Homogeneous continua that are not separated by arcs

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Abstract. We prove that if $X$ is a strongly locally homogeneous and locally compact separable metric space and $G$ is a region in $X$ with $\dim G = 2$, then $G$ is not separated by any arc in $G$.

1. Introduction

By a space we mean a separable metric space. Kallipoliti and Papasoglu [4] proved that any locally connected, simply connected, homogeneous metric continuum can not be separated by arcs, and asked if this is true without the assumption of simply connectedness. A partial answer of this question was provided in [8] for homogeneous metric continua of dimension two having a non-trivial second integral Čech cohomology group. In the present paper we prove the following partial answer to Kallipoliti and Papasoglu’s question.

Theorem 1.1. Let $X$ be a locally compact strongly locally homogeneous space and $G$ be a region in $X$ with $\dim G = n \geq 2$. Then $G$ is not separated by any arc $J \subset G$.

Recall that a space is strongly locally homogeneous if every point $x \in X$ has a local basis of open sets $U$ such that for every $y, z \in U$ there is a homeomorphism $h$ on $X$ with $h(y) = z$ and $h$ is identity on $X \setminus U$. Obviously, every open subset of a strongly locally homogeneous space is also strongly locally homogeneous. Since strongly locally homogeneous connected spaces are homogeneous, any region $G$ satisfying the hypotheses of Theorem 1.1 should be homogeneous. We claim that it is locally connected as well. Indeed, since any strongly homogeneous Polish space is countable dense homogeneous [1] and a locally compact countable dense homogeneous connected space is locally connected [3],

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we have that any region $G$ from Theorem 1.1 is locally connected. (There is also a simple direct proof of this fact.) According to [6], every region of homogeneous locally compact space of dimension $n \geq 1$ can not be separated by a closed set of dimension $\leq n - 2$. So, Theorem 1.1 is interesting only for regions $G$ of dimension two.

2. SOME PRELIMINARY RESULTS

**Lemma 2.1.** Let $A$ be a closed nowhere dense subset of $X$ such that $\dim X \setminus A = 0$. Then there is a retraction $r : X \to A$ such that $r(X \setminus A)$ is countable.

**Proof.** The technique is similar to that in [5]. In brief, one constructs a cover $V = \{V_n : n \in \mathbb{N}\}$ by disjoint nonempty clopen subsets of $X$ such that

1. $\text{diam } V_n < d(V_n, A)$ for each $n$,
2. there is a sequence $\{a_n : n \in \mathbb{N}\}$ in $A$ such that $\lim_{n \to \infty} d(a_n, V_n) = 0$.

Then define $r : X \to A$ as follows: $r(a) = a$ for every $a$ and $r(V_n) = \{a_n\}$ for every $n$. It is easy to check that $r$ is as required. $\Box$

If $J$ is an arc and $p, q \in J$, then $(p, q)$ and $[p, q]$ denote, respectively, the open and closed subintervals in $J$ with endpoints $p, q$.

**Proposition 2.2.** Let $J = [a, b]$ be an arc in a space $X$ which is everywhere 2-dimensional. Then $b$ has arbitrarily small open neighborhoods $U$ such that $\text{bd}(U)$ is at most 1-dimensional and intersects $J$ in exactly one point.

**Proof.** Fix $\varepsilon > 0$ and let $U$ be an open neighborhood of $b$ in $X$ such that $\text{diam } \overline{U} < \varepsilon$ and $\dim \text{bd } U \leq 1$. We may assume without loss of generality that $J \setminus U \neq \emptyset$ and $J \cap U$ is uncountable. Put $Y = J \cup \overline{U}$. Moreover, put $A = J \cup \text{bd } U$, $B = (J \setminus U) \cup \text{bd } U$ and $C = (J \cap \overline{U}) \cup \text{bd } U$, respectively.

Let $D$ be a zero-dimensional dense subset of $U$ such that $\dim U \setminus D = 1$. Since $\dim J = 1$, we may clearly assume that $D \cap J = \emptyset$.

Because $C$ is a closed nowhere dense subset of $C \cup D$, there is a retraction $r_1 : C \cup D \to C$ such that $r_1(D)$ is countable (Lemma 2.1). Let $r : A \cup D \to A$ be defined by $r(x) = r_1(x)$ if $x \in C \cup D$ and $r(x) = x$ if $x \notin C \cup D$. Obviously $r$ is a retraction such that $r(D)$ is countable. Pick an arbitrary $s \in U \cap J$ such that $s \neq b$, $[s, b] \subset U$ and $s \notin r(D)$. Choose also two points $s_1, s_2 \in J \cap U$ different from $s$ and $b$ such that $s \in (s_1, s_2)$, and let $V_1 = A \setminus [s_1, b]$ and $V_2 = (s_2, b]$. Obviously $V_1$ and
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$V_2$ are open subsets of $A$ containing $B$ and $\{b\}$, respectively. Moreover, $\nabla_1 = A \setminus (s_1, b]$ and $\nabla_2 = [s_2, b]$.

Claim 1. $\{s\}$ is a partition in $A$ between $\nabla_1$ and $\nabla_2$.

Indeed, put $P = [s, b]$ and $Q = [a, s] \cup \text{bd}U$. Then $P$ and $Q$ are closed subsets of $A$ such that $P \cup Q = A$, $\nabla_2 \subset P$, $\nabla_1 \subset Q$ and $P \cap Q = \{s\}$.

Claim 2. $\{s\}$ is a partition in $A \cup D$ between $r^{-1}(\nabla_1)$ and $r^{-1}(\nabla_2)$.

Since $r^{-1}(s) = \{s\}$, this is a direct consequence of Claim 1.

By [7, Lemma 3.1.4], there is a partition $S$ between $\{b\}$ and $B$ in $Y$ such that $S \cap (A \cup D) \subset \{s\}$. If $s \not\in S$, then $S \cup \{s\}$ is also a partition between $\{b\}$ and $B$ in $Y$, hence we may assume without loss of generality that $s \in S$. But then $S \cap J = \{s\}$. Write $Y \setminus S$ as $E \cup F$, where $E$ and $F$ are disjoint relatively open subsets of $Y$ such that $b \in E$ and $B \subset F$.

Claim 3. $E \subset U$.

Indeed, since $E \cap B = E \cap ((J \setminus U) \cup \text{bd}U) = \emptyset$, this is clear.

Since $E$ is open in $U$ and $U$ is open in $X$ we have that $E$ is open in $X$. Moreover, $\text{diam } E < \varepsilon$. Also, $E \cup S$ is closed in $Y$ and hence in $X$. As a consequence $\text{bd } E \subset S$. Since $S \subset U \setminus D$, we have $\dim S \leq 1$, as required.

It will be convenient to use additive notation for the topological group $\mathbb{S}^1$.

The following result can be proved by tools from algebraic topology. For the convenience of the reader, we include a simple direct proof.

**Proposition 2.3.** Let $X$ be a space and let $A$ be a closed subspace of it. Moreover, let $\gamma: A \to \mathbb{S}^1$ be continuous. Suppose that there are closed subsets $P_1, P_2$ of $X$ satisfying the following conditions:

- $P_1 \cup P_2 = X$ and if $C = P_1 \cap P_2$ then $C \cap A$ is a singleton, say $c$;
- $\gamma|P_i \cap A$ is extendable over $P_i$ for each $i = 1, 2$, but $\gamma$ is not extendable over $X$.

Then there is a continuous function $\beta: C \to \mathbb{S}^1$ such that $\beta(c) = 0$ and $\beta$ is not nullhomotopic.

**Proof.** Let $\alpha_i: P_i \to \mathbb{S}^1$ for $i = 1, 2$ be a continuous extension of $\gamma|P_i \cap A$. Define $\beta: C \to \mathbb{S}^1$ by $\beta(x) = \alpha_1(x) - \alpha_2(x)$ ($x \in C$). Then, clearly, $\beta(c) = 0$. We claim that $\beta$ is as required, and argue by contradiction. Assume that $\beta$ is nullhomotopic. Let $H: C \times I \to \mathbb{S}^1$ be a homotopy
such that $H_0 \equiv 0$ and $H_1 = \beta$. Define $S: C \times \mathbb{I} \to \mathbb{S}^1$ by $S(x,t) = H(x,t) - H(c,t)$. Then $S_0 \equiv 0$, $S_1 = \beta$ and $S(c,t) = 0$ for every $t$. Define a homotopy $T: (C \cup (P_2 \cap A)) \times \mathbb{I} \to \mathbb{S}^1$ by

$$T(x,t) = \begin{cases} S(x,t) & (x \in C, t \in \mathbb{I}), \\ 0 & (x \in P_2 \cap A, t \in \mathbb{I}). \end{cases}$$

Then $T_0 \equiv 0$ and hence can be extended to the constant function with value 0 on $P_2$. By the Borsuk Homotopy Extension Theorem [7, 1.4.2], the function $T_1$ can be extended to a continuous function $\delta: P_2 \to \mathbb{S}^1$.

Now define $\varepsilon: X \to \mathbb{S}^1$ as follows:

$$\varepsilon|P_1 = \alpha_1, \quad \varepsilon|P_2 = \delta + \alpha_2.$$ 

If $x \in C$, then $\varepsilon|P_1(x) = \alpha_1(x)$ and $\varepsilon|P_2(x) = \delta(x) + \alpha_2(x) = S_1(x) + \alpha_2(x) = \beta(x) + \alpha_2(x) = \alpha_1(x)$. Hence $\varepsilon$ is well defined and continuous. Also observe that if $x \in P_2 \cap A$, then $\varepsilon(x) = 0 + \alpha_2(x) = \alpha_2(x)$. Hence $\varepsilon$ extends $\gamma$, which is a contradiction. \hfill \Box

3. Proof of Theorem 1.1

Throughout, let $X$ be a locally compact and strongly locally homogeneous space, and $G$ be a region in $X$ of dimension 2. Suppose $G$ is separated by an arc $J = [a,b] \subset G$. Recall that $G$ is homogeneous and locally connected (see §1). Write $G \setminus J$ as $G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint nonempty open subsets of $G$. Everywhere below $\overline{K}$ denotes the closure of $K$ in $G$ for any set $K \subset G$.

We say that a space $Y$ has no local cut points if no connected open subset $U \subset Y$ has a cut point.

**Lemma 3.1.** $G$ has no local cutpoints.

**Proof.** By Kruspki [6, Theorem 2.1] it follows that every nonempty open connected subset $U$ of $G$ is a Cantor manifold of dimension 2. Hence $U$ cannot be separated by a zero-dimensional closed set. \hfill \Box

A space $X$ is **crowded** if it has no isolated points.

**Lemma 3.2.** The set $S = \overline{G_1} \cap \overline{G_2}$ is a 1-dimensional closed and crowded subspace of $J$ which separates $G$.

**Proof.** Assume first that $J \setminus (\overline{G_1} \cup \overline{G_2}) \neq \emptyset$. Then $G$ is somewhere at most 1-dimensional. Hence $G$ is at most 1-dimensional at every point by homogeneity. But this contradicts $G$ being 2-dimensional.
Hence \( J \subset \overline{G_1} \cup \overline{G_2} \) and so \( G = \overline{G_1} \cup \overline{G_2} \). If \( S \) is empty, then \( G \) is covered by the disjoint nonempty closed sets \( \overline{G_1} \) and \( \overline{G_2} \) which contradicts the connectivity of \( G \).

Now assume that \( x \) is an isolated point of \( S \). Let \( U \) be an open connected neighborhood of \( x \) in \( G \) such that \( U \cap S = \{x\} \). Then \( x \) is a cutpoint of \( U \). But this contradicts Lemma 3.1.

We conclude that \( S \) separates \( G \) and consequently has to be 1-dimensional by Krupski [6].

Let \( s \) be the maximum of \( S \) (as a subset of \([a, b]\)). Then \( J_s = [a, s] \) also separates \( G \) and \( G \setminus J_s \) is the union of the disjoint open sets \( G'_1 \) and \( G'_2 \), where \( G'_1 = \overline{G_i} \setminus J_s \). Moreover, \( s \in \overline{G_1} \cap \overline{G_2} \). Hence, we can assume without loss of generality that \( b \in \overline{G_1} \cap \overline{G_2} \).

**Lemma 3.3.** There is an open neighborhood \( U \subset G \) of \( b \) having compact closure and a compact set \( F \subset G \) such that for every open neighborhood \( V \) of \( b \) with \( \overline{V} \subset U \) there exist a compact set \( M_U \subset \overline{U} \) and a continuous function \( f : \partial F(U \cap F) \to S^1 \) such that:

1. \( b \in U \cap F \);
2. \( M_U \) is everywhere 2-dimensional and \( M_U \cap V \neq \emptyset \);
3. \( \dim \partial U \leq 1 \) and \( J \cap \partial U \) is a point;
4. \( f \) is not extendable over \( \partial F(U \cap F) \cup M_U \), but it is extendable over \( \partial F(U \cap F) \cup P \) for every proper closed set \( P \) of \( M_U \).

**Proof.** Choose a compact neighborhood \( O_b \) of \( b \) in \( G \). Since every neighborhood of \( b \) is of dimension 2, there is a compact subset \( Y \subset O_b \), a closed set \( A \subset Y \) and a continuous function \( g : A \to S^1 \) not extendable over \( Y \). Let \( F \) be a minimal closed subset of \( Y \) containing \( A \) such that \( g \) is not extendable over \( F \). Then for every open subset \( W \) of \( F \setminus A \) with \( \overline{W} \cap A = \emptyset \) there is a function \( f_W : F \setminus W \to S^1 \) extending \( g \) such that \( f_W \) can not be extended to a continuous function \( \overline{f_W} : F \to S^1 \). This means that \( f_W | \partial F \) is not extendable over \( \overline{W} \). Consequently, \( F \setminus A \) is everywhere two-dimensional. We can assume by homogeneity that \( b \in F \setminus A \). Indeed, by Effros’ theorem [2], we take \( O_b \) so small that for every point \( x \in O_b \) there is a homeomorphism \( h \) on \( G \) with \( h(b) = x \) and \( O_h \subset h(G) \). Then, consider the set \( h(G) \) instead of \( G \).

By Proposition 2.2, there are an open neighborhood \( U \) of \( b \) whose closure in \( G \) is a compact and a point \( c \in (a, b) \) such that \( \partial U \cap J = \{c\} \), \( \dim \partial U \leq 1 \) and \( \overline{U} \cap A = \emptyset \). Suppose \( V \) is an open neighborhood of \( b \) such that \( \overline{V} \subset U \), and consider a continuous function \( f_V : F \setminus V \to S^1 \) extending \( g \) which is not extendable over \( F \). Let \( f = f_V | \partial F(U \cap F) \). Clearly, \( f \) cannot be extended to a continuous function \( \overline{f} : \overline{U \cap F} \to S^1 \), but \( f \) can be extended to a continuous function from \((U \cap F) \setminus V \) into
$\mathbb{S}^1$. Let $M_U$ be a minimal closed subset of $\overline{U \cap F}$ with the property that $f$ cannot be extended to a continuous function $\tilde{f} : \text{bd}_F(U \cap F) \cup M_U \to \mathbb{S}^1$. The minimality of $M_U$ implies that $f$ is extendable over $\text{bd}_F(U \cap F) \cup P$ for any any closed set $P \not\subseteq M_U$. Because $f$ is extendable over $\overline{U \cap F} \setminus V$, $M_U \cap V \neq \emptyset$. It is clear that $M_U$ is a continuum.

Assume that $O$ is a nonempty open subset of $M_U$ such that $\dim O \leq 1$. Taking a smaller open subset of $O$, we may assume that $\dim \overline{O} \leq 1$. There are two possibilities, either $O \subset \text{bd}_F(U \cap F)$ or $O \setminus \text{bd}_F(U \cap F) \neq \emptyset$. If $O \subset \text{bd}_F(U \cap F)$, $M_U \setminus O$ is a proper closed subset of $M_U$ having the same properties as $M_U$, which contradicts minimality. If $O' = O \setminus \text{bd}_F(U \cap F) \neq \emptyset$, then $P = M_U \setminus O'$ is a proper closed subset of $M_U$. So, there is an extension $f_1 : \text{bd}_F(U \cap F) \cup P \to \mathbb{S}^1$ of $f$. Since $\dim \overline{O} \leq 1$, we can extend $f_1$ over $\text{bd}_F(U \cap F) \cup M_U$, a contradiction. Therefore, $M_U$ is everywhere 2-dimensional. 

Now, we can complete the proof of Theorem 1.1. Choose open neighborhoods $U$ and $V$ of $b$, closed sets $F \subset G$ and $M_U \subset \overline{U \cap F}$ and a continuous function $f : \text{bd}_F(U \cap F) \to \mathbb{S}^1$ satisfying the conditions (1) – (4) from Lemma 3.3. Let also $J \cap \text{bd} U = \{c\}$ and $C = [c, b]$. We can also assume that $V$ satisfies the additional property that for every two points $p, q \in V$ there is a homeomorphism $\varphi$ of $G$ supported on $V$ with $\varphi(p) = q$. We may consequently assume without loss of generality that $b \in M_U$. Indeed, if $b \notin M_U$ we take a point $x \in M_U \cap V$ and a homeomorphism $\varphi$ of $G$ supported on $V$ such that $\varphi(x) = b$. Then the set $\varphi(M_U)$ satisfies all condition from Lemma 3.3 and contains $b$. Since $M_U$ is everywhere 2-dimensional, $\dim(M_U \cap V) = 2$. Hence, $M_U \cap V$ meets at least one of the sets $G_i$, $i = 1, 2$.

Assume first that $M_U \cap V \cap G_1 \neq \emptyset$ but $M_U \cap V \cap G_2 = \emptyset$.

Then $M_U \cap W$ meets $G_1$ for every neighborhood $W$ of $b$ with $W \subset V$. Indeed, because $\dim M_U \cap W = 2$ and $M_U \cap W \cap G_2 = \emptyset$ it follows that $M_U \cap G_1 \cap W \neq \emptyset$. There consequently is a neighborhood $W$ of $b$ in $G$ such that

1. $W \subset V$, $(M_U \cap V) \cap (G_1 \setminus W) \neq \emptyset$ and $M_U \cap G_1 \cap W \neq \emptyset$;
2. For every $x, y \in W$ there is a homeomorphism $h$ of $G$ supported on $W$ with $h(x) = y$.

Finally, choose points $x \in M_U \cap G_1 \cap W$ and $y \in W \cap G_2$ and a homeomorphism $h : G \to G$ supported on $W$ with $h(x) = y$. Since $h(z) = z$ for all points $z \in (M_U \cap V) \cap (G_1 \setminus W)$, the set $\tilde{K} = h(M_U)$ meets both $G_1$ and $G_2$. Moreover, the function $f$ is not extendable over $\text{bd}_F(U \cap F) \cup \tilde{K}$ (otherwise $f$ would be extendable over $\text{bd}_F(U \cap F) \cup M_U$). On the other hand, since each of the sets $Q_i = h^{-1}(\tilde{K} \cap G_i)$,
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$i = 1, 2$, is a proper closed subset of $M_U$, $f$ is extendable over each of the sets $\text{bd}_F(U \cap F) \cup (\bar{K} \cap \bar{G}_i)$. Let $\gamma : \text{bd} U \to \mathbb{S}^1$ be an extension of $f$ (recall that $\dim \text{bd} U \leq 1$ and $\text{bd}_F(U \cap F)$ is a closed subset of $\text{bd} U$, so such $\gamma$ exists). Because $f$ is not extendable over $\text{bd}_F(U \cap F) \cup \bar{K}$, $\gamma$ is not extendable over the set $K = \text{bd} U \cup \bar{K} \cup C$. Denote $P_i = C \cup (K \cap \bar{G}_i)$, $i = 1, 2$. Obviously, $P_1 \cup P_2 = K$ and $P_1 \cap P_2 = C$. Then for each $i$ we have $P_i \cap \text{bd} U = \{c\} \cup (\text{bd} U \cap \bar{G}_i)$. So, the function $\gamma|_{(P_i \cap \text{bd} U)}$ is extendable over the set $P_i$ because $\dim C \cup \text{bd} U = 1$. Hence, we can apply Proposition 2.3 (with $A = \text{bd} U$) to conclude that there is a continuous function $\beta : C \to \mathbb{S}^1$ such that $\beta$ is not nullhomotopic, a contradiction.

Assume next that $M_U \cap V$ meets both $G_1$ and $G_2$. We can now proceed as above (considering $M_U$ instead of $\bar{K}$) to obtain the desired contradiction.

References


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