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Signed Meadow valued Probability Mass Functions

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Abstract

The Kolmogorov axioms for probability mass functions are phrased in the context of signed meadows.

Keywords and phrases: Boolean algebra, meadow, Bayes’ theorem, Bayesian reasoning.

1 Introduction

We will provide an axiomatization of a probability mass function (pmf) on a Boolean algebra, serving as the event space, producing elements of a signed meadow as “probabilities”. Reasons for writing this paper include these:

1. Developing an equational approach to the axiomatization of probability, which allows making a clear distinction between assumptions and consequences.

2. Identification of the (ground form of) the Bayesian approach to reasoning in case of uncertainty as a combination of the following phases:
   (a) Expressing one’s expert (or non-expert) opinion in terms of the specification of a probability mass function, the so-called prior pmf.
   (b) Using model theory or proof theory to establish the consistency of the prior pmf specification with the axioms for a probability mass function on a Boolean algebra.
   (c) Identification of events that may play the role of evidence and of pairs of events that may be understood as a pair of competing hypotheses that may each serve as an explanation of that evidence.
   (d) Application of classical (equational) logic to the prior pmf specification in combination with the axioms for a probability mass function for deriving various conditional probabilities.
   (e) Updating the prior pmf to a posterior pmf, depending on a hypothesis, as well as deriving various likelihood ratios.
   (f) Non-classical reasoning steps such as abduction and inference to the best explanation, in order to infer explanations of observed conditions.
The special focus of this paper concerns step \(2d\), and to a lesser extent step \(2b\).

3. Finding an attractive and productive application for the theory of signed meadows as outlined in [2].

We will produce 27 equational axioms covering Boolean algebra, meadows, the sign function, and the pmf. Then we will introduce several derived operators and prove a number of simple facts, including Bayes’ theorem.

These axioms constitute a finite equational basis for the class of Boolean algebra based, cancellation meadow valued pmfs. In other words the proof theoretic results of [2] extend to the case with Boolean algebra based pmfs. We understand this result to express that the set of 27 axioms is complete in a reasonable sense.

## 2 Boolean algebras and meadows

In this section we specify the mathematical context on which our axiomatization is based. In particular, we provide specifications for Boolean algebras and (signed) meadows.

### 2.1 Boolean algebras

A Boolean algebra \((B, +, -, 0, 1)\) may be defined as a system with at least two elements such that \(\forall x, y, z \in B\) the well-know postulates of Boolean algebra are valid. Because we want to avoid overlap with the operations of a meadow, we will consider Boolean algebras with notation from propositional logic, thus consider \((B, \lor, \land, \neg, \top, \bot)\) and adopt the axioms in Table 1. In [1] it was shown that the axioms in Table 1 constitute an equational basis.

### 2.2 Valuated Boolean algebras

A Boolean algebra can be equipped with a valuation \(v\) that assigns to its elements stalks in a signed meadow. By way of notational convention we will assume that \(E\) is the name of the carrier of a boolean algebra, and that \(S\) names the carrier of the meadow in a valuated Boolean algebra.
\begin{align*}
\quad (x + y) + z &= x + (y + z) \\
\quad x + y &= y + x \\
\quad x + 0 &= x \\
\quad x + (-x) &= 0 \\
\quad (x \cdot y) \cdot z &= x \cdot (y \cdot z) \\
\quad x \cdot y &= y \cdot x \\
\quad 1 \cdot x &= x \\
\quad x \cdot (y + z) &= x \cdot y + x \cdot z \\
\quad (x^{-1})^{-1} &= x \\
\quad x \cdot (x \cdot x^{-1}) &= x
\end{align*}

Table 2: Md: axioms for meadows

In this paper we will investigate the special case where the valuation function of a valuated Boolean algebra is a probability mass function by requiring that the valuation satisfies the Kolmogorov axioms for probability mass functions cast to the setting of signed meadows.

### 2.3 Events and cancellation meadows

In the setting of probability mass functions the elements of the underlying Boolean algebra are referred to as events.\(^1\) We will use “stalk” to refer to an element of a meadow\(^2\) and a probability mass function (pmf) is a valuation (from events to the stalks of a signed meadow)\(^3\).

An expression of type E is an event expression or an event term, an expression of type S is a stalk expression or equivalently a stalk term. In the signature of a valuated Boolean algebra there is just one notation for a pmf, the function symbol \(P\)\(^4\). We shall write

\[1_x \text{ for } x \cdot x^{-1} \quad \text{and} \quad 0_x \text{ for } 1 - 1_x.\]

The set of axioms in Table 2 specifies the class of meadows and the axioms in Table 3 specify the sign function. Together these tables contain the axioms for signed meadows. We will consider the subclass of signed cancellation meadows. A cancellation meadow satisfies the Inverse Law (IL) of Table 4.

---

\(^1\)Events are closed under \(- \lor -\) which represents alternative occurrence and \(- \land -\) which represents simultaneous occurrence.

\(^2\)Rational numbers and real numbers are instances of stalks.

\(^3\)We will exclude PMFs with negative values, leaving the exploration of that generalization to future work.

\(^4\)In some cases the restriction to a single pmf \(P\) is impractical and providing a dedicated sort for pmfs brings higher flexibility. This expansion may be achieved in different ways. In Appendix A we will specify the expansion of a Boolean algebra with a pmf-space.
\[
\begin{align*}
    s(1_x) &= 1_x 
    & \quad (17) \\
    s(0_x) &= 0_x 
    & \quad (18) \\
    s(-1) &= -1 
    & \quad (19) \\
    s(x^{-1}) &= s(x) 
    & \quad (20) \\
    s(x \cdot y) &= s(x) \cdot s(y) 
    & \quad (21) \\
    0_{s(x) - s(y)} \cdot (s(x + y) - s(x)) &= 0 
    & \quad (22)
\end{align*}
\]

Table 3: Sign: axioms for the sign operator

\[
x \neq 0 \implies x \cdot x^{-1} = 1.
\]

Table 4: Inverse Law (IL)

\section{Axioms for a signed meadow valued probability mass function (APMF)}

In this section we formulate axioms for a probability mass function. Following the methods of abstract data type specification we will focus on axioms in equational form. Then, we briefly discuss some derived operators and some properties thereof.

\subsection{Axioms of APMF}

Table 5 provides axioms for a pmf. In Table 5 we use inversive notation.\footnote{The term inversive notation was coined in \cite{3}. It stands in contrast with divisive notation that makes use of a two place division operator symbol.} Together with the axioms for signed meadows and for Boolean algebras we find the following set of axioms

\[\text{APMF} = \text{BA} + \text{Md} + \text{Sign} + \text{Pmf}\] \footnote{Notice that IL is not contained in APMF.}

A valuated Boolean algebra that satisfies the axioms of APMF will be called a pmf-structure. If its meadow is non-trivial it is a non-trivial pmf-structure and if its meadow is a cancellation meadow the pmf-structure is a cancellation pmf-structure.

These axioms capture Kolmogorov’s axioms in a setting of Boolean algebra (instead of set theory) and meadows (instead of fields). Axiom \ref{ax25} expresses that \(P(x) \geq 0\), axiom \ref{ax26} distributes \(P\) over finite unions. In the absence of an infinitary version of axiom \ref{ax26} we consider these axioms to constitute an axiomatization for the restricted concept of probability mass functions only, rather than for probability measures in general.
\[ P(\top) = 1 \]  \hspace{1cm} (23)
\[ P(\bot) = 0 \]  \hspace{1cm} (24)
\[ s(s(P(x)) + 1) = 1 \]  \hspace{1cm} (25)
\[ P(x \lor y) = P(x) + P(y) - P(x \land y) \]  \hspace{1cm} (26)
\[ P(x \land y) \cdot P(y) \cdot P(y)^{-1} = P(x \land y) \]  \hspace{1cm} (27)

Table 5: Pmf: axioms for a probability mass function

3.2 Derived operators and some derived facts

Variables \( p \) and \( q \) range over \( S \), while \( x \) and \( y \) range over \( E \). Two notations for a division operator (divisive notation) are added to the syntax with defining equations as below. Further an alternative notation for joint probability and a notation for conditional probability are defined. Bayes’ rule (Theorem 1 below, item 3) appears as a theorem of APMF in this setting.

Definition 1 (Ordering, division, joint probability and conditional probability).

1. \( x < y \equiv_{\text{def}} s(y - x) = 1 \),
2. \( x \leq y \equiv_{\text{def}} s(s(y - x) + 1) = 1 \),
3. \( \frac{p}{q} \equiv_{\text{def}} p \cdot q^{-1} \),
4. \( p/q \equiv_{\text{def}} \frac{p}{q} \),
5. \( P(x, y) \equiv_{\text{def}} P(x \land y) \),
6. \( P(x \mid y) \equiv_{\text{def}} \frac{P(x, y)}{P(y)} \).

The following properties are immediate:

- \( P(x) \geq 0 \),
- \( P(x) = P(x) \cdot P(x \mid x) \),
- \( P(x \mid x) = \frac{P(x)}{P(x)} \).

Theorem 1. Key properties of a pmf:

1. \( P(x) \leq 1 \) \hspace{0.5cm} (probability upper bound),
2. \( P(x, y) = P(x \mid y) \cdot P(y) \) \hspace{0.5cm} (joint probability factorization),
3. $P(x \mid y) = \frac{P(y \mid x) \cdot P(x)}{P(y)}$ (Bayes’ rule).

Proof. (1): First notice $1 = P(\top) = P(x \lor \neg x) = (P(x) + P(\neg x)) - P(\bot) = P(x) + P(\neg x)$, so that $P(x) = 1 - P(\neg x)$. With Definition 1 (clause 2) we derive $s(s(1 - P(x)) + 1) = s(s(P(\neg x) + 1) = 1 and conclude $P(x) \leq 1$.

(2): $P(x, y) = P(x \land y) = P(x \land y) \cdot P(y) \cdot P(y)^{-1} = P(x, y) \cdot P(y) \cdot P(y)^{-1} = (P(x, y)/P(y)) \cdot P(y) = P(x \mid y) \cdot P(y)$.

(3): $P(x \mid y) = P(x, y)/P(y) = P(x \land y)/P(y) = P(y \land x)/P(y) = P(y \land x) \cdot (P(x)/P(x))/P(y) = P(y, x)/P(x) \cdot P(x)/P(y) = P(y \mid x) \cdot P(x)/P(y)$.

### 3.3 Independence of events

Given a pmf-structure two events $x$ and $y$ are said to be independent relative to that structure if $P(x \land y) = P(x) \cdot P(y)$ is valid.

**Theorem 2.** Events $x$ and $y$ are independent if and only if $P(x \mid y) = P(x) \cdot P(y \mid y)$ and equivalently if and only if $P(y \mid x) = P(y) \cdot P(x \mid x)$.

**Proof.** Assume that $x$ and $y$ are independent, now $P(x \mid y) = P(x \land y)/P(y) = P(x) \cdot P(y)/P(y) = P(x) \cdot P(y \mid y)$, and similarly one finds $P(y \mid x) = P(y) \cdot P(x \mid x)$.

Conversely: from $P(x \mid y) = P(x) \cdot P(y \mid y)$ one finds $P(x \land y)/P(y) = P(x) \cdot P(y)/P(y)$ and after multiplying both sides by $P(y)$ one obtains $P(x \land y)/P(y) \cdot P(y) = P(x) \cdot P(y)/P(y) \cdot P(y)$ which implies $P(x \land y) = P(x) \cdot P(y)$.

### 3.4 Set notation for events

We assume that a set notation based event space $E$ exists as well, with variables $u, v$, and with a universum $\Omega$ (the sample space), which is cast into Boolean events by an embedding operator $e$. In practice $e$ will often be left implicit.

1. $e(\emptyset) = \bot$,
2. $e(\Omega) = \top$,
3. $e(u \cup v) = e(u) \lor e(v)$,
4. $e(u \cap v) = e(u) \land e(v)$

### 4 Logical aspects of APMF

We provide a completeness proof for the axiom system APMF. We comment on options for stronger forms of completeness and we provide a notational framework for dealing with the notion of a random variable.
4.1 Completeness of APMF

In [2] it is shown that Md+Sign constitutes a finite basis for the equational theory of cancellation meadows. That fact is understood as a completeness result because a stronger set of axioms would necessarily exclude some meadows that are expansions of ordered fields.

The basis theorem for signed meadows can be extended to the setting of pmf-structures thus obtaining a satisfactory completeness result for APMF.

**Theorem 3.** APMF is complete for the equational theory of the class of cancellation pmf-structures.

**Proof.** Assume for some equation \( t = r \), APMF \( \not\vdash t = r \). By contraposition we may assume that \( t = r \) is refuted in some model \( M \) of APMF. It suffices to show that \( t = r \) is also refuted in some (non-trivial) cancellation pmf-structure. Let \( x_1, \ldots, x_k \) be the set of variables ranging over \( E \) that occur in the equation and assume that \( y_1, \ldots, y_l \) is the collection of meadow variables in \( t = r \). Let \( \sigma \) be a valuation of these variables that refutes \( t = r \) in pmf-structure \( K \). \( K \) is a non-trivial pmf-structure because otherwise \( K \models t = r \).

Let \( e_1, \ldots, e_k \) constitute a sequence of events in \( E_K \) and let \( s_1, \ldots, s_l \) be a sequence of stalks in \( S_K \) such that \( K \not\models t(e_1, \ldots, e_k, s_1, \ldots, s_l) = r(e_1, \ldots, e_k, s_1, \ldots, s_l) \). With \( E_K(e_1, \ldots, e_k) \) we will denote the subalgebra of the Boolean algebra of \( K \) generated by \( e_1, \ldots, e_k \), and with \( K_S(e_1, \ldots, e_k) \) we will denote the subalgebra of \( K \) generated by \( e_1, \ldots, e_k \) and all stalks of \( S_K \).

Let \( m \) be the number of occurrences of \( P \) in the equation \( t = r \). We list these occurrences in a linear order as \( P(f_1(x_1, \ldots, x_k)), \ldots, P(f_m(x_1, \ldots, x_k)) \), with \( f_j \) appropriately chosen Boolean expressions. We choose new variables \( z_1, \ldots, z_m \) ranging over \( S \), and we obtain \( t' \) and \( r' \) by replacing in \( t \) and \( r \) each occurrence of \( P(f_j(x_1, \ldots, x_k)) \) by \( z_j \) for \( 1 \leq j \leq m \).

There are \( 2^m \) different conjunctions of \( x_j \) and \( \neg x_j \) for \( 1 \leq j \leq m \). Let \( \alpha_j \) be an enumeration of Boolean expressions for these conjunctions (\( 1 \leq j \leq 2^m \)). Next we choose \( 2^m \) different and new variables \( u_j \) (\( 1 \leq j \leq 2^m \)) ranging over \( S \) that will represent the values of \( P(\alpha_j(e_1, \ldots, e_k)) \).

Claim 1: \( K \models \sum_{j=1}^{2^m} P(\alpha_j) = 1 \). This immediately follows from \( BA \vdash \bigvee_{j=1}^{2^m} \alpha_j = \top \).

Each expression \( f_j \) can be written as a disjunctive normal form by means of a disjunction of expression of the form \( \alpha_j \). We write \( f_j = \bigvee_{i \in H_j} \alpha_i \) for appropriately chosen subsets \( H_j \) of \( \{1, \ldots, 2^m\} \).

Claim 2. APMF \( \vdash P(f_j) = \sum_{i \in H_j} P(\alpha_i) \) for \( 1 \leq j \leq m \).

The expression \( F \) is introduced as follows (with \( U \) abbreviating \( \sum_{i=1}^{2^m} u_i \)):

\[
F = 0_{1-U} \cdot \prod_{j=1}^{m} 0_{z_j - \sum_{i \in H_j} u_i} \cdot \prod_{i=1}^{2^m} 0_{1-s(u_i)+1}.
\]

We write \( K_{Md} \) for the reduct of \( K \) to the signature of meadows. \( K_{Md} \) is a non-trivial meadow. With valuation \( \sigma \) interpreting the \( y_i \) as \( s_i \), \( u_i \) as \( P(\alpha_i(e_1, \ldots, e_k)) \) and the \( z_i \) as \( P(f_i(e_1, \ldots, e_k)) \) for \( i \) in appropriate ranges we find that \( K_{Md}, \sigma \models F = 1 \) and therefore \( K_{Md}, \sigma \not\models F' \cdot t' = F \cdot r' \).

\[
\text{We notice that by definition APMF is sound for the class of cancellation pmf-structures.}
\]
Claim 3. There is a non-trivial cancellation meadow $\overline{M}$ that satisfies $\overline{M}, \rho \not\models F \cdot t' = F \cdot r'$ for some appropriate valuation, say $\rho$. This follows immediately from the basis theorem in [2] by contraposition.

In $\overline{M}$ the interpretations of the variables $u_i$ are non-negative and do not exceed 1, while summing up to 1, and the variables $z_i$ are equal to the interpretations of the same sums of $u_i$’s as in $K$.

Claim 4. The meadow $\overline{M}$ can be expanded to a pmf-structure refuting $t = r$. As a Boolean algebra component one may take the Boolean algebra reduct of $E_K(e_1, \ldots, e_k)$. A pmf, say $p$ must be defined for the interpretation of $P$. It suffices to define $p$ on $e_1, \ldots, e_k$ by $p(e_j) = \sum_{j \in H_j} \rho(u_j)$.

4.2 Stronger completeness results: pmf-structures expanding a fixed meadow

One may focus on pmf-structures that expand a specific cancellation meadow, say $M$, for instance the meadow of rational numbers and look for a finite basis of the equational theory of the corresponding restricted class of pmf-structures.

We will only consider the case of the meadow of rational numbers here. To that end consider the equation:

$$\frac{2 \cdot P(a)^2 - 1}{2 \cdot P(a)^2 - 1} = 1.$$  

This equation holds for each pmf-structure expanding the meadow of rational numbers. As it is invalid in the meadow of reals it does not follow from APMF. This consideration indicates that searching for a complete axiomatization of the equations true in the class of rational number based pmf-structures is problematic, in the sense that it will call for additional axioms that are entirely unrelated to probabilities.

In spite of this observation on the rationals, the idea of working with stronger completeness criteria is attractive.

4.3 On the definition of random variables

A random variable (r-variable, so denoted in order to avoid confusion with logical variables) consists of the following elements:

1. A name (some identifier), say $A$,
2. A domain (set) $D_A$,
3. A $D_A$-indexed family of event expressions: $V_A = \{\alpha^A \mid \alpha \in D_A\}$,
4. A theory consisting of $\neg \alpha^A \lor \neg \beta^A$ for $\alpha, \beta \in D_A, \alpha \neq \beta$, and $\sum_{\alpha \in D_A} P(\alpha^A) = 1$,
5. A mnemonic notation: $P(A = \alpha) = P(\alpha^A)$ for $\alpha \in D_A$.

---

8This explanation of random variables has a syntactical bias, a definition geared towards a specific cancellation pmf-structure phrased in terms of events rather than in terms of event expressions.
Independence of random variables. Two random variables $A$ and $B$ are independent if for all $\alpha \in D_A$ and $\beta \in D_B$ the event expressions $\alpha^A$ and $\beta^B$ are independent. That is:

$$P(\alpha^A, \beta^B) = P(\alpha^A) \cdot P(\beta^B).$$

In mnemonic notation this requirement reads:

$$P(A = \alpha, B = \beta) = P(A = \alpha) \cdot P(B = \beta).$$

5 Example I: “straightforward Bayes”

In this section we analyze a straightforward example of an application of Bayes’ rule.

5.1 The example

Here is a first example of reasoning with probabilities. We assume the following hypothetical but conceivable data:

1. A rare disease RD occurs with probability $1/100,000$ in the population of a country CO.
2. A potentially problematic nutritional habit NH is very widespread, in fact 4 out of 10 people in CO show NH.
3. It has been found that 8 out of 10 persons in CO who are suffering from RD show NH as well.

The question is to find the probability that someone showing NH suffers from RD. In order to answer that question the formalization of the three facts is as follows: we assume that RD and NH are names for events, and we assume that the pmf $P$ comprises the available probabilistic data: $P(RD) = 1/100,000$, $P(NH) = 4/10$, and $P(NH \mid RD) = 8/10$.

The question then is to compute $p = P(RD \mid NH)$. Using Bayes’ rule one finds:

$$p = \frac{P(NH \mid RD) \cdot P(RD)}{P(NH)} = \frac{8/10 \cdot 1/100,000}{4/10} = 0.2 \cdot 10^{-4}.$$ 

At face value this answer is valid.

One may notice that if NH were less widespread, say 1/500 only, this result changes significantly: for $q = P(RD \mid NH)$ we find

$$q = \frac{8/10 \cdot 1/100,000}{1/500} = 0.4 \cdot 10^{-2}.$$ 

This result is also credible.

---

9This example has been derived from Example 1.2 as presented in the freely accessible 2013-version of [1].
Now we may consider the case that the occurrence of NH is even more rare, say 1 out of 1,000,000. For \( r = P(\text{RD} \mid \text{NH}) \) we find

\[
r = \frac{8/10 \cdot 1/100,000}{1/1,000,000} = 8.
\]

This outcome is lacking credibility because probabilities are supposed not to exceed 1.

### 5.2 What is wrong, and how to improve?

The interesting aspect of this example is that while the first two equations and computations (for \( p \) and for \( q \)) correspond with conventional textbook examples, the third variation indicates that something went wrong in all three cases. We conclude this:

1. The production of a value for \( r \) that exceeds 1 constitutes a failure of the reasoning process at hand.
2. That failure may have been caused by an underlying fault. The failure is merely a symptom of that fault.
3. The fault may correlate with (go hand in hand with) errors taking place during the reasoning process that do not necessarily feature as failures at the end of the reasoning process.
4. In the determinations (or derivations) of \( p \) and of \( q \) similar errors might have occurred.
5. Therefore the validity of the values found for \( p \) and for \( q \) may be questioned.

In order to progress from this point we first notice that the data from which \( r \) has been computed allow the following proof:

\[ 1/1,000,000 = P(\text{NH}) \geq P(\text{NH} \land \text{RD}) = P(\text{NH} \mid \text{RD}) \cdot P(\text{RD}) = 8/10 \cdot 1/100,000 \]

from which one easily obtains 0 = 1. The data that underly the question for \( r \) are inconsistent. And for that reason unsurprisingly any result for \( r \) can be obtained including the otherwise meaningless value 8. Not having checked consistency before declaring the presence of a successful outcome is a fault.

It follows that in the cases of \( p \) and \( q \) consistency of the data must be established in order to confirm the credibility of the calculation of their respective values.

In this case a consistency proof can be provided by finding a pmf on the Boolean algebra generated by \( \text{RD} \) and \( \text{NH} \) which complies with the given data. That is an easy exercise in these cases for which we leave the details to the reader.

\[ ^{10} \]

A more convincing example of the problems at hand is obtained when the computation of a conditional probability leads to a seemingly credible value in a case where consistency fails. Then all other probabilities in the range \([0,1]\) could have been derived just as well, and the value that was actually found lacks reliability in spite of its credibility at face value.
5.3 Probabilistic reasoning versus classical reasoning about pmfs

Some authors claim that reasoning with Bayes’ rule constitutes probabilistic reasoning. That is not what we find: ordinary logic applied to instances of cancellation pmf-structures involve application of Bayes’ rule. Probabilistic reasoning comes in a later stage when non-monotonic reasoning steps are applied: for instance (i) abduction which infers from a relatively high likelihood of every $E$ to its factual status, or (ii) inference to the best explanation, which strengthens abduction by making use of confidence created from a judgement to the extent that $E$ is intrinsically simple or convincing as an explanation of observed events.

6 Example II: simulating classical logic

In this section we discuss an example that illustrates reasoning uniformly about a class of pmfs. This example also illustrates a simulation of reasoning in a classical two-valued logic.

6.1 The example

We assume the existence of three boxes $A$, $B$, and $C$, as well as the existence of two objects $a$ and $b$ and the following requirements on events:[11]

1. If $a$ is not in $A$ then $a$ is in $C$,
2. If $b$ is not in $B$ then $b$ is in $C$,
3. $a$ and $b$ are not both in $C$.

We want to infer the following: if $a$ is not in $A$ or $b$ is not in $B$ then $C$ is occupied. This fact can be derived by means of classical logic not making any reference to probabilities.

Our objective below is to reach this conclusion as an application of the equational logic of pmf-structures. In other words, an application of classical logic is simulated via a special case of equational reasoning about pmfs.

We use $D_A = D_B = D_C = \{\text{occ, empty}\}$, with occ abbreviating occupied. The requirements on $P$ are formalized as follows:

1. $P(C = \text{occ} | A = \text{empty}) = P(A = \text{empty} | A = \text{empty})$,
2. $P(C = \text{occ} | B = \text{empty}) = P(B = \text{empty} | B = \text{empty})$,
3. $P(A = \text{empty}, B = \text{empty}) = 0$,
4. $P(A = \text{empty} \lor B = \text{empty}) > 0$.

The required fact that we intend to establish is formalized as:

$$\Phi \equiv P(C = \text{occ} | A = \text{empty} \lor B = \text{empty}) = 1.$$[11]

[11]This example is based on Example 1.4 as presented in the 2013-version of [1].
6.2 Checking consistency

Consistency of the data (that is requirements) is established by considering any pmf \( P = P_{p_1,p_2} \) taking values as below:

- \( P(A = \text{occ}, B = \text{occ}, C = \text{occ}) = 0 \),
- \( P(A = \text{occ}, B = \text{occ}, C = \text{empty}) = 1 - p_1 - p_2 \),
- \( P(A = \text{occ}, B = \text{empty}, C = \text{occ}) = p_1 \) with \( 0 \leq p_1 \leq 1 \),
- \( P(A = \text{occ}, B = \text{empty}, C = \text{empty}) = 0 \),
- \( P(A = \text{empty}, B = \text{occ}, C = \text{occ}) = p_2 \) with \( 0 \leq p_1 \leq 1 - p_2 \) and \( p_1 + p_2 \neq 0 \),
- \( P(A = \text{empty}, B = \text{occ}, C = \text{empty}) = 0 \),
- \( P(A = \text{empty}, B = \text{empty}, C = \text{occ}) = 0 \),
- \( P(A = \text{empty}, B = \text{empty}, C = \text{empty}) = 0 \).

The requirement \( p_1 + p_2 \neq 0 \) implies that at least two different events (states) can exist.

We notice that if \( P \) is such that \( p_1 = 0 \), simplifying the second requirement to

\[
P(C = \text{occ} \mid B = \text{empty}) = 1
\]

fails. It fails in APMF because then

\[
P(B = \text{empty}) = 0 \rightarrow P(C = \text{occ} \mid B = \text{empty}) = 0,
\]

and it fails as well in a conventional approach because then the conditional probability

\[
P(C = \text{occ} \mid B = \text{empty})
\]

is undefined.

6.3 Simulated proof

Uniformly for all \( P \) we find by inspection of models that \( \Phi \) holds for all \( P_{p_1,p_2} \) and thus \( \Phi \) can be proven from the requirements on \( P_{p_1,p_2} \) and by completeness a formal proof from APMF can be found. Working from the four requirements on \( P \) listed above we find the simulated proof displayed in Table 6.
Φ = \( P(C = \text{occ} \land (A = \text{empty} \lor B = \text{empty})) \mid P(A = \text{empty} \lor B = \text{empty}) \)

\[
\begin{align*}
\Phi & = \frac{P((C = \text{occ} \land A = \text{empty}) \lor (C = \text{occ} \land B = \text{empty}))}{P(A = \text{empty} \lor B = \text{empty})} \\
& = \frac{P((C = \text{occ} \land A = \text{empty}) + P(C = \text{occ} \land B = \text{empty}) - Q}{P(A = \text{empty} \lor B = \text{empty})} \\
& = \frac{P((C = \text{occ} \land A = \text{empty}) + P(C = \text{occ} \land B = \text{empty}))}{P(A = \text{empty} \lor B = \text{empty})} \\
& = \frac{P((C = \text{occ} \mid A = \text{empty}) \cdot P(A = \text{empty}) + P(C = \text{occ} \mid B = \text{empty}) \cdot P(B = \text{empty})}{P(A = \text{empty} \lor B = \text{empty})} \\
& = \frac{P(A = \text{empty})}{P(A = \text{empty})} \cdot P(A = \text{empty}) + \frac{P(B = \text{empty})}{P(B = \text{empty})} \cdot P(B = \text{empty}) \\
& = P(A = \text{empty}) + P(B = \text{empty}) \\
& = P(A = \text{empty} \lor P(B = \text{empty}) \\
& = P(A = \text{empty}) + P(B = \text{empty}) - P(A = \text{empty} \land B = \text{empty}) \\
& = \frac{P(A = \text{empty} \lor B = \text{empty})}{P(A = \text{empty} \lor B = \text{empty})} \\
& = 1,
\end{align*}
\]

where

\[
Q = P((C = \text{occ}) \land (A = \text{empty} \land B = \text{empty}))
\]

\[
= P((C = \text{occ}) \land (A = \text{empty} \land B = \text{empty})) \cdot \frac{P(A = \text{empty} \land B = \text{empty})}{P(A = \text{empty} \land B = \text{empty})}
\]

\[
= P((C = \text{occ}) \land (A = \text{empty} \land B = \text{empty})) \cdot 0
\]

\[
= 0.
\]

Table 6: A simulated proof of Φ = 1
7 Concluding remarks

The incentive for this work came from a talk given by professor Ian Evett on the occasion of the retirement of dr. Huub Hardy as a driving force behind the MSc Forensic Science at the University of Amsterdam.\footnote{This meeting took place at Science Park Amsterdam, Friday June 7, 2013 under the heading “Frontiers of Forensic Science”, and was organized by Andrea Haker.} That talk both illustrated the headway that the Bayesian approach to reasoning in forensic matters has made in recent years and the conceptual and political problems that may still lie ahead of its universal adoption in the legal process.

In order to improve the understanding of these issues an elementary logical formalization of reasoning with probabilities might be useful. With that perspective in mind we came to the conclusion that the development a new approach from first principles to Bayesian reasoning was justified. The formalization of probabilities outlined above is supposed to be helpful when a formal and logically precise perspective on reasoning with probabilities is aimed at.

The long term perspective of this work is that it may lead to a novel approach for the formalization of Bayesian methods in evidence based reasoning. That, however, is still a remote perspective, and its viability cannot be assessed at this stage.

We acknowledge many discussions with Dr. Andrea Haker (University of Amsterdam) regarding the relevance of logically grounded reasoning methodologies in forensic science.

References


A PMF space

Updating a pmf by taking all probabilities conditional with respect to some event considered evidence produces a new pmf. That process can be repeated in principle and in order to investigate the merits of repeated updates working with a space (sort) PMF of pmfs is needed.

We introduce a set PMF for pmfs and we will read $P(x)$ as an abbreviation of $\text{apply}(P, x)$ where $\text{apply}$ is a new function in the signature and $P$ serves as a variable over the sort PMF.

The following pmf update function $\text{update}^x_y$ can be applied in order to determine the posterior pmf after evidence $y$ has become available.

$$\text{update}^y_x(P)(x) = P(x \mid y)$$

For purposes of modularization one may wish to split updates in different phases. However, we need that evidence $x$ and $y$ must be independent if successive updates for $x$ and for $y$ are supposed to be equivalent to a single step update corresponding with evidence $x \wedge y$, as is stated in the following proposition.

**Proposition 1.**

$$\frac{P(x \wedge y)}{P(x) \cdot P(y)} = \frac{P(x \wedge y)}{P(x \wedge y)} \iff \text{update}^x_y(P) = \text{update}^x_{y \wedge z}(P).$$

*Proof.* ($\rightarrow$) Assume $\frac{P(x \wedge y)}{P(x) \cdot P(y)} = \frac{P(x \wedge y)}{P(x \wedge y)}$. Then:

$$\text{update}^x_{y \wedge z}(P)(z) = \frac{P(z \wedge x \wedge y)}{P(x \wedge y)}$$

$$= P(z \wedge x \wedge y) \cdot \frac{P(x \wedge y)}{P(x \wedge y)} \cdot \frac{1}{P(x \wedge y)}$$

$$= P(z \wedge x \wedge y) \cdot \frac{P(x \wedge y)}{P(x \wedge y)} \cdot \frac{1}{P(x) \cdot P(y)}$$

$$= \frac{P(z \wedge x \wedge y)}{P(y) \cdot P(x)}$$

$$= (P(z \wedge x \wedge y) / P(y)) / P(x)$$

$$= (\text{update}^y_x(P)(z \wedge x)) / P(x)$$

$$= \text{update}^x_y(\text{update}^y_x(P))(z).$$

($\leftarrow$) Reasoning in the opposite direction one finds from $\text{update}^x_y(\text{update}^y_x(P)) = \text{update}^x_{y \wedge z}(P)$ that

$$\frac{P(z \wedge x \wedge y)}{P(x \wedge y)} = \frac{P(x \wedge z \wedge y)}{P(y) \cdot P(x)}.$$

Taking $z = \top$ we find that $\frac{P(x \wedge y)}{P(x) \cdot P(y)} = \frac{P(x \wedge y)}{P(x \wedge y)}$.  \hfill $\Box$
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