. If C is closed in X and A is nowhere dense in B, then $A \setminus C$ is nowhere dense in $B \setminus C$.*

(2) *Let X be a noncompact locally compact Hausdorff space and $A, B \subseteq \alpha X$. If A is nowhere dense in B, then $A \setminus \{\infty\}$ is nowhere dense in $B \setminus \{\infty\}$.*

Proof. (1) Let U be open in $B \setminus C$ and $U \subseteq \mathsf{c}_{B\setminus C}(A \setminus C)$. Since C is closed in X, we have that $B \setminus C$ is open in B. Therefore, U is open in B. Since A is nowhere dense in B, we have that $A \setminus C$ is nowhere dense in B. So $U \subseteq \mathsf{c}_{B\setminus C}(A \setminus C) \subseteq \mathsf{c}_B(A \setminus C)$ yields that $U = \varnothing$. Thus, $A \setminus C$ is nowhere dense in $B \setminus C$.

(2) Observe that $\{\infty\}$ is closed in αX and apply (1). $\square$

Lemma 4.2. *Let X be a noncompact locally compact Hausdorff space and $n \in \omega$. If* $\mathrm{mdim}(X) \le n$, *then* $\mathrm{mdim}(\alpha X) \le n + 1$.

Proof. Suppose $\mathrm{mdim}(\alpha X) > n + 1$. By Proposition 2.3, there are nonempty closed subsets $F_0, F_1, \dots, F_{n+2}$ of αX such that $F_0 = \alpha X$ and F_{i+1} is nowhere dense in F_i for $0 \le i < n+2$. Put $F_i' = F_i \setminus \{\infty\}$ for $0 \le i < n+2$. Then $F_i' = F_i \cap X$ is closed in X for $0 \le i < n+2$, $F_0' = X$, and by Lemma 4.1(2), F_{i+1}' is nowhere dense in F_i' for $0 \le i < n+1$. If $F_{n+1}' = \varnothing$, then $F_{n+1} = \{\infty\}$. However, $\varnothing \ne F_{n+2} \subseteq F_{n+1} = \{\infty\}$ gives that $F_{n+2} = \{\infty\} = F_{n+1}$, contradicting that F_{n+2} is nowhere dense in F_{n+1}. Therefore, $F_{n+1}' \ne \varnothing$, and hence $\mathrm{mdim}(X) > n$ by Proposition 2.3. $\square$

Remark 4.3. Let X be a noncompact locally compact Hausdorff space. Since X is a subspace of αX, by [3, Lemma 3.3], $\mathrm{mdim}(X) \le \mathrm{mdim}(\alpha X)$. Therefore, if $\mathrm{mdim}(X)$ is finite, then Lemma 4.2 yields that $\mathrm{mdim}(X) \le \mathrm{mdim}(\alpha X) \le \mathrm{mdim}(X) + 1$. However, $\mathrm{mdim}(\alpha X)$ could take on both values. For example, if X is the ordinal space ω, then $\mathrm{mdim}(X) = 0$ (since X is discrete) and $\mathrm{mdim}(\alpha X) = \mathrm{mdim}(\omega + 1) = 1$ by [3, Example 3.7.3]. On the other hand, if X is the ordinal $\omega 2$, then $\alpha X = \omega 2 + 1$. It follows from [3, Lemma 3.3 and Example 3.7] that $\mathrm{mdim}(X) = \mathrm{mdim}(\alpha X) = 1$ since $\omega + 1 \subseteq X \subseteq \alpha X \subseteq \omega^2$.

Lemma 4.4. *Let X be a noncompact locally compact Hausdorff space. If X is scattered, then αX is scattered.*

Proof. Let Y be a nonempty subspace of αX. If $\infty \notin Y$, then Y is a nonempty subspace of X, and since X is scattered, Y has an isolated point. Suppose $\infty \in Y$. If $Y = \{\infty\}$, then Y consists of a single isolated point. If $Y \ne \{\infty\}$, then $Y \setminus \{\infty\}$ is a nonempty subspace of X, and hence has an isolated point, say x. So there is U open in X such that $\{x\} = (Y \setminus \{\infty\}) \cap U$. Therefore, $\{x\} = Y \cap U$. But U open in X and X open in αX imply that U is open in αX. Thus, Y has an isolated point, and so αX is scattered. $\square$

Theorem 4.5. *Let X be locally compact Hausdorff.*

(1) *If X is of finite modal Krull dimension, then X is scattered.*
(2) *If X is scattered, then X is zero-dimensional.*

Proof. (1) If X is compact, then apply Theorem 3.5. Suppose X is noncompact. By Lemma 4.2, αX is of finite modal Krull dimension. Since αX is compact Hausdorff, αX is scattered by Theorem 3.5. Therefore, X is a scattered space as it is a subspace of αX.

(2) If X is compact, then it is well known that X is zero-dimensional (see, e.g., [22, Theorem 8.5.4]). Suppose X is noncompact. By Lemma 4.4, αX is scattered. Being a scattered compact Hausdorff space, αX is zero-dimensional. But then X is zero-dimensional as it is a subspace of αX. $\square$

The rest of this section relates finite modal Krull dimension and the Cantor-Bendixson rank. Let X be a topological space and let $A \subseteq X$. For an ordinal α define $\mathsf{d}^\alpha A$ by

$$\begin{aligned} \mathsf{d}^0 A &= A, \\ \mathsf{d}^{\alpha+1} A &= \mathsf{d}(\mathsf{d}^\alpha A), \\ \mathsf{d}^\alpha A &= \bigcap \{\mathsf{d}^\beta A \mid \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The *Cantor-Bendixson rank* of X is the least ordinal γ satisfying $\mathsf{d}^\gamma X = \mathsf{d}^{\gamma+1} X$. Setting $D = \mathsf{d}^\gamma X$ and $S = X \setminus D$ gives the *Cantor-Bendixson decomposition* of X into the dense-in-itself closed subspace D and the scattered open subspace S of X. If X is scattered, then $D = \varnothing$ and $X = S$. Similarly, if X is dense-in-itself, then $X = D$ and $S = \varnothing$.

Recall that if Y is a subspace of X, then $\mathsf{d}_Y^n A = \mathsf{d}_X^n(A) \cap Y$ for $A \subseteq Y$ and $n \in \omega$.

Lemma 4.6. *If Y is an open subspace of X, then $\mathsf{d}_X^n(X) \cap Y = \mathsf{d}_Y^n Y$ for each $n \in \omega$.*

Proof. By induction on $n \in \omega$. The base case follows from

$$\mathsf{d}_X^0(X) \cap Y = X \cap Y = Y = \mathsf{d}_Y^0 Y.$$

Suppose $\mathsf{d}_X^n(X) \cap Y = \mathsf{d}_Y^n Y$. We have:

$$\begin{aligned} \mathsf{d}_Y^{n+1} Y &= \mathsf{d}_Y(\mathsf{d}_Y^n Y) = \mathsf{d}_X(\mathsf{d}_Y^n Y) \cap Y \\ &= \mathsf{d}_X(\mathsf{d}_X^n(X) \cap Y) \cap Y \subseteq \mathsf{d}_X(\mathsf{d}_X^n X) \cap Y = \mathsf{d}_X^{n+1}(X) \cap Y. \end{aligned}$$

Let $x \in \mathsf{d}_X^{n+1}(X) \cap Y$. Suppose U is an open neighborhood of x in Y. Since Y is open, so is U in X. As $x \in \mathsf{d}_X(\mathsf{d}_X^n X)$, there is $y \in U \setminus \{x\}$ such that $y \in \mathsf{d}_X^n X$. Therefore, $y \in \mathsf{d}_X^n(X) \cap Y = \mathsf{d}_Y^n Y$, giving that $x \in \mathsf{d}_Y(\mathsf{d}_Y^n Y) = \mathsf{d}_Y^{n+1} Y$. Thus, $\mathsf{d}_X^{n+1}(X) \cap Y \subseteq \mathsf{d}_Y^{n+1} Y$, and the result follows. $\square$

Lemma 4.7. *Let X be weakly scattered.*

(1) *For each $n \in \omega$ and $A \subseteq X$, the set $\mathsf{d}^{n+1} A$ is nowhere dense in X.*
(2) *$\mathsf{d}X$ is the largest nowhere dense subset of X.*

Proof. (1) Since $\mathsf{d}^{n+1} A \subseteq \mathsf{d}^{n+1} X \subseteq \mathsf{d}X$, it is sufficient to show that $\mathsf{d}X$ is nowhere dense in X. Let U be a nonempty open set in X. Since X is weakly scattered, $\mathsf{Iso}(X)$ is dense, so $U \cap \mathsf{Iso}(X) \neq \varnothing$. Therefore, $U \not\subseteq X \setminus \mathsf{Iso}(X) = \mathsf{d}X$. Since $\mathsf{d}X$ is closed, it follows that $\mathsf{d}X$ is nowhere dense in X.

(2) Let N be nowhere dense in X. Then $\mathsf{Iso}(X) \cap N = \varnothing$. Therefore, $N \subseteq X \setminus \mathsf{Iso}(X) = \mathsf{d}X$, and so $\mathsf{d}X$ is the largest nowhere dense subset of X. $\square$

Remark 4.8. Since a scattered space is weakly scattered, Lemma 4.7 applies to scattered spaces.

Theorem 4.9. *Let X be a nonempty scattered Hausdorff space and $n \in \omega$. Then* $\mathrm{mdim}(X) = n$ *iff* $\mathsf{d}^{n+1}X = \varnothing$ *and* $\mathsf{d}^{n}X \neq \varnothing$.

Proof. By induction on $n \in \omega$.

Base case: Let $n = 0$. Since X is a nonempty Hausdorff space, by [3, Remark 4.8 and Theorem 4.9], $\mathrm{mdim}(X) = 0$ iff X is discrete, which happens iff $X = \mathsf{Iso}(X)$, which is equivalent to $\mathsf{d}^1 X = X \setminus \mathsf{Iso}(X) = \varnothing$ and $\mathsf{d}^0 X = X \neq \varnothing$.

Inductive case: Let $n \geq 0$ and for every nonempty scattered Hausdorff space Y, we have $\mathrm{mdim}(Y) = n$ iff $\mathsf{d}_Y^{n+1} Y = \varnothing$ and $\mathsf{d}_Y^{n} Y \neq \varnothing$.

Suppose $\mathrm{mdim}(X) = n+1$. Set $Y = \mathsf{d}_X X$. By Lemma 4.7(1), Y is nowhere dense in X, so $\mathrm{mdim}(Y) \leq n$. By Proposition 2.3, there are nonempty closed $F_0, \dots, F_n$ in X such that $F_0 = X$ and F_{i+1} is nowhere dense in F_i for $0 \leq i < n$. By Lemma 4.7(2), $F_1 \subseteq Y$, so F_2 is nowhere dense in Y. Therefore, $Y, F_2, \dots, F_n$ are closed in Y, F_2 is nowhere dense in Y, and F_{i+1} is nowhere dense in F_i for $2 \leq i < n$. Applying Proposition 2.3 again yields $\mathrm{mdim}(Y) \geq n$. Thus, $\mathrm{mdim}(Y) = n$. Since Y is a nonempty closed scattered subspace of X, by the inductive hypothesis, we have:

$$\mathsf{d}_X^{n+2} X = \mathsf{d}_X^{n+1}(\mathsf{d}_X X) = \mathsf{d}_X^{n+1} Y = \mathsf{d}_X^{n+1}(Y) \cap Y = \mathsf{d}_Y^{n+1} Y = \varnothing$$

and

$$\mathsf{d}_X^{n+1} X = \mathsf{d}_X^{n}(\mathsf{d}_X X) = \mathsf{d}_X^{n} Y = \mathsf{d}_X^{n}(Y) \cap Y = \mathsf{d}_Y^{n} Y \neq \varnothing.$$

Conversely, suppose $\mathsf{d}_X^{n+2} X = \varnothing$ and $\mathsf{d}_X^{n+1} X \neq \varnothing$. Set $F_i = \mathsf{d}_X^i X$ for $0 \leq i \leq n+1$. Then each F_i is a nonempty closed scattered subspace of X. Therefore, by Lemma 4.7(1), $F_{i+1} = \mathsf{d}_X(\mathsf{d}_X^i X) = \mathsf{d}_X F_i = \mathsf{d}_{F_i} F_i$ is nowhere dense in F_i. Thus, $X = F_0, \dots, F_{i+1}$ are nonempty closed subsets of X with F_{i+1} nowhere dense in F_i for $0 \leq i < n+1$. By Proposition 2.3, $\mathrm{mdim}(X) \geq n+1$. Since F_1 is closed in X, we have:

$$\mathsf{d}_{F_1}^{n+1} F_1 = \mathsf{d}_X^{n+1}(F_1) \cap F_1 = \mathsf{d}_X^{n+1} F_1 = \mathsf{d}_X^{n+1}(\mathsf{d}_X X) = \mathsf{d}_X^{n+2} X = \varnothing$$

and

$$\mathsf{d}_{F_1}^{n} F_1 = \mathsf{d}_X^{n}(F_1) \cap F_1 = \mathsf{d}_X^{n} F_1 = \mathsf{d}_X^{n}(\mathsf{d}_X X) = \mathsf{d}_X^{n+1} X \neq \varnothing.$$

So, by the inductive hypothesis, $\mathrm{mdim}(F_1) = n$. Let N be nowhere dense in X. By Lemma 4.7(2), $N \subseteq F_1$, so $\mathrm{mdim}(N) \leq \mathrm{mdim}(F_1) = n$ by [3, Lemma 3.3]. Thus, $\mathrm{mdim}(X) \leq n+1$, and so $\mathrm{mdim}(X) = n+1$. $\square$

Corollary 4.10. *Let X be a nonempty locally compact Hausdorff space of finite modal Krull dimension. Then the Cantor-Bendixson rank of X is* $\mathrm{mdim}(X) + 1$.

Proof. Let $\mathrm{mdim}(X) = n \in \omega$. By Theorem 4.5, X is scattered; and by Theorem 4.9, $\mathsf{d}^n X \neq \varnothing$ and $\mathsf{d}^{n+1} X = \varnothing$. Thus, the Cantor-Bendixson rank of X is $n + 1 = \mathrm{mdim}(X) + 1$. $\square$

5. A new logic arising from a scattered Stone space

If X is a scattered space, then $X \Vdash \mathsf{S4.Grz}$, so $\mathsf{S4.Grz} \subseteq \mathsf{L}(X)$. Moreover, $\mathsf{S4.Grz}$ and $\mathsf{S4.Grz}_n$ for each $n \geq 1$ arise as $\mathsf{L}(X)$ for some scattered Stone space X. In this section we construct a scattered Stone space whose logic is not one of these logics, thus obtaining an affirmative answer to [6, Question 6.2]. Our construction utilizes the work of Mrowka [19, 20]. Recall that a family $\mathscr{R}$ of infinite subsets of the natural numbers $\mathbb{N}$ is *almost disjoint* provided the intersection of any two distinct members of $\mathscr{R}$ is finite.

Definition 5.1. A *Mrowka* space is $X := \mathbb{N} \cup \mathscr{R}$ where $\mathscr{R}$ is almost disjoint and the topology on X is generated by the basis consisting of:

- $O(n) := \{n\}$ for $n \in \mathbb{N}$,
- $O(R, F) := \{R\} \cup (R \setminus F)$ for $R \in \mathscr{R}$ and $F \subset \mathbb{N}$ finite.

It is a consequence of [19] that every Mrowka space X has the following properties:

Proposition 5.2.
(1) $\mathbb{N}$ *is open and dense in* X.
(2) $\mathscr{R}$ *is closed and discrete in* X.
(3) $O(R) := O(R, \varnothing)$ *is a clopen subset of* X.
(4) $O(R)$ *is homeomorphic to the one-point compactification* $\alpha\mathbb{N}$ *of* $\mathbb{N}$.

Consequently, a Mrowka space X is a scattered locally compact Hausdorff space. If $\mathscr{R}$ is infinite, then X is not compact. By [20], there is an infinite almost disjoint family $\mathscr{R}$ such that the Čech-Stone compactification βX of X is the one-point compactification αX of X. From now on, we will assume that X is a Mrowka space such that $\beta X = \alpha X$, see Figure 3.

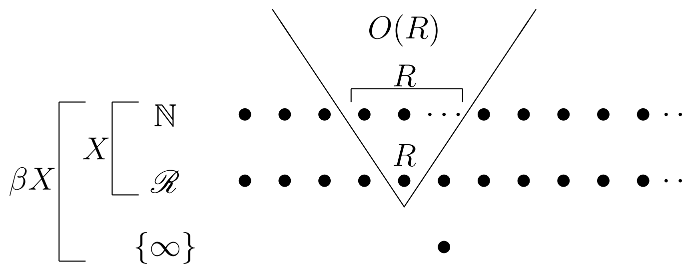

FIGURE 3. Depiction of $\beta X = \alpha X$ for a Mrowka space X, and of $O(R)$ for $R \in \mathscr{R}$.

Lemma 5.3. *The space* $\beta X = X \cup \{\infty\}$ *is compact, scattered, and has modal Krull dimension* 2.

Proof. Clearly βX is compact. Since X is scattered, αX is scattered by Lemma 4.4. So $\beta X = \alpha X$ is scattered. Let d be the derivative operator in βX. Because $\mathbb{N}$ is the set of isolated points of βX, we have $\mathsf{d}(\beta X) = \mathscr{R} \cup \{\infty\}$. Since $\mathscr{R}$ is discrete in X and ∞ is a limit point of $\mathscr{R}$, the set of isolated points of $\mathsf{d}(\beta X)$ is $\mathscr{R}$. Therefore, $\mathsf{d}^2(\beta X) = \{\infty\}$ and $\mathsf{d}^3(\beta X) = \varnothing$. Thus, $\mathrm{mdim}(\beta X) = 2$ by Theorem 4.9. □

Recall that a function $f : X \to Y$ between topological spaces is *interior* if f is continuous and open. Equivalently, f is interior provided $f^{-1}[\mathsf{c}_Y A] = \mathsf{c}_X f^{-1}[A]$ for each $A \subseteq Y$ (see, e.g., [21, Section III.3]). We say that Y is an *interior image* of X provided that f is an interior surjection. We call a function $f : X \to \mathfrak{F}$ from a topological space X to an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ *interior* provided that $f : X \to (W, \tau_R)$ is interior, where (W, τ_R) is the Alexandroff space associated with $\mathfrak{F}$.

Lemma 5.4. *Let* Y *be a space and* $\mathfrak{F} = (W, R)$ *a partially ordered* $\mathsf{S4}$*-frame such that there is* $m \in \max(\mathfrak{F})$*. If* $\mathfrak{F}$ *is an interior image of a clopen subspace* Z *of* Y*, then* $\mathfrak{F}$ *is an interior image of* Y*.*

Proof. Let Z be a clopen subspace of Y and $f : Z \to \mathfrak{F}$ an interior surjection. Extend f to $g : Y \to \mathfrak{F}$ by setting $g(x) = m$ for $x \in Y \setminus Z$. Clearly g is a well-defined surjection. Let U be open in Y. Then both $U \cap Z$ and $U \setminus Z$ are open in Y, and $U \cap Z$ is open in Z. So $g[U] = g[U \cap Z] \cup g[U \setminus Z] = f[U \cap Z] \cup g[U \setminus Z]$. Now $f[U \cap Z]$ is open in $\mathfrak{F}$ since f is interior and $U \cap Z$ is open in Z. Also $g[U \setminus Z]$ is either $\varnothing$ or $\{m\}$, both of which are open in $\mathfrak{F}$. Thus, $g[U]$ is open in $\mathfrak{F}$, and hence g is an open mapping.

Let U be an open subset of $\mathfrak{F}$. Then $f^{-1}[U]$ is open in Z, and so is open in Y. If $m \notin U$, then $g^{-1}[U] = f^{-1}[U]$ is open in Y. If $m \in U$, then $g^{-1}[U] = (Y \setminus Z) \cup f^{-1}[U]$ is a union of two open subsets of Y, and hence is open in Y. Thus, g is continuous, and so $\mathfrak{F}$ is an interior image of Y. □

Lemma 5.5. *Suppose that X is a Mrowka space such that $\beta X = \alpha X$ and $\mathfrak{F}$ is a finite rooted partially ordered $\mathsf{S4}$-frame. Then $\mathfrak{F}$ is an interior image of βX iff $\mathfrak{F}$ is an interior image of an open subspace of βX.*

Proof. One implication is obvious. For the other, suppose that $\mathfrak{F}$ is an interior image of an open subspace U of βX, say via $f : U \to \mathfrak{F}$. Let $x \in f^{-1}(r)$ where r is the root of $\mathfrak{F}$. Since $\beta X = \alpha X$ is scattered, it is zero-dimensional. So there is V clopen in βX such that $x \in V \subseteq U$. Then V is open in U, and hence $f|_V$ is interior. It is onto since $f[V]$ is open in $\mathfrak{F}$ and $r = f(x) \in f[V]$, giving that $f[V] = \mathfrak{F}$. Applying Lemma 5.4 yields that $\mathfrak{F}$ is an interior image of βX. □

Let $\mathfrak{F}$ be a finite rooted partially ordered $\mathsf{S4}$-frame of depth 2. Then $\mathfrak{F}$ is isomorphic to a *k-fork* $\mathfrak{F}_k$ depicted in Figure 4.

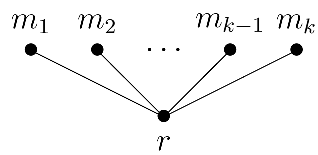

FIGURE 4. The k-fork $\mathfrak{F}_k$.

Lemma 5.6. *For any $k \in \mathbb{N}$, the k-fork $\mathfrak{F}_k$ is an interior image of βX.*

Proof. Choose and fix $R \in \mathscr{R}$. The subspace $O(R)$ is homeomorphic with the ordinal space $\omega + 1$. For each $k \in \mathbb{N}$, the k-fork $\mathfrak{F}_k$ is an interior image of $\omega + 1$ (see, e.g., [7, Lemma 3.4]), and hence $\mathfrak{F}_k$ is an interior image of $O(R)$. Since $O(R)$ is open in βX and $\mathfrak{F}_k$ is a finite poset, the result follows from Lemma 5.5. □

Consider the tree $\mathfrak{F}$ depicted in Figure 5.

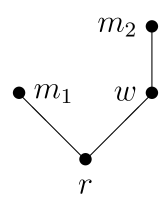

FIGURE 5. The tree $\mathfrak{F}$.

Lemma 5.7. *Let $\mathfrak{F}$ be as in Figure 5.*

(1) *$\mathfrak{F}$ is not an interior image of βX.*
(2) *$\mathfrak{F}$ is not an interior image of any open subspace of βX.*

Proof. (1) Suppose there is an onto interior map $f : \beta X \to \mathfrak{F}$. Since $\mathbb{N}$ consists of isolated points and f is open, we have $f[\mathbb{N}] \subseteq \{m_1, m_2\}$. Let $R \in \mathscr{R}$. Since $O(R)$ is open in βX, we have that $f[O(R)]$ is open in $\mathfrak{F}$. If $f(R) = r$, then

$$f[O(R)] = f[\{R\} \cup R] = \{f(R)\} \cup f[R] \subseteq \{r\} \cup f[\mathbb{N}] \subseteq \{r\} \cup \{m_1, m_2\} = \mathfrak{F} \setminus \{w\},$$

which is not open in $\mathfrak{F}$. Therefore, $f(R) \neq r$, and so $f^{-1}(r) = \{\infty\}$.

Put $A = f^{-1}(m_1)$ and $B = f^{-1}[\{w, m_2\}]$. Then A and B are disjoint open subsets of X such that $A \cup B = X$. Thus, A and B are clopen, and hence completely separated subsets of X. By [10, Corollary 3.6.2], $\mathsf{c}A \cap \mathsf{c}B = \varnothing$, where c is closure in βX. But

$$\mathsf{c}A = \mathsf{c}f^{-1}(m_1) = f^{-1}[\mathsf{c}_{\mathfrak{F}}\{m_1\}] = f^{-1}[{\downarrow} m_1] = f^{-1}[\{m_1, r\}] = A \cup \{\infty\}$$

and

$$\mathsf{c}B = \mathsf{c}f^{-1}[\{w, m_2\}] = f^{-1}[\mathsf{c}_{\mathfrak{F}}\{w, m_2\}] = f^{-1}[\downarrow\{w, m_2\}] = f^{-1}[\{r, w, m_2\}] = B \cup \{\infty\},$$

yielding a contradiction as $\infty \in \mathsf{c}A \cap \mathsf{c}B = \varnothing$. Thus, no such f exists.

(2) follows immediately from (1) and Lemma 5.5. □

We will utilize the above lemmas to show that if X is a Mrowka space such that $\beta X = \alpha X$, then $\mathsf{L}(X)$ is different from $\mathsf{S4.Grz}$ and $\mathsf{S4.Grz}_n$ for every $n \geq 1$. For this we recall the so-called Fine-Jankov formula $\chi_{\mathfrak{F}}$ of a finite rooted **S4**-frame $\mathfrak{F} = (W, R)$ (see [11]). Suppose that W has n elements, say $w_1, \dots, w_n$ where w_1 is a root of $\mathfrak{F}$, and define $\chi_{\mathfrak{F}}$ as the conjunction of the formulas:

- p_1
- $\Box(p_1 \vee \cdots \vee p_n)$
- $\Box(p_i \to \neg p_j)$ for distinct $1 \leq i, j \leq n$
- $\Box(p_i \to \Diamond p_j)$ when $w_i R w_j$
- $\Box(p_i \to \neg\Diamond p_j)$ when $\neg(w_i R w_j)$.

The formula $\chi_{\mathfrak{F}}$ encodes the structure of the frame $\mathfrak{F}$ in such a way that for any $\mathsf{S4}$-frame $\mathfrak{G}$ we have $\mathfrak{G} \Vdash \neg\chi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not a p-morphic image of a generated subframe of $\mathfrak{G}$ [11, Section 2, Lemma I]. The following generalizes Fine's result to the topological setting (see [3, Lemma 3.5]):

Proposition 5.8. *For a topological space X and a finite rooted $\mathsf{S4}$-frame $\mathfrak{F}$ we have $X \Vdash \neg\chi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is not an interior image of any open subspace of X.*

We are ready to give an affirmative answer to [6, Question 6.2].

Theorem 5.9. *For any Mrowka space X such that $\beta X = \alpha X$ we have that*

$$\mathsf{S4.Grz}_3 + \neg\chi_{\mathfrak{F}} \subseteq \mathsf{L}(\beta X) \subset \mathsf{S4.Grz}_2$$

where $\chi_{\mathfrak{F}}$ is the Fine-Jankov formula of the tree $\mathfrak{F}$ depicted in Figure 5.

Proof. Since βX is scattered, $\mathsf{S4.Grz} \subseteq \mathsf{L}(\beta X)$. By Lemma 5.3, $\mathrm{mdim}(\beta X) = 2$. So by Proposition 2.3, $\beta X \Vdash \mathsf{bd}_3$ and $\beta X \nVdash \mathsf{bd}_2$. It follows from Lemma 5.7(2) and Proposition 5.8 that $\beta X \Vdash \neg\chi_{\mathfrak{F}}$. Therefore, $\mathsf{S4.Grz}_3 + \neg\chi_{\mathfrak{F}} \subseteq \mathsf{L}(\beta X)$.

Since $\mathsf{S4.Grz}_2$ is the logic of the k-forks $\mathfrak{F}_k$, $k \geq 1$, and by Lemma 5.6, each $\mathfrak{F}_k$ is an interior image of βX, we have that $\mathsf{L}(\beta X) \subseteq \mathsf{S4.Grz}_2$. The containment is strict since $\mathsf{L}(\beta X) \nvdash \mathsf{bd}_2$. □

Remark 5.10. It is well known (see, e.g., [9, Sec. 9.4]) that in the intuitionistic setting, the negation of the Fine-Jankov formula of the tree $\mathfrak{F}$ depicted in Figure 5 axiomatizes the *Scott logic* obtained by adding to the intuitionistic propositional calculus the *Scott axiom*

$$((\neg\neg p \to p) \to (p \vee \neg p)) \to (\neg p \vee \neg\neg p).$$

Thus, the logic $\mathsf{S4.Grz}_3 + \neg\chi_{\mathfrak{F}}$ can alternatively be axiomatized by adding to $\mathsf{S4.Grz}_3$ the Gödel translation of the Scott axiom.

The remainder of the paper shows that no new logics arise upon imposing an additional condition on a scattered Stone space. In particular, the logic arising from a scattered hereditarily paracompact Stone space is either $\mathsf{S4.Grz}$ or $\mathsf{S4.Grz}_n$ for some $n \geq 1$. In fact, we prove a stronger result by relaxing compact to locally compact and hereditarily paracompact to open hereditarily collectionwise normal and open hereditarily strongly zero-dimensional.

6. Basic cardinality results about locally compact Hausdorff spaces

In this section we present basic cardinality results about locally compact Hausdorff spaces that will be utilized in Section 7. In what follows we will freely use the Axiom of Choice and view cardinal numbers as initial ordinal numbers. For a topological space X and $x \in X$, let $\chi(x)$ be the least cardinal number of a local base at x. The following result is well known.

Theorem 6.1 (Alexandroff and Urysohn [1]). *Let X be locally compact Hausdorff. Then for every $x \in X$ and every open neighborhood U of x, we have $\chi(x) \le |U|$.*

The next lemma follows from the well-known technique in the theory of resolvability developed by Hewitt [13] (see Theorems 42, 46, and 47). For convenience, we present a sketch of the proof.

Lemma 6.2. *Let X be a locally compact Hausdorff space, $x \in \mathsf{d}X$, and $n \ge 2$. Then there exist pairwise disjoint $A_1, \dots, A_n \subseteq X \setminus \{x\}$ such that $x \in \mathsf{d}A_i$ for each $i = 1, \dots, n$.*

Proof. (Sketch) If X is finite, then since X is Hausdorff, $\mathsf{d}X = \varnothing$, and there is nothing to prove. Suppose X is infinite. Let $\gamma := \chi(x)$. Then there is a local base at x which can be enumerated as $\{U_\alpha \mid \alpha < \gamma\}$. Since X is Hausdorff and $x \in \mathsf{d}X$, each U_α is infinite, and by the Alexandroff and Urysohn theorem, $\gamma \le |U_\alpha|$ for each $\alpha < \gamma$. We build the A_i by transfinite recursion.

Base step ($\alpha = 0$)**:** Since $U_0 \setminus \{x\}$ is infinite, choose distinct $a_1^0, \dots, a_n^0 \in U_0 \setminus \{x\}$, and let $A_1^0 = \{a_1^0\}, \dots, A_n^0 = \{a_n^0\}$. Then $A_1^0, \dots, A_n^0 \subseteq X \setminus \{x\}$ are pairwise disjoint and $|A_i^0| = 1 < \gamma$ for each $i = 1, \dots, n$.

Recursive step: Let $\beta < \gamma$ be nonzero. Assume for each $\alpha < \beta$ that the pairwise disjoint sets $A_1^\alpha, \dots, A_n^\alpha \subseteq X \setminus \{x\}$ have already been chosen so that $|A_i^\alpha| < \gamma$ for each $i = 1, \dots, n$. For each $i = 1, \dots, n$, we have that $\left|\bigcup_{\alpha<\beta} A_i^\alpha\right| < \gamma$ because $\beta < \gamma$ and $|A_i^\alpha| < \gamma$ for each $\alpha < \beta$. Since $\gamma \le |U_\beta|$, we may choose distinct

$$a_1^\beta, \dots, a_n^\beta \in (U_\beta \setminus \{x\}) \setminus \left(\bigcup_{\alpha<\beta} A_1^\alpha \cup \dots \cup \bigcup_{\alpha<\beta} A_n^\alpha\right),$$

and set

$$A_1^\beta = \left(\bigcup_{\alpha<\beta} A_1^\alpha\right) \cup \{a_1^\beta\}, \dots, A_n^\beta = \left(\bigcup_{\alpha<\beta} A_n^\alpha\right) \cup \{a_n^\beta\}.$$

We then have that $|A_i^\beta| < \gamma$ for each $i = 1, \dots, n$. Define

$$A_1 = \bigcup_{\beta<\gamma} A_1^\beta, \dots, A_n = \bigcup_{\beta<\gamma} A_n^\beta.$$

Then $A_1, \dots, A_n \subseteq X \setminus \{x\}$ are pairwise disjoint.

Let U be an open neighborhood of x. Because $\{U_\alpha \mid \alpha < \gamma\}$ is a local base at x, there is $\alpha < \gamma$ such that $U_\alpha \subseteq U$. For each i we have that $a_i^\alpha \in (U_\alpha \setminus \{x\}) \cap A_i$. This yields that $(U \setminus \{x\}) \cap A_i \ne \varnothing$. Thus, $x \in \mathsf{d}A_i$ for each $i = 1, \dots, n$. □

Remark 6.3. In Lemma 6.2, we can replace n by an arbitrary cardinal κ strictly less than γ.

Recall that a family $\mathscr{F}$ of subsets of a space X is *discrete* provided for each $x \in X$ there is an open neighborhood that has nonempty intersection with at most one member of $\mathscr{F}$. Note that a discrete family is pairwise disjoint. Also recall that a T_1-space X is *collectionwise normal* provided if $\{F_i \mid i \in I\}$ is a discrete family of closed subsets of X, then there is a discrete family $\{U_i \mid i \in I\}$ of open subsets of X such that $F_i \subseteq U_i$ for all $i \in I$. Clearly a collectionwise normal space is normal (and hence also Hausdorff).

Lemma 6.4. *Let X be a locally compact collectionwise normal space of modal Krull dimension $n \in \omega$. Then there is a family $\{B_x \mid x \in \mathsf{d}^n X\}$ of pairwise disjoint clopen subsets of X such that $B_x \cap \mathsf{d}^n X = \{x\}$ for each $x \in \mathsf{d}^n X$.*

Proof. Theorem 4.5 yields that X is scattered and zero-dimensional. By Theorem 4.9, $\mathsf{d}^n X \neq \varnothing$ and $\mathsf{d}^{n+1} X = \varnothing$. Therefore, $\mathsf{d}^n X$ is discrete in X, and so $\{\{x\} \mid x \in \mathsf{d}^n X\}$ is a discrete family of closed subsets of X. Since X is collectionwise normal, there is a discrete family $\{U_x \mid x \in \mathsf{d}^n X\}$ of open subsets of X such that $\{x\} \subseteq U_x$ for each $x \in \mathsf{d}^n X$. Being discrete, $\{U_x \mid x \in \mathsf{d}^n X\}$ is pairwise disjoint. Because X is zero-dimensional, there is a family $\{B_x \mid x \in \mathsf{d}^n X\}$ of clopen subsets of X such that $x \in B_x$ and $B_x \subseteq U_x$ for each $x \in \mathsf{d}^n X$. Clearly $\{B_x \mid x \in \mathsf{d}^n X\}$ is pairwise disjoint since $\{U_x \mid x \in \mathsf{d}^n X\}$ is pairwise disjoint. Let $x \in \mathsf{d}^n X$. Obviously $B_x \cap \mathsf{d}^n X \supseteq \{x\}$. Let $y \in B_x \cap \mathsf{d}^n X$. Then $y \in B_x \subseteq U_x$, giving that $U_x \cap U_y \neq \varnothing$. Thus, $x = y$ and so $B_x \cap \mathsf{d}^n X = \{x\}$. $\square$

7. Logics arising from scattered locally compact HP spaces

The main results of this section are a mapping theorem for scattered locally compact open hereditarily collectionwise normal and open hereditarily strongly zero-dimensional spaces and a classification of the logics arising as $\mathsf{L}(X)$ for such an X. As a corollary, we classify the logics arising as $\mathsf{L}(X)$ for X a scattered locally compact hereditarily paracompact space.

We recall that a Tychonoff space X is *strongly zero-dimensional* if βX is zero-dimensional (see, e.g., [10, Section 6.2]). Clearly being zero-dimensional is a hereditary property, but strong zero-dimensionality is not hereditary. We call a strongly zero-dimensional space X *open hereditarily strongly zero-dimensional* (OHSZ) provided every nonempty open subspace of X is strongly zero-dimensional. Similarly, we call a T_1-space X *open hereditarily collectionwise normal* (OHCN) whenever each open subspace of X is collectionwise normal.

Theorem 7.1. *Let $n \in \omega$, X be a locally compact OHCN OHSZ space of modal Krull dimension n, and $\mathfrak{F}$ be a finite tree of depth at most $n+1$. Then there is an interior surjection $f : X \to \mathfrak{F}$ that maps each $x \in \mathsf{d}^n X$ to the root of $\mathfrak{F}$.*

Proof. Proof by induction on $n \in \omega$. If $n = 0$, then $\mathrm{mdim}(X) = 0$, giving that X is discrete. Since $\mathfrak{F}$ consists of only the root, there is only one mapping of X onto $\mathfrak{F}$ (sending every element of X to the root of $\mathfrak{F}$), and it is clearly interior. This establishes the base case.

Let $n > 0$. Suppose for every locally compact OHCN OHSZ space Y of modal Krull dimension $n-1$ and every finite tree $\mathfrak{F}$ of depth at most n, there is an interior mapping of Y onto $\mathfrak{F}$ sending $\mathsf{d}_Y^{n-1} Y$ to the root of $\mathfrak{F}$.

Let X be a locally compact OHCN OHSZ space of modal Krull dimension n. Then X is scattered by Theorem 4.5. Let $\mathfrak{F}$ be a finite tree of depth at most $n+1$ and let r be the root of $\mathfrak{F}$. If r has no children, then there is only one mapping of X onto $\mathfrak{F}$, and it is clearly interior. Suppose $c_1, \dots, c_m$ are the children of r. For $i = 1, \dots, m$, let $\mathfrak{F}_i$ be the subtree of $\mathfrak{F}$ whose underlying set is ${\uparrow} c_i$, see Figure 6. Then the depth of each $\mathfrak{F}_i$ is at most n.

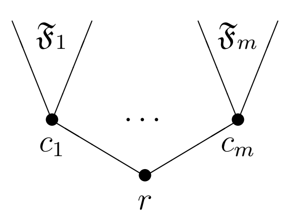

Figure 6. The subtrees $\mathfrak{F}_i$ of $\mathfrak{F}$.

By Lemma 6.4, there is a pairwise disjoint family $\{B_x \mid x \in \mathsf{d}^n X\}$ of clopens in X such that $B_x \cap \mathsf{d}^n X = \{x\}$ for each $x \in \mathsf{d}^n X$. Since X is locally compact, so is each subspace $B_x \cap \mathsf{d}^{n-1} X$. By Lemma 6.2, there are pairwise disjoint $A_1^x, \dots, A_m^x \subseteq B_x \cap \mathsf{d}^{n-1} X$ such that $x \in \mathsf{d}(A_i^x)$ for each $i = 1, \dots, m$. Set $F_i = \bigcup_{x \in \mathsf{d}^n X} A_i^x$ for $i = 1, \dots, m$, see Figure 7.

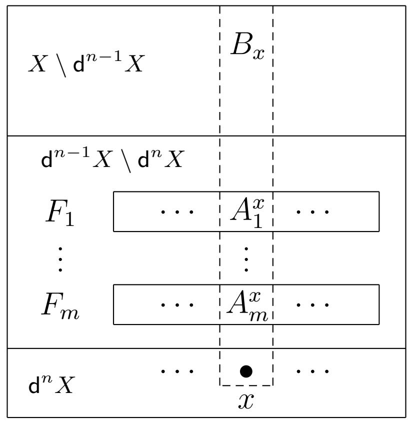

FIGURE 7. Depiction of the families B_x, A_i^x, and F_i.

Clearly $F_i \subseteq \mathsf{d}^{n-1} X$ and $F_i \cap \mathsf{d}^n X = \varnothing$ for each i. It is also obvious that $\mathsf{d}^n X \subseteq \mathsf{c} F_i$ which yields that $F_i \cup \mathsf{d}^n X \subseteq \mathsf{c} F_i$. To see the reverse inclusion, suppose $x \notin F_i \cup \mathsf{d}^n X$. Then $x \notin \mathsf{d}(\mathsf{d}^{n-1} X)$, so there is an open neighborhood U of x such that $(U \setminus \{x\}) \cap \mathsf{d}^{n-1} X = \varnothing$. Therefore, $U \cap F_i \subseteq U \cap \mathsf{d}^{n-1} X \subseteq \{x\}$, yielding that $U \cap F_i = \varnothing$. Thus, $x \notin \mathsf{c} F_i$, and so $\mathsf{c} F_i = F_i \cup \mathsf{d}^n X$. Because F_i and $\mathsf{d}^n X$ are disjoint, we have that $F_i = (F_i \cup \mathsf{d}^n X) \setminus \mathsf{d}^n X = \mathsf{c} F_i \cap (X \setminus \mathsf{d}^n X)$ is closed in $X \setminus \mathsf{d}^n X$. Consequently, $F_1, \dots, F_m$ is a pairwise disjoint family of nonempty closed subsets of the subspace $X \setminus \mathsf{d}^n X$ of X.

Since X is a locally compact OHCN OHSZ space, so is $X \setminus \mathsf{d}^n X$ (since $X \setminus \mathsf{d}^n X$ is open in X). Therefore, $X \setminus \mathsf{d}^n X$ is a normal strongly zero-dimensional space. Thus, [5, Lemma 3.2] is applicable, and so there is a clopen partition $\{Y_i \mid 1 \le i \le m\}$ of $X \setminus \mathsf{d}^n X$ such that $F_i \subseteq Y_i$ for each $i = 1, \dots, m$, see Figure 8.

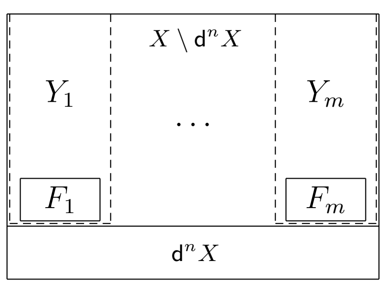

FIGURE 8. Depiction of the family Y_i.

Fix $i = 1, \dots, m$. Clearly $\mathsf{d}^n X \subseteq \mathsf{c} F_i \subseteq \mathsf{c} Y_i$, which gives that $Y_i \cup \mathsf{d}^n X \subseteq \mathsf{c} Y_i$. Also $U_i := \bigcup_{j \ne i} Y_j$ is open in X since U_i is open in $X \setminus \mathsf{d}^n X$ and $X \setminus \mathsf{d}^n X$ is open in X. Because $\{Y_1, \dots, Y_m, \mathsf{d}^n X\}$ is a partition of X, we have that $Y_i \cup \mathsf{d}^n X = X \setminus U_i$. Therefore, $Y_i \cup \mathsf{d}^n X$ is a closed subset of X containing Y_i and contained in $\mathsf{c} Y_i$. Thus, $\mathsf{c} Y_i = Y_i \cup \mathsf{d}^n X$.

Because Y_i is clopen in $X \setminus \mathsf{d}^n X$, which is open in X, we have that Y_i is open in X. Since X is a locally compact OHCN OHSZ space, so is Y_i. By Lemma 4.6,

$$\mathsf{d}^n_{Y_i} Y_i = \mathsf{d}^n(X) \cap Y_i \subseteq \mathsf{d}^n(X) \cap (X \setminus \mathsf{d}^n X) = \varnothing$$

and

$$\mathsf{d}_{Y_i}^{n-1} Y_i = \mathsf{d}^{n-1}(X) \cap Y_i \supseteq F_i \neq \varnothing.$$

Since X is scattered Hausdorff, so is Y_i. Therefore, Theorem 4.9 yields that $\mathrm{mdim}(Y_i) = n-1$, and so the inductive hypothesis is applicable to Y_i. Let $f_i : Y_i \to \mathfrak{F}_i$ be an onto interior mapping sending $\mathsf{d}_{Y_i}^{n-1} Y_i$ to c_i.

Define $f : X \to \mathfrak{F}$ by $f(x) = r$ when $x \in \mathsf{d}^n X$ and $f(x) = f_i(x)$ when $x \in Y_i$, see Figure 9. Then f is a well defined surjection since $\{Y_1, \dots, Y_m, \mathsf{d}^n X\}$ is a partition of X, each f_i maps Y_i onto $\mathfrak{F}_i$, and $f[\mathsf{d}^n X] = \{r\}$. It is left to show that f is interior.

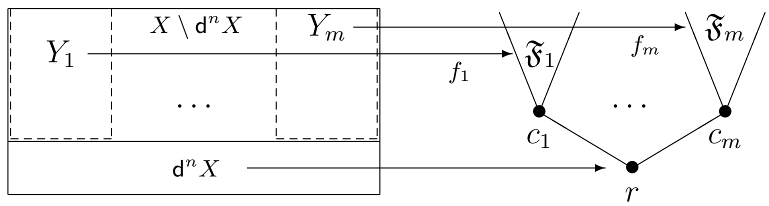

FIGURE 9. Depiction of $f : X \to \mathfrak{F}$.

First we show that f is continuous. Let $w \in \mathfrak{F}$. If $w = r$, then $f^{-1}[{\uparrow}w] = f^{-1}[\mathfrak{F}] = X$ is open in X. If $w \neq r$, then ${\uparrow}w \subseteq \mathfrak{F}_i$ for some $i = 1, \dots, m$. Therefore, $f^{-1}[{\uparrow}w] = (f_i)^{-1}[{\uparrow}w]$ is open in Y_i. But Y_i is open in X, and so $f^{-1}[{\uparrow}w]$ is open in X. Thus, f is continuous.

Next we show that f is open. Let U be open in X. We have:

$$\begin{aligned}
f[U] &= f[U \cap X] = f\left[U \cap (Y_1 \cup Y_2 \cup \cdots \cup Y_m \cup \mathsf{d}^n X)\right] \\
&= f\left[(U \cap Y_1) \cup (U \cap Y_2) \cup \cdots \cup (U \cap Y_m) \cup (U \cap \mathsf{d}^n X)\right] \\
&= f[U \cap Y_1] \cup f[U \cap Y_2] \cup \cdots \cup f[U \cap Y_m] \cup f[U \cap \mathsf{d}^n X] \\
&= f_1[U \cap Y_1] \cup f_2[U \cap Y_2] \cup \cdots \cup f_m[U \cap Y_m] \cup f[U \cap \mathsf{d}^n X].
\end{aligned}$$

Since $U \cap Y_i$ is open in Y_i, we have $f_i[U \cap Y_i]$ is open in $\mathfrak{F}_i$, and hence is open in $\mathfrak{F}$. If $U \cap \mathsf{d}^n X = \varnothing$, then $f[U \cap \mathsf{d}^n X] = \varnothing$, and so $f[U]$ is a union of open subsets of $\mathfrak{F}$, hence an open subset of $\mathfrak{F}$. Suppose $x \in U \cap \mathsf{d}^n X$. Then for each $i = 1, \dots, m$ we have that $x \in \mathsf{c}F_i$, giving that $\varnothing \neq U \cap F_i \subseteq U \cap \mathsf{d}_{Y_i}^{n-1} Y_i$. Therefore, $c_i \in f_i[U \cap \mathsf{d}_{Y_i}^{n-1} Y_i] \subseteq f_i[U \cap Y_i]$, yielding that $f_i[U \cap Y_i] = \mathfrak{F}_i$. Thus, $f[U] = \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \cdots \cup \mathfrak{F}_m \cup \{r\} = \mathfrak{F}$ is open in $\mathfrak{F}$. Consequently, f is open. □

Corollary 7.2. *Let $n \in \omega$, X be a scattered locally compact OHCN OHSZ space, and $\mathfrak{F}$ a finite tree of depth at most $n+1$. If $\mathsf{d}^n X \neq \varnothing$, then there is an interior surjection $f : X \setminus \mathsf{d}^{n+1} X \to \mathfrak{F}$ that maps each $x \in \mathsf{d}^n X$ to the root of $\mathfrak{F}$.*

Proof. Let $Y = X \setminus \mathsf{d}^{n+1} X$. Then Y is an open scattered locally compact OHCN OHSZ subspace of X. By Lemma 4.6,

$$\mathsf{d}_Y^n Y = \mathsf{d}^n(X) \cap Y = \mathsf{d}^n(X) \cap (X \setminus \mathsf{d}^{n+1} X) = \mathsf{d}^n X \setminus \mathsf{d}^{n+1} X \neq \varnothing$$

because $\mathsf{d}^n X \setminus \mathsf{d}^{n+1} X = \mathsf{Iso}(\mathsf{d}^n X)$, and since $\mathsf{d}^n X$ is a nonempty subspace of a scattered space, $\mathsf{Iso}(\mathsf{d}^n X) \neq \varnothing$. Also,

$$\mathsf{d}_Y^{n+1} Y = \mathsf{d}^{n+1}(X) \cap Y = \mathsf{d}^{n+1}(X) \cap (X \setminus \mathsf{d}^{n+1} X) = \varnothing.$$

Therefore, $\mathrm{mdim}(Y) = n$ by Theorem 4.9. Now apply Theorem 7.1 to Y. □

Theorem 7.3. *Let X be a nonempty scattered locally compact OHCN OHSZ space.*

1. *If $\mathrm{mdim}(X) = \infty$, then $\mathsf{L}(X) = \mathsf{S4.Grz}$.*
2. *If $\mathrm{mdim}(X) = n \in \omega$, then $\mathsf{L}(X) = \mathsf{S4.Grz}_{n+1}$.*

Proof. Since X is scattered, $\mathsf{S4.Grz} \subseteq \mathsf{L}(X)$. To prove (1), let φ be a modal formula such that $\mathsf{S4.Grz} \not\vdash \varphi$. Then there is a finite tree $\mathfrak{F}$ refuting φ. Suppose the depth of $\mathfrak{F}$ is $n \geq 1$. Since $\mathrm{mdim}(X) = \infty$, by Theorem 4.9, $\mathsf{d}^n X \neq \varnothing$ for all $n \in \omega$. As $\mathsf{d}^{n-1} X \neq \varnothing$, Corollary 7.2 yields that $\mathfrak{F}$ is an interior image of the open subspace $X \setminus \mathsf{d}^n X$ of X. Because interior images reflect refutations (see, e.g., [2, Proposition 2.9]), $X \setminus \mathsf{d}^n X$ refutes φ. Since open subspaces reflect refutations (see, e.g., [2, Proposition 2.9]), X refutes φ. Thus, $\mathsf{L}(X) = \mathsf{S4.Grz}$.

To prove (2), suppose $\mathrm{mdim}(X) = n \in \omega$. Then $X \Vdash \mathsf{bd}_{n+1}$ by Proposition 2.3. Therefore, $\mathsf{S4.Grz}_{n+1} \subseteq \mathsf{L}(X)$. Conversely, let φ be a modal formula such that $\mathsf{S4.Grz}_{n+1} \not\vdash \varphi$. Then there is a finite tree $\mathfrak{F}$ of depth at most $n+1$ refuting φ. By Theorem 7.1, $\mathfrak{F}$ is an interior image of X. Thus, X refutes φ, and hence $\mathsf{L}(X) = \mathsf{S4.Grz}_{n+1}$. □

An important class of spaces that simultaneously generalizes both the class of metrizable spaces and the class of compact Hausdorff spaces is that of paracompact spaces (see, e.g., [10, Section 5.1] for a detailed account). A space X is *hereditarily paracompact* (HP) if each subspace of X is paracompact. It turns out that a space is HP iff it is open hereditarily paracompact.

Lemma 7.4. *A scattered locally compact HP space X is both OHCN and OHSZ.*

Proof. Since every paracompact space is collectionwise normal [10, Theorem 5.1.18], we have that an HP space is hereditarily collectionwise normal, and hence OHCN. Let Y be a nonempty open subspace of X. Then Y is a scattered locally compact paracompact space. It follows from Theorem 4.5 that Y is zero-dimensional. By [10, Theorem 6.2.10], Y is strongly zero-dimensional. Thus, X is OHSZ. □

We use Lemma 7.4 to obtain the following corollaries to Theorem 7.1, Corollary 7.2, and Theorem 7.3.

Corollary 7.5. *If in Theorem 7.1 OHCN OHSZ is replaced by HP, then the conclusion still holds.*

Proof. A locally compact HP space of finite modal Krull dimension is scattered by Theorem 4.5. The result now follows from Lemma 7.4 and Theorem 7.1. □

Corollary 7.6. *If in Corollary 7.2 OHCN OHSZ is replaced by HP, then the conclusion still holds.*

Proof. The result follows immediately from Lemma 7.4 and Corollary 7.2. □

Corollary 7.7. *If in Theorem 7.3 we replace OHCN OHSZ by HP, the conclusion still holds.*

Proof. The result follows immediately from Lemma 7.4 and Theorem 7.3. □

As follows from the next example, there are some well studied spaces to which Theorem 7.3 applies but Corollary 7.7 does not.

Example 7.8. Let ω_1 be the least uncountable ordinal with the interval topology. It follows from [23] that ω_1 is OHCN, and it follows from [12, Theorem 5.1] that ω_1 is OHSZ. On the other hand, ω_1 is not paracompact (see, e.g., [10, Example 5.1.21]).

8. Concluding remarks

We conclude the paper with some, seemingly challenging, open problems:

- Is there a Mrowka space X satisfying $\beta X = \alpha X$ such that $\mathsf{L}(\beta X) = \mathsf{S4.Grz}_3 + \neg\chi_{\mathfrak{F}}$? We conjecture there is such a Mrowka space. To prove this conjecture, we point out that $\mathsf{S4.Grz}_3 + \neg\chi_{\mathfrak{F}}$ is complete with respect to the following class $\mathcal{K}$ of frames. Recall that an $\mathsf{S4}$-frame $\mathfrak{F} = (W, R)$ is *path connected* provided for any $w, v \in W$ there are $w_1, \dots, w_n \in W$ such that $w_1 = w$, $w_n = v$, and either $w_i R w_{i+1}$ or $w_{i+1} R w_i$ for each $1 \le i < n$. The Alexandroff space of a path connected $\mathsf{S4}$-frame is a connected topological space. The class $\mathcal{K}$ consists of finite rooted posets of depth ≤ 3 such that those of depth 3 satisfy

  (†) the subframe obtained by deleting the root is path connected.

  Thus, it is enough to show that every finite rooted poset of depth 3 satisfying (†) is an interior image of βX. While we have a candidate for X and can construct interior mappings for a number of examples, the task in general remains elusive due to the combinatorial complexity of these posets.
- Classify the logics arising as $\mathsf{L}(\beta X)$ where X is an arbitrary Mrowka space (satisfying $\beta X = \alpha X$).
- What is the logic of an arbitrary scattered Stone space?
- What is the logic of an arbitrary Stone space?
- What is the logic of an arbitrary compact Hausdorff space?
- What is the logic of an arbitrary locally compact Hausdorff space?

The same questions can be asked in the intuitionistic setting. Note that the logics $\mathsf{S4}, \mathsf{S4.1}$, and $\mathsf{S4.Grz}$ are modal companions of the intuitionistic propositional calculus IPC. Thus, in the intuitionistic setting we obtain IPC and the logics IPC_n ($n \ge 1$), which are the intuitionistic analogues of the logics $\mathsf{S4.Grz}_n$. In addition, as follows from Remark 5.10, we obtain the Scott logic of depth 3. A complete classification remains a challenging open problem in the intuitionistic setting as well.

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