Online Appendix of "A generalization of the Aumann-Shapley value for risk capital allocation problems"

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This is the Online Appendix of the paper entitled "A generalization of the Aumann-Shapley value for risk capital allocation problems". This Online Appendix contains all proofs.

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A Proofs of Subsections 2.1, 3.1 and 3.2

Proof of Proposition 2.2 Let Q be the generating probability measure set of ρ that is defined in (3), i.e.,

$$Q = \{\mathbb{Q} \in \mathcal{P}(\Omega, \mathcal{F}) : \mathbb{Q}(A) \geq v(A) \text{ for all } A \in \mathcal{F}\},$$

where $v : \mathcal{F} \to \mathbb{R}_+$ is supermodular, $v(\emptyset) = 0$ and $v(\Omega) = 1$. Note that as the state space Ω is finite, the σ-algebra $\mathcal{F}$ is finite as well. Because $\mathcal{F}$ is finite, Q is defined via a finite number of linear inequalities on $[0,1]^\Omega$. So, Q is a convex polytope. Let $\widetilde{Q}$ be the finite collection of extreme points of this convex polytope. Because $\mathbb{Q} \to E_{\mathbb{Q}}[X]$ is a linear map on Q for every $X \in \mathbb{R}^\Omega$, (1) is a linear programming problem and, therefore, we have

$$\rho(X) = \sup\{E_{\mathbb{Q}}[X] : \mathbb{Q} \in Q\} = \max\left\{E_{\mathbb{Q}}[X] : \mathbb{Q} \in \widetilde{Q}\right\}, \qquad \text{for all } X \in \mathbb{R}^\Omega.$$

Hence, $\rho(X)$ equals the maximum of all expectations of X under the probability measures in $\widetilde{Q}$. Hence, $\widetilde{Q}$ is finite a generating probability measure set. This concludes the proof. □

Proof of Proposition 2.3 The set Q^v defined in (3) is the core of the Transferable Utility game (Ω, v), where the state space Ω is now interpreted as a "player" set. Supermodularity of the function v is equivalent to convexity of the game (Ω, v) (Shapley, 1971). Moreover, Shapley (1971) shows that the core of a convex game is the convex hull of the marginal vectors. The marginal vectors of the game are the vectors $m^{\sigma,v} \in \mathbb{R}^\Omega$ with $m^{\sigma,v}_{\sigma(j)} := \mathbb{Q}^{\sigma,v}(\omega_{\sigma(j)})$, for all $\sigma \in \Pi(\Omega)$.

Proof of Proposition 3.2 For all $R \in \mathcal{R}$, we have

$$\begin{aligned} r(\lambda) &= \max\left\{E_{\mathbb{Q}}\left[\sum_{i \in N} \lambda_i X_i\right] : \mathbb{Q} \in Q(\rho)\right\} \\ &= \max\left\{\sum_{i \in N} \lambda_i E_{\mathbb{Q}}[X_i] : \mathbb{Q} \in Q(\rho)\right\} \\ &= \max\{f_{\mathbb{Q}}(\lambda) : \mathbb{Q} \in Q(\rho)\}, \end{aligned} \tag{A.1}$$

for all $\lambda \in [0,1]^N$. This concludes the proof. □

Proof of Proposition 3.4 (i) Follows directly from the fact that r is the maximum of finitely many linear (hence partially differentiable) functions $f_{\mathbb{Q}_m}, m \in \{1, \dots, p\}$.

We continue with the proof of (ii). We obtain for all $\ell, m \in \{1, \dots, p\}$ that

$$\begin{aligned} A_{\mathbb{Q}_\ell} \cap A_{\mathbb{Q}_m} &= \left\{\lambda \in [0,1]^N : r(\lambda) = f_{\mathbb{Q}_\ell}(\lambda) = f_{\mathbb{Q}_m}(\lambda)\right\} \\ &\subseteq \left\{\lambda \in [0,1]^N : f_{\mathbb{Q}_\ell}(\lambda) = f_{\mathbb{Q}_m}(\lambda)\right\} \\ &= \left\{\lambda \in [0,1]^N : \sum_{i \in N} \lambda_i \left(E_{\mathbb{Q}_\ell}\left[X_i\right] - E_{\mathbb{Q}_m}\left[X_i\right]\right) = 0\right\}. \end{aligned} \tag{A.2}$$

If $E_{\mathbb{Q}_\ell}[X_i] = E_{\mathbb{Q}_m}[X_i]$ for all $i \in N$, we have $A_{\mathbb{Q}_\ell} = A_{\mathbb{Q}_m}$ which implies $\ell = m$. So, the set $A_{\mathbb{Q}_\ell} \cap A_{\mathbb{Q}_m}$ is a (possibly empty) subset of a hyperplane passing through $\lambda = e_\emptyset$ for all $\ell, m \in \{1, \dots, p\}$ such that $\ell \neq m$. We have by construction that

$$[0,1]^N \backslash L(R) = \bigcup_{\ell, m \in \{1, \dots, p\} : \ell \neq m} A_{\mathbb{Q}_\ell} \cap A_{\mathbb{Q}_m}, \qquad \text{for all } R \in \mathcal{R}. \tag{A.3}$$

From this it follows that the collection of profiles where the risk capital function r is not partially differentiable is a subset of the collection of a finite number of hyperplanes passing trough $\lambda = e_\emptyset$.
□

Proof of Proposition 3.6 Let $n \in \mathbb{N}$ and $P \in \mathcal{P}^n$. The result follows directly from

$$\sum_{i \in N} K_i^{path,P}(R) = \sum_{i \in N} \sum_{k=0}^{|N|n-1} \left[r(P(k+1)) - r(P(k))\right] \mathbb{1}_{i(P,k)=i} \tag{A.4}$$

$$= \sum_{k=0}^{|N|n-1} \left[r(P(k+1)) - r(P(k))\right] \sum_{i \in N} \mathbb{1}_{i(P,k)=i} \tag{A.5}$$

$$= \sum_{k=0}^{|N|n-1} \left[r(P(k+1)) - r(P(k))\right] \tag{A.6}$$

$$= r(P(|N|n)) - r(P(0))$$

$$= r(e_N), \tag{A.7}$$

where $\mathbb{1}_{i(P,k)=i} = 1$ if $i(P,k) = i$ and $\mathbb{1}_{i(P,k)=i} = 0$ otherwise. Here, (A.4) follows from (21), (A.5) follows by interchanging the summations, (A.6) follows from the fact that there is precisely one $i \in N$ such that $i(P,k) = i$ for all $k \in \{0, \dots, |N|n-1\}$ and (A.7) follows from Definition 3.5(i). This concludes the proof. □

B Proof of Proposition 3.8

To prove Proposition 3.8, we first prove the following lemma.

Lemma B.1 *Let $R \in \mathcal{R}$ and $n \in \mathbb{N}$. We have for all $i \in N$ that*

$$K_i^{avg,n}(R) = \sum_{\lambda \in G^n : \lambda_i < 1} t^n(\lambda) p_i^n(\lambda) [r(\lambda + (1/n) \cdot e_i) - r(\lambda)], \tag{B.1}$$

where

$$t^n(\lambda) = \frac{\prod_{j \in N} \begin{pmatrix} n \\ n\lambda_j \end{pmatrix}}{\begin{pmatrix} |N|n \\ |N|n\bar{\lambda} \end{pmatrix}}, \tag{B.2}$$

and

$$p_i^n(\lambda) = \frac{1 - \lambda_i}{\sum_{j \in N}(1 - \lambda_j)}, \tag{B.3}$$

for all $\lambda \in G^n \backslash \{e_N\}$, $\bar{\lambda} = \frac{1}{|N|} \sum_{i \in N} \lambda_i$, for all $\lambda \in \mathbb{R}^N$, and where the risk capital function r is defined in (8).

Proof of Lemma B.1 In this proof, we use the following notation. The set $\widetilde{G}_k^n$ is given by

$$\widetilde{G}_k^n = \left\{ \lambda \in G^n : \sum_{i \in N} \lambda_i = \frac{k}{n} \right\}, \qquad \text{for all } n \in \mathbb{N} \text{ and } k \in \{0, \dots, |N|n\}. \tag{B.4}$$

The set $\widetilde{G}_k^n$ consists of all participation profiles on the grid where the sum of the coordinates is constant. Note that we have

$$\widetilde{G}_k^n = \{P(k) : P \in \mathcal{P}^n\}, \qquad \text{for all } n \in \mathbb{N} \text{ and } k \in \{0, \dots, |N|n\}. \tag{B.5}$$

Next, we show (B.1). The rule $K^{avg,n}(R)$ can be rewritten as

$$K^{avg,n}(R) = \frac{1}{|\mathcal{P}^n|} \sum_{P \in \mathcal{P}^n} K^{path,P}(R) \tag{B.6}$$

$$= \frac{1}{|\mathcal{P}^n|} \sum_{P \in \mathcal{P}^n} \sum_{k=0}^{|N|n-1} [r(P(k+1)) - r(P(k))] \cdot e_{i(P,k)} \tag{B.7}$$

$$= \sum_{k=0}^{|N|n-1} \sum_{P \in \mathcal{P}^n} \frac{1}{|\mathcal{P}^n|} [r(P(k+1)) - r(P(k))] \cdot e_{i(P,k)}, \tag{B.8}$$

where (B.6) follows from Definition 3.1 and (B.7) follows from (21). Let $i \in N$. Then, we obtain

$$K_i^{avg,n}(R) = \sum_{k=0}^{|N|n-1} \sum_{P \in \mathcal{P}^n : i(P,k)=i} \frac{1}{|\mathcal{P}^n|} [r(P(k+1)) - r(P(k))] \tag{B.9}$$

$$= \sum_{k=0}^{|N|n-1} \sum_{P \in \mathcal{P}^n : i(P,k)=i} \frac{1}{|\mathcal{P}^n|} [r(P(k) + (1/n) \cdot e_i) - r(P(k))] \tag{B.10}$$

$$= \sum_{k=0}^{|N|n-1} \sum_{\lambda \in \widetilde{G}_k^n} \sum_{\substack{P \in \mathcal{P}^n: \\ i(P,k)=i, P(k)=\lambda}} \frac{1}{|\mathcal{P}^n|} [r(\lambda + (1/n) \cdot e_i) - r(\lambda)] \tag{B.11}$$

$$= \sum_{k=0}^{|N|n-1} \sum_{\lambda \in \widetilde{G}_k^n : \lambda_i < 1} [r(\lambda + (1/n) \cdot e_i) - r(\lambda)] \sum_{\substack{P \in \mathcal{P}^n: \\ i(P,k)=i, P(k)=\lambda}} \frac{1}{|\mathcal{P}^n|} \tag{B.12}$$

$$= \sum_{k=0}^{|N|n-1} \sum_{\lambda \in \widetilde{G}_k^n : \lambda_i < 1} [r(\lambda + (1/n) \cdot e_i) - r(\lambda)] \, t^n(\lambda) p_i^n(\lambda) \tag{B.13}$$

$$= \sum_{\lambda \in G^n : \lambda_i < 1} [r(\lambda + (1/n) \cdot e_i) - r(\lambda)] \, t^n(\lambda) p_i^n(\lambda), \tag{B.14}$$

where we define

$$t^n(\lambda) = \frac{\left|\left\{P \in \mathcal{P}^n : P\left(|N|n\bar{\lambda}\right) = \lambda\right\}\right|}{|\mathcal{P}^n|},$$

as the fraction of paths in $\mathcal{P}^n$ that pass through λ and

$$p_i^n(\lambda) = \frac{\left|\left\{P \in \mathcal{P}^n : P\left(|N|n\bar{\lambda}\right) = \lambda, i\left(P, |N|n\bar{\lambda}\right) = i\right\}\right|}{\left|\left\{P \in \mathcal{P}^n : P\left(|N|n\bar{\lambda}\right) = \lambda\right\}\right|},$$

as the fraction of the paths in $\mathcal{P}^n$ passing through λ, that pass through $\lambda + \frac{1}{n} \cdot e_i$ as well. Here, (B.9) follows from (B.8), (B.10) follows from (20), (B.11) follows from (B.5), (B.12) follows from the fact that if $k \in \{0, \ldots, |N|n-1\}$ and $\lambda \in \widetilde{G}_k^n$ are such that $\lambda_i = 1$ then no path $P \in \mathcal{P}^n$ exists with $i(P,k) = i$ and $P(k) = \lambda$, (B.13) follows from the fact that if $k \in \{0, \ldots, |N|n-1\}$ and $\lambda \in \widetilde{G}_k^n$ are such that $P(k) = \lambda$ then $k = |N|n\bar{\lambda}$ and (B.14) follows from the fact that $\bigcup_{k=1}^{|N|n-1} G_k^n = G^n$ and

$G^n_{k_1} \cap G^n_{k_2} = \emptyset$ if $k_1 \neq k_2$.

Next, we show (B.2). Any path can be regarded as an ordered sequence of $|N|n$ steps, where for every division $i \in N$ precisely n steps are made in the direction of division i. Hence,

$$|\mathcal{P}^n| = \frac{(|N|n)!}{(n!)^{|N|}}. \tag{B.15}$$

Let $\lambda \in G^n \backslash \{e_N\}$. The number of paths P in $\mathcal{P}^n$ such that $P\left(|N|n\bar{\lambda}\right) = \lambda$ is given by

$$\left|\left\{P \in \mathcal{P}^n : P\left(|N|n\bar{\lambda}\right) = \lambda\right\}\right| = \frac{\left(|N|n\bar{\lambda}\right)!\left(|N|n(1-\bar{\lambda})\right)!}{\prod_{j\in N}(n\lambda_j)!(n(1-\lambda_j))!}. \tag{B.16}$$

Hence, one can verify that dividing (B.16) by (B.15) yields (B.2). Note that, keeping $\bar{\lambda}$ constant, the various values of $t^n(\lambda)$ constitute a density function of some multivariate hypergeometric distribution.

Finally, we show (B.3). The number of paths P in $\mathcal{P}^n$ with $P(|N|n\bar{\lambda}) = \lambda$ and $i(P, |N|n\bar{\lambda}) = i$ (i.e. passing through λ and $\lambda + (1/n)e_i$) is given by:

$$\left|\left\{P \in \mathcal{P}^n : P\left(|N|n\bar{\lambda}\right) = \lambda, i\left(P, |N|n\bar{\lambda}\right) = i\right\}\right| = \frac{(|N|n\bar{\lambda})!(|N|n(1-\bar{\lambda})-1)!}{\left[\prod_{j\in N}(n\lambda_j)!\right]\cdot\left[\prod_{j\in N\backslash\{i\}}(n(1-\lambda_j))!\right]\cdot(n(1-\lambda_i)-1)!}. \tag{B.17}$$

Dividing (B.17) by (B.16) yields (B.3) in a straightforward way. □

Proof of Proposition 3.8 It follows immediately from the proof of Lemma B.1 that the function $t^n(\lambda)$ represents the probability that λ lies on a path, if we randomly select a path from $\mathcal{P}^n$ according to the discrete uniform distribution. Moreover, $p^n_i(\lambda)$ is the conditional probability that $\lambda + (1/n)\cdot e_i$ lies on a path, provided that the path passes through λ.

C Proof of Theorem 3.9

We use the following notation.

- We use the Bachmann-Landau notation. Let $f, g : \mathbb{N} \to \mathbb{R}$ be two real-valued functions. Then, we write $f(n) = \mathcal{O}(g(n))$ if there is a $K > 0$ such that $|f(n)| \leq K|g(n)|$ for every $n \in \mathbb{N}$. If $f : \mathbb{N} \to \mathbb{R}$ is such that $f(n) = \mathcal{O}(n^{-p})$ for every $p > 0$, we write $f(n) = \mathcal{O}(n^{-\infty})$. Moreover, if $g : \mathbb{R}_{++} \to \mathbb{R}$ is such that there is a $K > 0$ such that $|g(\varepsilon)| \leq K\varepsilon$ for every $\varepsilon > 0$, we write $g(\varepsilon) = \mathcal{O}(\varepsilon)$. Here, $\mathbb{R}_{++} = (0, \infty)$ is the set of all positive, real numbers.

- Let $f : \mathbb{R}_{++} \times \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$. Then, we write $f(\varepsilon, n) = \mathcal{O}^{\varepsilon}(g(n))$ if for every $\varepsilon > 0$, there is a $K_\varepsilon > 0$ such that $|f(\varepsilon, n)| \leq K_\varepsilon |g(n)|$ for all $n \in \mathbb{N}$. This notation is an extension of the standard Bachmann-Landau notation.
- For all $\lambda \in \mathbb{R}^N$, we write $\|\lambda\| = \sqrt{\sum_{i \in N} \lambda_i^2}$ as the Euclidean norm of λ.
- We define the set of participation profiles that are not nearby $\lambda = e_\emptyset$ and e_N as follows. For all $n \in \mathbb{N}$ and $\varepsilon > 0$, we define $G_\varepsilon = \left\{\lambda \in [0,1]^N : \varepsilon \leq \bar{\lambda} \leq 1 - \varepsilon\right\}$, and $G_\varepsilon^n = G^n \cap G_\varepsilon$.
- We define D^d as the set of participation profiles in the d-environment of the diagonal, i.e., for all $d > 0$, we have $D^d = \{\lambda \in [0,1]^N : \|\lambda - \bar{\lambda} \cdot e_N\| < d\}$. Moreover, we define for all $n \in \mathbb{N}$ the set $D(n) = D^{d_n}$, where $d_n = n^{-\frac{1}{2} + \frac{1}{8|N|}}$.[1]

To prove Theorem 3.9, we will prove the following three propositions. The proofs of these propositions are in Subsections C.1, C.2, and C.3, respectively.

Proposition C.1 *Let $i \in N$ and define $Dom = \{(\varepsilon, n, \lambda) : \varepsilon > 0, n \in \mathbb{N}, \lambda \in G_\varepsilon^n\}$. Then, we have*

$$t^n(\lambda) = \left(e^{-c(\bar{\lambda}) n \|\lambda - \bar{\lambda} \cdot e_N\|^2}\right) b(n, \bar{\lambda})[1 + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}})], \qquad \text{if } \lambda \in D(n), \tag{C.1}$$

$$= \mathcal{O}^{\varepsilon}(n^{-\infty}), \qquad \text{if } \lambda \notin D(n), \tag{C.2}$$

and

$$p_i^n(\lambda) = \frac{1}{|N|}[1 + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}})], \qquad \text{if } \lambda \in D(n), \tag{C.3}$$

$$= \mathcal{O}(1), \qquad \text{if } \lambda \notin D(n), \tag{C.4}$$

for all $(\varepsilon, n, \lambda) \in Dom$, where

$$c(\bar{\lambda}) = \frac{1}{2\bar{\lambda}(1 - \bar{\lambda})} > 0, \tag{C.5}$$

and

$$b(n, \bar{\lambda}) = (2\pi n)^{\frac{1}{2}(1-|N|)} \sqrt{|N|} (\bar{\lambda}(1 - \bar{\lambda}))^{\frac{1}{2}(1-|N|)}. \tag{C.6}$$

[1] Theorem 3.9 can be proven by using a diagonal width $d_n = n^{-\frac{1}{2}+\delta}$ for some $\delta \in \left(0, \frac{1}{2(|N|+2)}\right)$. The proofs are based on $\delta = \frac{1}{8|N|}$.

For large n, we get that $t^n(\lambda)$ only depends on λ via $\bar{\lambda}$ and $\|\lambda - \bar{\lambda} \cdot e_N\|$ and that $p_i^n(\lambda)$ is symmetric close to the diagonal. For a given $n \in \mathbb{N}$ and $\bar{\lambda} \in \left\{0, \frac{1}{n}, \ldots, 1\right\}$, the function $b(n, \bar{\lambda})$ is approximately the probability that a path goes through the diagonal (i.e., through $\bar{\lambda} \cdot e_N$) and $c(\bar{\lambda})$ indicates a speed at which $t^n(\lambda)$ converges to zero for participation profiles away from the diagonal. The function $t^n(\lambda)$ is exponentially small in n if λ is not nearby to the diagonal, i.e., $\lambda \notin D(n)$. Moreover, $p_i^n(\lambda)$ is bounded. Therefore, only participation profiles very close to the diagonal are relevant for $K^{avg,n}$ if n converges to infinity.

To proceed with the proof, we define the function $h^n : [0,1]^N \backslash \{e_\emptyset, e_N\} \to \mathbb{R}_{++}$ as follows:

$$h^n(\lambda) = \left(e^{-c(\bar{\lambda})n\|\lambda - \bar{\lambda} \cdot e_N\|^2}\right) b(n, \bar{\lambda}) \frac{1}{|N|}, \tag{C.7}$$

for all $\lambda \in [0,1]^N \backslash \{e_\emptyset, e_N\}$ and $n \in \mathbb{N}$, where $c(\bar{\lambda})$ is defined in (C.5) and $b(n, \bar{\lambda})$ in (C.6).

It follows from Proposition C.1 that

$$t^n(\lambda) p_i^n(\lambda) = h^n(\lambda)[1 + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}})], \tag{C.8}$$

for all $(\varepsilon, n, \lambda) \in Dom$ such that $\lambda \in D(n)$. This leads to the following approximation.

Proposition C.2 *Let $R \in \mathcal{R}$. Then, for all $i \in N$ we have*

$$K_i^{avg,n}(R) = \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] \phi_m^{n,\varepsilon} + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}}),$$

where

$$\phi_m^{n,\varepsilon} = \frac{1}{n} \sum_{\lambda \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}} h^n(\lambda), \tag{C.9}$$

with p^* and $\mathbb{Q}_m$ as defined in Proposition 3.2.

The expression $\phi_m^{n,\varepsilon}$ is a weight for a gradient of the risk capital function r "nearby" the diagonal, namely $(E_{\mathbb{Q}_m}[X_i])_{i \in N}$. Next, we show that we can replace this weight by an expression that has a geometric interpretation and is not dependent on n or ε anymore. This result is obtained by replacing the sum in (C.9) by an integral (see Lemma C.20 and Lemma C.21) and, thereafter, solving this integral.

Proposition C.3 *For all $R \in \mathcal{R}$, it holds that*

$$K_i^{avg,n}(R) = \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i]\phi_m + \mathcal{O}(\varepsilon) + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}}), \qquad \textit{for all } i \in N,$$

where ϕ_m for $m \in \{1, \dots, p^*\}$ is as defined in (27).

Proof of Theorem 3.9 Let $R \in \mathcal{R}$. From Proposition C.3, we get for all $n \in \mathbb{N}$ and $\varepsilon > 0$ that

$$\left| K_i^{avg,n}(R) - \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i]\phi_m \right| < K\varepsilon + L_{\varepsilon} n^{-\frac{1}{4}}, \qquad \text{where } K, L_{\varepsilon} > 0.$$

Pick an $\eta > 0$. Let $\varepsilon = \frac{\eta}{2K}$ and N_η such that $L_\varepsilon N_\eta^{-\frac{1}{4}} = \frac{1}{2}\eta$. Then, we have for all $n > N_\eta$ that

$$\left| K_i^{avg,n}(R) - \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i]\phi_m \right| < \eta.$$

This concludes the proof. □

In the remaining three subsections of this Online Appendix, we present the proofs of Propositions C.1, C.2, and C.3, respectively.

C.1 Proof of Proposition C.1

We use the following definitions, notation and properties:

- The function $g : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} x \ln(x) & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- The function $G : [0,1]^N \to \mathbb{R}$ is given by

$$G(\lambda) = |N| g\left(\bar{\lambda}\right) - \sum_{i \in N} g(\lambda_i) + |N| g\left(1 - \bar{\lambda}\right) - \sum_{i \in N} g(1 - \lambda_i), \qquad \text{for all } \lambda \in [0,1]^N. \quad \text{(C.10)}$$

- For all $\lambda \in [0,1]^N$, we define

$$N_1^\lambda = \{i \in N : \lambda_i > 0\} \text{ and } N_2^\lambda = \{i \in N : \lambda_i < 1\}. \qquad \text{(C.11)}$$

- For $x, y \in \mathbb{R}$ we denote $[x; y]$ as the interval $[\min\{x, y\}, \max\{x, y\}]$, i.e., $[x; y] = [x, y]$ if $x \leq y$ and $[x; y] = [y, x]$ if $x > y$.

- Some arithmetic rules of the Bachmann-Landau notation are given by:

$$\begin{aligned} f(n) = \mathcal{O}(n^a), g(n) = \mathcal{O}(n^b) &\to f(n) + g(n) = \mathcal{O}(n^a), && \text{for all } a \geq b, \\ f(n) = \mathcal{O}(n^a), g(n) = \mathcal{O}(n^{-\infty}) &\to f(n) + g(n) = \mathcal{O}(n^a), && \text{for all } a \in \mathbb{R}, \\ f(n) = \mathcal{O}(n^a), g(n) = \mathcal{O}(n^b) &\to f(n)g(n) = \mathcal{O}(n^{a+b}), && \text{for all } a, b \in \mathbb{R}, \\ f(n) = \mathcal{O}(n^a) &\to f(n) = \mathcal{O}(n^b), && \text{for all } a \leq b. \end{aligned}$$

  Moreover, we have

$$f(n) = \mathcal{O}(n^a), g(n) = \mathcal{O}^{\varepsilon}(n^b) \to f(n) + g(n) = \mathcal{O}^{\varepsilon}(n^a), \qquad \text{for all } a \geq b.$$

- It is well known that for any $k \in \mathbb{R}$, $\delta > 0$ and $c \in (0, 1)$ the function $f : \mathbb{N} \to \mathbb{R}_{++}$, defined by $f(n) = n^k c^{n^\delta}$, is such that $f(n) = \mathcal{O}(n^{-\infty})$.

C.1.1 Some preliminary lemmas

Lemma C.4 *The function g is continuous and strictly convex, i.e., if $x, y \in \mathbb{R}_+$, $x \neq y$ and $\lambda \in (0, 1)$, then $g(\lambda x + (1 - \lambda)y) < \lambda g(x) + (1 - \lambda)g(y)$.*

Proof Continuity of f follows from continuity of $x \to x\ln(x)$ for $x > 0$ and the fact that $\lim_{x \downarrow 0} x\ln(x) = 0$. Strict convexity follows from $g''(x) = \frac{1}{x} > 0$ for every $x > 0$. □

Lemma C.5 *For the function G the following holds:*

1. *G is continuous;*
2. *$G(\lambda) \leq 0$ for all $\lambda \in [0, 1]^N$; moreover, $G(\lambda) = 0$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{|N|}$;*
3. *for all $\lambda \in (0, 1)^N$, we have*

$$G(\lambda) = -c(\bar{\lambda})\|\lambda - \bar{\lambda} \cdot e_N\|^2 + R,$$

   where $|R| \leq \frac{1}{3}|N| \min\{\lambda_1, \ldots, \lambda_{|N|}, 1 - \lambda_1, \ldots, 1 - \lambda_{|N|}\}^{-2}\|\lambda - \bar{\lambda} \cdot e_N\|^3$.

Proof 1. This follows from continuity of g (Lemma C.4).

2. This follows from strict convexity of g (Lemma C.4).

3. Let $\lambda \in (0,1)^N$ and $i \in N$. Then, there exists a $\xi_{i,1} \in [\lambda_i; \bar{\lambda}]$ such that

$$g(\lambda_i) = g(\bar{\lambda}) + g'(\bar{\lambda})(\lambda_i - \bar{\lambda}) + \frac{g''(\bar{\lambda})}{2}(\lambda_i - \bar{\lambda})^2 + \frac{g'''(\xi_{i,1})}{6}(\lambda_i - \bar{\lambda})^3 \tag{C.12}$$

$$= g(\bar{\lambda}) + (\ln(\bar{\lambda}) + 1)(\lambda_i - \bar{\lambda}) + \frac{1}{2\bar{\lambda}}(\lambda_i - \bar{\lambda})^2 - \frac{1}{6\xi_{i,1}^2}(\lambda_i - \bar{\lambda})^3, \tag{C.13}$$

where (C.12) follows from Taylor's theorem. Note that

$$\sum_{i \in N}(\lambda_i - \bar{\lambda}) = 0. \tag{C.14}$$

Then, summing the expression (C.13) of $g(\lambda_i)$ for all $i \in N$ yields

$$\sum_{i \in N} g(\lambda_i) = |N| g(\bar{\lambda}) + \frac{1}{2\bar{\lambda}}\|\lambda - \bar{\lambda} \cdot e_N\|^2 - \sum_{i \in N} \frac{1}{6\xi_{i,1}^2}(\lambda_i - \bar{\lambda})^3.$$

Similarly, we obtain

$$\sum_{i \in N} g(1 - \lambda_i) = |N| g(1 - \bar{\lambda}) + \frac{1}{2(1 - \bar{\lambda})}\|\lambda - \bar{\lambda} \cdot e_N\|^2 + \sum_{i \in N} \frac{1}{6\xi_{i,2}^2}(\lambda_i - \bar{\lambda})^3.$$

where $\xi_{i,2} \in [1 - \lambda_i; 1 - \bar{\lambda}]$ for all $i \in N$. Now the upper bound of $|R|$ follows from $\xi_{i,1} \geq \min\{\lambda_1, \ldots, \lambda_{|N|}\}$, $\xi_{i,2} \geq \min\{1 - \lambda_1, \ldots, 1 - \lambda_{|N|}\}$ and $|(\lambda_i - \bar{\lambda})^3| \leq \|\lambda - \bar{\lambda} \cdot e_N\|^3$ for all $i \in N$. □

Lemma C.6 *Let $d, \varepsilon > 0$. Then, for all $\lambda \in G_\varepsilon \cap D^d$, we have*

$$\min\{\lambda_1, \ldots, \lambda_{|N|}, 1 - \lambda_1, \ldots, 1 - \lambda_{|N|}\} > \varepsilon - d.$$

Proof Let $\lambda \in G_\varepsilon \cap D^d$. Since $\left|\lambda_i - \bar{\lambda}\right| \leq \left\|\lambda - \bar{\lambda} \cdot e_N\right\| < d$, we obtain $\lambda_i > \bar{\lambda} - d$ and $1 - \lambda_i > 1 - \bar{\lambda} - d$ for all $i \in N$. Moreover, we have $\varepsilon \leq \bar{\lambda} \leq 1 - \varepsilon$. Hence, we obtain $\lambda_i > \varepsilon - d$ and $1 - \lambda_i > \varepsilon - d$ for all $i \in N$. This concludes the proof. □

Lemma C.7 *For all (n, λ) such that $n \in \mathbb{N}$ and $\lambda \in G^n \backslash \{e_\emptyset, e_N\}$, we have*

$$t^n(\lambda) = \left(e^{G(\lambda)}\right)^n (2\pi n)^{\frac{1}{2}(1 + |N| - |N_1^\lambda| - |N_2^\lambda|)} \sqrt{|N|} \frac{(\bar{\lambda}(1 - \bar{\lambda}))^{\frac{1}{2}}}{\prod_{i \in N_1^\lambda} \sqrt{\lambda_i} \prod_{i \in N_2^\lambda} \sqrt{1 - \lambda_i}}$$

$$\cdot \left[1 + \mathcal{O}\left(\frac{1}{n\min(\{\lambda_j : j \in N_1^\lambda\} \cup \{1-\lambda_j : j \in N_2^\lambda\})}\right)\right], \tag{C.15}$$

where N_1^λ and N_2^λ are defined in (C.11).

Proof Using (B.2), we obtain for all (n, λ) such that $n \in \mathbb{N}$ and $\lambda \in G^n \backslash \{e_\emptyset, e_N\}$ that

$$\begin{aligned}
t^n(\lambda) &= \frac{\prod_{i\in N}\begin{pmatrix} n \\ n\lambda_i \end{pmatrix}}{\begin{pmatrix} |N|n \\ |N|n\bar{\lambda} \end{pmatrix}} \\
&= \frac{(n!)^{|N|}(|N|n\bar{\lambda})!(|N|n(1-\bar{\lambda}))!}{(|N|n)!\prod_{i\in N}[(n\lambda_i)!(n(1-\lambda_i))!]} \\
&= \frac{(n!)^{|N|}(|N|n\bar{\lambda})!(|N|n(1-\bar{\lambda}))!}{(|N|n)!\prod_{i\in N_1^\lambda}(n\lambda_i)!\prod_{i\in N_2^\lambda}(n(1-\lambda_i))!}.
\end{aligned}$$

Taking the logarithm yields

$$\begin{aligned}
\ln(t^n(\lambda)) &= |N|\ln(n!) + \ln((|N|n\bar{\lambda})!) + \ln((|N|n(1-\bar{\lambda}))!) - \ln((|N|n)!) \\
&\quad - \sum_{i\in N_1^\lambda}\ln((n\lambda_i)!) - \sum_{i\in N_2^\lambda}\ln((n(1-\lambda_i))!).
\end{aligned} \tag{C.16}$$

Now, using Stirling's approximation, which is given by

$$\ln(n!) = g(n) - n + \frac{1}{2}\ln(2\pi n) + \mathcal{O}\left(\frac{1}{n}\right), \qquad \text{for all } n \in \mathbb{N},$$

formula (C.16) can be written as

$$\begin{aligned}
\ln(t^n(\lambda)) &= |N|g(n) - |N|n + \frac{1}{2}|N|\ln(2\pi n) + \mathcal{O}\left(\frac{1}{n}\right) \\
&\quad + g(|N|n\bar{\lambda}) - |N|n\bar{\lambda} + \frac{1}{2}\ln(2\pi|N|n\bar{\lambda}) + \mathcal{O}\left(\frac{1}{|N|n\bar{\lambda}}\right) \\
&\quad + g(|N|n(1-\bar{\lambda})) - |N|(n(1-\bar{\lambda})) + \frac{1}{2}\ln(2\pi|N|n(1-\bar{\lambda})) + \mathcal{O}\left(\frac{1}{|N|n(1-\bar{\lambda})}\right) \\
&\quad - \left[g(|N|n) - |N|n + \frac{1}{2}\ln(2\pi|N|n) + \mathcal{O}\left(\frac{1}{|N|n}\right)\right] \\
&\quad - \sum_{i\in N_1^\lambda}\left[g(n\lambda_i) - n\lambda_i + \frac{1}{2}\ln(2\pi n\lambda_i) + \mathcal{O}\left(\frac{1}{n\lambda_i}\right)\right]
\end{aligned}$$

$$- \sum_{i \in N_2^\lambda} \left[g(n(1-\lambda_i)) - n(1-\lambda_i) + \frac{1}{2}\ln(2\pi n(1-\lambda_i)) + \mathcal{O}\left(\frac{1}{n(1-\lambda_i)}\right)\right].$$

Now, using that $g(xy) = xg(y) + yg(x)$ for all $x, y \geq 0$, $g(0) = 0$, $\sum_{i \in N_1^\lambda} \lambda_i = |N|\bar{\lambda}$ and $\sum_{i \in N_2^\lambda}(1 - \lambda_i) = |N|(1-\bar{\lambda})$, we get

$$\begin{aligned}
\ln(t^n(\lambda)) &= |N|g(n) - |N|n + \frac{1}{2}|N|\ln(2\pi n) + \mathcal{O}\left(\frac{1}{n}\right) \\
&+ \bar{\lambda}g(|N|n) + |N|ng(\bar{\lambda}) - |N|n\bar{\lambda} + \frac{1}{2}\ln(2\pi n) + \frac{1}{2}\ln(|N|) + \frac{1}{2}\ln(\bar{\lambda}) + \mathcal{O}\left(\frac{1}{n\bar{\lambda}}\right) \\
&+ (1-\bar{\lambda})g(|N|n) + |N|ng(1-\bar{\lambda}) - |N|n(1-\bar{\lambda}) + \frac{1}{2}\ln(2\pi n) + \frac{1}{2}\ln(|N|) + \frac{1}{2}\ln(1-\bar{\lambda}) \\
&+ \mathcal{O}\left(\frac{1}{n(1-\bar{\lambda})}\right) \\
&- g(|N|n) + |N|n - \frac{1}{2}\ln(2\pi n) - \frac{1}{2}\ln(|N|) + \mathcal{O}\left(\frac{1}{n}\right) \\
&- |N|\bar{\lambda}g(n) - \sum_{i \in N} ng(\lambda_i) + |N|n\bar{\lambda} - \frac{1}{2}|N_1^\lambda|\ln(2\pi n) - \frac{1}{2}\sum_{i \in N_1^\lambda}\ln(\lambda_i) + \sum_{i \in N_1^\lambda}\mathcal{O}\left(\frac{1}{n\lambda_i}\right) \\
&- |N|(1-\bar{\lambda})g(n) - \sum_{i \in N} ng(1-\lambda_i) + |N|n(1-\bar{\lambda}) - \frac{1}{2}|N_2^\lambda|\ln(2\pi n) - \frac{1}{2}\sum_{i \in N_2^\lambda}\ln(1-\lambda_i) \\
&+ \sum_{i \in N_2^\lambda}\mathcal{O}\left(\frac{1}{n(1-\lambda_i)}\right).
\end{aligned}$$

From $|N|g(n) - |N|\bar{\lambda}g(n) - |N|(1-\bar{\lambda})g(n) = 0$, $-|N|n - |N|n\bar{\lambda} - |N|(n(1-\bar{\lambda})) + |N|n + |N|n\bar{\lambda} + |N|n(1-\bar{\lambda}) = 0$, $\bar{\lambda}g(|N|n) + (1-\bar{\lambda})g(|N|n) - g(|N|n) = 0$ and rearranging and collecting some terms it follows that

$$\begin{aligned}
\ln(t^n(\lambda)) &= n\left[|N|g(\bar{\lambda}) - \sum_{i \in N} g(\lambda_i) + |N|g(1-\bar{\lambda}) - \sum_{i \in N} g(1-\lambda_i)\right] \\
&+ \left[\frac{1}{2}(1 + |N| - |N_1^\lambda| - |N_2^\lambda|)\right]\ln(2\pi n) + \frac{1}{2}\ln(|N|) \\
&+ \frac{1}{2}\ln(\bar{\lambda}) + \frac{1}{2}\ln(1-\bar{\lambda}) - \frac{1}{2}\sum_{i \in N_1^\lambda}\ln(\lambda_i) - \frac{1}{2}\sum_{i \in N_2^\lambda}\ln(1-\lambda_i) \\
&+ \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{1}{n\bar{\lambda}}\right) + \mathcal{O}\left(\frac{1}{n(1-\bar{\lambda})}\right) + \sum_{i \in N_1^\lambda}\mathcal{O}\left(\frac{1}{n\lambda_i}\right) + \sum_{i \in N_2^\lambda}\mathcal{O}\left(\frac{1}{n(1-\lambda_i)}\right).
\end{aligned}$$

Then, recall the function G from (C.10). We get

$$\ln(t^n(\lambda)) = nG(\lambda) + \left[\frac{1}{2}(1 + |N| - |N_1^\lambda| - |N_2^\lambda|)\right]\ln(2\pi n) + \frac{1}{2}\ln(|N|) + \frac{1}{2}\ln(\bar{\lambda}(1-\bar{\lambda}))$$

$$- \frac{1}{2} \sum_{i \in N_1^\lambda} \ln(\lambda_i) - \frac{1}{2} \sum_{i \in N_2^\lambda} \ln(1 - \lambda_i) + \mathcal{O}\left(\frac{1}{n}\right) + \mathcal{O}\left(\frac{1}{n\bar{\lambda}}\right) + \mathcal{O}\left(\frac{1}{n(1-\bar{\lambda})}\right)$$
$$+ \sum_{i \in N_1^\lambda} \mathcal{O}\left(\frac{1}{n\lambda_i}\right) + \sum_{i \in N_2^\lambda} \mathcal{O}\left(\frac{1}{n(1-\lambda_i)}\right).$$

So, taking the exponent and using the fact that $e^x = 1 + \mathcal{O}(x)$ if $x \in [0, K]$ for some constant $K > 0$, yields

$$t^n(\lambda) = \left(e^{G(\lambda)}\right)^n (2\pi n)^{\frac{1}{2}(1+|N|-|N_1^\lambda|-|N_2^\lambda|)} \sqrt{|N|} \frac{(\bar{\lambda}(1-\bar{\lambda}))^{\frac{1}{2}}}{\prod_{i \in N_1^\lambda} \sqrt{\lambda_i} \prod_{i \in N_2^\lambda} \sqrt{1-\lambda_i}} \cdot \left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right]$$
$$\cdot \left[1 + \mathcal{O}\left(\frac{1}{n\bar{\lambda}}\right)\right] \cdot \left[1 + \mathcal{O}\left(\frac{1}{n(1-\bar{\lambda})}\right)\right] \prod_{i \in N_1^\lambda} \left[1 + \mathcal{O}\left(\frac{1}{n\lambda_i}\right)\right] \prod_{i \in N_2^\lambda} \left[1 + \mathcal{O}\left(\frac{1}{n(1-\lambda_i)}\right)\right].$$

Then, as $\lambda_i \geq \min\{\lambda_j : j \in N_1^\lambda\}$ for all $i \in N_1^\lambda$,$1 - \lambda_i \geq \min\{1 - \lambda_j : j \in N_2^\lambda\}$ for all $i \in N_2^\lambda$, $\bar{\lambda} \geq \frac{1}{|N|} \min\{\lambda_j : j \in N_1^\lambda\}$ and $1 - \bar{\lambda} \geq \frac{1}{|N|} \min\{1 - \lambda_j : j \in N_2^\lambda\}$, the result follows in a straightforward way. □

Lemma C.8 *We have for all* $(\varepsilon, n, \lambda) \in Dom$ *with* $d_n < \frac{1}{2}\varepsilon$ *and* $\lambda \in D(n)$ *that*

$$\frac{(\bar{\lambda}(1-\bar{\lambda}))^{\frac{1}{2}|N|}}{\prod_{i \in N} \sqrt{\lambda_i} \prod_{i \in N} \sqrt{1-\lambda_i}} = 1 + \mathcal{O}^\varepsilon(n^{-1+\frac{1}{4|N|}}). \tag{C.17}$$

Proof According to Lemma C.6 we have $\lambda_i \geq \frac{1}{2}\varepsilon$ and $1 - \lambda_i \geq \frac{1}{2}\varepsilon$ for all $i \in N$. Consequently, we have $\bar{\lambda} \geq \frac{1}{2}\varepsilon$ and $1 - \bar{\lambda} \geq \frac{1}{2}\varepsilon$. According to Taylor's theorem, we have

$$\ln(\lambda_i) = \ln(\bar{\lambda}) + \frac{1}{\bar{\lambda}}(\lambda_i - \bar{\lambda}) - \frac{1}{2\xi_{i,1}^2}(\lambda_i - \bar{\lambda})^2, \tag{C.18}$$

for some $\xi_{i,1} \in [\lambda_i; \bar{\lambda}]$ and for all $i \in N$. From (C.14) and (C.18) it follows that

$$\frac{1}{2} \sum_{i \in N} \ln(\lambda_i) = \frac{1}{2}|N| \ln(\bar{\lambda}) - \sum_{i \in N} \frac{1}{4\xi_{i,1}^2}(\lambda_i - \bar{\lambda})^2. \tag{C.19}$$

Similarly, we obtain

$$\frac{1}{2} \sum_{i \in N} \ln(1 - \lambda_i) = \frac{1}{2}|N| \ln(1 - \bar{\lambda}) - \sum_{i \in N} \frac{1}{4\xi_{i,2}^2}(\bar{\lambda} - \lambda_i)^2, \tag{C.20}$$

where $\xi_{i,2} \in [1 - \lambda_i; 1 - \bar{\lambda}]$ for all $i \in N$. Since $\xi_{i,1} \geq \frac{1}{2}\varepsilon$, $\xi_{i,2} \geq \frac{1}{2}\varepsilon$ and $(\lambda_i - \bar{\lambda})^2 \leq \|\lambda - \bar{\lambda} \cdot e_N\|^2$ for all $i \in N$, we get

$$\sum_{i\in N}\frac{1}{4\xi_{i,1}^2}(\lambda_i-\bar\lambda)^2+\sum_{i\in N}\frac{1}{4\xi_{i,2}^2}(\bar\lambda-\lambda_i)^2 \le 2|N|\varepsilon^{-2}\|\lambda-\bar\lambda\cdot e_N\|^2$$
$$\le 2|N|\varepsilon^{-2}d_n^2$$
$$= 2|N|\varepsilon^{-2}n^{-1+\frac{1}{4|N|}}$$
$$= \mathcal{O}^\varepsilon(n^{-1+\frac{1}{4|N|}}).$$

Using the fact that $e^x = 1+\mathcal{O}(x)$ if $x \in [0,K]$ for some constant $K > 0$ yields

$$e^{\mathcal{O}^\varepsilon(n^{-1+\frac{1}{4|N|}})} = 1+\mathcal{O}^\varepsilon(n^{-1+\frac{1}{4|N|}}).$$

Now taking the exponent in (C.19) and (C.20) yields the desired result. □

C.1.2 Proof of Proposition C.1

We now use Lemmas C.4 to C.8 to prove Proposition C.1. We do this in several steps: (C.1) is shown in Lemma C.9, (C.2) in Lemma C.10, (C.3) in Lemma C.11 and (C.4) in Lemma C.12. We implicitly use in the statement of this proposition that if $g(n) = \mathcal{O}(n^c)$ for some $c \le -\frac{1}{4}$, we have $g(n) = \mathcal{O}(n^{-\frac{1}{4}})$.

Note that the result follows directly if $|N| = 1$, so we let $|N| \ge 2$.

Lemma C.9 *We have for all $(\varepsilon, n, \lambda) \in Dom$ that*

$$t^n(\lambda) = \left(e^{-c(\bar\lambda)n\|\lambda-\bar\lambda\cdot e_N\|^2}\right) b(n,\bar\lambda)\left[1+\mathcal{O}^\varepsilon\left(n^{-\frac{1}{2}+\frac{3}{8|N|}}\right)\right], \qquad \textit{if } \lambda \in D(n).$$

Proof It is sufficient to show this result for all $n \in \mathbb{N}$ such that $d_n < \frac{1}{2}\varepsilon$. From Lemma C.6, we then get

$$\min\{\lambda_1,\ldots,\lambda_{|N|},1-\lambda_1,\ldots,1-\lambda_{|N|}\} > \frac{1}{2}\varepsilon, \tag{C.21}$$

and, so,

$$N_1^\lambda = N_2^\lambda = N. \tag{C.22}$$

Using Lemma C.5.3 and the fact that $\|\lambda - \bar{\lambda} \cdot e_N\|^3 = \mathcal{O}(n^{-\frac{3}{2}+\frac{3}{8|N|}})$, we get that

$$G(\bar{\lambda}) = -c(\bar{\lambda})\|\lambda - \bar{\lambda} \cdot e_N\|^2 + \mathcal{O}^{\varepsilon}(n^{-\frac{3}{2}+\frac{3}{8|N|}}).$$

Hence,

$$\begin{aligned} e^{nG(\lambda)} &= e^{-c(\bar{\lambda})n\|\lambda-\bar{\lambda}\cdot e_N\|^2} \cdot e^{\mathcal{O}^{\varepsilon}(n^{-\frac{3}{2}+\frac{3}{8|N|}})} \\ &= e^{-c(\bar{\lambda})n\|\lambda-\bar{\lambda}\cdot e_N\|^2} \left[1 + \mathcal{O}^{\varepsilon}\left(n^{-\frac{1}{2}+\frac{3}{8|N|}}\right)\right]. \qquad \text{(C.23)} \end{aligned}$$

where (C.23) follows from the fact that $e^x = 1 + \mathcal{O}(x)$ if $x \in [0, K]$ for some constant $K > 0$. Substituting (C.17), (C.21), (C.22) and (C.23) in (C.15) yields the desired result. □

Lemma C.10 *We have for all $(\varepsilon, n, \lambda) \in Dom$ that*

$$t^n(\lambda) = \mathcal{O}^{\varepsilon}(n^{-\infty}), \qquad \text{if } \lambda \notin D(n).$$

Proof Let $\varepsilon \in (0, 1)$, denote $d = \frac{1}{3|N|}\varepsilon^2$ and recall the function G in (C.10). The set $G_\varepsilon \backslash D^d$ is compact. Moreover, the function G is continuous (Lemma C.5.1). Hence, the function G takes a maximum value m_ε on $G_\varepsilon \backslash D^d$. As $\lambda \in D^d$ if $\lambda_1 = \cdots = \lambda_{|N|}$, we obtain from Lemma C.5.2 that $m_\varepsilon < 0$. Let (n, λ) be such that $n \in \mathbb{N}$ and $\lambda \in G_\varepsilon^n \backslash D^d$. Since $\lambda_i \geq \frac{1}{n}$ for all $i \in N_1^\lambda$, $1 - \lambda_i \geq \frac{1}{n}$ for all $i \in N_2^\lambda$ and $\bar{\lambda}(1 - \bar{\lambda}) < 1$, we get from Lemma C.7 that

$$t^n(\lambda) = \mathcal{O}^{\varepsilon}(n^{\frac{1}{2}(1+|N|)}(e^{m_\varepsilon})^n).$$

Since $e^{m_\varepsilon} \in (0, 1)$ and $\lim_{n\to\infty} c^n n^d = 0$ for $c \in (0, 1)$ and $d \in \mathbb{R}$, we have for all $(\varepsilon, n, \lambda) \in Dom$ that

$$t^n(\lambda) = \mathcal{O}^{\varepsilon}(n^{-\infty}), \qquad \text{if } \lambda \notin D^d.$$

Next, we show this result for all (n, λ) such that $n \in \mathbb{N}$ and $\lambda \in (G_\varepsilon^n \cap D^d) \backslash D(n)$. We obtain from Lemma C.5.3 that

$$\begin{aligned} G(\lambda) &= -c(\bar{\lambda})\|\lambda - \bar{\lambda} \cdot e_N\|^2 + R \\ &= -c(\bar{\lambda})\|\lambda - \bar{\lambda} \cdot e_N\|^2 \left[1 - R(c(\bar{\lambda}))^{-1}\|\lambda - \bar{\lambda} \cdot e_N\|^{-2}\right], \end{aligned}$$

where $|R| \leq \frac{1}{3}|N| \min\{\lambda_1, \ldots, \lambda_{|N|}, 1-\lambda_1, \ldots, 1-\lambda_{|N|}\}^{-2} \|\lambda - \bar{\lambda} \cdot e_N\|^3$. From Lemma C.6, we get

$$\min\{\lambda_1, \ldots, \lambda_{|N|}, 1-\lambda_1, \ldots, 1-\lambda_{|N|}\} > \varepsilon - d > \frac{3|N|-1}{3|N|}\varepsilon > \frac{1}{2}\varepsilon.$$

Moreover, we have $(c(\bar{\lambda}))^{-1} = 2\bar{\lambda}(1-\bar{\lambda}) \leq \frac{1}{2}$ and $\|\lambda - \bar{\lambda} \cdot e_N\| < d$. Therefore, we have

$$|R(c(\bar{\lambda}))^{-1}\|\lambda - \bar{\lambda} \cdot e_N\|^{-2}| \leq |R|(c(\bar{\lambda}))^{-1}\|\lambda - \bar{\lambda} \cdot e_N\|^{-2} \leq \frac{1}{6}|N|\left(\frac{1}{2}\varepsilon\right)^{-2} d < \frac{1}{2}.$$

So, then, we obtain that

$$nG(\lambda) < -c(\bar{\lambda})n\|\lambda - \bar{\lambda} \cdot e_N\|^2 \frac{1}{2} \leq -n\|\lambda - \bar{\lambda} \cdot e_N\|^2 \leq -n^{\frac{1}{4|N|}},$$

which follows from $c(\bar{\lambda}) \geq 2$, and, hence,

$$e^{nG(\lambda)} < e^{-n^{\frac{1}{4|N|}}}. \tag{C.24}$$

We get

$$t^n(\lambda) = \mathcal{O}^{\varepsilon}(e^{nG(\lambda)} n^{\frac{1}{2}(1-|N|)}) \tag{C.25}$$
$$= \mathcal{O}^{\varepsilon}((e^{-1})^{n^{\frac{1}{4|N|}}} n^{\frac{1}{2}(1-|N|)}) \tag{C.26}$$
$$= \mathcal{O}^{\varepsilon}(n^{-\infty}), \tag{C.27}$$

where (C.25) follows from Lemma C.7, (C.26) follows from (C.24) and (C.27) follows from the fact that $\lim_{n\to\infty} n^k c^{n^\delta} = 0$ for all $k \in \mathbb{R}$, $c \in (0,1)$ and $\delta > 0$. □

Lemma C.11 *We have for all $i \in N$ and $(\varepsilon, n, \lambda) \in Dom$ that*

$$p_i^n(\lambda) = \frac{1}{|N|}\left[1 + \mathcal{O}^{\varepsilon}\left(n^{-\frac{1}{2}+\frac{1}{8|N|}}\right)\right], \qquad \textit{if } \lambda \in D(n).$$

Proof Note that from $\lambda \in G_\varepsilon^n$ it follows that $\lambda \neq e_N$, so $\bar{\lambda} < 1$. Then, the result follows directly from

$$\left|\frac{1-\lambda_i}{\sum_{j\in N}(1-\lambda_j)} - \frac{1}{|N|}\right| = \left|\frac{1-\lambda_i}{(1-\bar{\lambda})|N|} - \frac{1-\bar{\lambda}}{(1-\bar{\lambda})|N|}\right|$$
$$= \frac{|\bar{\lambda} - \lambda_i|}{(1-\bar{\lambda})|N|}$$

$$< \frac{n^{-\frac{1}{2}+\frac{1}{8|N|}}}{(1-\bar{\lambda})|N|} \tag{C.28}$$

$$\leq \frac{n^{-\frac{1}{2}+\frac{1}{8|N|}}}{\varepsilon|N|}, \tag{C.29}$$

for all $(\varepsilon, n, \lambda) \in Dom$ such that $\lambda \in D(n)$. Here, (C.28) follows from $|\bar{\lambda} - \lambda_i| \leq \|\lambda - \bar{\lambda} \cdot e_N\| < d_n = n^{-\frac{1}{2}+\frac{1}{8|N|}}$ and (C.29) follows from $1 - \bar{\lambda} \geq \varepsilon$. This concludes the proof. □

Lemma C.12 *We have for all $i \in N$ and $(\varepsilon, n, \lambda) \in Dom$ that $p_i^n(\lambda) = \mathcal{O}(1)$.*

Proof This follows directly from $0 \leq p_i^n(\lambda) \leq 1$. □

C.2 Proof of Proposition C.2

We use the following notation:

- For all $x \in \mathbb{R}$, we write $\lfloor x \rfloor$ as the largest integer not greater than x and $\lceil x \rceil$ as the smallest integer not less than x.
- For all $n \in \mathbb{N}$ and $\lambda \in G^n$, the set $C^n(\lambda)$ is given by

$$C^n(\lambda) = \left\{ \lambda + \frac{1}{n} x : x \in [0,1]^N \right\}. \tag{C.30}$$

- The set $D'(n)$ is given by

$$D'(n) = D^{d'_n}, \qquad \text{where } d'_n = d_n + (\sqrt{|N|}/n) = n^{-\frac{1}{2}+\frac{1}{8|N|}} + (\sqrt{|N|}/n). \tag{C.31}$$

- If there might be confusion about the notation $|\cdot|$ for the absolute value of a real number and the cardinality of a set, we sometimes write $\sharp(A)$ as the cardinality of the set A.
- We write $\nu(B)$ as the Lebesgue measure of the set B. Note that

$$\nu(C^n(\lambda)) = n^{-|N|}, \qquad \text{for all } \lambda \in G^n, \tag{C.32}$$

and

$$\nu(D'(n)) = \mathcal{O}(d_n'^{|N|-1}) = \mathcal{O}(n^{-\frac{1}{2}+\frac{1}{8|N|}}), \qquad \text{for all } n \in \mathbb{N}. \tag{C.33}$$

- Let $R \in \mathcal{R}$ and $\varepsilon > 0$. We define the set $B(R, n)$ by

$$B(R,n) = \left\{ \lambda \in [0,1]^N : \exists \hat{\lambda} \in [0,1]^N \backslash L(R) : \|\lambda - \hat{\lambda}\| < \frac{1}{n} \right\}, \tag{C.34}$$

for all $R \in \mathcal{R}$ and $n \in \mathbb{N}$, where $L(R)$ is defined in (18). This is the set of all participation profiles close to a participation profile that is an element of multiple sets $A_{\mathbb{Q}_m}$. As the risk capital allocation problem is always clear from the context, we write $B(n) = B(R, n)$.

First, we show that only the participation profiles in G^n_ε have a non-negligible aggregate contribution.

Lemma C.13 *For all $i \in N$, we have*

$$n^{-1} \sum_{\lambda \in G^n \backslash G^n_\varepsilon : \lambda_i < 1} t^n(\lambda) p^n_i(\lambda) = \mathcal{O}(\varepsilon) + \mathcal{O}(n^{-1}).$$

Proof Recall (B.4) for the definition of $\widetilde{G}^n_k$. We obtain

$$\begin{aligned}
\sum_{\lambda \in G^n \backslash G^n_\varepsilon : \lambda_i < 1} t^n(\lambda) p^n_i(\lambda) &= \sum_{k=0}^{\lceil \varepsilon |N| n \rceil - 1} \sum_{\lambda \in \widetilde{G}^n_k : \lambda_i < 1} t^n(\lambda) p^n_i(\lambda) + \sum_{k=\lfloor (1-\varepsilon)|N|n \rfloor + 1}^{|N|n-1} \sum_{\lambda \in \widetilde{G}^n_k : \lambda_i < 1} t^n(\lambda) p^n_i(\lambda) && \text{(C.35)} \\
&\leq \sum_{k=0}^{\lceil \varepsilon |N| n \rceil - 1} \sum_{\lambda \in \widetilde{G}^n_k : \lambda_i < 1} t^n(\lambda) + \sum_{k=\lfloor (1-\varepsilon)|N|n \rfloor + 1}^{|N|n-1} \sum_{\lambda \in \widetilde{G}^n_k : \lambda_i < 1} t^n(\lambda) && \text{(C.36)} \\
&\leq \sum_{k=0}^{\lceil \varepsilon |N| n \rceil - 1} 1 + \sum_{k=\lfloor (1-\varepsilon)|N|n \rfloor + 1}^{|N|n-1} 1 && \text{(C.37)} \\
&= \lceil \varepsilon |N| n \rceil + \lceil \varepsilon |N| n \rceil - 1 \\
&< 2\varepsilon |N| n + 1 && \text{(C.38)} \\
&= \mathcal{O}(\varepsilon) n + \mathcal{O}(1).
\end{aligned}$$

Here, (C.35) follows from (B.4) and (B.13), (C.36) follows from $0 \leq p^n_i(\lambda) \leq 1$ for all $\lambda \in G^n \backslash \{e_N\}$, (C.37) follows from $\sum_{\lambda \in \widetilde{G}^n_k} t^n(\lambda) = 1$ for all $k \in \{0, \ldots, |N|n - 1\}$ and (C.38) follows from the fact that $\lceil x \rceil < x + 1$ for all $x \in \mathbb{R}$. $\square$

The following result follows almost directly from Proposition C.1.

Lemma C.14 *For all $i \in N$, we have*

$$\sum_{\lambda \in [G^n_\varepsilon \setminus D(n)]:\lambda_i<1} t^n(\lambda) p^n_i(\lambda) = \mathcal{O}^\varepsilon(n^{-\infty}).$$

Proof This result follows directly from

$$\sum_{\lambda \in [G^n_\varepsilon \setminus D(n)]:\lambda_i<1} t^n(\lambda) p^n_i(\lambda) = \sum_{\lambda \in [G^n_\varepsilon \setminus D(n)]:\lambda_i<1} \mathcal{O}^\varepsilon(n^{-\infty}) \tag{C.39}$$

$$< (n+1)^{|N|} \mathcal{O}^\varepsilon(n^{-\infty}) \tag{C.40}$$

$$= \mathcal{O}^\varepsilon(n^{-\infty}),$$

where (C.39) follows from Proposition C.1 and (C.40) follows from $\sharp(\{\lambda \in [G^n_\varepsilon \setminus D(n)] : \lambda_i < 1\}) < \sharp(G^n) = (n+1)^{|N|}$. □

Lemma C.15 *Let $R \in \mathcal{R}$. Then, we have*

$$r\left(\lambda + (1/n)\cdot e_i\right) - r(\lambda) = \mathcal{O}\left(n^{-1}\right),$$

for all $i \in N$ and (n, λ) such that $n \in \mathbb{N}$, $\lambda \in G^n$ and $\lambda_i < 1$.

Proof Denote $c = \max\{|f_{\mathbb{Q}}(e_j)| : \mathbb{Q} \in Q(\rho), j \in N\}$. Let $\mathbb{Q}_1, \mathbb{Q}_2 \in Q(\rho)$ be such that $r\left(\lambda + (1/n)\cdot e_i\right) = f_{\mathbb{Q}_1}\left(\lambda + (1/n)\cdot e_i\right)$ and $r(\lambda) = f_{\mathbb{Q}_2}(\lambda)$. Then, we have

$$\begin{aligned} r\left(\lambda + (1/n)\cdot e_i\right) - r(\lambda) &= f_{\mathbb{Q}_1}\left(\lambda - (1/n)\cdot e_i\right) - f_{\mathbb{Q}_2}\left(\lambda\right) \\ &\leq f_{\mathbb{Q}_1}\left(\lambda + (1/n)\cdot e_i\right) - f_{\mathbb{Q}_1}\left(\lambda\right) \\ &= \frac{1}{n} f_{\mathbb{Q}_1}\left(e_i\right) \\ &\leq \frac{1}{n} c \end{aligned}$$

and

$$\begin{aligned} r\left(\lambda + (1/n)\cdot e_i\right) - r(\lambda) &= f_{\mathbb{Q}_1}\left(\lambda + (1/n)\cdot e_i\right) - f_{\mathbb{Q}_2}\left(\lambda\right) \\ &\geq f_{\mathbb{Q}_2}\left(\lambda + (1/n)\cdot e_i\right) - f_{\mathbb{Q}_2}\left(\lambda\right) \\ &= \frac{1}{n} f_{\mathbb{Q}_2}\left(e_i\right) \\ &\geq -\frac{1}{n} c. \end{aligned}$$

This concludes the proof. ☐

Lemma C.16 *For all $i \in N$, we have*

$$\sum_{\lambda \in G_\varepsilon^n \cap D(n)} |t^n(\lambda) p_i^n(\lambda) - h^n(\lambda)| = \mathcal{O}^\varepsilon(n^{\frac{3}{4}}).$$

Proof It is sufficient to show this result only for $n \in \mathbb{N}$ such that $d_n < \frac{1}{2}\varepsilon$. If $|N| = 1$ the result is trivial as $t^n(\lambda) p_i^n(\lambda) = h^n(\lambda) = 1$ for all $\lambda \in G_\varepsilon^n$. Next, we let $|N| \geq 2$. For all $\lambda \in G_\varepsilon^n \cap D(n)$, we have

$$\begin{aligned} |t^n(\lambda) p_i^n(\lambda) - h^n(\lambda)| &= \left| h^n(\lambda)[1 + \mathcal{O}^\varepsilon(n^{-\frac{1}{2}+\frac{3}{8|N|}})] \cdot [1 + \mathcal{O}^\varepsilon(n^{-\frac{1}{2}+\frac{1}{8|N|}})] - h^n(\lambda) \right| && \text{(C.41)} \\ &= \left| h^n(\lambda) \mathcal{O}^\varepsilon(n^{-\frac{1}{2}+\frac{3}{8|N|}}) \right| \\ &= \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+\frac{3}{8|N|}}), && \text{(C.42)} \end{aligned}$$

where (C.41) follows from Lemma C.9 and Lemma C.11 and (C.42) follows from $h^n(\lambda) = \mathcal{O}^\varepsilon(n^{\frac{1}{2}(1-|N|)})$. If $y \in C^n(\lambda)$ for a $\lambda \in G_\varepsilon^n \cap D(n)$, we have

$$\begin{aligned} \|y - \bar{y} \cdot e_N\| &\leq \|y - \bar{\lambda} \cdot e_N\| && \text{(C.43)} \\ &\leq \|y - \lambda\| + \|\lambda - \bar{\lambda} \cdot e_N\| && \text{(C.44)} \\ &< (\sqrt{|N|}/n) + n^{-\frac{1}{2}+\frac{1}{8|N|}}, && \text{(C.45)} \end{aligned}$$

where (C.43) and (C.44) follow from the triangular inequality and (C.45) follows from the fact that $\|y - \lambda\| \leq (\sqrt{|N|}/n)$ for all $y \in C^n(\lambda)$. So, we get

$$\bigcup_{\lambda \in G_\varepsilon^n \cap D(n)} C^n(\lambda) \subset D'(n). \quad \text{(C.46)}$$

and, so,

$$\begin{aligned} n^{-|N|} \sharp(G_\varepsilon^n \cap D(n)) &\leq \nu\left(D'(n)\right) && \text{(C.47)} \\ &= \mathcal{O}(d_n'^{|N|-1}) && \text{(C.48)} \\ &= \mathcal{O}\left(\left(n^{-\frac{1}{2}+\frac{1}{8|N|}} + (\sqrt{|N|}/n)\right)^{(|N|-1)}\right) \\ &= \mathcal{O}(n^{-\frac{1}{2}|N|+\frac{5}{8}-\frac{1}{8|N|}}), && \text{(C.49)} \end{aligned}$$

where (C.47) follows from (C.32) and (C.46), and (C.48) follows from (C.33). From this, we get

$$\sum_{\lambda\in G_\varepsilon^n\cap D(n)} |t^n(\lambda)p_i^n(\lambda)-h^n(\lambda)| \leq \sharp(G_\varepsilon^n\cap D(n))\mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+\frac{3}{8|N|}})$$
$$= \mathcal{O}^\varepsilon(n^{\frac{5}{8}+\frac{2}{8|N|}}).$$

As $|N|\geq 2$, this concludes the proof. □

Lemma C.17 *Let $R\in\mathcal{R}$. Then, for all $\varepsilon>0$ and all $m\in\{p^*+1,\dots,p\}$, we have for sufficiently large n that*

$$G_\varepsilon\cap D(n)\cap A_{\mathbb{Q}_m}=\emptyset.$$

Proof If $p^*=p$, the result follows directly and, so, we let $p^*<p$. Denote

$$\alpha = r(e_N) - \max_{m'\in\{p^*+1,\dots,p\}} f_{\mathbb{Q}_{m'}}(e_N) > 0,$$

and let $\ell\in\{1,\dots,p^*\}$ and $m\in\{p^*+1,\dots,p\}$. Then, we have

$$f_{\mathbb{Q}_\ell}(e_N)\geq f_{\mathbb{Q}_m}(e_N)+\alpha.$$

By linearity of $f_{\mathbb{Q}_\ell}$, we have

$$f_{\mathbb{Q}_\ell}(t\cdot e_N)-f_{\mathbb{Q}_m}(t\cdot e_N)=t(f_{\mathbb{Q}_\ell}(e_N)-f_{\mathbb{Q}_m}(e_N))\geq t\alpha, \qquad \text{for all } t\in[0,1]. \tag{C.50}$$

If $f_{\mathbb{Q}_{m'}}(e_i)=0$ for all $m'\in\{1,\dots,p\}$ and for all $i\in N$, we have $p=p^*=1$, which contradicts the assumption that $p^*<p$. So, let $M=\max_{m'\in\{1,\dots,p\}}\|(f_{\mathbb{Q}_{m'}}(e_i))_{i\in N}\|>0$ and $\varepsilon>0$. Then, define $N_\varepsilon=\left(\frac{2M}{\alpha\varepsilon}\right)^4$ and let $n>N_\varepsilon$. Then, we obtain for every $\lambda\in G_\varepsilon\cap D(n)$ that

$$f_{\mathbb{Q}_\ell}(\lambda)-f_{\mathbb{Q}_m}(\lambda)=f_{\mathbb{Q}_\ell}(\bar\lambda\cdot e_N)-f_{\mathbb{Q}_m}(\bar\lambda\cdot e_N)+f_{\mathbb{Q}_\ell}(\lambda-\bar\lambda\cdot e_N)-f_{\mathbb{Q}_m}(\lambda-\bar\lambda\cdot e_N) \tag{C.51}$$
$$\geq \bar\lambda\alpha+f_{\mathbb{Q}_\ell}(\lambda-\bar\lambda\cdot e_N)-f_{\mathbb{Q}_m}(\lambda-\bar\lambda\cdot e_N), \tag{C.52}$$

where (C.51) follows from linearity of $f_{\mathbb{Q}_\ell}$ and $f_{\mathbb{Q}_m}$ and (C.52) follows from (C.50). Moreover, we obtain that

$$
\begin{aligned}
|f_{\mathbb{Q}_{m'}}(\lambda - \bar{\lambda}\cdot e_N)| &\leq \|(f_{\mathbb{Q}_{m'}}(e_i))_{i\in N}\| \cdot \|\lambda - \bar{\lambda}\cdot e_N\| && \text{(C.53)}\\
&\leq M n^{-\frac{1}{4}} && \text{(C.54)}\\
&< M N_\varepsilon^{-\frac{1}{4}} && \text{(C.55)}\\
&= \frac{1}{2}\varepsilon\alpha && \text{(C.56)}\\
&\leq \frac{1}{2}\bar{\lambda}\alpha, && \text{(C.57)}
\end{aligned}
$$

for all $m' \in \{1,\ldots,p\}$, where (C.53) follows from the Cauchy-Schwartz inequality applied to $\sum_{i\in N} f_{\mathbb{Q}_m}(e_i)(\lambda_i - \bar{\lambda})$, (C.54) follows from $m' \in \{1,\ldots,p\}$ and $\lambda \in D(n)$, (D.2) follows from $n > N_\varepsilon$, (C.56) follows from substituting the definition of N_ε, follows from and (C.57) follows from $\lambda \in G_\varepsilon$. Hence, substituting (C.57) in (C.52) yields that $f_{\mathbb{Q}_\ell}(\lambda) - f_{\mathbb{Q}_m}(\lambda) > 0$. Therefore, we have $\lambda \notin A_{\mathbb{Q}_m}$ for every $\lambda \in G_\varepsilon \cap D(n)$ and, hence,

$$
G_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m} = \emptyset. \qquad \text{(C.58)}
$$

□

Note that from (16) and Lemma C.17 it follows for all $\varepsilon > 0$ that

$$
G_\varepsilon \cap D(n) \subset \bigcup_{m\in\{1,\ldots,p^*\}} A_{\mathbb{Q}_m}, \qquad \text{for large } n.
$$

We next show that we can neglect participation profiles close to profiles where the function r is non-differentiable. Note that $B(n)$, as defined in (C.34), is the set of participation profiles close to a participation profile where the function r is non-differentiable. For all $n \in \mathbb{N}$ we have that if $\lambda \in A_{\mathbb{Q}_m} \backslash B(n)$ for some $m \in \{1,\ldots,p\}$, then $\lambda + (1/n)\cdot e_i \in A_{\mathbb{Q}_m}$ for all $i \in N$ and, by linearity of $f_{\mathbb{Q}_m}$, $r(\lambda + (1/n)\cdot e_i) - r(\lambda) = \frac{1}{n}E_{\mathbb{Q}_m}[X_i]$.

Lemma C.18 *Let $R \in \mathcal{R}$. Then, we have*

$$
\sum_{\lambda\in G_\varepsilon^n \cap D(n)\cap B(n)} h^n(\lambda) = \mathcal{O}^\varepsilon(n^{\frac{5}{8}}).
$$

Proof If $p = 1$, we have that $B(n) = \emptyset$ for all $n \in \mathbb{N}$ and, so, the result follows directly. Next, let

$p > 1$. Recall (A.3), i.e.,

$$[0,1]^N \backslash L(R) = \bigcup_{\ell,m\in\{1,\dots,p\}:\ell\neq m} A_{\mathbb{Q}_\ell} \cap A_{\mathbb{Q}_m}.$$

Let $\varepsilon > 0$, $\ell, m \in \{1,\dots,p\}$, $\ell \neq m$ and $n > \frac{2}{\varepsilon}$. We define

$$H^n(\ell,m) = \left\{\lambda \in G_\varepsilon^n \cap D(n) : \exists\hat{\lambda} \in A_{\mathbb{Q}_\ell} \cap A_{\mathbb{Q}_m} : \|\lambda - \hat{\lambda}\| \leq \frac{1}{n}\right\},$$

and $D_\varepsilon = \{\lambda \in G_\varepsilon : \lambda = \bar{\lambda} \cdot e_N\}$. According to Lemma C.17 we have for all $m \in \{p^*+1,\dots,p\}$ that $D_\varepsilon \cap A_{\mathbb{Q}_m} = \emptyset$. Since D_ε and $A_{\mathbb{Q}_m}$ are both compact we can define $\alpha_{\varepsilon,m} = \text{dist}(D_\varepsilon, A_{\mathbb{Q}_m}) = \min\{\|x-y\| : x \in D_\varepsilon, y \in A_{\mathbb{Q}_m}\}$. Obviously, $\alpha_{\varepsilon,m} > 0$. So, if $\ell \notin \{1,\dots,p^*\}$ or $m \notin \{1,\dots,p^*\}$ we get $H^n(\ell,m) = \emptyset$ for large n. If $p^* = 1$ it follows from this that $H^n(\ell,m) = \emptyset$ for all $\ell, m \in \{1,\dots,p\}$. Next, let $p^* > 1$ and $\ell, m \in \{1,\dots,p^*\}$. Recall (A.2) from the proof of Proposition 3.4(ii), i.e.,

$$A_{\mathbb{Q}_\ell} \cap A_{\mathbb{Q}_m} \subset \left\{\lambda \in \mathbb{R}^N : \sum_{i\in N} \lambda_i(E_{\mathbb{Q}_\ell}[X_i] - E_{\mathbb{Q}_m}[X_i]) = 0\right\} := V(\ell,m).$$

Note that $V(\ell,m)$ is an $(|N|-1)$-dimensional linear space where $\{t \cdot e_N : t \in \mathbb{R}\} \subset V(\ell,m)$. To obtain an upper bound of the cardinality of $H^n(\ell,m)$, we first derive the Lebesgue measure of the following Euclidean set

$$\widetilde{H}^n(\ell,m) = \left\{\lambda \in G_{\frac{1}{2}\varepsilon} \cap D'(n) : \exists\hat{\lambda} \in V(\ell,m) : \|\lambda - \hat{\lambda}\| \leq \frac{\sqrt{|N|}}{n} + \frac{1}{n}\right\}.$$

We describe this set via the Gram-Schmidt process. Choose an orthonormal basis $u_1,\dots,u_{|N|}$ of $\mathbb{R}^N$ such that $u_1 = \frac{e_N}{\sqrt{|N|}}$, $u_1,\dots,u_{|N|-1}$ is an orthonormal basis of the $(|N|-1)$-dimensional space $V(\ell,m)$ and $u_{|N|}$ is a unit normal vector of the $(|N|-1)$-dimensional space $V(\ell,m)$. So $u_{|N|}$ is a multiple of the vector $(E_{\mathbb{Q}_\ell}[X_i] - E_{\mathbb{Q}_m}[X_i])_{i\in N}$. Now let $\lambda \in \widetilde{H}^n(\ell,m)$. Let λ_1 be the unique element in $V(\ell,m)$ that is closest to λ. Obviously $\|\lambda - \lambda_1\| \leq \frac{\sqrt{|N|}}{n} + \frac{1}{n}$. Let $\lambda_2 = \bar{\lambda}_1 \cdot e_N$ $(= \bar{\lambda} \cdot e_N)$ be the unique element in $\{t \cdot e_N : t \in \mathbb{R}\}$ that is closest to λ_1 (and hence closest to λ). We provide an overview of the construction of λ_1 and λ_2 in Figure 1. Obviously $\|\lambda - \lambda_2\|^2 = \|\lambda - \lambda_1\|^2 + \|\lambda_1 - \lambda_2\|^2$ and hence $\|\lambda_1 - \lambda_2\| \leq \|\lambda - \lambda_2\| = \|\lambda - \bar{\lambda} \cdot e_N\| < d'_n$. Now we can write $\lambda = \alpha_1 u_1 + \dots + \alpha_{|N|} u_{|N|}$ where $\lambda_2 = \alpha_1 u_1$, $\lambda_1 - \lambda_2 = \alpha_2 u_2 + \dots + \alpha_{|N|-1} u_{|N|-1}$ and $\lambda - \lambda_1 = \alpha_{|N|} u_{|N|}$. From this it follows that $|\alpha_1| = \|\lambda_2\| = \bar{\lambda}\sqrt{|N|} < \sqrt{|N|}$, $|\alpha_k| \leq \sqrt{\alpha_2^2 + \dots + \alpha_{|N|-1}^2} = \|\lambda_1 - \lambda_2\| < d'_n = \mathcal{O}(n^{-\frac{1}{2}+\frac{1}{8|N|}})$

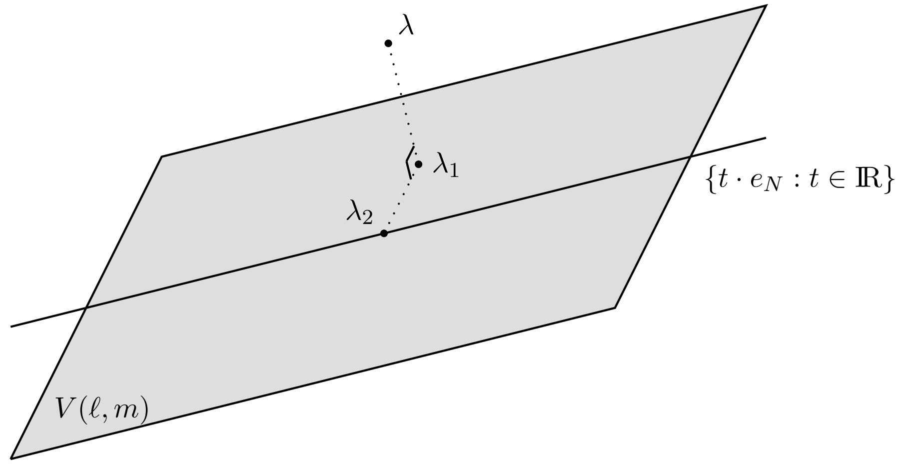

Figure 1: Illustration of λ_1 and λ_2 corresponding to the proof of Lemma C.18.

for all $k \in \{2, \ldots, |N|-1\}$ and $|\alpha_{|N|}| = \|\lambda - \lambda_1\| \leq \frac{1}{n} + \frac{\sqrt{|N|}}{n} = \mathcal{O}(n^{-1})$. Hence,

$$\begin{aligned}\nu(\widetilde{H}^n(\ell,m)) &= \mathcal{O}(1)\mathcal{O}(n^{(-\frac{1}{2}+\frac{1}{8|N|})(|N|-2)})\mathcal{O}(n^{-1}) \\ &= \mathcal{O}(n^{-\frac{1}{2}|N|+\frac{1}{8}}). \end{aligned} \tag{C.59}$$

For all $\lambda \in G^n_\varepsilon$ and $y \in C^n(\lambda)$, we get from

$$\bar{y} = \bar{\lambda} + (\bar{y} - \bar{\lambda}) \begin{cases} \geq \varepsilon - (1/n) > \frac{1}{2}\varepsilon, \\ \leq 1 - \varepsilon + (1/n) < 1 - \frac{1}{2}\varepsilon, \end{cases} \tag{C.60}$$

that $y \in G_{\frac{1}{2}\varepsilon}$. Moreover, we get

$$\min_{\hat{\lambda} \in V(\ell,m)} \|y - \hat{\lambda}\| \leq \|y - \lambda\| + \min_{\hat{\lambda} \in V(\ell,m)} \|\lambda - \hat{\lambda}\| \leq \frac{\sqrt{|N|}}{n} + \frac{1}{n}, \qquad \text{for all } \lambda \in H^n(\ell,m) \text{ and } y \in C^n(\lambda).$$

From this, (C.46) and (C.60), we get

$$\bigcup_{\lambda \in H^n(\ell,m)} C^n(\lambda) \subset \widetilde{H}^n(\ell,m), \qquad \text{for all } n \in \mathbb{N} \text{ such that } n > \frac{2}{\varepsilon}. \tag{C.61}$$

From (C.32) and (C.61) we get

$$n^{-|N|}\sharp(H^n(\ell,m)) \leq \nu(\widetilde{H}^n(\ell,m)). \tag{C.62}$$

Substituting (C.59) in (C.62) yields

$$\sharp(H^n(\ell,m)) = \mathcal{O}(n^{\frac{1}{2}|N|+\frac{1}{8}}). \tag{C.63}$$

Then, we obtain

$$\sum_{\lambda\in G_\varepsilon^n\cap D(n)\cap B(n)} h^n(\lambda) \le \sum_{\ell,m\in\{1,\dots,p\}:\ell\neq m}\ \sum_{\lambda\in H^n(\ell,m)} h^n(\lambda) \tag{C.64}$$

$$\le \begin{pmatrix} p \\ 2 \end{pmatrix} \max_{\ell,m\in\{1,\dots,p\}:\ell\neq m} \sharp(H^n(\ell,m)) \max_{\lambda\in G_\varepsilon} h^n(\lambda) \tag{C.65}$$

$$= \begin{pmatrix} p \\ 2 \end{pmatrix} \mathcal{O}(n^{\frac{1}{2}|N|+\frac{1}{8}})\mathcal{O}^\varepsilon(n^{\frac{1}{2}(1-|N|)}) \tag{C.66}$$

$$= \mathcal{O}^\varepsilon(n^{\frac{5}{8}}),$$

where (C.64) follows from (A.3), (C.65) follows from $\sharp(\{\ell,m\in\{1,\dots,p\}:\ell\neq m\}) = \begin{pmatrix} p \\ 2 \end{pmatrix}$ and (C.66) follows from (C.63) and $h^n(\lambda) = \mathcal{O}^\varepsilon(n^{\frac{1}{2}(1-|N|)})$ for all $\lambda\in G_\varepsilon$. This concludes the proof. □

Proof of Proposition C.2 It is sufficient to show this result for sufficiently large n. We get

$$K_i^{avg,n}(R) = \sum_{\lambda\in G^n:\lambda_i<1} t^n(\lambda)p_i^n(\lambda)\left[r\left(\lambda+(1/n)\cdot e_i\right)-r(\lambda)\right] \tag{C.67}$$

$$= \sum_{\lambda\in G_\varepsilon^n:\lambda_i<1} t^n(\lambda)p_i^n(\lambda)\left[r\left(\lambda+(1/n)\cdot e_i\right)-r(\lambda)\right] + \mathcal{O}(\varepsilon)+\mathcal{O}(n^{-1}) \tag{C.68}$$

$$= \sum_{\lambda\in G_\varepsilon^n\cap D(n)} t^n(\lambda)p_i^n(\lambda)\left[r\left(\lambda+(1/n)\cdot e_i\right)-r(\lambda)\right] + \mathcal{O}(\varepsilon)+\mathcal{O}^\varepsilon\left(n^{-1}\right) \tag{C.69}$$

$$= \sum_{\lambda\in G_\varepsilon^n\cap D(n)} h^n(\lambda)\left[r\left(\lambda+(1/n)\cdot e_i\right)-r(\lambda)\right] + \mathcal{O}(\varepsilon)+\mathcal{O}^\varepsilon(n^{-\frac{1}{4}}) \tag{C.70}$$

$$= \sum_{\lambda\in[G_\varepsilon^n\cap D(n)]\setminus B(n)} h^n(\lambda)\left[r\left(\lambda+(1/n)\cdot e_i\right)-r(\lambda)\right] + \mathcal{O}(\varepsilon)+\mathcal{O}^\varepsilon(n^{-\frac{1}{4}}) \tag{C.71}$$

$$= \sum_{m=1}^{p}\ \sum_{\lambda\in[G_\varepsilon^n\cap D(n)\cap A_{\mathbb{Q}_m}]\setminus B(n)} h^n(\lambda)\frac{1}{n}E_{\mathbb{Q}_m}[X_i] + \mathcal{O}(\varepsilon)+\mathcal{O}^\varepsilon(n^{-\frac{1}{4}}) \tag{C.72}$$

$$= \sum_{m=1}^{p^*}\ \sum_{\lambda\in[G_\varepsilon^n\cap D(n)\cap A_{\mathbb{Q}_m}]\setminus B(n)} h^n(\lambda)\frac{1}{n}E_{\mathbb{Q}_m}[X_i] + \mathcal{O}(\varepsilon)+\mathcal{O}^\varepsilon(n^{-\frac{1}{4}}) \tag{C.73}$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] \frac{1}{n} \sum_{\lambda \in [G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}] \setminus B(n)} h^n(\lambda) + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}})$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] \frac{1}{n} \sum_{\lambda \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}} h^n(\lambda) + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}}), \tag{C.74}$$

where (C.67) follows from Proposition B.1, (C.68) follows from Lemma C.13 and Lemma C.15, (C.69) follows from Lemma C.14 and Lemma C.15, (C.70) follows from Lemma C.15 and Lemma C.16, (C.71) follows from Lemma C.15 and Lemma C.18, (C.72) follows from $[0,1]^N \setminus L(R) \subset B(n)$, (C.73) follows from Lemma C.17 and (C.74) follows from Lemma C.18. This concludes the proof.□

C.3 Proof of Proposition C.3

Lemma C.19 *The function h^n is differentiable for a fixed $n \in \mathbb{N}$, and, moreover, we have for all $i \in N$ and $(\varepsilon, n, \lambda) \in Dom$ that*

$$\frac{\partial h^n}{\partial \lambda_i}(\lambda) = \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+1+\frac{1}{8|N|}}), \qquad \text{if } \lambda \in D'(n),$$

where $D'(n)$ is defined in (C.31).

Proof Define the functions $f^n(\lambda) = -c(\bar{\lambda})n\|\lambda - \bar{\lambda} \cdot e_N\|^2$ and $g(\lambda) = (\bar{\lambda}(1-\bar{\lambda}))^{\frac{1}{2}(1-|N|)}$ for all $\lambda \in [0,1]^N$. Then, we obtain

$$\frac{\partial h^n}{\partial \lambda_i}(\lambda) = \frac{\partial f^n}{\partial \lambda_i}(\lambda) \cdot h^n(\lambda) + \frac{\partial g}{\partial \lambda_i}(\lambda) \cdot \frac{h^n(\lambda)}{g(\lambda)}, \qquad \text{for all } \lambda \in [0,1]^N \setminus \{e_\emptyset, e_N\}. \tag{C.75}$$

Moreover, we obtain the following approximations for all $\lambda \in G_\varepsilon \cap D'(n)$:

$$\begin{aligned}
\frac{\partial f^n}{\partial \lambda_i}(\lambda) &= -c(\bar{\lambda})n \left[\sum_{k \neq i} 2\left(\lambda_k - \bar{\lambda}\right) \cdot -\frac{1}{|N|} + 2(\lambda_i - \bar{\lambda}) \left(1 - \frac{1}{|N|}\right) \right] \\
&\quad + \frac{1 - 2\bar{\lambda}}{2|N|(\bar{\lambda}(1-\bar{\lambda}))^2} n \|\lambda - \bar{\lambda} \cdot e_N\|^2 \\
&= -c(\bar{\lambda})n2\left(\lambda_i - \bar{\lambda}\right) + \frac{1 - 2\bar{\lambda}}{2|N|(\bar{\lambda}(1-\bar{\lambda}))^2} n \|\lambda - \bar{\lambda} \cdot e_N\|^2 \\
&= \mathcal{O}^\varepsilon(n^{\frac{1}{2}+\frac{1}{8|N|}}) + \mathcal{O}^\varepsilon(n^{\frac{1}{4|N|}}) \qquad (C.76) \\
&= \mathcal{O}^\varepsilon(n^{\frac{1}{2}+\frac{1}{8|N|}}), \\
\frac{\partial g}{\partial \lambda_i}(\lambda) &= \mathcal{O}^\varepsilon(1),
\end{aligned}$$

$$(g(\lambda))^{-1} = \mathcal{O}(1),$$
$$h^n(\lambda) = \mathcal{O}^\varepsilon(n^{\frac{1}{2}(1-|N|)}),$$

where (C.76) follows from $|\lambda_i - \bar{\lambda}| \leq \|\lambda - \bar{\lambda} \cdot e_N\| \leq d'_n = \mathcal{O}(n^{-\frac{1}{2}+\frac{1}{8|N|}})$. Then, the result follows from substituting these equations in (C.75). □

Lemma C.20 *Let $R \in \mathcal{R}$. Then, we have for all $m \in \{1, \ldots, p\}$ that*

$$\sum_{\lambda \in G^n_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m}} h^n(\lambda) = n^{|N|} \sum_{\lambda \in G^n_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m}} \int_{C^n(\lambda)} h^n(\lambda^*) d\lambda^* + \mathcal{O}^\varepsilon(n^{\frac{5}{8}}),$$

where $C^n(\lambda)$ is defined in (C.30)

Proof Let $\varepsilon > 0$. It is sufficient to show this result for all $n \in \mathbb{N}$ such that $n > \frac{2}{\varepsilon}$. Let $\lambda \in G^n_\varepsilon \cap D(n)$. From (C.46) and (C.60) it follows that

$$C^n(\lambda) \subset G_{\frac{1}{2}\varepsilon} \cap D'(n). \tag{C.77}$$

We get from (C.77) and Lemma C.19 that h^n is differentiable in λ^* for all $\lambda^* \in C^n(\lambda)$. Applying Taylor's theorem yields that

$$h^n(\lambda) - h^n(\lambda^*) = \sum_{i \in N} \frac{\partial h}{\partial \lambda_i}(\chi)(\lambda_i - \lambda^*_i), \text{ for all } \lambda^* \in C^n(\lambda), \text{ where } \chi \in \text{conv}\{\lambda, \lambda^*\}. \tag{C.78}$$

Here, as $\chi \in C^n(\lambda)$, we get from Lemma C.19 that

$$\frac{\partial h}{\partial \lambda_i}(\chi) = \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+1+\frac{1}{8|N|}}), \qquad \text{for all } i \in N. \tag{C.79}$$

So, as $|\lambda_i - \lambda^*_i| \leq n^{-1}$ for all $\lambda^* \in C^n(\lambda)$ and $i \in N$, we get from (C.78) and (C.79) that

$$\begin{aligned} h^n(\lambda) - h^n(\lambda^*) &= |N| \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+1+\frac{1}{8|N|}}) n^{-1} \\ &= \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+\frac{1}{8|N|}}), \end{aligned}$$

for all $\lambda^* \in C^n(\lambda)$. From this, we directly get

$$h^n(\lambda) - n^{|N|} \int_{C^n(\lambda)} h^n(\lambda^*) d\lambda^* = \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+\frac{1}{8|N|}}), \qquad \text{for all } \lambda \in G^n_\varepsilon \cap D(n). \tag{C.80}$$

Moreover, from (C.49) we get

$$
\begin{aligned}
\sharp(G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}) &\leq \sharp(G_\varepsilon^n \cap D(n)) \\
&= \mathcal{O}(n^{\frac{1}{2}|N|+\frac{5}{8}-\frac{1}{8|N|}}).
\end{aligned} \tag{C.81}
$$

Hence, from (C.80) and (C.81) it follows that

$$
\begin{aligned}
\left| \sum_{\lambda \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}} \left(h^n(\lambda) - n^{|N|} \int_{C^n(\lambda)} h^n(\lambda^*) d\lambda^* \right) \right| &\leq \sharp(G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}) \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+\frac{1}{8|N|}}) \\
&= \mathcal{O}^\varepsilon(n^{\frac{5}{8}}).
\end{aligned}
$$

This concludes the result. □

Lemma C.21 *Let $R \in \mathcal{R}$. Then, we have for all $m \in \{1, \ldots, p\}$ that*

$$
\sum_{\lambda^* \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}} \int_{C^n(\lambda^*)} h^n(\lambda) d\lambda = \int_{G_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m}} h^n(\lambda) d\lambda + \mathcal{O}^\varepsilon(n^{-|N|+\frac{5}{8}}).
$$

Proof Let $\varepsilon > 0$ and define $D''(n) = D^{d''_n}$, where $d''_n = d_n - (\sqrt{|N|}/n)$. It is sufficient to show this result for all $n \in \mathbb{N}$ such that $n > \frac{2}{\varepsilon}$. Define $A = \bigcup_{\lambda^* \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}} C^n(\lambda^*)$ and $B = G_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m}$. Moreover, define

$$
\begin{aligned}
E_1^n &= B(n/(\sqrt{|N|}+1)) \cap G_{\frac{1}{2}\varepsilon} \\
E_2^n &= [G_{\varepsilon-(1/n)} \cap D'(n)] \backslash D''(n) \\
E_3^n &= [D'(n) \cap G_{\varepsilon-(1/n)}] \backslash G_{\varepsilon+(1/n)},
\end{aligned}
$$

where the set $B(n)$ is defined in (C.34). We first show

$$
(A \backslash B) \cup (B \backslash A) \subset E_1^n \cup E_2^n \cup E_3^n. \tag{C.82}
$$

Let $y_1 \in A \backslash B$, so we have $y_1 \in C^n(\lambda)$ for some $\lambda \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}$. If $y_1 \notin A_{\mathbb{Q}_m}$, there is a $\lambda' \in [0,1]^N \backslash L(R)$ such that $\lambda' \in \text{conv}\{\lambda, y_1\}$ and, so, $y_1 \in E_1^n$. If $y_1 \notin D(n)$, we have according to (C.45) that $\|y_1 - \bar{y}_1 \cdot e_N\| < (\sqrt{|N|}/n) + d_n = d'_n$ and, so, $y_1 \in E_2^n$. If $y_1 \notin G_\varepsilon^n$, then $\bar{y}_1 < \varepsilon$ or $\bar{y}_1 > 1 - \varepsilon$ and hence we have according to (C.60) that $\varepsilon - (1/n) \leq \bar{y}_1 \leq 1 - (\varepsilon - (1/n))$ and, so, $y_1 \in E_3^n$. Now, let $y_2 \in B \backslash A$, so we have $y_2 \in G_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m}$ and there does not exist a

$\lambda \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}$ such that $y_2 \in C^n(\lambda)$. Let λ such that $y_2 \in C^n(\lambda)$. If $\lambda \notin A_{\mathbb{Q}_m}$, there exists an $\lambda' \in [0,1]^N \backslash L(R)$ such that $\lambda' \in \operatorname{conv}\{\lambda, y_2\}$ and, so, $y_2 \in E_1^n$. If $\lambda \notin D(n)$, we get from the triangle inequality that $\|y_2 - \bar{y}_2 \cdot e_N\| \geq \|\lambda - \bar{\lambda} \cdot e_N\| - \|y_2 - \lambda\| \geq d_n - (\sqrt{|N|}/n) = d_n''$ and, so, $y_2 \notin D''(n)$. So, $y_2 \in E_2^n$. If $\lambda \notin G_\varepsilon^n$, then $\bar{\lambda} < \varepsilon$ or $\bar{\lambda} > 1-\varepsilon$ and hence $\bar{y}_2 = \bar{\lambda} + (\bar{y}_2 - \bar{\lambda}) < \varepsilon + (1/n)$ or $\bar{y}_2 < 1 - (\varepsilon + (1/n))$ and so, $y_2 \notin G_{\varepsilon+(1/n)}$. So, $y_2 \in E_3^n$. Hence, we have shown (C.82). Then, we get

$$\begin{aligned}
\left| \int_A h^n(\lambda) d\lambda - \int_B h^n(\lambda) d\lambda \right| &\leq \int_{A\backslash B} h^n(\lambda) d\lambda + \int_{B\backslash A} h^n(\lambda) d\lambda && \text{(C.83)} \\
&\leq \int_{E_1^n \cup E_2^n \cup E_3^n} h^n(\lambda) d\lambda && \text{(C.84)} \\
&\leq \sum_{k=1}^{3} \int_{E_k^n} h^n(\lambda) d\lambda && \text{(C.85)} \\
&\leq \sum_{k=1}^{3} \nu(E_k^n) \mathcal{O}^\varepsilon(n^{\frac{1}{2}(1-|N|)}) && \text{(C.86)} \\
&= \mathcal{O}^\varepsilon(n^{-|N|+\frac{5}{8}}). && \text{(C.87)}
\end{aligned}$$

Here, (C.83) follows from $\int_A h^n(\lambda)d\lambda - \int_B h^n(\lambda)d\lambda = \int_{A\backslash B} h^n(\lambda)d\lambda - \int_{B\backslash A} h^n(\lambda)d\lambda$, (C.84) follows from (C.82), (C.85) is a standard rule of integration, (C.86) follows from $h^n(\lambda) = \mathcal{O}^\varepsilon(n^{\frac{1}{2}(1-|N|)})$ for all $\lambda \in G_{\frac{1}{2}\varepsilon}$ and (C.87) follows from $\nu(E_1^n) = \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+\frac{1}{8}})$ (see (C.59)) and we get in a similar fashion as for (C.59) via a Gram-Schmidt process that

$$\begin{aligned}
\nu(E_2^n) &= \mathcal{O}\left(\left(n^{-\frac{1}{2}+\frac{1}{8|N|}} + (\sqrt{|N|}/n)\right)^{|N|-1} - \left(n^{-\frac{1}{2}+\frac{1}{8|N|}} - (\sqrt{|N|}/n)\right)^{|N|-1} \right) \\
&= \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|+\frac{1}{8}}), \\
\nu(E_3^n) &= \mathcal{O}((n^{-\frac{1}{2}+\frac{1}{8|N|}} + (\sqrt{|N|}/n))^{|N|-1} n^{-1}) \\
&= \mathcal{O}^\varepsilon(n^{-\frac{1}{2}|N|-\frac{3}{8}}).
\end{aligned}$$

This concludes the proof. □

Lemma C.22 *For all $t \in (0,1)$ it holds that*

$$\int_0^{n^{\frac{1}{4|N|}}/2t(1-t)} e^{-s} s^{\frac{1}{2}(|N|-3)} ds = \Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right) + \mathcal{O}(n^{-\infty}),$$

where Γ *is the Gamma function:*

$$\Gamma(\kappa) = \int_0^\infty e^{-t} t^{\kappa-1} dt, \qquad \textit{for all } \kappa > 0. \tag{C.88}$$

Proof We get

$$\Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right) - \int_0^{n^{\frac{1}{4|N|}}/2t(1-t)} e^{-s} s^{\frac{1}{2}(|N|-3)} ds = \int_{n^{\frac{1}{4|N|}}/2t(1-t)}^\infty e^{-s} s^{\frac{1}{2}(|N|-3)} ds \tag{C.89}$$

$$\leq K \int_{n^{\frac{1}{4|N|}}/2t(1-t)}^\infty e^{-\frac{1}{2}s} ds \tag{C.90}$$

$$= K2e^{-n^{\frac{1}{4|N|}}/4t(1-t)} \tag{C.91}$$

$$\leq K2e^{-n^{\frac{1}{4|N|}}} \tag{C.92}$$

$$= \mathcal{O}(n^{-\infty}), \tag{C.93}$$

where $K > 0$. Here, (C.89) is a standard integration rule, (C.90) follows from that there exists a constant $K > 0$ such that $e^{-s} s^{\frac{1}{2}(|N|-3)} < Ke^{-\frac{1}{2}s}$ for all $s > 1$, (C.91) follows from $\int_a^b e^{-\frac{1}{2}s} ds = -2(e^{-\frac{1}{2}b} - e^{-\frac{1}{2}a})$ for all $a \leq b$, (C.92) follows from $4t(1-t) \leq 1$ for all $t \in (0,1)$ and (C.93) follows from the fact that $(e^{-1})^{n^{\frac{1}{4|N|}}} = \mathcal{O}(n^{-\infty})$. This concludes the proof. □

Proof of Proposition C.3 We get

$$K_i^{avg,n}(R) = \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] \frac{1}{n} \sum_{\lambda \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}} h^n(\lambda) + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}}) \tag{C.94}$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] n^{|N|-1} \sum_{\lambda \in G_\varepsilon^n \cap D(n) \cap A_{\mathbb{Q}_m}} \int_{C^n(\lambda)} h^n(\lambda^*) d\lambda^* + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}}) \tag{C.95}$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] n^{|N|-1} \int_{G_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m}} h^n(\lambda) d\lambda + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}}) \tag{C.96}$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] n^{\frac{1}{2}(|N|-1)} (2\pi)^{\frac{1}{2}(1-|N|)} |N|^{-\frac{1}{2}} \tag{C.97}$$

$$\cdot \int_{G_\varepsilon \cap D(n) \cap A_{\mathbb{Q}_m}} \left(e^{-\frac{1}{2\bar{\lambda}(1-\bar{\lambda})} n \|\lambda - \bar{\lambda} \cdot e_N\|^2}\right) (\bar{\lambda}(1-\bar{\lambda}))^{\frac{1}{2}(1-|N|)} d\lambda + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}})$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] n^{\frac{1}{2}(|N|-1)} (2\pi)^{\frac{1}{2}(1-|N|)} \tag{C.98}$$

$$\cdot \int_\varepsilon^{1-\varepsilon} \int_0^{d_n} \int_{S_m} e^{-\frac{1}{2t(1-t)} r^2 n} (t(1-t))^{\frac{1}{2}(1-|N|)} r^{|N|-2} d\omega dr dt + \mathcal{O}(\varepsilon) + \mathcal{O}^\varepsilon(n^{-\frac{1}{4}})$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] n^{\frac{1}{2}(|N|-1)} (2\pi)^{\frac{1}{2}(1-|N|)} \mu(S_m) \tag{C.99}$$

$$\cdot \int_{\varepsilon}^{1-\varepsilon} \int_0^{d_n} e^{-\frac{1}{2t(1-t)} r^2 n} (t(1-t))^{\frac{1}{2}(1-|N|)} r^{|N|-2} dr dt + \mathcal{O}(\varepsilon) + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}})$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] n^{\frac{1}{2}(|N|-1)} (2\pi)^{\frac{1}{2}(1-|N|)} \phi_m 2 \frac{\pi^{-\frac{1}{2}(1-|N|)}}{\Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right)} \tag{C.100}$$

$$\cdot \int_{\varepsilon}^{1-\varepsilon} \int_0^{d_n} e^{-\frac{1}{2t(1-t)} r^2 n} (t(1-t))^{\frac{1}{2}(1-|N|)} r^{|N|-2} dr dt + \mathcal{O}(\varepsilon) + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}})$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] \phi_m n^{\frac{1}{2}(|N|-1)} 2^{1\frac{1}{2}-\frac{1}{2}|N|} \frac{1}{\Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right)} \left(\frac{2}{n}\right)^{\frac{1}{2}(|N|-2)} \tag{C.101}$$

$$\cdot \int_{\varepsilon}^{1-\varepsilon} \int_0^{n^{\frac{1}{4|N|}}/2t(1-t)} e^{-s} s^{\frac{1}{2}(|N|-2)} (t(1-t))^{-\frac{1}{2}} \sqrt{\frac{t(1-t)}{2ns}} ds dt + \mathcal{O}(\varepsilon) + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}})$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] \phi_m \frac{1}{\Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right)} \tag{C.102}$$

$$\cdot \int_{\varepsilon}^{1-\varepsilon} \int_0^{n^{\frac{1}{4|N|}}/2t(1-t)} e^{-s} s^{\frac{1}{2}(|N|-3)} ds dt + \mathcal{O}(\varepsilon) + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}})$$

$$= \sum_{m=1}^{p^*} E_{\mathbb{Q}_m}[X_i] \phi_m \frac{1}{\Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right)} \int_{\varepsilon}^{1-\varepsilon} \left(\Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right) + \mathcal{O}(n^{-\infty}) \right) dt + \mathcal{O}(\varepsilon) + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}}) \tag{C.103}$$

$$= \sum_{m=1}^{p^*} \phi_m E_{\mathbb{Q}_m}[X_i] + \mathcal{O}(\varepsilon) + \mathcal{O}^{\varepsilon}(n^{-\frac{1}{4}}) \tag{C.104}$$

Here, (C.94) follows from Proposition C.2, (C.95) follows from Lemma C.20 and (C.96) follows from Lemma C.21, (C.97) follows from substitution of (C.7), (C.98) follows from the polar coordinate transformation $\lambda = t \cdot e_N + r\omega$ and $d\lambda = r^{|N|-2}|N|^{\frac{1}{2}} d(t, r, \omega)$, (C.99) follows from the fact that $\int_{S_m} d\omega = \mu(S_m)$, (C.100) follows from $\mu(S_m) = \phi_m \mu(S)$ and the well-known result that the hypersurface measure of an $|N|$-dimensional ball is given by

$$\mu(S) = 2 \frac{\pi^{-\frac{1}{2}(1-|N|)}}{\Gamma\left(\frac{1}{2}|N| - \frac{1}{2}\right)},$$

where Γ is defined in (C.88), (C.101) follows from the transformation $s = \frac{r^2 n}{2t(1-t)}$ and $dr = \sqrt{\frac{t(1-t)}{2ns}} ds$, (C.102) follows from canceling of some terms and (C.103) follows from Lemma C.22. This concludes the proof. □

D Proofs of Section 4

Proof of Proposition 4.1 (i) Follows immediately from (11), (26) and (27).

(ii) Follows immediately from (i) and the fact that the Aumann-Shapley value, if it exists, is the unique element of the fuzzy core (Aubin, 1981). □

Proof of Corollary 4.2 If $|N| = 2$, we get

$$S = \{(-\sqrt{0.5}, \sqrt{0.5}), (\sqrt{0.5}, -\sqrt{0.5})\}$$
$$S_m = \{z \in S : z_1 E_{\mathbb{Q}_m}[X_1] + z_2 E_{\mathbb{Q}_m}[X_2] \geq z_1 E_{\mathbb{Q}_\ell}[X_1] + z_2 E_{\mathbb{Q}_\ell}[X_2] \text{ for all } \ell \in \{1, \ldots, p^*\}\}.$$

So, $\mu(S) = |S| = 2$, and $\mu(S_m) = |S_m| = |\{z \in S : z_1 E_{\mathbb{Q}_m}[X_1] + z_2 E_{\mathbb{Q}_m}[X_2] \geq z_1 E_{\mathbb{Q}_\ell}[X_1] + z_2 E_{\mathbb{Q}_\ell}[X_2] \text{ for all } \ell \in \{1, \ldots, p^*\}\}|$. Note that for all $\ell \in \{1, \ldots, p^*\}$, it holds by construction that $E_{\mathbb{Q}_\ell}[X_1] + E_{\mathbb{Q}_\ell}[X_2]$ is the same (and equal to $r(e_N)$). So, if $z = (-\sqrt{0.5}, \sqrt{0.5})$, then $z_1 E_{\mathbb{Q}_m}[X_1] + z_2 E_{\mathbb{Q}_m}[X_2] \geq z_1 E_{\mathbb{Q}_\ell}[X_1] + z_2 E_{\mathbb{Q}_\ell}[X_2]$ for all $\ell \in \{1, \ldots, p^*\}$ holds when $E_{\mathbb{Q}_m}[X_2] = \max\{E_{\mathbb{Q}_\ell}[X_2] : \ell \in \{1, \ldots, p^*\}\}$ or, equivalently, $E_{\mathbb{Q}_m}[X_1] = \min\{E_{\mathbb{Q}_\ell}[X_1] : \ell \in \{1, \ldots, p^*\}\}$. Likewise, if $z = (\sqrt{0.5}, -\sqrt{0.5})$, then $z_1 E_{\mathbb{Q}_m}[X_1] + z_2 E_{\mathbb{Q}_m}[X_2] \geq z_1 E_{\mathbb{Q}_\ell}[X_1] + z_2 E_{\mathbb{Q}_\ell}[X_2]$ for all $\ell \in \{1, \ldots, p^*\}$ holds when $E_{\mathbb{Q}_m}[X_1] = \max\{E_{\mathbb{Q}_\ell}[X_1] : \ell \in \{1, \ldots, p^*\}\}$ or, equivalently, $E_{\mathbb{Q}_m}[X_2] = \min\{E_{\mathbb{Q}_\ell}[X_2] : \ell \in \{1, \ldots, p^*\}\}$. This concludes the proof. □

Proof of Theorem 4.3 It follows immediately from Definition 3.7 and Definition 3.10 that it is sufficient to show that for all $n \in \mathbb{N}$, all $P \in \mathcal{P}^n$, and all $R \in \mathcal{R}$, the properties *Translation Invariance*, *Scale Invariance* and *Monotonicity* are satisfied for the allocation rule $K^{path,P}(R)$ defined in (21).

We start with showing the property *Translation Invariance*. Let $P \in \mathcal{P}^n$, $n \in \mathbb{N}$, $j \in N$, $R = ((X_i)_{i \in N}, \rho) \in \mathcal{R}$ and $\widetilde{R} = ((\widetilde{X}_i)_{i \in N}, \rho) \in \mathcal{R}$ such that $(\widetilde{X}_i)_{i \in N} = (X_j + c \cdot e_\Omega, X_{-j})$ for some $c \in \mathbb{R}$. Let r ($\tilde{r}$) be the fuzzy game corresponding to R ($\tilde{R}$), as defined in (8). Then, we get

$$\begin{aligned}
\widetilde{r}(\lambda) &= \rho\left(\sum_{i \in N} \lambda_i \widetilde{X}_i\right) \\
&= \rho\left(\sum_{i \in N} \lambda_i \cdot X_i + c \cdot \lambda_j \cdot e_\Omega\right) \\
&= \rho\left(\sum_{i \in N} \lambda_i \cdot X_i\right) + c \cdot \lambda_j && \text{(D.1)} \\
&= r(\lambda) + c \cdot \lambda_j, && \text{(D.2)}
\end{aligned}$$

for all $\lambda \in [0,1]^N$, where (D.1) follows from *Translation Invariance* of ρ. We get

$$K^{path,P}(\widetilde{R}) = \sum_{k=0}^{|N|n-1} [\widetilde{r}(P(k+1)) - \widetilde{r}(P(k))] \cdot e_{i(P,k)} \tag{D.3}$$

$$= \sum_{k=0}^{|N|n-1} [r(P(k+1)) + c \cdot P_j(k+1) - r(P(k)) - c \cdot P_j(k)] \cdot e_{i(P,k)} \tag{D.4}$$

$$= K^{path,P}(R) + c \cdot \sum_{k=0}^{|N|n-1} [P_j(k+1) - P_j(k)] \cdot e_{i(P,k)} \tag{D.5}$$

$$= K^{path,P}(R) + c \cdot \sum_{k=0}^{|N|n-1} [P_j(k+1) - P_j(k)] \cdot e_j \tag{D.6}$$

$$= K^{path,P}(R) + c \cdot [P_j(|N|n) - P_j(0)] \cdot e_j$$

$$= K^{path,P}(R) + c \cdot e_j, \tag{D.7}$$

where $P_j(k)$ is the j-th element of $P(k)$. Here, (D.3) follows from (21), (D.4) follows from (D.2), (D.5) follows from (21), (D.6) follows from $P_j(k+1) - P_j(k) = 0$ if $i(P,k) \neq j$ (see (20)) and (D.7) follows from Definition 3.5(i). This concludes the proof of *Translation Invariance*.

The proof of *Scale Invariance* is similar to the proof of *Translation Invariance*.

Next, we show *Monotonicity*. Let the risk measure ρ be non-decreasing in the sense that $\rho\left(\sum_{i \in N} \lambda_i X_i\right) \leq \rho\left(\sum_{i \in N} \lambda_i^* X_i\right)$ whenever $\lambda, \lambda^* \in [0,1]^N$ and $\lambda \leq \lambda^*$. Combined with (8) and (20), this implies that $r(P(k+1)) - r(P(k)) \geq 0$ for all $k \in \{0, \dots, |N|n-1\}$. It now follows immediately from (21) that $K^{path,P}(R) \geq 0$. This concludes the proof of *Monotonicity*. □

References

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