Supplementary appendices to Calvert Jump, Hommes, and Levine (2019): Learning, heterogeneity, and complexity in the New Keynesian model.

D The Full Non-Linear New Keynesian Model

In this appendix we make set out full details of the non-linear New Keynesian model. We proceed from rational expectations to AU learning in stages.

D.1 The Rational Expectations Model

D.1.1 Households

Household j chooses between work and leisure. Let $C_t(j)$ be consumption and $H_t(j)$ be hours worked. The within-period utility function is,

$$U_t(j) = U(C_t(j), H_t(j)) = \log(C_t(j)) - \frac{H_t(j)^{1+\phi}}{1+\phi}, \tag{D.1}$$

where a multiplicative constant which usually defines the units of the sub-utility function of hours worked has been normalised to unity. Given (D.1), the value function of the representative household at time t is,

$$V_t(j) = V_t(B_{t-1}(j)) = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s U(C_{t+s}(j), H_{t+s}(j))\right]. \tag{D.2}$$

The household's problem at time t is to choose paths for consumption $\{C_t(j)\}$, labour supply $\{H_t(j)\}$, and holdings of financial savings to maximize $V_t(j)$ given by (D.2) given its flow budget constraint in period t,

$$B_t(j) = R_t B_{t-1}(j) + W_t H_t(j) + \Gamma_t - C_t(j), \tag{D.3}$$

where $B_t(j)$ is the given net stock of financial assets at the end of period t, W_t is the wage rate and R_t is the ex post real interest rate paid on assets held at the beginning of period t. The ex post real interest rate is given by,

$$R_t = \frac{R_{n,t-1}}{\Pi_t}, \tag{D.4}$$

where $R_{n,t}$ and Π_t are the nominal interest and inflation rates respectively and Γ_t are profits from wholesale and retail firms owned by households. W_t, R_t, and Γ_t are all exogenous to household j. As usual all variables are expressed in real terms relative to the price of final output.

The first order conditions are,

$$U_{C,t}(j) = \beta\mathbb{E}_t\left[R_{t+1}U_{C,t+1}(j)\right], \tag{D.5}$$

$$\frac{U_{L,t}(j)}{U_{C,t}(j)} = W_t. \tag{D.6}$$

An equivalent representation of the Euler consumption equation (D.5), which will be useful when we consider the behaviour of firms, is,

$$1 = \mathbb{E}_t\left[\Lambda_{t,t+1}(j)R_{t+1}\right], \tag{D.7}$$

where $\Lambda_{t,t+1}(j) \equiv \beta\frac{U_{C,t+1}(j)}{U_{C,t}(j)}$ is the real stochastic discount factor for household j, over the interval $[t, t+1]$.

For our choice of utility function, $U_{C,t} = \frac{1}{C_t}$ and $U_{H,t} = -H_t^{\phi}$, so the household's first order conditions become,

$$\frac{1}{C_t(j)} = \beta\mathbb{E}_t\left[\frac{R_{t+1}}{C_{t+1}(j)}\right], \tag{D.8}$$

$$C_t(j)H_t(j)^{\phi} = W_t \Rightarrow H_t = \left(\frac{W_t}{C_t(j)}\right)^{\frac{1}{\phi}}. \tag{D.9}$$

In a symmetric equilibrium of identical households, $C_t(j) = C_t$, aggregate per household consumption, and $H_t(j) = H_t$, average hours worked.

D.1.2 Firms in the Wholesale Sector

Wholesale firms employ a Cobb-Douglas production function to produce a homogeneous output,

$$Y_t^W = F(A_t, H_t) = A_t H_t^{\alpha}, \tag{D.10}$$

where A_t is total factor productivity. Profit-maximizing demand for labour results in the first order condition,

$$W_t = \frac{P_t^W}{P_t}F_{H,t} = \alpha\frac{P_t^W}{P_t}\frac{Y_t^W}{H_t}. \tag{D.11}$$

D.1.3 Firms in the Retail Sector

The retail sector uses a homogeneous wholesale good to produce a basket of differentiated goods for aggregate consumption,

$$C_t = \left(\int_0^1 C_t(m)^{(\zeta-1)/\zeta}dm\right)^{\zeta/(\zeta-1)}, \tag{D.12}$$

where ζ is the elasticity of substitution. For each m, the consumer chooses $C_t(m)$ at a price $P_t(m)$ to maximize (D.12) given total expenditure $\int_0^1 P_t(m)C_t(m)dm$. This results in a set of consumption demand equations for each differentiated good m with price $P_t(m)$ of the form,

$$C_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} C_t \Rightarrow Y_t(m) = \left(\frac{P_t(m)}{P_t}\right)^{-\zeta} Y_t, \tag{D.13}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta}dm\right]^{\frac{1}{1-\zeta}}$. P_t is the aggregate price index. C_t and P_t are Dixit-Stigliz aggregates - see Dixit and Stiglitz (1977).

For each variety m the retail good is produced costlessly from wholesale production according to

$$Y_t(m) = Y_t^W = A_t H_t(m)^\alpha. \tag{D.14}$$

Following Calvo (1983), we now assume that there is a probability of $1-\xi$ at each period that the price of each retail good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is held fixed.[1] For each retail producer m, given its real marginal cost MC_t, the objective is at time t to choose $\{P_t^0(m)\}$ to maximize discounted profits,

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - P_{t+k} MC_{t+k}\right], \tag{D.15}$$

subject to (D.13), where $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{C,t+k}/P_{t+k}}{U_{C,t}/P_t}$ is now the *nominal* stochastic discount factor over the interval $[t, t+k]$. The solution to this is,

$$\mathbb{E}_t \sum_{k=0}^{\infty} \xi^k \Lambda_{t,t+k} Y_{t+k}(m) \left[P_t^0(m) - \frac{1}{(1-1/\zeta)} P_{t+k} MC_{t+k}\right] = 0, \tag{D.16}$$

and by the law of large numbers the evolution of the price index is given by,

$$P_{t+1}^{1-\zeta} = \xi P_t^{1-\zeta} + (1-\xi)(P_{t+1}^0)^{1-\zeta}. \tag{D.17}$$

In order to set up the model in non-linear form as a set of difference equations, we need to represent the price dynamics as difference equations. First define k period ahead inflation as,

$$\Pi_{t,t+k} \equiv \frac{P_{t+k}}{P_t} = \frac{P_{t+1}}{P_t}\frac{P_{t+2}}{P_{t+1}} \cdot \cdot \frac{P_{t+k-1}}{P_{t+k}} = \Pi_{t,t+1}\Pi_{t+1,t+2} \cdot \cdot \Pi_{t+k-1,t+k},$$

noting that $\Pi_{t,t+1} = \Pi_{t+1}$ and $\Pi_{t,t} = 1$.

Next, using (D.13) with $P_{t+k}(m) = P_0(m)$, the price set at time t which survives with probability ξ, we have that,

[1]Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

$$\Lambda_{t,t+k}Y_{t+k}(m) = \beta^k \frac{U_{C,t+k}}{U_{C,t}}\frac{P_t}{P_{t+k}}\left(\frac{P_0(m)}{P_{t+k}}\right)^{-\zeta} Y_{t+k} = \beta^k \frac{U_{C,t+k}}{U_{C,t}}\Pi_{t,t+k}^{\zeta-1}\left(\frac{P_0(m)}{P_t}\right)^{-\zeta} Y_{t+k}.$$

Hence, cancelling out $\left(\frac{P_0(m)}{P_t}\right)^{-\zeta}$ and multiplying by $\frac{U_{C,t}}{P_t}$, we can write (D.16) as,

$$E_t \sum_{k=0}^{\infty}(\xi\beta)^k U_{C,t+k}\Pi_{t,t+k}^{\zeta-1}Y_{t+k}\left[\frac{P_t^0(m)}{P_t} - \Pi_{t+k}MC_{t+k}MS_{t+k}\right] = 0. \tag{D.18}$$

We seek a symmetric equilibrium where firms who are either re-setting their prices or are locked into a contract are identical. In such an equilibrium, the price dynamics can be written as difference equations as follows:

$$\frac{P_t^0}{P_t} = \frac{J_t}{JJ_t}, \tag{D.19}$$

$$JJ_t - \xi\mathbb{E}_t\left[\Pi_{t+1}^{\zeta-1}JJ_{t+1}\Lambda_{t,t+1}\right] = Y_t, \tag{D.20}$$

$$J_t - \xi\mathbb{E}_t\left[\Pi_{t+1}^{\zeta}J_{t+1}\Lambda_{t,t+1}\right] = \left(\frac{1}{1-\frac{1}{\zeta}}\right)Y_t MC_t MS_t, \tag{D.21}$$

$$1 = \xi\Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{1-\zeta}, \tag{D.22}$$

$$\Delta_t = \xi\Pi_t^{\frac{\zeta}{\alpha}}\Delta_{t-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{\frac{\zeta}{\alpha}}, \tag{D.23}$$

(D.24)

$$MC_t = \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}}, \tag{D.25}$$

where (D.34) uses (D.11). Note that we have introduced a mark-up shock MS_t, and that the real marginal cost, MC_t, is variable.

Price dispersion lowers aggregate output as follows. Market clearing in the labour market gives,

$$H_t = \sum_{m=1}^{n} H_t(m) = \sum_{m=1}^{n}\left(\frac{Y_t(m)}{A_t}\right)^{\frac{1}{\alpha}} = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{\alpha}}\sum_{m=1}^{n}\left(\frac{P_t(m)}{P_t}\right)^{-\frac{\zeta}{\alpha}}, \tag{D.26}$$

using (D.13). Hence equilibrium for good m gives,

$$Y_t = \frac{Y_t^W}{\Delta_t^{\alpha}}, \tag{D.27}$$

where price dispersion is defined by,

$$\Delta_t \equiv \left(\sum_{m=1}^{n}\left(\frac{P_t(m)}{P_t}\right)^{-\frac{\zeta}{\alpha}}\right). \tag{D.28}$$

Price dispersion is linked to inflation as follows. Assuming as before that the number of firms is large, we obtain the following dynamic relationship:

$$\Delta_t = \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{-\frac{\zeta}{\alpha}} . \tag{D.29}$$

D.1.4 Profits

To close the model in a manner that will be useful when we come to consider AU learning, we require total profits from retail and wholesale firms, Γ_t, remitted to households. This is given in real terms by,

$$\Gamma_t = \underbrace{Y_t - \frac{P_t^W}{P_t} Y_t^W}_{\text{retail}} + \underbrace{\frac{P_t^W}{P_t} Y_t^W - W_t H_t}_{\text{Wholesale}} = Y_t - \alpha \frac{P_t^W}{P_t} Y_t^W , \tag{D.30}$$

using the first-order condition (D.11).

D.1.5 Closing the Model

The model is closed with a resource constraint,

$$Y_t = C_t , \tag{D.31}$$

and a monetary policy rule for the nominal interest rate given by the following Taylor-type rule,

$$\log \left(\frac{R_{n,t}}{R_n} \right) = \rho_r \log \left(\frac{R_{n,t-1}}{R_n} \right) + (1-\rho_r) \left(\theta_\theta \log \left(\frac{\Pi_t}{\Pi} \right) + \theta_y \log \left(\frac{Y_t}{Y} \right) \right) . \tag{D.32}$$

Finally, there is an exogenous AR1 shock process to marginal cost (e.g. a mark-up shock):

$$\log MS_t - \log MS = \rho_{MS} (\log MS_{t-1} - \log MS) + \epsilon_{MS,t} . \tag{D.33}$$

D.1.6 Summary of Model

Households:

$$\begin{aligned}
U_t &= U(C_t, H_t) = \log C_t - \frac{H_t^{1+\phi}}{1+\phi} \\
U_{C,t} &= \beta \mathbb{E}_t \left[R_{t+1} U_{C,t+1} \right] \\
R_t &= \frac{R_{n,t-1}}{\Pi_t} \\
U_{C,t} &= \frac{1}{C_t} \\
U_{H,t} &= -H_t^\phi \\
\frac{U_{L,t}}{U_{C,t}} &= W_t
\end{aligned}$$

Firms:

$$
\begin{aligned}
Y_t^W &= F(A_t, H_t) = A_t H_t^{\alpha} \\
Y_t &= \frac{Y_t^W}{\Delta_t^{\alpha}} \\
\frac{P_t^W}{P_t} F_{H,t} &= \frac{P_t^W}{P_t} \frac{\alpha Y_t^W}{H_t} = W_t \\
\frac{P_t^0}{P_t} &= \frac{J_t}{JJ_t} \\
JJ_t &= \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta-1} JJ_{t+1} \Lambda_{t,t+1} \right] + Y_t \\
J_t &= \xi \mathbb{E}_t \left[\Pi_{t+1}^{\zeta} J_{t+1} \Lambda_{t,t+1} \right] + \left(\frac{1}{1-\frac{1}{\zeta}} \right) Y_t MC_t MS_t \\
1 &= \xi \Pi_t^{\zeta-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{1-\zeta} \\
\Delta_t &= \xi \Pi_t^{\frac{\zeta}{\alpha}} \Delta_{t-1} + (1-\xi) \left(\frac{J_t}{JJ_t} \right)^{\frac{\zeta}{\alpha}} \\
MC_t &= \frac{P_t^W}{P_t} = \frac{W_t}{F_{H,t}}
\end{aligned}
$$

Closure:

$$
\begin{aligned}
Y_t &= C_t \\
\log\left(\frac{R_{n,t}}{R_n}\right) &= \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\theta \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right)\right) + \epsilon_{M,t} \\
\log MS_t - \log MS &= \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t}
\end{aligned}
$$

D.1.7 Steady State

In recursive form the zero-growth zero-inflation ($\Pi = 1$) steady state of can be written,

$$
\begin{aligned}
R &= \frac{1}{\beta} \\
\Lambda &= \beta \\
\frac{P^W}{P} &= 1 - \frac{1}{\zeta} \\
\frac{C}{Y} &= 1 \\
H &= \alpha^{\frac{1}{1+\phi}} \\
Y^W &= (AH)^{\alpha} \\
Y &= Y^W \\
W &= \alpha \frac{P^W}{P} \frac{Y^W}{H} \\
J &= JJ = \frac{Y}{1-\beta\xi} \\
\Delta &= 1
\end{aligned}
$$

using $P^W Y^W = PY$ by the free entry condition.

For a particular steady state inflation rate $\Pi > 1$ the New Keynesian features of the steady state become,

$$
\begin{aligned}
\frac{J}{JJ} &= \left(\frac{1-\xi\Pi^{\zeta-1}}{1-\xi} \right)^{\frac{1}{1-\zeta}} \\
MC = \frac{P^W}{P} &= \left(1 - \frac{1}{\zeta} \right) \frac{J(1-\beta\xi\Pi^{\zeta})}{JJ(1-\beta\xi\Pi^{\zeta-1})} \\
\Delta &= \frac{(1-\xi)^{\alpha} \left(\frac{J}{JJ} \right)^{-\zeta}}{1-\xi\Pi^{\zeta}}
\end{aligned}
$$

Then $P^W Y^W / PY = MC\Delta \neq 1$.

D.2 Exogenous Point Expectations

As a first step towards AU learning we now formulate the consumption and pricing decision of the household and firms respectively in terms of current and expected future aggregate variables exogenous these agents.

D.2.1 Households

For households, solving (D.3) forward in time and imposing the transversality condition on debt we can write,

$$
B_{t-1}(j) = \mathrm{PV}_t(C_t(j)) - \mathrm{PV}_t(W_t H_t(j)) - \mathrm{PV}_t(\Gamma_t), \tag{D.34}
$$

where the present (expected) value of a series $\{X_{t+i}\}_{i=0}^{\infty}$ at time t is defined by,

$$\mathrm{PV}_t(X_t) \equiv \mathbb{E}_t \sum_{i=0}^{\infty} \frac{X_{t+i}}{R_{t,t+i}} = \frac{X_t}{R_t} + \frac{1}{R_t}\mathrm{PV}_{t+1}(X_{t+1}), \tag{D.35}$$

where $R_{t,t+1} \equiv R_t R_{t+1} R_{t+2} \cdots R_{t+i}$ is the real interest rate over the interval $[t, t+i]$.

The forward-looking budget constraint (D.34) holds for the representative household. In aggregate there is no net debt so $B_{t-1} = 0$. Then in a symmetric equilibrium, substituting for H_t from (D.9) we have,

$$\mathrm{PV}_t(C_t) = \mathrm{PV}_t\left(\frac{W_t^{1+\frac{1}{\phi}}}{C_t^{\frac{1}{\phi}}}\right) + \mathrm{PV}_t(\Gamma_t). \tag{D.36}$$

Solving (D.8) forward in time we have, for $i \geq 1$,

$$\frac{1}{C_t} = \beta^i \mathbb{E}_t\left[\frac{R_{t+1,t+i}}{C_{t+i}}\right]. \tag{D.37}$$

The internally rational solution to the household optimization problem seeks a solution to its decision functions for C_t and H_t that are functions of *non-rational point expectations* $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$, treated as exogenous processes given at time t as opposed to rational model-consistent expectations $\{\mathbb{E}_t W_{t+i}\}_{i=0}^{\infty}$, etc[2]. With point expectations we use (D.37) to obtain,

$$\mathbb{E}_t^* C_{t+i} = C_t \beta^i \mathbb{E}_t^* R_{t+1,t+i} \; ; \; i \geq 1, \tag{D.38}$$

$$\mathbb{E}_t^*(W_{t+i} H_{t+i}) = \frac{(\mathbb{E}_t^* W_{t+i})^{1+\frac{1}{\phi}}}{(\mathbb{E}_t^* C_{t+i})^{\frac{1}{\phi}}}. \tag{D.39}$$

Substituting (D.38) and (D.39) into the forward-looking household budget constraint, and using $\sum_{i=0}^{\infty} \beta^i = \frac{1}{1-\beta}$, we arrive at,

$$\frac{C_t}{R_t(1-\beta)} = \frac{1}{R_t C_t^{\frac{1}{\phi}}}\left(W_t^{1+\frac{1}{\phi}} + \sum_{i=1}^{\infty}(\beta^{\frac{1}{\phi}})^{-i}\left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}}\right)^{1+\frac{1}{\phi}}\right) + \sum_{i=0}^{\infty}\frac{\mathbb{E}_t^* \Gamma_{t+i}}{\mathbb{E}_t^* R_{t,t+i}}, \tag{D.40}$$

$$H_t = \left(\frac{W_t}{C_t}\right)^{\frac{1}{\phi}}. \tag{D.41}$$

(D.40) and (D.41) constitute the consumption and hours decision rules given point expectations of $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$.

[2] With point expectations agents treat $\mathbb{E}_t^*(\cdot)$ as certain, although the environment is stochastic (see Evans and Honkapohja (2001), page 61). Since $\mathbb{E}_t f(X_t) \approx f(\mathbb{E}_t(X_t))$ and $\mathbb{E}_t f(X_t Y_t)) \approx f(\mathbb{E}_t(X_t Y_t))$ up to a first-order Taylor-series expansion, assuming point expectations is equivalent to using a linear approximation of (D.36) and (D.37) as is usually done in the literature.

D.2.2 Retail Firms

Turning next to price-setting by retail firms, write (D.20) and (D.21) as,

$$J_t = \left(\frac{1}{1-\frac{1}{\zeta}}\right) Y_t MC_t MS_t + \mathbb{E}_t \sum_{k=1}^{\infty} \xi^k \Lambda_{t,t+k} \Pi_{t,t+k}^{\zeta} Y_{t+k} MC_{t+k} MS_{t+k}, \tag{D.42}$$

$$JJ_t = Y_t + \mathbb{E}_t \sum_{k=1}^{\infty} \xi^k \Lambda_{t,t+k} \Pi_{t,t+k}^{\zeta-1} Y_{t+k}. \tag{D.43}$$

Assuming point expectations, as for households, we have,

$$\begin{aligned} J_t &= \left(\frac{1}{1-\frac{1}{\zeta}}\right) \left(Y_t MC_t MS_t + \sum_{k=1}^{\infty} \xi^k \mathbb{E}_t^* \Lambda_{t,t+k} (\mathbb{E}_t^* \Pi_{t,t+k})^{\zeta} \mathbb{E}_t^* Y_{t+k} \mathbb{E}_t^* MC_{t+k} \mathbb{E}_t^* MS_{t+k} \right) \\ &= \left(\frac{1}{1-\frac{1}{\zeta}}\right) (Y_t MC_t MS_t + \Omega_{3,t}), \end{aligned} \tag{D.44}$$

$$\begin{aligned} JJ_t &= Y_t + \sum_{k=1}^{\infty} \xi^k \mathbb{E}_t^* \Lambda_{t,t+k} (\mathbb{E}_t^* \Pi_{t,t+k})^{\zeta-1} \mathbb{E}_t^* Y_{t+k} \\ &= Y_t + \Omega_{4,t}, \end{aligned} \tag{D.45}$$

where, noting that $\mathbb{E}_t^* \Lambda_{t,t+1} = \frac{1}{\mathbb{E}_t^* R_{t+1}}$ and $\Pi_{t,t+1} = \Pi_{t+1}$, we have,

$$\Omega_{3,t} = \xi \frac{(\mathbb{E}_t^* \Pi_{t+1})^{\zeta} \mathbb{E}_t^* Y_{t+1} \mathbb{E}_t^* MC_{t+1} \mathbb{E}_t^* MS_{t+1}}{\mathbb{E}_t^* R_{t+1}} + \xi \frac{\mathbb{E}_t^* \Pi_{t+1}^{\zeta}}{\mathbb{E}_t^* R_{t+1}} \Omega_{3,t+1}, \tag{D.46}$$

$$\Omega_{4,t} = \xi (\frac{\mathbb{E}_t^* \Pi_{t+1})^{\zeta-1} \mathbb{E}_t^* Y_{t+1}}{\mathbb{E}_t^* R_{t+1}} + \xi \frac{\mathbb{E}_t^* \Pi_{t+1}^{\zeta-1}}{\mathbb{E}_t^* R_{t+1}} \Omega_{4,t+1}. \tag{D.47}$$

Recalling that the optimal price re-setting decision rule is given by $\frac{P_t^O}{P_t} = \frac{J_t}{JJ_t}$, (D.44) and (D.45) now give us the this rule given exogenous expectations of $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^{\infty}$, and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^{\infty}$.

D.3 AU learning in the NK Model

The final step to complete the AU equilibrium is to choose the learning rule for $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$ for households and $\{\mathbb{E}_t^* \Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* MC_{t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^* MS_{t+i}\}_{i=0}^{\infty}$ for retail firms.

We assume general bounded rational expectations rules so that,

$$\mathbb{E}_t^*[W_{t+i}] = \mathbb{E}_t^*[W_{t+1}] \text{ for } i \geq 1, \tag{D.48}$$

and similarly for $\{\mathbb{E}_t^*\Gamma_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^*\Pi_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^*Y_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^*MC_{t+i}\}_{i=0}^{\infty}$ and $\{\mathbb{E}_t^*MS_{t+i}\}_{i=0}^{\infty}$, whilst,

$$\mathbb{E}_t^* R_{t,t+i} = R_t \frac{R_{n,t}}{\mathbb{E}_t^*\Pi_{t+1}} (\mathbb{E}_t^* R_{t+1})^{i-1}, \tag{D.49}$$

which takes into account the observation of $R_{n,t}$ at time t. One-period ahead forecasts are given in the main body of the text.

With adaptive expectations, (D.40) becomes,

$$\frac{C_t}{R_t(1-\beta)} = \frac{1}{R_t C_t^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \frac{(\mathbb{E}_t^* W_{t+1})^{1+\frac{1}{\phi}}}{\beta^{\frac{1}{\phi}} (\mathbb{E}_t^* R_{t+1})^{1+\frac{1}{\phi}} - 1} \right) + \frac{\mathbb{E}_t^*\Gamma_{t+1}}{\mathbb{E}_t^* R_{t+1} - 1},$$

whilst (D.46) and (D.47) now become,

$$\begin{aligned} \Omega_{3,t} &= \frac{\xi(\mathbb{E}_t^*\Pi_{t+1})^{\zeta}\mathbb{E}_t^*Y_{t+1}\mathbb{E}_t^*MC_{t+1}\mathbb{E}_t^*MS_{t+1}}{\mathbb{E}_t^*R_{t+1} - \xi(\Pi_{t+1})^{\zeta}} \\ \Omega_{4,t} &= \frac{\xi(\mathbb{E}_t^*\Pi_{t+1})^{\zeta-1}\mathbb{E}_t^*Y_{t+1}}{\mathbb{E}_t^*R_{t+1} - \xi(\mathbb{E}_t^*\Pi_{t+1})^{\zeta-1}}. \end{aligned}$$

This completes the internally rational equilibrium with point adaptive expectations.

D.4 Proof of Lemma

In the first order conditions for Calvo contracts and expressions for value functions we are confronted with expected discounted sums of the general form,

$$\Omega_t = \mathbb{E}_t \left[\sum_{k=0}^{\infty} \beta^k X_{t,t+k} Y_{t+k} \right], \tag{D.50}$$

where $X_{t,t+k}$ has the property $X_{t,t+k} = X_{t,t+1}X_{t+1,t+k}$ and $X_{t,t} = 1$ (for example an inflation, interest or discount rate over the interval $[t, t+k]$).

Lemma

Ω_t can be expressed as,

$$\Omega_t = Y_t + \beta \mathbb{E}_t \left[X_{t,t+1}\Omega_{t+1} \right]. \tag{D.51}$$

Proof

$$\begin{aligned}
\Omega_t &= X_{t,t}Y_t + \mathbb{E}_t\left[\sum_{k=1}^{\infty}\beta^k X_{t,t+k}Y_{t+k}\right] \\
&= Y_t + \mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'+1} X_{t,t+k'+1}Y_{t+k'+1}\right] \\
&= Y_t + \beta\mathbb{E}_t\left[\sum_{k'=0}^{\infty}\beta^{k'} X_{t,t+1}X_{t+1,t+k'+1}Y_{t+k'+1}\right] \\
&= Y_t + \beta\mathbb{E}_t\left[X_{t,t+1}\Omega_{t+1}\right]. \qquad \square
\end{aligned}$$

D.5 Proof of Equation D.29

In the next period, ξ of these firms will keep their old prices, and $(1-\xi)$ will change their prices to P^O_{t+1}. By the law of large numbers, we assume that the distribution of prices among those firms that do not change their prices is the same as the overall distribution in period t. It follows that we may write,

$$\begin{aligned}
\Delta_{t+1} &= \xi \sum_{j_{no\ change}} \left(\frac{P_t(j)}{P_{t+1}}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi\left(\frac{P_t}{P_{t+1}}\right)^{-\zeta} \sum_{j_{no\ change}} \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi\left(\frac{P_t}{P_{t+1}}\right)^{-\zeta} \sum_{j} \left(\frac{P_t(j)}{P_t}\right)^{-\zeta} + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta} \\
&= \xi\Pi^{\zeta}_{t+1}\Delta_t + (1-\xi)\left(\frac{J_{t+1}}{JJ_{t+1}}\right)^{-\zeta}. \qquad \square
\end{aligned}$$

E Linearization of Appendix D Model about the Deterministic Steady State

E.1 Households

The Euler equation and choice of hours supplied,

$$\begin{aligned} U_{C,t} &= \beta \mathbb{E}_t^* \left[R_{t+1} U_{C,t+1} \right], && \text{(E.1)} \\ -\frac{U_{H,t}}{U_{C,t}} &= W_t, && \text{(E.2)} \end{aligned}$$

which with choice of utility function,

$$U_t = U(C_t, H_t^s) = \log(C_t) - \frac{(H_t^s)^{1+\phi}}{1+\phi}, \tag{E.3}$$

gives,

$$\begin{aligned} \frac{1}{C_t} &= \beta \mathbb{E}_t^* \left[\frac{R_{t+1}}{C_{t+1}} \right], && \text{(E.4)} \\ H_t^s &= \left(\frac{W_t}{C_t} \right)^{\frac{1}{\phi}}. && \text{(E.5)} \end{aligned}$$

Let $c_t \equiv \log(C_t/C)$ and $r_t \equiv \log(R_t/R)$. Then the log-linearization of (E.4) (E.5) and the Fischer equation gives,

$$\begin{aligned} c_t &= \mathbb{E}_t^* \left[c_{t+1} - r_{t+1} \right], \\ h_t^s &= \frac{1}{\phi}(w_t - c_t), \\ r_t &= r_{n,t-1} - \pi_t. \end{aligned}$$

The forward-looking consumption equation under perfect foresight (or assuming point expectations) is,

$$\begin{aligned} \frac{C_t}{R_t(1-\beta)} &= \frac{1}{R_t C_t^{\frac{1}{\phi}}} \left(W_t^{1+\frac{1}{\phi}} + \Omega_{1,t} \right) + \Omega_{2,t}, && \text{(E.6)} \\ \Omega_{1,t} &\equiv \sum_{i=1}^{\infty} (\beta^{\frac{1}{\phi}})^{-i} \left(\frac{\mathbb{E}_t^* W_{t+i}}{\mathbb{E}_t^* R_{t+1,t+i}} \right)^{1+\frac{1}{\phi}}, \\ \Omega_{2,t} &\equiv \sum_{i=0}^{\infty} \frac{\mathbb{E}_t^* \Gamma_{t+i}}{\mathbb{E}_t^* R_{t,t+i}}, \\ H_t &= \left(\frac{W_t}{C_t} \right)^{\frac{1}{\phi}}. && \text{(E.7)} \end{aligned}$$

Hence,

$$\Omega_{1,t} = \frac{1}{\beta^{\frac{1}{\phi}}} \left(\left(\frac{\mathbb{E}_t^* W_{t+1}}{\mathbb{E}_t^* R_{t+1}} \right)^{1+\frac{1}{\phi}} + \frac{\mathbb{E}_t^* \Omega_{1,t+1}}{\mathbb{E}_t^* R_{t+1}^{1+\frac{1}{\phi}}} \right), \tag{E.8}$$

$$\Omega_{2,t} = \frac{1}{R_t} \left(\Gamma_t + \mathbb{E}_t^* \Omega_{2,t+1} \right). \tag{E.9}$$

(E.6) and (E.7) constitute the consumption and hours decision rules given expectations $\{\mathbb{E}_t^* W_{t+i}\}_{i=0}^{\infty}$, $\{\mathbb{E}_t^* R_{t,t+i}\}_{i=0}^{\infty}$, and $\{\mathbb{E}_t^* \Gamma_{t+i}\}_{i=0}^{\infty}$.

Let $c_t \equiv \log(C_t/C)$, $w_t \equiv \log(W_t/W)$, $r_t \equiv \log(R_t/R)$, $\gamma_t \equiv \log(\Gamma_t/\Gamma)$, $h_t \equiv \log(H_t/H)$, $\omega_{1,t} \equiv \log(\Omega_{1,t}/\Omega_1)$, and $\omega_{2,t} \equiv \log(\Omega_{2,t}/\Omega_2)$. Then the log-linearization of (E.6) and (E.7) gives,

$$\alpha_1 c_t = \alpha_2 w_t + \alpha_3(\omega_{2,t} + r_t) + \alpha_4 \omega_{1,t}, \tag{E.10}$$

$$\omega_{1,t} = \alpha_5 \mathbb{E}_t^* w_{t+1} - \alpha_6 \mathbb{E}_t^* r_{t+1} + \beta \mathbb{E}_t^* \omega_{1,t+1}, \tag{E.11}$$

$$\omega_{2,t} = (1-\beta)(\gamma_t - r_t) - \beta r_t + \beta \mathbb{E}_t^* \omega_{2,t+1}, \tag{E.12}$$

$$\gamma_t = \frac{c_y}{\gamma_y} c_t - \frac{\alpha}{\gamma_y}(w_t + h_t), \tag{E.13}$$

where the (positive) coefficients are given by,

$$\begin{aligned}
\alpha_1 &\equiv 1 + \frac{\alpha}{\phi c_y}, \\
\alpha_2 &\equiv (1-\beta)\left(1 + \frac{1}{\phi}\right)\frac{\alpha}{c_y}, \\
\alpha_3 &\equiv \frac{\gamma_y}{c_y}, \\
\alpha_4 &\equiv \frac{\beta\alpha}{c_y}, \\
\alpha_5 &\equiv (1-\beta)\left(1 + \frac{1}{\phi}\right), \\
\alpha_6 &\equiv 1 + \frac{1}{\phi}
\end{aligned}$$

where $c_y = 1$ and $\gamma_y = 1 - \alpha$

E.2 Firms

The non-linear price dynamics are given by,

$$JJ_t - \xi\beta\mathbb{E}_t[\Pi_{t+1}^{\zeta-1} JJ_{t+1}] = Y_t U_{C,t}, \tag{E.14}$$

$$J_t - \xi\beta\mathbb{E}_t[\Pi_{t+1}^{\zeta} J_{t+1}] = \left(\frac{1}{1-\frac{1}{\zeta}}\right) Y_t U_{C,t}(MC_t + MS_t), \tag{E.15}$$

$$1 = \xi\Pi_t^{\zeta-1} + (1-\xi)\left(\frac{J_t}{JJ_t}\right)^{1-\zeta}. \tag{E.16}$$

The zero growth and positive inflation rate steady state, Π, for the NK features are,

$$
\begin{aligned}
J(1-\beta\xi\Pi^{\zeta}) &= \frac{YU_C MC}{\left(1-\frac{1}{\zeta}\right)}, && \text{(E.17)}\\
JJ(1-\beta\xi\Pi^{\zeta-1}) &= YU_C, && \text{(E.18)}\\
\frac{J}{JJ} &= \left(\frac{1-\xi\Pi^{\zeta-1}}{1-\xi}\right)^{\frac{1}{1-\zeta}}, && \text{(E.19)}\\
MC &= \left(1-\frac{1}{\zeta}\right)\frac{J(1-\beta\xi\Pi^{\zeta})}{JJ(1-\beta\xi\Pi^{\zeta-1})}, && \text{(E.20)}\\
\Delta &= \frac{(1-\xi)^{\frac{1}{1-\zeta}}(1-\xi\Pi^{\zeta-1})^{\frac{-\zeta}{1-\zeta}}}{1-\xi\Pi^{\zeta}}. && \text{(E.21)}
\end{aligned}
$$

For a zero-inflation steady state $\Pi = 1$, we arrive $\frac{J}{JJ} = \Delta = 1$ and $MC = \left(1-\frac{1}{\zeta}\right)$, but in general there is a long-run inflation-output trade-off in the choice of the steady-state inflation rate. The implications of introducing a non-zero inflation steady state into the standard New Keynesian model are explored by Ascari and Ropele (2007).

Expanding (E.14) as a Taylor series yields,

$$
\begin{aligned}
& JJ + JJ_t - JJ - \xi\beta\mathbb{E}_t[\Pi^{\zeta-1}JJ + (\zeta-1)\Pi^{\zeta-2}JJ(\Pi_{t+1}-\Pi) \\
+ \quad & \Pi^{\zeta-1}(JJ_{t+1}-JJ)] = YU_C + U_C(Y_t - Y) + Y(U_{C,t}-U_C). \qquad \text{(E.22)}
\end{aligned}
$$

Cancelling out the constants on both sides, putting $\Pi = 1$ and dividing by JJ, we have,

$$
jj_t \equiv \frac{JJ_t - JJ}{JJ} = \xi\beta E_t\left[(\zeta-1)\pi_{t+1} + jj_{t+1}\right] + \frac{YU_C}{JJ}(y_t + u_{C,t}). \qquad \text{(E.23)}
$$

Similarly linearizing (E.15), we arrive at,

$$
j_t \equiv \frac{J_t - J}{J} = \xi\beta E_t\left[\zeta\pi_{t+1} + j_{t+1}\right] + \frac{YU_C MC}{J\left(1-\frac{1}{\zeta}\right)}(y_t + u_{C,t} + mc_t + ms_t). \qquad \text{(E.24)}
$$

Next we linearize (E.16) and put $\Pi = 1$ to obtain,

$$
\xi(\zeta-1)\pi_t + (1-\xi)(1-\zeta)\left(\frac{J}{JJ}\right)^{-\zeta}(j_t - jj_t) = 0. \qquad \text{(E.25)}
$$

Using the steady state relationships with $\Pi = 1$, we have that $\frac{YU_C}{JJ} = \frac{YU_C MC}{J} = 1-\beta\xi$, and (E.24) and (E.25) give,

$$
\begin{aligned}
jj_t &= \xi\beta E_{t+1}\left[(\zeta-1)\pi_{t+1} + jj_{t+1}\right] + (1-\beta\xi)(y_t + u_{C,t}), && \text{(E.26)}\\
j_t &= \xi\beta E_{t+1}\left[\zeta\pi_{t+1} + j_{t+1}\right] + y_t + (1-\beta\xi)(u_{C,t} + mc_t + ms_t), && \text{(E.27)}\\
\xi\pi_t &= (1-\xi)(j_t - jj_t). && \text{(E.28)}
\end{aligned}
$$

Finally, subtracting (E.27) from (E.26), and using (E.28), we arrive at the *linear NK Phillips Curve*,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} mc_t. \tag{E.29}$$

This can be solved forward in time to give,

$$\pi_t = \frac{(1-\xi)(1-\beta\xi)}{\xi} \sum_{i=0}^{\infty} \beta^i mc_{t+i}, \tag{E.30}$$

telling us that in the NK model in proportional deviation terms about the steady state, inflation is proportional to the discounted sum of expected future deviations of marginal costs.

The rest of the supply sides consists of a first-order demand for hours and a Cobb-Douglas production function:

$$\begin{aligned} W_t &= \alpha \frac{P_t^W}{P_t} \frac{Y_t^W}{H_t^d}, \\ Y_t &= Y_t^W = A_t (H_t^d)^\alpha, \\ MC &\equiv \frac{P_t^W}{P_t}, \end{aligned}$$

from which we arrive at the log-linearization,

$$y_t = a_t + \alpha h_t^d mc_t = w_t - y_t + h_t^d. \tag{E.31}$$

E.3 Monetary Rule, equilibrium, and shock process

We consider a monetary policy rules for the nominal interest rate given by the following Taylor-type rule:

$$\log\left(\frac{R_{n,t}}{R_n}\right) = \rho_r \log\left(\frac{R_{n,t-1}}{R_n}\right) + (1-\rho_r)\left(\theta_\theta \log\left(\frac{\Pi_t}{\Pi}\right) + \theta_y \log\left(\frac{Y_t}{Y}\right)\right), \tag{E.32}$$

The log-linear form is,

$$r_{n,t} = \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_y y_t). \tag{E.33}$$

Equilibria in the output and labour markets are given by,

$$Y_t = C_t, \tag{E.34}$$

$$H_t^s = H_t^d = H_t, \tag{E.35}$$

which have the log-linear forms,

$$\begin{aligned} y_t &= c_t, && \text{(E.36)} \\ h_t^s &= h_t^d = h_t. && \text{(E.37)} \end{aligned}$$

Finally, the AR1 shock process us already in log-linear form if $ms_t \equiv \log MS_t - \log MS = \log MS_t/MS$:

$$\log MS_t - \log MS = \rho_{MS}(\log MS_{t-1} - \log MS) + \epsilon_{MS,t}. \tag{E.38}$$

E.4 Summary of Linearized RE-AU Model

E.4.1 RE Model

To summarise, the linearised rational expectations model is given by,

$$\begin{aligned} c_t &= \mathbb{E}_t\left[c_{t+1} - r_{t+1}\right] \\ \textbf{or } \alpha_1 c_t &= \alpha_2 w_t + \alpha_3(\omega_{1,t} + r_t) + \alpha_4 \omega_{2,t} \\ \omega_{1,t} &= \alpha_5 \mathbb{E}_t w_{t+1} - \alpha_6 \mathbb{E}_t r_{t+1} + \beta \mathbb{E}_t \omega_{1,t+1} \\ \omega_{2,t} &= (1-\beta)(\gamma_t - r_t) - \beta r_t + \beta \mathbb{E}_t \omega_{2,t+1} \\ \gamma_t &= \frac{c_y}{\gamma_y} c_t - \frac{\alpha}{\gamma_y}(w_t + h_t^s) \\ h_t^s &= \frac{1}{\phi}(w_t - c_t) \end{aligned}$$

$$\begin{aligned} r_t &= r_{n,t-1} - \pi_t \\ \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}(mc_t + ms_t) \\ y_t &= a_t + \alpha h_t^d \\ mc_t &= w_t - y_t + h_t^d \\ y_t &= c_t \\ h_t^s &= h_t^d \end{aligned}$$

plus a policy rule,

$$r_{n,t} = \rho_r r_{n,t-1} + (1-\rho_r)(\theta_\pi \pi_t + \theta_y y_t) + mps_t \tag{E.39}$$

giving 9 (or 11) equations in c_t (or c_t, $\omega_{1,t}$, $\omega_{2,t}$), y_t, h_t^s, h_t^d, w_t, r_t, $r_{n,t}$, π_t and mc_t given the AR1 exogenous process for ms_t.

E.4.2 AU Model

The linearised model with AU learning is given by,

$$
\begin{aligned}
\alpha_1 c_t &= \alpha_2 w_t + \alpha_3(\omega_{1,t} + r_t) + \alpha_4 \omega_{2,t} \\
\gamma_t &= \frac{c_y}{\gamma_y} c_t - \frac{\alpha}{\gamma_y}(w_t + h_t^s) \\
\omega_{1,t} &= \frac{1}{1-\beta}\left[\alpha_5 \mathbb{E}_t^* w_{t+1} + \alpha_6 \mathbb{E}_{h,t}^* \pi_{t+1}\right] - \alpha_6 (r_{n,t} + \frac{\beta}{1-\beta}\mathbb{E}_t^* r_{n,t+1}) \\
\omega_{2,t} &= (1-\beta)\gamma_t + \beta \mathbb{E}_t^* \gamma_{t+1} - (r_{n,t-1} + \beta r_{n,t} + \frac{\beta^2}{1-\beta}\mathbb{E}_t^* r_{n,t+1}) + \pi_t + \frac{\beta}{1-\beta}\mathbb{E}_{h,t}^* \pi_{t+1} \\
h_t^s &= \frac{1}{\phi}(w_t - c_t)
\end{aligned}
$$

$$
\begin{aligned}
\pi_t &= \frac{(1-\xi)}{\xi}(p_t^o - p_t) \\
&= \frac{(1-\xi)}{\xi}\left(\frac{1}{1-\beta\xi}\mathbb{E}_{f,t}^* \pi_{t+1} + (1-\beta\xi)\left((mc_t + ms_t) + \frac{\beta\xi}{1-\beta\xi}\mathbb{E}_t^*(mc_{t+1} + ms_{t+1})\right)\right)
\end{aligned}
$$

with point expectations given in the main body of the text.

E.4.3 Composite RE-AU Model

Finally, the composite RE-AU model is as above, but with the following aggregation conditions included,

$$
\begin{aligned}
h_t^d &= n_t \left(h_t^s\right)^{RE} + (1-n_t)\left(h_t^s\right)^{AU} = h_t \\
y_t &= a_t + \alpha h_t \\
c_t &= n_t \left(c_t\right)^{RE} + (1-n_t)\left(c_t\right)^{AU} = y_t \\
p_t^o &= n_t \left(p_t^o\right)^{RE} + (1-n_t)\left(p_t^o\right)^{AU} \\
\pi_t &= \frac{(1-\xi)}{\xi}(p_t^o - p_t)
\end{aligned}
$$

where n_t is the proportion of agents with rational expectations at time t, and with an appropriate rule for the distribution of profits.

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