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HOMOGENEOUS CONTINUA THAT ARE NOT SEPARATED BY ARCS

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Abstract. We prove that if $X$ is a strongly locally homogeneous and locally compact separable metric space and $G$ is a region in $X$ with dim $G = 2$, then $G$ is not separated by any arc in $G$.

1. Introduction

By a space we mean a separable metric space. Kallipoliti and Papasoglu \[4\] proved that any locally connected, simply connected, homogeneous metric continuum can not be separated by arcs, and asked if this is true without the assumption of simply connectedness. A partial answer of this question was provided in \[8\] for homogeneous metric continua of dimension two having a non-trivial second integral Čech cohomology group. In the present paper we prove the following partial answer to Kallipoliti and Papasoglu’s question.

**Theorem 1.1.** Let $X$ be a locally compact strongly locally homogeneous space and $G$ be a region in $X$ with dim $G = n \geq 2$. Then $G$ is not separated by any arc $J \subset G$.

Recall that a space is strongly locally homogeneous if every point $x \in X$ has a local basis of open sets $U$ such that for every $y, z \in U$ there is a homeomorphism $h$ on $X$ with $h(y) = z$ and $h$ is identity on $X \setminus U$. Obviously, every open subset of a strongly locally homogeneous space is also strongly locally homogeneous. Since strongly locally homogeneous connected spaces are homogeneous, any region $G$ satisfying the hypotheses of Theorem 1.1 should be homogeneous. We claim that it is locally connected as well. Indeed, since any strongly homogeneous Polish space is countable dense homogeneous \[1\] and a locally compact countable dense homogeneous connected space is locally connected \[3\],

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we have that any region $G$ from Theorem 1.1 is locally connected. (There is also a simple direct proof of this fact.) According to [6], every region of homogeneous locally compact space of dimension $n \geq 1$ cannot be separated by a closed set of dimension $\leq n - 2$. So, Theorem 1.1 is interesting only for regions $G$ of dimension two.

2. SOME PRELIMINARY RESULTS

**Lemma 2.1.** Let $A$ be a closed nowhere dense subset of $X$ such that $\dim X \setminus A = 0$. Then there is a retraction $r: X \to A$ such that $r(X \setminus A)$ is countable.

**Proof.** The technique is similar to that in [5]. In brief, one constructs a cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ by disjoint nonempty clopen subsets of $X$ such that

1. diam $V_n < d(V_n, A)$ for each $n$,
2. there is a sequence $\{a_n : n \in \mathbb{N}\}$ in $A$ such that $\lim_{n \to \infty} d(a_n, V_n) = 0$.

Then define $r: X \to A$ as follows: $r(a) = a$ for every $a$ and $r(V_n) = \{a_n\}$ for every $n$. It is easy to check that $r$ is as required. \qed

If $J$ is an arc and $p, q \in J$, then $(p, q)$ and $[p, q]$ denote, respectively, the open and closed subintervals in $J$ with endpoints $p, q$.

**Proposition 2.2.** Let $J = [a, b]$ be an arc in a space $X$ which is everywhere 2-dimensional. Then $b$ has arbitrarily small open neighborhoods $U$ such that bd$(U)$ is at most 1-dimensional and intersects $J$ in exactly one point.

**Proof.** Fix $\varepsilon > 0$ and let $U$ be an open neighborhood of $b$ in $X$ such that diam $U < \varepsilon$ and dim bd$U \leq 1$. We may assume without loss of generality that $J \setminus U \neq \emptyset$ and $J \cap U$ is uncountable. Put $Y = J \cup \overline{U}$. Moreover, put $A = J \cup \text{bd}U$, $B = (J \setminus U) \cup \text{bd}U$ and $C = (J \cap \overline{U}) \cup \text{bd}U$, respectively.

Let $D$ be a zero-dimensional dense subset of $U$ such that dim $U \setminus D = 1$. Since dim $J = 1$, we may clearly assume that $D \cap J = \emptyset$.

Because $C$ is a closed nowhere dense subset of $C \cup D$, there is a retraction $r_1: C \cup D \to C$ such that $r_1(D)$ is countable (Lemma 2.1).

Let $r: A \cup D \to A$ be defined by $r(x) = r_1(x)$ if $x \in C \cup D$ and $r(x) = x$ if $x \notin C \cup D$. Obviously $r$ is a retraction such that $r(D)$ is countable. Pick an arbitrary $s \in U \cap J$ such that $s \neq b$, $[s, b] \subset U$ and $s \notin r(D)$. Choose also two points $s_1, s_2 \in J \cap U$ different from $s$ and $b$ such that $s \in (s_1, s_2)$, and let $V_1 = A \setminus [s_1, b]$ and $V_2 = (s_2, b]$. Obviously $V_1$ and
$V_2$ are open subsets of $A$ containing $B$ and $\{b\}$, respectively. Moreover, $\overline{V}_1 = A \setminus (s_1, b]$ and $\overline{V}_2 = [s_2, b]$.

Claim 1. $\{s\}$ is a partition in $A$ between $\overline{V}_1$ and $\overline{V}_2$.

Indeed, put $P = [s, b]$ and $Q = [a, s] \cup \text{bd}U$. Then $P$ and $Q$ are closed subsets of $A$ such that $P \cup Q = A$, $\overline{V}_2 \subset P$, $\overline{V}_1 \subset Q$ and $P \cap Q = \{s\}$.

Claim 2. $\{s\}$ is a partition in $A \cup D$ between $r^{-1}(\overline{V}_1)$ and $r^{-1}(\overline{V}_2)$.

Since $r^{-1}(s) = \{s\}$, this is a direct consequence of Claim 1.

By [7, Lemma 3.1.4], there is a partition $S$ between $\{b\}$ and $B$ in $Y$ such that $S \cap (A \cup D) \subset \{s\}$. If $s \notin S$, then $S \cup \{s\}$ is also a partition between $\{b\}$ and $B$ in $Y$, hence we may assume without loss of generality that $s \in S$. But then $S \cap J = \{s\}$. Write $Y \setminus S$ as $E \cup F$, where $E$ and $F$ are disjoint relatively open subsets of $Y$ such that $b \in E$ and $B \subset F$.

Claim 3. $E \subset U$.

Indeed, since $E \cap B = E \cap ((J \setminus U) \cup \text{bd}U) = \emptyset$, this is clear.

Since $E$ is open in $U$ and $U$ is open in $X$ we have that $E$ is open in $X$. Moreover, $\text{diam} E < \varepsilon$. Also, $E \cup S$ is closed in $Y$ and hence in $X$. As a consequence $\text{bd}E \subset S$. Since $S \subset U \setminus D$, we have $\dim S \leq 1$, as required.

It will be convenient to use additive notation for the topological group $S^1$.

The following result can be proved by tools from algebraic topology. For the convenience of the reader, we include a simple direct proof.

Proposition 2.3. Let $X$ be a space and let $A$ be a closed subspace of it. Moreover, let $\gamma: A \to S^1$ be continuous. Suppose that there are closed subsets $P_1, P_2$ of $X$ satisfying the following conditions:

- $P_1 \cup P_2 = X$ and if $C = P_1 \cap P_2$ then $C \cap A$ is a singleton, say $c$;
- $\gamma|P_i \cap A$ is extendable over $P_i$ for each $i = 1, 2$, but $\gamma$ is not extendable over $X$.

Then there is a continuous function $\beta: C \to S^1$ such that $\beta(c) = 0$ and $\beta$ is not nullhomotopic.

Proof. Let $\alpha_i: P_i \to S^1$, for $i = 1, 2$, be a continuous extension of $\gamma|P_i \cap A$. Define $\beta: C \to S^1$ by $\beta(x) = \alpha_1(x) - \alpha_2(x)$ ($x \in C$). Then, clearly, $\beta(c) = 0$. We claim that $\beta$ is as required, and argue by contradiction. Assume that $\beta$ is nullhomotopic. Let $H: C \times I \to S^1$ be a homotopy
such that $H_0 \equiv 0$ and $H_1 = \beta$. Define $S: C \times \mathbb{I} \to \mathbb{S}^1$ by $S(x,t) = H(x,t) - H(c,t)$. Then $S_0 \equiv 0$, $S_1 = \beta$ and $S(c,t) = 0$ for every $t$. Define a homotopy $T: (C \cup (P_2 \cap A)) \times \mathbb{I} \to \mathbb{S}^1$ by

$$T(x,t) = \begin{cases} S(x,t) & (x \in C, t \in \mathbb{I}), \\ 0 & (x \in P_2 \cap A, t \in \mathbb{I}). \end{cases}$$

Then $T_0 \equiv 0$ and hence can be extended to the constant function with value 0 on $P_2$. By the Borsuk Homotopy Extension Theorem [7, 1.4.2], the function $T_1$ can be extended to a continuous function $\delta: P_2 \to \mathbb{S}^1$.

Now define $\varepsilon: X \to \mathbb{S}^1$ as follows:

$$\varepsilon|P_1 = \alpha_1, \quad \varepsilon|P_2 = \delta + \alpha_2.$$ 

If $x \in C$, then $\varepsilon|P_1(x) = \alpha_1(x)$ and $\varepsilon|P_2(x) = \delta(x) + \alpha_2(x) = S_1(x) + \alpha_2(x) = \beta(x) + \alpha_2(x) = \alpha_1(x)$. Hence $\varepsilon$ is well defined and continuous. Also observe that if $x \in P_2 \cap A$, then

$$\varepsilon(x) = 0 + \alpha_2(x) = \alpha_2(x).$$

Hence $\varepsilon$ extends $\gamma$, which is a contradiction. \hfill \Box

3. Proof of Theorem 1.1

Throughout, let $X$ be a locally compact and strongly locally homogeneous space, and $G$ be a region in $X$ of dimension 2. Suppose $G$ is separated by an arc $J = [a,b] \subset G$. Recall that $G$ is homogeneous and locally connected (see §1). Write $G \setminus J$ as $G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint nonempty open subsets of $G$. Everywhere below $\overline{K}$ denotes the closure of $K$ in $G$ for any set $K \subset G$.

We say that a space $Y$ has no local cut points if no connected open subset $U \subset Y$ has a cut point.

**Lemma 3.1.** $G$ has no local cutpoints.

**Proof.** By Kruspiki [6, Theorem 2.1] it follows that every nonempty open connected subset $U$ of $G$ is a Cantor manifold of dimension 2. Hence $U$ cannot be separated by a zero-dimensional closed set. \hfill \Box

A space $X$ is crowded if it has no isolated points.

**Lemma 3.2.** The set $S = \overline{G_1} \cap \overline{G_2}$ is a 1-dimensional closed and crowded subspace of $J$ which separates $G$.

**Proof.** Assume first that $J \setminus (\overline{G_1} \cup \overline{G_2}) \neq \emptyset$. Then $G$ is somewhere at most 1-dimensional. Hence $G$ is at most 1-dimensional at every point by homogeneity. But this contradicts $G$ being 2-dimensional.
Hence $J \subset \overline{G_1} \cup \overline{G_2}$ and so $G = \overline{G_1} \cup \overline{G_2}$. If $S$ is empty, then $G$ is covered by the disjoint nonempty closed sets $\overline{G_1}$ and $\overline{G_2}$ which contradicts the connectivity of $G$.

Now assume that $x$ is an isolated point of $S$. Let $U$ be an open connected neighborhood of $x$ in $G$ such that $U \cap S = \{x\}$. Then $x$ is a cutpoint of $U$. But this contradicts Lemma 3.1.

We conclude that $S$ separates $G$ and consequently has to be 1-dimensional by Krupski [6].

Let $s$ be the maximum of $S$ (as a subset of $[a, b]$). Then $J_s = [a, s]$ also separates $G$ and $G \setminus J_s$ is the union of the disjoint open sets $G'_1$ and $G'_2$, where $G'_i = \overline{G_i} \setminus J_s$. Moreover, $s \in \overline{G_1} \cap \overline{G_2}$. Hence, we can assume without loss of generality that $b \in \overline{G_1} \cap \overline{G_2}$.

**Lemma 3.3.** There is an open neighborhood $U \subset G$ of $b$ having compact closure and a compact set $F \subset G$ such that for every open neighborhood $V$ of $b$ with $\overline{V} \subset U$ there exist a compact set $M_U \subset \overline{U}$ and a continuous function $f : \bd(U \cap F) \to S^1$ such that:

1. $b \in U \cap F$;
2. $M_U$ is everywhere 2-dimensional and $M_U \cap V \neq \emptyset$;
3. $\dim \bd U \leq 1$ and $J \cap \bd U$ is a point;
4. $f$ is not extendable over $\bd(U \cap F) \cup M_U$, but it is extendable over $\bd(U \cap F) \cup P$ for every proper closed set $P$ of $M_U$.

**Proof.** Choose a compact neighborhood $O_b$ of $b$ in $G$. Since every neighborhood of $b$ is of dimension 2, there is a compact subset $Y \subset O_b$, a closed set $A \subset Y$ and a continuous function $g : A \to S^1$ not extendable over $Y$. Let $F$ be a minimal closed subset of $Y$ containing $A$ such that $g$ is not extendable over $F$. Then for every open subset $W$ of $F \setminus A$ with $\overline{W} \cap A = \emptyset$ there is a function $f_W : F \setminus W \to S^1$ extending $g$ such that $f_W$ can not be extended to a continuous function $\tilde{f}_W : F \to S^1$. This means that $f_W|\bd F W$ is not extendable over $\overline{W}$. Consequently, $F \setminus A$ is everywhere two-dimensional. We can assume by homogeneity that $b \in F \setminus A$. Indeed, by Effros’ theorem [2], we take $O_b$ so small that for every point $x \in O_b$ there is a homeomorphism $h$ on $G$ with $h(b) = x$ and $O_h \subset h(G)$. Then, consider the set $h(G)$ instead of $G$.

By Proposition 2.2 there are an open neighborhood $U$ of $b$ whose closure in $G$ is a compact and a point $c \in (a, b)$ such that $\bd U \cap J = \{c\}$, $\dim \bd U \leq 1$ and $\overline{U} \cap A = \emptyset$. Suppose $V$ is an open neighborhood of $b$ such that $\overline{V} \subset U$, and consider a continuous function $f_V : F \setminus V \to S^1$ extending $g$ which is not extendable over $F$. Let $f = f_V|\bd(U \cap F)$. Clearly, $f$ cannot be extended to a continuous function $\tilde{f} : U \cap F \to S^1$, but $f$ can be extended to a continuous function from $(U \cap F) \setminus V$ into
$S^1$. Let $M_U$ be a minimal closed subset of $\overline{U \cap F}$ with the property that $f$ cannot be extended to a continuous function $\tilde{f} : \bd F (U \cap F) \cup M_U \to S^1$. The minimality of $M_U$ implies that $f$ is extendable over $\bd F (U \cap F) \cup P$ for any any closed set $P \not\subseteq M_U$. Because $f$ is extendable over $(U \cap F) \setminus V$, $M_U \cap V \neq \emptyset$. It is clear that $M_U$ is a continuum.

Assume that $O$ is a nonempty open subset of $M_U$ such that $\dim O \leq 1$. Taking a smaller open subset of $O$, we may assume that $\dim \overline{O} \leq 1$. There are two possibilities, either $O \subset \bd F (U \cap F)$ or $O \setminus \bd F (U \cap F) \neq \emptyset$. If $O \subset \bd F (U \cap F)$, $M_U \setminus O$ is a proper closed subset of $M_U$ having the same properties as $M_U$, which contradicts minimality. If $O' = O \setminus \bd F (U \cap F) \neq \emptyset$, then $P = M_U \setminus O'$ is a proper closed subset of $M_U$. So, there is an extension $\tilde{f}_1 : \bd F (U \cap F) \cup P \to S^1$ of $f$. Since $\dim \overline{O'} \leq 1$, we can extend $\tilde{f}_1$ over $\bd F (U \cap F) \cup M_U$, a contradiction. Therefore, $M_U$ is everywhere 2-dimensional.

Now, we can complete the proof of Theorem 1.1. Choose open neighborhoods $U$ and $V$ of $b$, closed sets $F \subset G$ and $M_U \subset U \cap F$ and a continuous function $f : \bd F (U \cap F) \to S^1$ satisfying the conditions (1) $-$ (4) from Lemma 3.3. Let also $J \cap \bd U = \{c\}$ and $C = [c, b]$. We can also assume that $V$ satisfies the additional property that for every two points $p, q \in V$ there is a homeomorphism $\varphi$ of $G$ supported on $V$ with $\varphi(p) = q$. We may consequently assume without loss of generality that $b \in M_U$. Indeed, if $b \notin M_U$ we take a point $x \in M_U \cap V$ and a homeomorphism $\varphi$ of $G$ supported on $V$ such that $\varphi(x) = b$. Then the set $\varphi(M_U)$ satisfies all condition from Lemma 3.3 and contains $b$. Since $M_U$ is everywhere 2-dimensional, $\dim (M_U \cap V) = 2$. Hence, $M_U \cap V$ meets at least one of the sets $G_i, i = 1, 2$.

Assume first that $M_U \cap V \cap G_1 \neq \emptyset$ but $M_U \cap V \cap G_2 = \emptyset$.

Then $M_U \cap W$ meets $G_1$ for every neighborhood $W$ of $b$ with $W \subset V$. Indeed, because $\dim M_U \cap W = 2$ and $M_U \cap W \cap G_2 = \emptyset$ it follows that $M_U \cap G_1 \cap W \neq \emptyset$. There consequently is a neighborhood $W$ of $b$ in $G$ such that

(5) $\overline{W} \subset V$, $(M_U \cap V) \cap (G_1 \setminus \overline{W}) \neq \emptyset$ and $M_U \cap G_1 \cap W \neq \emptyset$

(6) For every $x, y \in W$ there is a homeomorphism $h$ of $G$ supported on $W$ with $h(x) = y$.

Finally, choose points $x \in M_U \cap G_1 \cap W$ and $y \in W \cap G_2$ and a homeomorphism $h : G \to G$ supported on $W$ with $h(x) = y$. Since $h(z) = z$ for all points $z \in (M_U \cap V) \cap (G_1 \setminus \overline{W})$, the set $\tilde{K} = h(M_U)$ meets both $G_1$ and $G_2$. Moreover, the function $f$ is not extendable over $\bd F (U \cap F) \cup \tilde{K}$ (otherwise $f$ would be extendable over $\bd F (U \cap F) \cup M_U$). On the other hand, since each of the sets $Q_i = h^{-1}(\tilde{K} \cap \overline{G_i})$, 

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\( i = 1, 2 \), is a proper closed subset of \( M_U \), \( f \) is extendable over each of the sets \( \text{bd}_F(U \cap F) \cup (\bar{\bar{K}} \cap \bar{G}_i) \). Let \( \gamma : \text{bd} U \to S^1 \) be an extension of \( f \) (recall that \( \dim \text{bd} U \leq 1 \) and \( \text{bd}_F(U \cap F) \) is a closed subset of \( \text{bd} U \), so such \( \gamma \) exists). Because \( f \) is not extendable over \( \text{bd}_F(U \cap F) \cup \bar{\bar{K}} \), \( \gamma \) is not extendable over the set \( K = \text{bd} U \cup \bar{\bar{K}} \cup C \). Denote \( P_i = C \cup (K \cap \bar{G}_i) \), \( i = 1, 2 \). Obviously, \( P_1 \cup P_2 = K \) and \( P_1 \cap P_2 = C \). Then for each \( i \) we have \( P_i \cap \text{bd} U = \{c\} \cup (\text{bd} U \cap \bar{G}_i) \). So, the function \( \gamma|_{(P_i \cap \text{bd} U)} \) is extendable over the set \( P_i \) because \( \dim C \cup \text{bd} U = 1 \). Hence, we can apply Proposition 2.3 (with \( A = \text{bd} U \)) to conclude that there is a continuous function \( \beta : C \to S^1 \) such that \( \beta \) is not nullhomotopic, a contradiction.

Assume next that \( M_U \cap V \) meets both \( G_1 \) and \( G_2 \). We can now proceed as above (considering \( M_U \) instead of \( \bar{\bar{K}} \)) to obtain the desired contradiction.

References


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