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HOMOGENEOUS CONTINUA THAT ARE NOT SEPARATED BY ARCS

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Abstract. We prove that if $X$ is a strongly locally homogeneous and locally compact separable metric space and $G$ is a region in $X$ with $\dim G = 2$, then $G$ is not separated by any arc in $G$.

1. Introduction

By a space we mean a separable metric space. Kallipoliti and Papasoglu [4] proved that any locally connected, simply connected, homogeneous metric continuum cannot be separated by arcs, and asked if this is true without the assumption of simply connectedness. A partial answer of this question was provided in [8] for homogeneous metric continua of dimension two having a non-trivial second integral Čech cohomology group. In the present paper we prove the following partial answer to Kallipoliti and Papasoglu’s question.

Theorem 1.1. Let $X$ be a locally compact strongly locally homogeneous space and $G$ be a region in $X$ with $\dim G = n \geq 2$. Then $G$ is not separated by any arc $J \subset G$.

Recall that a space is strongly locally homogeneous if every point $x \in X$ has a local basis of open sets $U$ such that for every $y, z \in U$ there is a homeomorphism $h$ on $X$ with $h(y) = z$ and $h$ is identity on $X \setminus U$. Obviously, every open subset of a strongly locally homogeneous space is also strongly locally homogeneous. Since strongly locally homogeneous connected spaces are homogeneous, any region $G$ satisfying the hypotheses of Theorem 1.1 should be homogeneous. We claim that it is locally connected as well. Indeed, since any strongly homogeneous Polish space is countable dense homogeneous [1], and a locally compact countable dense homogeneous connected space is locally connected [3],

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we have that any region $G$ from Theorem 1.1 is locally connected. (There is also a simple direct proof of this fact.) According to [6], every region of homogeneous locally compact space of dimension $n \geq 1$ can not be separated by a closed set of dimension $\leq n-2$. So, Theorem 1.1 is interesting only for regions $G$ of dimension two.

2. SOME PRELIMINARY RESULTS

**Lemma 2.1.** Let $A$ be a closed nowhere dense subset of $X$ such that $\dim X \setminus A = 0$. Then there is a retraction $r: X \to A$ such that $r(X \setminus A)$ is countable.

**Proof.** The technique is similar to that in [5]. In brief, one constructs a cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ by disjoint nonempty clopen subsets of $X$ such that

1. $\text{diam } V_n < d(V_n, A)$ for each $n$,
2. there is a sequence $\{a_n : n \in \mathbb{N}\}$ in $A$ such that $\lim_{n \to \infty} d(a_n, V_n) = 0$.

Then define $r: X \to A$ as follows: $r(a) = a$ for every $a$ and $r(V_n) = \{a_n\}$ for every $n$. It is easy to check that $r$ is as required. $\square$

If $J$ is an arc and $p, q \in J$, then $(p, q)$ and $[p, q]$ denote, respectively, the open and closed subintervals in $J$ with endpoints $p, q$.

**Proposition 2.2.** Let $J = [a, b]$ be an arc in a space $X$ which is everywhere 2-dimensional. Then $b$ has arbitrarily small open neighborhoods $U$ such that $\text{bd}(U)$ is at most 1-dimensional and intersects $J$ in exactly one point.

**Proof.** Fix $\varepsilon > 0$ and let $U$ be an open neighborhood of $b$ in $X$ such that $\text{diam } U < \varepsilon$ and $\dim \text{bd} U \leq 1$. We may assume without loss of generality that $J \setminus U \neq \emptyset$ and $J \cap U$ is uncountable. Put $Y = J \cup \overline{U}$. Moreover, put $A = J \cup \text{bd} U$, $B = (J \setminus U) \cup \text{bd} U$ and $C = (J \cap \overline{U}) \cup \text{bd} U$, respectively.

Let $D$ be a zero-dimensional dense subset of $U$ such that $\dim U \setminus D = 1$. Since $\dim J = 1$, we may clearly assume that $D \cap J = \emptyset$.

Because $C$ is a closed nowhere dense subset of $C \cup D$, there is a retraction $r_1: C \cup D \to C$ such that $r_1(D)$ is countable (Lemma 2.1). Let $r: A \cup D \to A$ be defined by $r(x) = r_1(x)$ if $x \in C \cup D$ and $r(x) = x$ if $x \notin C \cup D$. Obviously $r$ is a retraction such that $r(D)$ is countable.

Pick an arbitrary $s \in U \cap J$ such that $s \neq b$, $[s, b] \subset U$ and $s \notin r(D)$. Choose also two points $s_1, s_2 \in J \cap U$ different from $s$ and $b$ such that $s \in (s_1, s_2)$, and let $V_1 = A \setminus [s_1, b]$ and $V_2 = (s_2, b]$. Obviously $V_1$ and
V_2 are open subsets of A containing B and \{b\}, respectively. Moreover, \( V_1 = A \setminus (s_1, b] \) and \( V_2 = [s_2, b] \).

Claim 1. \( \{s\} \) is a partition in A between \( V_1 \) and \( V_2 \).

Indeed, put \( P = [s, b] \) and \( Q = [a, s] \cup \text{bd} U \). Then \( P \) and \( Q \) are closed subsets of A such that \( P \cup Q = A, V_2 \subset P, V_1 \subset Q \) and \( P \cap Q = \{s\} \).

Claim 2. \( \{s\} \) is a partition in \( A \cup D \) between \( r^{-1}(V_1) \) and \( r^{-1}(V_2) \).

Since \( r^{-1}(s) = \{s\} \), this is a direct consequence of Claim 1.

By [7, Lemma 3.1.4], there is a partition \( S \) between \( \{b\} \) and \( B \) in \( Y \) such that \( S \cap (A \cup D) \subset \{s\} \). If \( s \notin S \), then \( S \cup \{s\} \) is also a partition between \( \{b\} \) and \( B \) in \( Y \), hence we may assume without loss of generality that \( s \in S \). But then \( S \cap J = \{s\} \). Write \( Y \setminus S \) as \( E \cup F \), where \( E \) and \( F \) are disjoint relatively open subsets of \( Y \) such that \( b \in E \) and \( B \subset F \).

Claim 3. \( E \subset U \).

Indeed, since \( E \cap B = E \cap ((J \setminus U) \cup \text{bd} U) = \emptyset \), this is clear.

Since \( E \) is open in \( U \) and \( U \) is open in \( X \) we have that \( E \) is open in \( X \). Moreover, \( \text{diam} E < \varepsilon \). Also, \( E \cup S \) is closed in \( Y \) and hence in \( X \). As a consequence \( \text{bd} E \subset S \). Since \( S \subset U \setminus D \), we have \( \dim S \leq 1 \), as required.

It will be convenient to use additive notation for the topological group \( S^1 \).

The following result can be proved by tools from algebraic topology. For the convenience of the reader, we include a simple direct proof.

**Proposition 2.3.** Let \( X \) be a space and let \( A \) be a closed subspace of it. Moreover, let \( \gamma \colon A \to S^1 \) be continuous. Suppose that there are closed subsets \( P_1, P_2 \) of \( X \) satisfying the following conditions:

- \( P_1 \cup P_2 = X \) and if \( C = P_1 \cap P_2 \) then \( C \cap A \) is a singleton, say \( c \);
- \( \gamma|P_i \cap A \) is extendable over \( P_i \) for each \( i = 1, 2 \), but \( \gamma \) is not extendable over \( X \).

Then there is a continuous function \( \beta \colon C \to S^1 \) such that \( \beta(c) = 0 \) and \( \beta \) is not nullhomotopic.

**Proof.** Let \( \alpha_i \colon P_i \to S^1 \) for \( i = 1, 2 \) be a continuous extension of \( \gamma|P_i \cap A \). Define \( \beta \colon C \to S^1 \) by \( \beta(x) = \alpha_1(x) - \alpha_2(x) \) (\( x \in C \)). Then, clearly, \( \beta(c) = 0 \). We claim that \( \beta \) is as required, and argue by contradiction. Assume that \( \beta \) is nullhomotopic. Let \( H \colon C \times I \to S^1 \) be a homotopy
such that $H_0 \equiv 0$ and $H_1 = \beta$. Define $S: C \times I \to \mathbb{S}^1$ by $S(x, t) = H(x, t) - H(c, t)$. Then $S_0 \equiv 0$, $S_1 = \beta$ and $S(c, t) = 0$ for every $t$. Define a homotopy $T: (C \cup (P_2 \cap A)) \times I \to \mathbb{S}^1$ by

$$T(x, t) = \begin{cases} S(x, t) & (x \in C, t \in I), \\ 0 & (x \in P_2 \cap A, t \in I). \end{cases}$$

Then $T_0 \equiv 0$ and hence can be extended to the constant function with value 0 on $P_2$. By the Borsuk Homotopy Extension Theorem [7, 1.4.2], the function $T_1$ can be extended to a continuous function $\delta: P_2 \to \mathbb{S}^1$.

Now define $\varepsilon: X \to \mathbb{S}^1$ as follows:

$$\varepsilon|_{P_1} = \alpha_1, \quad \varepsilon|_{P_2} = \delta + \alpha_2.$$ 

If $x \in C$, then $\varepsilon|_{P_1}(x) = \alpha_1(x)$ and $\varepsilon|_{P_2}(x) = \delta(x) + \alpha_2(x) = S_1(x) + \alpha_2(x) = \beta(x) + \alpha_2(x) = \alpha_1(x)$. Hence $\varepsilon$ is well defined and continuous. Also observe that if $x \in P_2 \cap A$, then

$$\varepsilon(x) = 0 + \alpha_2(x) = \alpha_2(x).$$

Hence $\varepsilon$ extends $\gamma$, which is a contradiction. □

3. Proof of Theorem 1.1

Throughout, let $X$ be a locally compact and strongly locally homogeneous space, and $G$ be a region in $X$ of dimension 2. Suppose $G$ is separated by an arc $J = [a, b] \subset G$. Recall that $G$ is homogeneous and locally connected (see §1). Write $G \setminus J$ as $G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint nonempty open subsets of $G$. Everywhere below $\overline{K}$ denotes the closure of $K$ in $G$ for any set $K \subset G$.

We say that a space $Y$ has no local cut points if no connected open subset $U \subset Y$ has a cut point.

**Lemma 3.1.** $G$ has no local cut points.

**Proof.** By Kruspki [6, Theorem 2.1] it follows that every nonempty open connected subset $U$ of $G$ is a Cantor manifold of dimension 2. Hence $U$ cannot be separated by a zero-dimensional closed set. □

A space $X$ is crowded if it has no isolated points.

**Lemma 3.2.** The set $S = \overline{G_1} \cap \overline{G_2}$ is a 1-dimensional closed and crowded subspace of $J$ which separates $G$.

**Proof.** Assume first that $J \setminus (\overline{G_1} \cup \overline{G_2}) \neq \emptyset$. Then $G$ is somewhere at most 1-dimensional. Hence $G$ is at most 1-dimensional at every point by homogeneity. But this contradicts $G$ being 2-dimensional.
Hence $J \subset \overline{G_1} \cup \overline{G_2}$ and so $G = \overline{G_1} \cup \overline{G_2}$. If $S$ is empty, then $G$ is covered by the disjoint nonempty closed sets $\overline{G_1}$ and $\overline{G_2}$ which contradicts the connectivity of $G$.

Now assume that $x$ is an isolated point of $S$. Let $U$ be an open connected neighborhood of $x$ in $G$ such that $U \cap S = \{x\}$. Then $x$ is a cutpoint of $U$. But this contradicts Lemma 3.1.

We conclude that $S$ separates $G$ and consequently has to be 1-dimensional by Krupski [6].

Let $s$ be the maximum of $S$ (as a subset of $[a, b]$). Then $J_s = [a, s]$ also separates $G$ and $G \setminus J_s$ is the union of the disjoint open sets $G_1'$ and $G_2'$, where $G_i' = \overline{G_i} \setminus J_s$. Moreover, $s \in \overline{G_1} \cap \overline{G_2}$. Hence, we can assume without loss of generality that $b \in \overline{G_1} \cap \overline{G_2}$.

**Lemma 3.3.** There is an open neighborhood $U \subset G$ of $b$ having compact closure and a compact set $F \subset G$ such that for every open neighborhood $V$ of $b$ with $\overline{V} \subset U$ there exist a compact set $M_U \subset U$ and a continuous function $f : \text{bd}(U \cap F) \to S^1$ such that:

1. $b \in U \cap F$;
2. $M_U$ is everywhere 2-dimensional and $M_U \cap V \neq \emptyset$;
3. $\dim \text{bd}U \leq 1$ and $J \cap \text{bd}U$ is a point;
4. $f$ is not extendable over $\text{bd}(U \cap F) \cup M_U$, but it is extendable over $\text{bd}(U \cap F) \cup P$ for every proper closed set $P$ of $M_U$.

**Proof.** Choose a compact neighborhood $O_b$ of $b$ in $G$. Since every neighborhood of $b$ is of dimension 2, there is a compact subset $Y \subset O_b$, a closed set $A \subset Y$ and a continuous function $g : A \to S^1$ not extendable over $Y$. Let $F$ be a minimal closed subset of $Y$ containing $A$ such that $g$ is not extendable over $F$. Then for every open subset $W$ of $F \setminus A$ with $\overline{W} \cap A = \emptyset$ there is a function $f_W : F \setminus W \to S^1$ extending $g$ such that $f_W$ can not be extended to a continuous function $f_W' : F \to S^1$. This means that $f_W|_{\text{bd}F}W$ is not extendable over $\overline{W}$. Consequently, $F \setminus A$ is everywhere two-dimensional. We can assume by homogeneity that $b \in F \setminus A$. Indeed, by Effros’ theorem [2], we take $O_b$ so small that for every point $x \in O_b$ there is a homeomorphism $h$ on $G$ with $h(b) = x$ and $O_b \subset h(G)$. Then, consider the set $h_i(G)$ instead of $G$.

By Proposition 2.2, there are an open neighborhood $U$ of $b$ whose closure in $G$ is a compact and a point $c \in (a, b)$ such that $\text{bd}U \cap J = \{c\}$, $\dim \text{bd}U \leq 1$ and $U \cap A = \emptyset$. Suppose $V$ is an open neighborhood of $b$ such that $\overline{V} \subset U$, and consider a continuous function $f_V : F \setminus V \to S^1$ extending $g$ which is not extendable over $F$. Let $f = f_V|_{\text{bd}(U \cap F)}$. Clearly, $f$ cannot be extended to a continuous function $f : U \cap F \to S^1$, but $f$ can be extended to a continuous function from $(U \cap F) \setminus V$ into
Let \( M_U \) be a minimal closed subset of \( \overline{U \cap F} \) with the property that \( f \) cannot be extended to a continuous function \( \tilde{f} : \overline{U \cap F} \cup M_U \to S^1 \). The minimality of \( M_U \) implies that \( f \) is extendable over \( \overline{U \cap F} \cup P \) for any any closed set \( P \not\subset M_U \). Because \( f \) is extendable over \( (U \cap F) \setminus V \), \( M_U \cap V \neq \emptyset \). It is clear that \( M_U \) is a continuum.

Assume that \( O \) is a nonempty open subset of \( M_U \) such that \( \dim O \leq 1 \). Taking a smaller open subset of \( O \), we may assume that \( \dim \overline{O} \leq 1 \). There are two possibilities, either \( O \subset \overline{U \cap F} \) or \( O \setminus \overline{U \cap F} \neq \emptyset \). If \( O \subset \overline{U \cap F} \), \( M_U \setminus O \) is a proper closed subset of \( M_U \) having the same properties as \( M_U \), which contradicts minimality. If \( O' = O \setminus \overline{U \cap F} \neq \emptyset \), then \( P = M_U \setminus O' \) is a proper closed subset of \( M_U \). So, there is an extension \( f_1 : \overline{U \cap F} \cup P \to S^1 \) of \( f \). Since \( \dim \overline{O} \leq 1 \), we can extend \( f_1 \) over \( \overline{U \cap F} \cup M_U \), a contradiction. Therefore, \( M_U \) is everywhere 2-dimensional.

Now, we can complete the proof of Theorem 1.1. Choose open neighborhoods \( U \) and \( V \) of \( b \), closed sets \( F \subset G \) and \( M_U \subset \overline{U \cap F} \) and a continuous function \( f : \overline{U \cap F} \to S^1 \) satisfying the conditions (1) – (4) from Lemma 3.3. Let also \( J \cap \overline{bdU} = \{ c \} \) and \( C = [c, b] \). We can also assume that \( V \) satisfies the additional property that for every two points \( p, q \in V \) there is a homeomorphism \( \varphi \) of \( G \) supported on \( V \) with \( \varphi(p) = q \). We may consequently assume without loss of generality that \( b \in M_U \). Indeed, if \( b \notin M_U \) we take a point \( x \in M_U \cap V \) and a homeomorphism \( \varphi \) of \( G \) supported on \( V \) such that \( \varphi(x) = b \). Then the set \( \varphi(M_U) \) satisfies all condition from Lemma 3.3 and contains \( b \). Since \( M_U \) is everywhere 2-dimensional, \( \dim(M_U \cap V) = 2 \). Hence, \( M_U \cap V \) meets at least one of the sets \( G_i \), \( i = 1, 2 \).

Assume first that \( M_U \cap V \cap G_1 \neq \emptyset \) but \( M_U \cap V \cap G_2 = \emptyset \).

Then \( M_U \cap W \) meets \( G_1 \) for every neighborhood \( W \) of \( b \) with \( W \subset V \). Indeed, because \( \dim M_U \cap W = 2 \) and \( M_U \cap W \cap G_2 = \emptyset \) it follows that \( M_U \cap G_1 \cap W \neq \emptyset \). There consequently is a neighborhood \( W \) of \( b \) in \( G \) such that

1. \( \overline{W} \subset V, (M_U \cap V) \cap (G_1 \setminus \overline{W}) \neq \emptyset \) and \( M_U \cap G_1 \cap W \neq \emptyset \);
2. For every \( x, y \in W \) there is a homeomorphism \( h \) of \( G \) supported on \( W \) with \( h(x) = y \).

Finally, choose points \( x \in M_U \cap G_1 \cap W \) and \( y \in W \cap G_2 \) and a homeomorphism \( h : G \to G \) supported on \( W \) with \( h(x) = y \). Since \( h(z) = z \) for all points \( z \in (M_U \cap V) \cap (G_1 \setminus \overline{W}) \), the set \( \overline{K} = h(M_U) \) meets both \( G_1 \) and \( G_2 \). Moreover, the function \( f \) is not extendable over \( \overline{bd}(U \cap F) \cup \overline{K} \) (otherwise \( f \) would be extendable over \( \overline{bd}(U \cap F) \cup M_U \)). On the other hand, since each of the sets \( Q_i = h^{-1}(\overline{K} \cap \overline{G_i}) \),
i = 1, 2, is a proper closed subset of $M_U$, $f$ is extendable over each of the sets $\text{bd}_F(U \cap F) \cup (\overline{K} \cap \overline{G}_i)$. Let $\gamma : \text{bd} U \to \mathbb{S}^1$ be an extension of $f$ (recall that $\dim \text{bd} U \leq 1$ and $\text{bd}_F(U \cap F)$ is a closed subset of $\text{bd} U$, so such $\gamma$ exists). Because $f$ is not extendable over $\text{bd}_F(U \cap F) \cup \overline{K}$, $\gamma$ is not extendable over the set $K = \text{bd} U \cup \overline{K} \cup C$. Denote $P_i = C \cup (K \cap \overline{G}_i)$, $i = 1, 2$. Obviously, $P_1 \cup P_2 = K$ and $P_1 \cap P_2 = C$. Then for each $i$ we have $P_i \cap \text{bd} U = \{c\} \cup (\text{bd} U \cap \overline{G}_i)$. So, the function $\gamma|_{(P_i \cap \text{bd} U)}$ is extendable over the set $P_i$ because $\dim C \cup \text{bd} U = 1$. Hence, we can apply Proposition 2.3 (with $A = \text{bd} U$) to conclude that there is a continuous function $\beta : C \to \mathbb{S}^1$ such that $\beta$ is not nullhomotopic, a contradiction.

Assume next that $M_U \cap V$ meets both $G_1$ and $G_2$. We can now proceed as above (considering $M_U$ instead of $\overline{K}$) to obtain the desired contradiction.

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