Supplementary Material

APPENDIX I: NON-EXTREMAL THERMODYNAMIC POTENTIALS

Eqs. (10), (12) can be obtained by extremising an appropriate effective potential, which can be derived from thermodynamic considerations (see e.g. [26, 27]). Since the local temperature of the NS5 brane does not vanish at extremality, it is appropriate to choose an effective potential that holds some other quantity fixed, e.g. the global entropy $S = \int_{\mathcal{B}_5} \sqrt{-\gamma} s/\mathbf{k} = 8\pi^2 (g_s M b_0^2)^{5/2} \mathcal{C} r_0^3 \cosh\alpha \sin^2\psi$, where $\mathcal{B}_5$ is the spatial part of the worldvolume $\mathcal{M}_6$, s is the local entropy density defined above and $\mathbf{k}$ the Killing vector in the direction of the velocity vector u. Under the assumptions that the $\overline{\text{D3}}$ is trivially embedded within the NS5, that the $\overline{\text{D3}}$ directions are aligned with spatial background isometries, and of constant dilaton and vanishing 5-form field strength, the potential at fixed global entropy and charges Q_3, Q_5 is

$$V_S[\psi] = -\int_{\mathcal{M}_6} d^6\sigma \sqrt{-\gamma}\varepsilon + Q_5 \int_{\mathcal{M}_6} \mathbb{P}[B_6] \ , \qquad (13)$$

with $\varepsilon = \frac{3}{2}\mathcal{C} r_0^2 + |Q_5 \sqrt{1+\tan^2\theta}| \tanh\alpha$ the local energy density (see (3)) and $\mathbb{P}[B_6]$ the pullback of the background B_6-field [28]. $V_S[\psi]$ is the total energy in the system and can be obtained from a Legendre transform of the Euclidean onshell action of the $\overline{\text{D3}}$-NS5 bound state.

The potential (13) when Wick rotating along the time direction has the interpretation of an equilibrium partition function for a higher-form fluid. Rewriting the charge Q_3 as

$$Q_3 = Q_5 \int_{S^2} \left(\sqrt{\gamma_\perp} \tan\theta + \mathbb{P}^\perp[C_2] \right) \ , \qquad (14)$$

where $\gamma^\perp_{ab}$ is the metric on S^2 and $\mathbb{P}^\perp[C_2]$ the pullback of C_2 onto the S^2, direct variation with respect to the sources γ_{ab}, C_3 and C_6 yields the stress tensor (3) and the currents J_2 and $\mathfrak{j}_6$ in (4), respectively [30]. The global temperature of the system T_{H} and chemical potentials $\Phi_{\text{H}}^{(3)}$ and $\Phi_{\text{H}}^{(5)}$ can be obtained from (13) according to

$$T_{\text{H}} = -\frac{1}{\sqrt{-\gamma}} \frac{\delta V_S}{\delta S} = \mathcal{T}\mathbf{k} \ , \ \Phi_{\text{H}}^{(3)} = \frac{1}{\sqrt{-\gamma}} \frac{\delta V_S}{\delta Q_3} \ , \quad \Phi_{\text{H}}^{(5)} = \frac{1}{\sqrt{-\gamma}} \frac{\delta V_S}{\delta Q_5} \ , \qquad (15)$$

and have the expected form that arises from a general analysis of higher-form fluids [27].

APPENDIX II: REGIMES OF VALIDITY

Validity of the blackfold expansion requires a large separation of scales $r_b \ll \mathcal{R}, L$. In the case at hand, the characteristic length scale r_b is the largest scale among the energy density radius ($r_\varepsilon \sim r_0 \sinh\alpha$) and the scales associated to the NS5 and $\overline{\text{D3}}$ charge respectively. The scale $\mathcal{R}$ is controlled by the size of the S^2 that the NS5 wraps, while the background scale L is set by the size of the S^3. We also use the fact that in all configurations of interest the two terms on the RHS of (10) are either comparable or the $\pi p/M$ term dominates. For our purposes it is sufficient to consider $r_h^{(NS5)} \ll \mathcal{R}$ and $r_h^{(\overline{\text{D3}})} \ll \mathcal{R}$ respectively leading to

$$\sqrt{\frac{N_5}{M}} \ll g_s \sin\psi \ , \quad \sqrt{\frac{p}{M}} \ll g_s \sqrt{M} \sin^2\psi . \qquad (16)$$

Both equations fail at the North pole, $\psi = 0$. For sufficiently large M, however, our calculations will be valid everywhere except a small region around the North pole. In turn, the constraint $r_\varepsilon \ll \mathcal{R}$ leads to the requirement $d \ll g_s \sqrt{p} (\sqrt{N_5} \sinh\alpha)^{-1}$.

In addition, since the NS5 brane has a running dilaton one may worry whether regions of spacetime with large values of string coupling e^ϕ invalidate our analysis. We note that the running of the dilaton is capped off at the horizon for non-extremal solutions at the value $e^\phi(r_0) = g_s \sqrt{\sin^2\theta + \cosh^2\alpha \cos^2\theta}$. Hence, by suitably tuning the asymptotic value of g_s we can achieve wide areas in parameter space where our solutions are everywhere weakly coupled. Admittedly, this tuning is not possible for extremal solutions. However, since it is understood how to treat the strong coupling singularity of NS5 branes in flat space, and since the constraint (blackfold) equations can be obtained in a far-zone analysis of the solution, where the string coupling is weak, we anticipate that a large dilaton in the bulk of the solution does not invalidate the conclusions of our analysis even at extremality.