Fracpairs: fractions over a reduced commutative ring

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Fracpairs: fractions over a reduced commutative ring

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Abstract
In the well-known construction of the field of fractions, division by zero is excluded. We introduce “fracpairs” as pairs subject to laws consistent with the use of the pair as a fraction, but do not exclude denominators to be zero. We investigate fracpairs over a reduced commutative ring (that is, a commutative ring that has no nonzero nilpotent elements) and find that these constitute a “common meadow” (a field equipped with a multiplicative inverse and an additional element a that is the inverse of zero and propagates through all operations). We prove that fracpairs over \( \mathbb{Z} \) constitute a homomorphic pre-image of the common meadow \( \mathbb{Q}_a \), the field \( \mathbb{Q} \) of rational numbers expanded with an a-totalized inverse. Moreover, we characterize the initial common meadow as an initial algebra of fracpairs. Next, we define canonical term algebras (and therewith normal forms) for fracpairs over the integers and some related structures that model the rational numbers, and we provide negative results concerning their associated term rewriting properties. Then we define “rational fracpairs” that constitute an initial algebra that is isomorphic to \( \mathbb{Q}_a \). Finally, we express some negative conjectures concerning alternative specifications for these (concrete) datatypes.

Keywords and phrases: Fraction as a pair, meadow, common meadow, division by zero, abstract datatype, concrete datatype, rational numbers, term rewriting.

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\[(x + y) + z = x + (y + z) \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)\]
\[x + y = y + x \quad x \cdot y = y \cdot x\]
\[x + 0 = x \quad 1 \cdot x = x\]
\[x + (-x) = 0 \quad x \cdot (y + z) = (x \cdot y) + (x \cdot z)\]

Table 1: CR, axioms for commutative rings with 0 as the zero and 1 as the multiplicative unit

\[\frac{p}{q}\]

with \(p\) and \(q\) in \(R\) and \(q \neq 0\). The equivalence \(\sim\) is defined by
\[\frac{p}{q} \sim \frac{r}{s} \text{ if and only if } p \cdot s = q \cdot r \text{ holds in } R.\]

For defining fracpairs we start from a commutative ring \(R\) that is reduced \(^1\) (see \([10]\)), i.e., \(R\) has no nonzero nilpotent elements, or equivalently, \(R\) has the property
\[x \cdot x = 0 \rightarrow x = 0.\] \((*)\)

The integral domain \(\mathbb{Z}\) of integers is a prime example of a reduced commutative ring \(^2\) and other examples that are not an integral domain are the ring \(\mathbb{Z}/6\mathbb{Z}\) and the ring \(\mathbb{Z} \times \mathbb{Z}\). We recall that a commutative ring is a structure satisfying the axioms CR in Table 1 and that some familiar consequences of these axioms are the following:
\[-0 = 0, \quad 0 \cdot 0 = 0, \quad -(x) = x, \quad \text{and} \quad -(x \cdot y) = x \cdot (-y).\]

As is common, we assume that \(\cdot\) binds stronger than \(+\) and we will often omit brackets (as in \(x \cdot y + x \cdot z\)).

We will lift the above construction of the field of fractions of an integral domain to a structure of fractions over a reduced commutative ring by dropping the requirement that \(q\) in \(\frac{p}{q}\) must not be zero, and we will identify such structures as common meadows.

The first main result of this paper (Corollary \([2]\) Section [2.3]) establishes that by extending the signature of fracpairs over \(\mathbb{Z}\) to the signature of common meadows and defining the operations of common meadows on these fracpairs in a natural way, one obtains a common meadow

---

\(^1\) Integral domain: a nonzero commutative ring in which the product of any two nonzero elements is nonzero.

\(^2\) Terminology: Lam \([13]\ p.194\) uses “commutative reduced ring” and “noncommutative reduced ring”.
that is a homomorphic pre-image of $\mathbb{Q}_a$, the field $\mathbb{Q}$ of rational numbers expanded with an $a$-totalized inverse (that is, $0^{-1} = a$). Secondly, we characterize the initial common meadow as an initial algebra of fracpairs.

The paper is structured as follows. In Section 2 we introduce fracpairs and prove our main results. In Section 3 we discuss some term rewriting issues for various meadows in the context of fracpairs, and define canonical term algebras that represent these meadows, including a representation of $\mathbb{Q}_a$ as an initial algebra of “rational fracpairs”. In Section 4 we end the paper with some conclusions and a brief digression.

2 Fracpairs

In Section 2.1 we define fracpairs and fracterms, and establish some elementary properties. In Section 2.2 we relate fracpairs to the setting of common meadows, and in Section 2.3 we present our main results.

2.1 Fracpairs, fracterms, and some elementary properties

Fracpairs are pairs of elements of a reduced commutative ring. With $R$ the sort of the ring and $P$ the sort of fracpairs we have as an only constructor for $P$ the fracpairing operator:

\[ \vdots : R \times R \rightarrow P. \]

For reasoning about fracpairs we introduce the notion of a fracterm as a syntactic object that represents a fracpair, and for fracterms we use some common terminology:

\[ \frac{p}{q} \text{ has numerator } p \text{ and denominator } q. \]

We denote with “$\frac{p}{q}$” (as such) the fracpair $\frac{p}{q}$ modulo the congruence considered and we will always take care that this congruence is clear from the context. With “fracterm $\frac{p}{q}$” we refer to the representing syntactic object with numerator $p$ and denominator $q$.

Hence, a “fracpair representation” is a fracterm. Fracpairs satisfy the “fracpair representation” property [FRS] in Table 2 Using the axioms for commutative rings, a consequence of [FRS] is

\[ \frac{x}{0} = \frac{x \cdot 0}{0 \cdot 0} = \frac{0}{0}, \]

and another particular consequence of [FRS] is

\[ \frac{x}{-y} = \frac{x \cdot (-y)}{(-y) \cdot (-y)} = \frac{(-x) \cdot y}{y \cdot y} = \frac{-x}{y}. \]

A fracterm can be simplified by means of FRS as long as its numerator has a nonzero factor $p$ such that $p \cdot p$ is a factor of its denominator.

\[ \frac{x \cdot z}{y \cdot (z \cdot z)} = \frac{x}{y \cdot z} \]
The initial algebra of fracpairs over a reduced commutative ring $R$ is denoted with $FP(R)$. One may view $FP(R)$ as an equivalence relation $\equiv_{FR}$ over pairs $\frac{p}{q}$ with $p, q \in R$. In the absence of operations with arguments in $P$, $\equiv_{FR}$ is a congruence as well.

Consistency of the construction of fracpairs amounts to the absence of unexpected identifications in $FP(R)$ for a nontrivial reduced commutative ring $R$. Below we prove the separation of various fracpairs and discuss some expected identifications.

Let a reduced commutative ring $R$ be given and let $Deq(R)$ be the equational diagram of $R$, that is, define the signature $\Sigma_r(C) = \Sigma_r \cup C$ by having a constant $c_p$ for each element $p \in R$ and $C = \{c_p \mid p \in R\}$, and define $Deq(R) = \{s = t \mid s, t$ closed terms over $\Sigma_r(C)$ and $R \models s = t\}$. For simplicity we will often write $p$ instead of $c_p$. Using this notation we know that $FP(R)$ is the initial algebra of $Deq(R) + CR + FRS$. In other words:

$$FP(R) \models s = t \text{ if and only if } Deq(R) + CR + FRS \vdash s = t.$$ 

**Proposition 1.** For fracpairs defined over a nontrivial reduced commutative ring $R$ it holds that for all $x, y$ and all nonzero $z$, 

$$\frac{x}{0} \not\equiv \frac{y}{z}.$$ 

**Proof.** Assume $FP(R)$ is a free (initial) model for the fracpairs over $R$ and suppose that $FP(R) \models \frac{p}{q} = \frac{r}{s}$ for $p, q \in R$ and some nonzero $r \in R$. The identity must follow from diagram equations and FRS-identifications in either direction.

Now each instance of these equations leaves the denominator 0 of $\frac{p}{q}$ invariant: if $s \cdot t = 0$, then $s \cdot (t \cdot t) = 0$ by CR, and if $s \cdot (t \cdot t) = 0$, then $(s \cdot t) \cdot (s \cdot t) = 0$ by CR and thus $s \cdot t = 0$ by the property \(\Box\) that defines reduced rings. Hence, during a sequence of proof steps this denominator cannot transform from zero to nonzero or from nonzero to zero, which completes the proof. \[\square\]

Recall that an element $a$ is invertible if $a$ divides 1, and that $a$ is a zero divisor if $a$ is nonzero and for some nonzero $b$, $a \cdot b = 0$. We conclude this section by exemplifying some expected identifications of fracpair representations and properties thereof.

**Proposition 2.** Let $R$ be a reduced commutative ring. For $p, q, r, s \in R$ such that 

$$FP(R) \models \frac{p}{q} = \frac{r}{s},$$ 

if $q$ is invertible then so is $s$.

**Proof.** Immediate by induction on the length of a proof for $\frac{p}{q} = \frac{r}{s}$. \[\square\]

**Proposition 3.** Let $R$ be a nontrivial reduced commutative ring. For $q, r, s \in R$ such that 

$$FP(R) \models \frac{0}{q} = \frac{r}{s},$$ 

if $q$ is invertible then $R \models r = 0$. 

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Proof. By Proposition 2, \( s \) is invertible. There must be a single proof step in which the numerator turns from zero to nonzero. Assume that the proof is by a single step of FRS in either direction. An expanding instance of FRS adds a factor to the numerator which leaves it zero. A simplifying instance of FRS, however, requires a factorization of 0 in two nonzero factors (both zero divisors) in such a way that at least one of the factors is a factor of \( q \) as well. But that implies that \( q \) itself is a zero divisor which contradicts its invertibility. \( \square \)

Propositions 1 – 3 and non-trivialness of \( R \) imply the following corollary.

Corollary 1. Let \( R \) be a nontrivial reduced commutative ring, then in \( \mathbb{FP}(R) \), \( 0, 1, \frac{1}{1}, \frac{1}{0} \) are pairwise distinct.

Now assume that \( D_{eq}(R) + CR + FRS \vdash \frac{p}{q} = \frac{r}{s} \). In the proof all simplifying instances of FRS may be deleted with the effect that for some values \( v_1, \ldots, v_k \) one obtains a proof for

\[
\frac{p}{q} = \frac{r \cdot (v_1 \cdot \ldots \cdot v_k)}{s \cdot (v_1 \cdot \ldots \cdot v_k)}
\]

that makes use of expanding instances of FRS only. By induction on the length of this proof one notices that if for some \( h \), \( h \cdot q = p \), for each step in the proof one finds that the denominator multiplied by \( h \) yields the numerator. In particular taking the special case that \( q = s = 1 \) we find that \( r \cdot (v_1 \cdot \ldots \cdot v_k) = h \cdot (v_1 \cdot \ldots \cdot v_k) \) and \( h = p \). Using Proposition 2 and \( s = 1 \) it follows that \( v_1 \cdot \ldots \cdot v_k \) is invertible, say with inverse \( v \). Therefore

\[
r = r \cdot ((v_1 \cdot \ldots \cdot v_k) \cdot v) = h \cdot ((v_1 \cdot \ldots \cdot v_k) \cdot v) = p.
\]

With a similar argument, but now considering the special case \( p = r = 1 \) one finds that for invertible \( q \) and \( s \), \( D_{eq}(R) + CR + FRS \vdash \frac{1}{q} = \frac{1}{s} \) implies \( q = s \). This proves the following proposition, which is formulated in diagram notation.

Proposition 4. For a nontrivial reduced commutative ring \( R \) and \( p, q \in R \),

1. If \( \mathbb{FP}(R) \models \frac{c_p}{1} = c_q \), then \( R \models c_p = c_q \).
2. If \( p \) and \( q \) are invertible and \( \mathbb{FP}(R) \models \frac{1}{c_p} = \frac{1}{c_q} \), then \( R \models c_p = c_q \).

2.2 Fracpairs: constants and operations

In Table 3 we define some constants and operations for fracpairs, tailored to the setting of common meadows 5, that is, structures over the signature

\[
\Sigma_{cm} = \{0, 1, a, -(\_), (\_)^{-1}, +, \cdot\}.
\]

Note that the defining equation \( \text{FP5} \) for addition has a familiar form, and that the defining equation of the inverse \( \text{FP7} \) also takes zero denominators into account. We shall omit brackets in sums and products of fracpairs and write \( \frac{a}{q} + \frac{b}{s} \) and \( \frac{a}{q} \cdot \frac{b}{s} \).

Proposition 5. Let \( R \) be a reduced commutative ring, then the relation \( =_{FR} \) over \( \mathbb{FP}(R) \) is a congruence with respect to \( \Sigma_{cm} \).
\[
\begin{align*}
0 &= \frac{0}{1} \quad \text{(FP1)} \\
1 &= \frac{1}{1} \quad \text{(FP2)} \\
a &= \frac{1}{0} \quad \text{(FP3)} \\
\left(\frac{x}{y}\right) \cdot \left(\frac{u}{v}\right) &= \frac{x \cdot u}{y \cdot v} \quad \text{(FP4)} \\
\left(\frac{x}{y}\right) + \left(\frac{u}{v}\right) &= \frac{(x \cdot v) + (u \cdot y)}{y \cdot v} \quad \text{(FP5)} \\
-\left(\frac{x}{y}\right) &= -\frac{x}{y} \quad \text{(FP6)} \\
\left(\frac{x}{y}\right)^{-1} &= \frac{y}{x} \quad \text{(FP7)} \\
\left(\frac{x}{y}\right) \cdot \left(\frac{u}{v}\right) &= \frac{x \cdot u}{y \cdot v} \quad \text{(FP4)}
\end{align*}
\]

Table 3: Fracpairs over the common meadow signature with defining equations

**Proof.** It suffices to show that if \( \frac{p}{q} \) can be proven equal to \( \frac{r}{s} \) with finitely many instances of the property FRS, then the same holds for their image under the meadow operations as defined in Table 3. Let \( A = \frac{p \cdot r}{q \cdot (r \cdot r)} \) and \( B = \frac{p \cdot r}{q \cdot r} \), so \( A =_{\text{FR}} B \). Then

- \( A \cdot \frac{t}{s} =_{\text{FR}} B \cdot \frac{t}{s} \) follows immediately from \( \text{FP4} \), and so does \( \frac{t}{s} \cdot A =_{\text{FR}} \frac{t}{s} \cdot B \),
- \( A + \frac{t}{s} =_{\text{FR}} B + \frac{t}{s} \) because

\[
\begin{align*}
\frac{p}{q} \cdot \left(\frac{r}{r \cdot r}\right) + \frac{s}{t} &= \frac{(p \cdot r) \cdot t + s \cdot (q \cdot (r \cdot r))}{(q \cdot (r \cdot r)) \cdot t} \\
&= \frac{(p \cdot t + s \cdot (q \cdot (r \cdot r))) \cdot (r \cdot r)}{(q \cdot (r \cdot r)) \cdot t} =_{\text{FR}} \frac{p \cdot t + s \cdot (q \cdot r)}{(q \cdot r) \cdot t} \\
&= \frac{p \cdot t + s \cdot (q \cdot r)}{(q \cdot r) \cdot t} =_{\text{FR}} \frac{p}{q} + \frac{s}{t},
\end{align*}
\]

and \( \frac{t}{s} + A =_{\text{FR}} \frac{t}{s} + B \) follows in a similar way,

- \( -A = -B \): trivial (by \( \text{FP6} \)),
- \( A^{-1} = B^{-1} \) because

\[
\begin{align*}
\left(\frac{p}{q} \cdot \left(\frac{r}{r \cdot r}\right)\right)^{-1} &= \frac{(q \cdot (r \cdot r)) \cdot (q \cdot (r \cdot r))}{(p \cdot r) \cdot (q \cdot (r \cdot r))} =_{\text{FR}} \frac{q \cdot (r \cdot r) \cdot (q \cdot r)}{(p \cdot r) \cdot (r \cdot r)} \\
&= \frac{(q \cdot (r \cdot r)) \cdot (q \cdot r)}{(p \cdot r) \cdot (r \cdot r)} =_{\text{FR}} \frac{q \cdot (r \cdot r) \cdot q}{(p \cdot r) \cdot q} \\
&= \frac{(q \cdot r) \cdot (q \cdot r)}{p \cdot (q \cdot r)} =_{\text{FR}} \frac{p}{q},
\end{align*}
\]

\[\square\]
(x + y) + z = x + (y + z)  
2x + y = y + x  
x + 0 = x  
x + (−x) = 0 · x  
(x · y) · z = x · (y · z)  
x · y = y · x  
1 · x = x  
x · (y + z) = (x · y) + (x · z)

<table>
<thead>
<tr>
<th>Expression</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>(−x) = x</td>
<td>0 · (x · y) = 0 · (x + y)</td>
</tr>
<tr>
<td>(x⁻¹)⁻¹ = x + (0 · (x⁻¹))</td>
<td>x · (x⁻¹) = 1 + (0 · (x⁻¹))</td>
</tr>
<tr>
<td>(x · y)⁻¹ = (x⁻¹) · (y⁻¹)</td>
<td>1⁻¹ = 1</td>
</tr>
<tr>
<td>0⁻¹ = a</td>
<td>x + a = a</td>
</tr>
<tr>
<td>x · a = a</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Mdₐ, a set of axioms for common meadows

2.3 Fracpairs constitute a common meadow

With the aim of regarding the multiplicative inverse as a full-fledged operation, meadows were introduced in [8] as fields equipped with a multiplicative inverse, and a common meadow [5] is an extension of a meadow with an additional element a that serves as the inverse of zero and propagates through all operations. Common meadows are formally defined as structures over the signature Σcm = \{0, 1, a, −(−), ·, +(·)} that satisfy Mdₐ, that is, the axioms in Table 4.

We further assume that the inverse operation (·)⁻¹ binds stronger than · and omit brackets whenever possible, e.g., x · (y⁻¹) is written as x · y⁻¹. With Σ ısı we denote the signature of common meadows with the sort named S. The axioms of Mdₐ that feature a (sub)term of the form 0 · t cover the case that t equals a, and some typical Mdₐ-consequences are these:

−0 = 0, 0 · 0 = 0, (−(−x) = x, x = x + 0 · x, and (−x · y) = x · (−y).

Corollary 1 already suggests a strong connection between fracpairs and common meadows. For a reduced commutative ring R we write $FP_{cm}(R)$ for the initial algebra of fracpairs over R equipped with the constants and operations for common meadows, and we write P for the sort of fracpairs. Now one may forget about the underlying ring R and the fracpairing operation. In the notation of module algebra (see [3]) we can express this kind of hiding in a concise way:

$$\Sigma^P_{cm} \square (FP_{cm}(R)).$$

Here the export operator $\Sigma \square M$ is the operation that exports equations over the signature $\Sigma$ from module M while declaring other signature elements auxiliary (or “hidden”). In this case we declare fracpairing to be an auxiliary operator and this yields the following elementary result, which together with the next corollary we see as our first main result.

**Theorem 1.** Let R be a reduced commutative ring and let P be the sort of fracpairs over R, then $\Sigma^P_{cm} \square (FP_{cm}(R))$ is a common meadow.

**Proof.** By Proposition 5, $=_{FR}$ is a congruence with respect to $\Sigma^P_{cm}$. Therefore, showing that $\Sigma^P_{cm} \square (FP_{cm}(R))$ is a common meadow only requires proof checking of all Mdₐ-axioms (see
Table [4]. We consider four cases, all other cases being equally straightforward:

\[
\frac{p}{q} \cdot \left( \frac{p}{q} \right)^{-1} = \frac{p}{q} \cdot \frac{q}{p} \cdot \frac{q}{p} = \frac{p \cdot (q \cdot q)}{q \cdot (p \cdot q)} = \frac{(p \cdot q) \cdot q}{p \cdot (q \cdot q)}
\]

\[
= \frac{p \cdot q}{p \cdot q} \cdot \frac{1 \cdot (p \cdot q) + 0 \cdot 1}{1 \cdot (p \cdot q)} = \frac{1 + 0}{1 + 0}
\]

\[
= 1 + \frac{0}{1} \cdot \frac{q \cdot q}{p \cdot q} = 1 + 0 \cdot \frac{q \cdot q}{p \cdot q} = 1 + 0 \cdot \left( \frac{p}{q} \right)^{-1},
\]

\[
\left( \frac{p}{q} \right)^{-1} = \left( \frac{q \cdot q}{p \cdot q} \right)^{-1} = \frac{(p \cdot q) \cdot (q \cdot q)}{(p \cdot q) \cdot (q \cdot q)} = \frac{(p \cdot q) \cdot p}{q \cdot (p \cdot q)} = \frac{p + 0 \cdot q}{q \cdot p}
\]

\[
= \frac{p}{q} + \frac{0}{q} \cdot \frac{q \cdot q}{p \cdot q} = \frac{p}{q} + 0 \cdot \left( \frac{p}{q} \right)^{-1},
\]

\[
0^{-1} = \left( \frac{0}{1} \right)^{-1} = \frac{1 \cdot 1}{0 \cdot 1} = \frac{1}{0} = a, \quad \text{and} \quad \frac{p}{q} + a = \frac{p}{q} + \frac{1}{0} = \frac{q}{0} \cdot \frac{1}{0} = a.
\]

Expanding the initial algebra of fracpairs over a reduced commutative ring with the operations of meadows is arguably the most straightforward construction of a common meadow. With \( Q_a \) we denote the common meadow with signature \( \Sigma_{cm}^a \) that is defined as the field \( Q \) of rational numbers expanded with an \( a \)-totalized inverse (that is, \( 0^{-1} = a \)). We have the following corollary of Theorem [1]

**Corollary 2.** The structure \( \Sigma_{cm}^p \square (FP_{cm}(Z)) \) is a common meadow, and moreover it is a proper homomorphic pre-image of \( Q_a \).

**Proof.** First, by Corollary [1] \( \Sigma_{cm}^p \square (FP_{cm}(Z)) \) is nontrivial (0 = \( \frac{0}{1} \), 1 = \( \frac{1}{1} \), and \( a = \frac{a}{1} \) are pairwise distinct). Define \( f : (\Sigma_{cm}^p \square (FP_{cm}(Z))) \rightarrow Q_a \) by \( f\left(\frac{a}{m}\right) = n \cdot m^{-1} \). Then \( f \) is a homomorphism, in particular \( f(x + y) = f(x) + f(y) \) because \( Q_a \models x \cdot y^{-1} + u \cdot v^{-1} = (x \cdot v + u \cdot y) \cdot (y \cdot v)^{-1} \) (see [5] Prop.2.2.2), and

\[
f\left(\frac{a}{1}\right) = 0 \cdot 1^{-1} = 0, \quad f\left(\frac{1}{1}\right) = 1 \cdot 1^{-1} = 1, \quad f\left(\frac{1}{0}\right) = 1 \cdot 0^{-1} = a.
\]

Furthermore, \( \Sigma_{cm}^p \square (FP_{cm}(Z)) \) is a proper (thus non-isomorphic) homomorphic pre-image of \( Q_a \) because, for example, \( \frac{0}{1} = \frac{0}{1} \) cannot be proved while the \( f \)-image of both is 0 (for another example, consider \( \frac{1}{1} \cdot \frac{1}{1} = \frac{1}{1} \)).

Our second main result is a characterization of the initial common meadow.

**Theorem 2.** The initial common meadow \( I(\Sigma_{cm}, Md_a) \) is isomorphic to \( FP_{cm}(Z) \).

**Proof.** We use the following two properties of common meadows. First, for each closed term \( t \) over the meadow signature, there exist closed terms \( p \) and \( q \) over the signature \( \Sigma_2 \) of rings such that \( Md_a \models t = p \cdot q^{-1} \) (see [5] Prop.2.2.3). Secondly, \( Md_a \models x \cdot (x^{-1} \cdot x^{-1}) = x^{-1} \) (see [5]...
Prop.2.2.1), and hence \( \text{Md}_a \vdash (x \cdot z) \cdot (y \cdot (z \cdot z)^{-1}) = x \cdot (y \cdot z)^{-1} \), which can be seen as a characterization of \( \text{FRS} \).

Because \( \text{FP}_{cm}(\mathbb{Z}) \) is a model of \( \text{Md}_a \), there exists a homomorphism

\[ \phi : I(\Sigma_{cm}, \text{Md}_a) \to \text{FP}_{cm}(\mathbb{Z}) \]

that satisfies \( \phi(0) = 0 \) and \( \phi(1) = 1 \).

For \( p \) a closed term over \( \Sigma_r \), we find \( \phi(p) = \frac{p}{1} \) (this follows by structural induction on \( p \)), and thus

\[ \phi((p)^{-1}) = \left( \frac{p}{1} \right)^{-1} = \frac{1}{p} \cdot \frac{1}{p} = \frac{1}{p} \cdot \frac{1}{p} = \frac{1}{p}. \]

Hence, for \( p, q \) closed terms over \( \Sigma_r \), \( \phi(p \cdot q^{-1}) = \frac{p}{q} \), and thus \( \phi(0) = \phi(1 \cdot 0^{-1}) = 1 \).

It follows immediately that \( \phi \) is surjective. Also, \( \phi \) is injective: if for closed terms \( p, q \) over \( \Sigma_r \), \( \phi(p \cdot q^{-1}) = \phi(r \cdot s^{-1}) \), then we can find a proof using \( \text{FRS} \) and the CR-axioms. For closed terms over \( \Sigma_r \), \( \text{Md}_a \) implies the CR-identities \(^3\) and each \( \text{FRS} \) instance in this proof can be mimicked in \( I(\Sigma_{cm}, \text{Md}_a) \) with an instance of the equation \( (x \cdot z) \cdot (y \cdot (z \cdot z)^{-1}) = x \cdot (y \cdot z)^{-1} \). Hence, \( \text{Md}_a \vdash p \cdot q^{-1} = r \cdot s^{-1} \), so \( p \cdot q^{-1} = r \cdot s^{-1} \) in \( I(\Sigma_{cm}, \text{Md}_a) \).

\[ \blacksquare \]

3 Term rewriting for meadows

In Section 3.1 we provide details about canonical terms for involutive meadows, for common meadows, and for fracpairs. Until now we have not been successful in resolving questions about the existence of specifications for meadows with nice term rewriting properties, and we provide in Section 3.2 a survey of relevant negative results. In Section 3.3 we define “rational fracpairs” that constitute an initial algebra isomorphic to \( \mathbb{Q}_a \).

3.1 DDRS’s and canonical terms

A so-called DDRS (datatype defining rewrite system, see \(^6\)) is an equational specification over some given signature that, interpreted as a rewrite system by orienting the equations from left-to-right, is ground complete and thus defines (unique) normal forms for closed terms. Given some DDRS, its canonical term algebra (CTA) is determined as the algebra over that signature with the set of normal forms as its domain, and in the context of CTA’s we will often speak about canonical terms instead of normal forms. An abstract datatype (ADT) may be understood as the isomorphism class of its instantiations which are in our case CTA’s.

In Table 5 we define a DDRS for the abstract datatype \( \mathbb{Z} \) over \( \Sigma_r \), the signature of rings. Observe that the “symmetric variant” of equation (5), that is, \( (-x) + (y + 1) = ((-x) + y) + 1 \), is an instance of equation (4).

\(^3\)In particular, \( p + (-p) = 0 \) (or equivalently, \( 0 \cdot p = 0 \)); this follows easily by structural induction on \( p \).
\[ -0 = 0 \]  \hspace{2cm} (2)  
\[ -(x) = x \]  \hspace{2cm} (3)  
\[ x + (y + z) = (x + y) + z \]  \hspace{2cm} (4)  
\[ x + 0 = x \]  \hspace{2cm} (5)  
\[ 1 + (-1) = 0 \]  \hspace{2cm} (6)  
\[ (x + 1) + (-1) = x \]  \hspace{2cm} (7)  
\[ x + (- (y + 1)) = (x + (-y)) + (-1) \]  \hspace{2cm} (8)  
\[ 0 + x = x \]  \hspace{2cm} (9)  
\[ (-1) + 1 = 0 \]  \hspace{2cm} (10)  
\[ (- (x + 1)) + 1 = -x \]  \hspace{2cm} (11)  
\[ (-x) + (-y) = -(x + y) \]  \hspace{2cm} (12)  
\[ x \cdot 0 = 0 \]  \hspace{2cm} (13)  
\[ x \cdot 1 = x \]  \hspace{2cm} (14)  
\[ x \cdot (-y) = (-x) \cdot y \]  \hspace{2cm} (15)  
\[ x \cdot (y + z) = (x \cdot y) + (x \cdot z) \]  \hspace{2cm} (16)  

Table 5: A DDDS for \( \mathbb{Z} \)

**Definition 1.** Positive numerals for \( \mathbb{Z} \) are defined inductively: 1 is a positive numeral, and \( n + 1 \) is a positive numeral if \( n \) is one. Negative numerals for \( \mathbb{Z} \) have the form \(- (n)\) with \( n \) a positive numeral. A numeral for \( \mathbb{Z} \) is either a positive or a negative numeral, or 0. Canonical terms for \( \mathbb{Z} \) are the numerals for \( \mathbb{Z} \).

We write \( \hat{\mathbb{Z}} \) for the canonical term algebra for integers with these canonical terms.

Thus, \( \hat{\mathbb{Z}} \) constitutes a datatype that implements (realizes) the ADT \( \mathbb{Z} \) by the DDDS specified in Table 5. Some other specifications of \( \mathbb{Z} \) “in the language of rings” are discussed in [1], but these have negative normal forms \(-1, \ (-1) + (-1), \ (((-1) + (-1)) + (-1)), \) etcetera.

Below we define three more types of canonical terms and their associated canonical term algebras. The (involutive) meadow \( \mathbb{Q}_0 \) is defined as the field \( \mathbb{Q} \) of rational numbers with a zero-totalized inverse (so \( 0^{-1} = 0 \) and \((-)^{-1}\) is an involution; see, e.g., [8, 4, 2]).

**Definition 2.** Canonical terms for \( \mathbb{Q}_0 \) are the canonical terms for \( \mathbb{Z} \) (see Definition 1) and closed expressions of the form \( n \cdot m^{-1} \) and \( (-n) \cdot m^{-1} \) such that

* \( n \) is a positive numeral, and  
* \( m \) is a positive numeral larger than 1, and  
* \( n \) and \( m \) (viewed as natural numbers) are relatively prime.

With \( \hat{\mathbb{Q}_0} \) we denote the canonical term algebra for the abstract datatype \( \mathbb{Q}_0 \) with these canonical terms.
Thus \( \hat{Q}_0 \) is a datatype that implements the ADT \( Q_0 \).

**Definition 3.** Canonical terms for \( Q_a \) are the canonical terms for \( Q_0 \) and the additional constant \( a \).

With \( \hat{Q}_a \) we denote the canonical term algebra for the abstract datatype \( Q_a \) with these canonical terms.

Thus \( \hat{Q}_a \) is a datatype that implements the ADT \( Q_a \).

**Definition 4.** Canonical terms for \( FP_{cm}(\hat{Z}) \) are all fracpairs \( \frac{n}{m} \) with \( n \) and \( m \) canonical terms for \( Z \) (see Definition 1) and \( m \) not a negative numeral, such that one of the following conditions is met, where we write \( mZ \) for the integer denoted by \( m \):

- \( n = 0 \), and \( mZ \) is squarefree, or
- \( m = 0 \) and \( n = 1 \), or
- \( m = 1 \), or
- \( m \neq 0 \) and \( n \neq 0 \) and \( m \neq 1 \) and for every prime \( p \), if \( mZ \) is a multiple of \( p \cdot p \) then \( nZ \) is not a multiple of \( p \).

\( \hat{FP}_{cm}(\hat{Z}) \) is the canonical term algebra with these canonical terms.

So, \( \hat{FP}_{cm}(\hat{Z}) \) constitutes a datatype that implements the ADT \( FP_{cm}(\hat{Z}) \).

### 3.2 Nonexistence of DDRS’s for \( \hat{FP}_{cm}(\hat{Z}) \), for \( \hat{Q}_0 \), and for \( \hat{Q}_a \)

In this section we prove some negative results concerning the existence of certain DDRS’s.

**Theorem 3.** There is no DDRS for \( \hat{FP}_{cm}(\hat{Z}) \).

**Proof.** Suppose \( E \) is a finite set of rewrite rules for the signature of \( \hat{FP}_{cm}(\hat{Z}) \) that constitutes a DDRS. Notice that if \( m \) is a positive numeral with \( mZ \) not squarefree, then the term

\[
\frac{0}{m}
\]

is not a normal form. Assume that \( m \) exceeds the length of all left-hand sides of equations in \( E \) (for some suitable measure), thus \( \frac{0}{m} \) must match with a left-hand side of say equation \( e \in E \) that has the form

\[
\frac{0}{X+k} \quad \text{or} \quad \frac{Y}{X+k}
\]

where we assume the following notational convention: \( X+0 \equiv X \), and for all natural numbers \( n \), \( X+(n+1) \equiv (X+n)+1 \).

Now choose a canonical term \( \ell \) with \( \ell Z \) squarefree and larger than \( mZ \). It follows that \( e \) rewrites \( \frac{0}{\ell} \) so that this term cannot be a normal form which contradicts the definition of canonical terms (Definition 4). \( \square \)
\[
\frac{x \cdot (((z \cdot z) + (u \cdot u)) + 1)}{y \cdot (((z \cdot z) + (u \cdot u)) + 1)} = \frac{x}{y}
\]

(RFRS)

Table 6: RFRS, the Rational Fracpair Representation Simplification property

This proof works just as well if a DDRS is allowed to make use of auxiliary operations. Moreover, very similar proofs work for \( \hat{Q}_0 \) and \( \hat{Q}_a \), as we state in the next theorem.

**Theorem 4.** There is no DDRS for \( \hat{Q}_0 \) and for \( \hat{Q}_a \).

**Proof.** Suppose \( E \) is a finite set of rewrite rules for \( \hat{Q}_0 \) that constitutes a DDRS and consider a term \( \frac{1+1}{m} \) with \( m \mathbb{Z} \) a multiple of 2 that exceeds the largest equation in \( E \) (for some suitable measure). Because \( \frac{1+1}{m} \) is not a canonical term it is rewritten by say equation \( e \in E \). The left-hand side of \( e \) must have the form \( \frac{t}{X+k} \) for some \( t \) and \( k \) so that \( t \) matches with \( 1+1 \).

From this condition it follows that \( X \) is not a variable in \( t \) and without loss of generality we may assume that \( t \in \{1+1,Y,1+Y,Y+1,Y+Y\} \). Now let \( \ell \) be a \( \hat{Q}_0 (\hat{Q}_a) \) canonical term so that \( \ell \mathbb{Z} \) is odd and exceeds \( m \mathbb{Z} \). We find that \( \frac{1+1}{m} \) is a canonical term acceding to the definition thereof but at the same time it is not a normal form because it can be rewritten by means of \( e \). Thus, such \( E \) does not exist.

Finally, observe that the above reasoning also applies for the case of \( \hat{Q}_a \). \( \square \)

The above proof also demonstrates that auxiliary functions won’t help, not even auxiliary sorts will enable the construction of a DDRS for \( \hat{Q}_0 \) or for \( \hat{Q}_a \). We notice that without the constraint that the normal forms are given in advance (by way of a choice of canonical terms) the matter is different because according to \cite{7}, a DDRS can be found with auxiliary functions for each computable datatype. We return to the question of DDRS’s for rational numbers in Section 4 where we express some (negative) conjectures about their existence.

### 3.3 An initial algebra of fracpairs for rational numbers

We define *rational fracpairs* over a reduced commutative ring \( R \) as those fracpairs that satisfy the “rational fracpair representation simplification” property [RFRS] defined in Table 6 and the fracpair representation simplification property [FRS] (see Table 2). We define the relation

\[ \approx_{\text{RFR}} \]

over pairs \( \frac{p}{q} \) with \( p, q \in R \) as the equivalence relation generated by both [RFRS] and [FRS].

It easily follows that \( \approx_{\text{RFR}} \) is a congruence with respect to the signature of common meadows (cf. Proposition 4).

Rational fracpairs are tailored to an initial specification of the rational numbers in the style of \( \mathbb{FP}_{cm}(\mathbb{Z}) \). We write \( \mathbb{FP}_{cm}^r(\mathbb{Z}) \) for the initial algebra of rational fracpairs over \( \mathbb{Z} \) equipped with the constants and operations for common meadows. We have the following elementary result.
Theorem 5. Let \( P \) be the sort of fracpairs over \( \mathbb{Z} \). The structure \( \Sigma_{cm} \square (FP_{cm}(\mathbb{Z})) \) is a common meadow that is isomorphic to \( \mathbb{Q}_a \).

Proof. By \( \square \) Corollary 8, we have that for each prime number \( p \) there exist \( a, b, m \in \mathbb{N} \) such that \( m \cdot p = a^2 + b^2 + 1 \). Now let such \( p, a, b, m \) be given and derive for arbitrary \( c, d \in \mathbb{N} \),

\[
\frac{c \cdot p}{d \cdot p} =_{\text{RFR}} \frac{c \cdot m \cdot p}{d \cdot m \cdot p} \quad \text{(by RFRS)}
\]

\[
=_{\text{RFR}} \frac{c \cdot m}{d \cdot m} \quad \text{(by FRS)}
\]

Hence, for \( n, m \in \mathbb{N} \) it follows that:

* \( \frac{n}{m} =_{\text{RFR}} \frac{p}{q} \) with \( p, q \) relative prime if \( p \neq 0 \neq q \),
* \( \frac{n}{m} =_{\text{RFR}} \frac{0}{1} \) if \( n = 0 \) and \( m \neq 0 \),
* \( \frac{n}{m} =_{\text{RFR}} \frac{1}{0} \) if \( m = 0 \) (cf. identity \( \square \)).

So, with \( =_{\text{RFR}} \) we can reduce each rational fraction to one that matches the definition of canonical terms for \( \mathbb{Q}_a \), identifying \( \frac{n}{m} \) with \( n \cdot m^{-1} \) if \( n \neq 0 \) and \( m \not\in \{0, 1\} \), with \( n \) if \( m = 1 \) or \( \{n = 0 \text{ and } m \neq 0\} \) and with \( a \) if \( m = 0 \) (cf. Definition 2).

The observation that the defining equations for the constants and operators of common meadows on fracpairs given in Table 3 match those for \( \mathbb{Q}_a \) finishes the proof. \( \square \)

4 Conclusions and digression

We lifted the notion of a quotient field construction by dropping the requirement that in a “fraction \( \frac{p}{q} \)” (over some integral domain) the \( q \) must not be equal to zero and came up with the notion of fracpairs defined over reduced commutative rings. Defining the constants and functions of a meadow on the sort of fracpairs in a natural way yields a common meadow (Thm 1), and this is arguably the most straightforward construction of a common meadow. In particular, we present a common meadow that is a proper homomorphic pre-image of \( \mathbb{Q}_a \) (Cor 2) and we characterize the initial common meadow as the initial algebra \( FP_{cm}(\mathbb{Z}) \) of fracpairs over \( \mathbb{Z} \) (Thm 2; confer the characterization of the involutive meadow in \( \square \)).

Then, in Section 3, we considered canonical terms and term rewriting for integers and for some meadows that model expanded versions of the rational numbers, and proved the nonexistence of DDRS’s (datatype defining rewrite systems) for the associated canonical term algebras of \( FP_{cm}(\mathbb{Z}), \mathbb{Q}_0 \) and \( \mathbb{Q}_a \) (Thm 3 and Thm 4), each of which is based on a DDRS in which the integers are represented as 0, the positive numerals 1, 1 + 1, (1 + 1) + 1, ..., and the negations thereof. Moreover, we defined “rational fracpairs” that constitute an initial algebra that is isomorphic to \( \mathbb{Q}_a \) (Thm 5).

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4This is Corollary 1 in the report version arXiv:0907.0540v3 of this paper.
We have the following four conjectures about the nonexistence of DDRS specifications for rational numbers:

1. The meadow of rationals $\mathbb{Q}_0$ admits an equational initial algebra specification (see [7] and a subsequent simplification in [4]). Now the conjecture is that no finite equational initial algebra specification of $\mathbb{Q}_0$ is both confluent and strongly terminating (interpreting the equations as left-to-right rewrite rules). This is irrespective of the choice of normal forms.

   Another formulation of this conjecture: $\mathbb{Q}_0$ cannot be specified by means of a DDRS.

2. We conjecture that for $\mathbb{Q}_a$ the same situation applies as for $\mathbb{Q}_0$: No DDRS for it can be found irrespective of the normal forms one intends the DDRS to have.

3. The following conjecture (if true) seems to be simpler to prove: $\mathbb{FP}_{cm}(\mathbb{Z})$ cannot be specified by means of a DDRS.

4. The above negative conjectures remain if one allows the DDRS to be modulo associativity of $+$ and $\cdot$, commutativity of $+$ and $\cdot$, or both associativity and commutativity $+$ and $\cdot$.

Concerning these matters, we should mention the work [11] of Contejean et al in which normal forms for rational numbers are specified by a complete term rewriting system modulo commutativity and associativity of $+$ and $\cdot$. The associated datatype $\mathbf{Rat}$ comprises two functions $\mathbf{rat}$, $\mathbf{/}$: $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbf{Rat}$, where the symbol $\mathbf{rat}$ denotes any fraction, while $\mathbf{/}$ denotes irreducible fractions. Also in this work, division by zero is allowed, “but such alien terms can be avoided by introducing a sort for non-null integers” and is not considered any further. The main purpose of this work is to use the resulting datatype for computing Gröbner bases of polynomial ideals over $\mathbb{Q}$.

As a final note, we observe that fracpairs over a reduced commutative ring extended with the meadow operations also have an interest of their own. We avoided the use of the word “fraction” because this term is sometimes used in the semantic sense, as in the field of fractions, and sometimes in the syntactic sense, as a fraction having a numerator and a denominator. For the first category, we introduced “fracpair” and for the second category “fracterm”. It is our understanding of Rollnik [13] that he prefers to view fractions in educational mathematics as fracpairs rather than as fracterms (phrased in our terminology). Furthermore, a special case for the addition of fracterms with equal denominators is the following law on addition:

\[
\frac{x}{y} + \frac{z}{y} = \frac{x + z}{y} \tag{17}
\]

With FRS and the defining equation for $+$ (see Table 3) a proof of this law is immediate:

\[
\frac{x}{y} + \frac{z}{y} = \frac{(x \cdot y) + (z \cdot y)}{y \cdot y} = \frac{(x + z) \cdot y}{y \cdot y} = \frac{x + z}{y}.
\]

Taking $\mathbb{Z}$ as the underlying reduced commutative ring, this relates to the notion of quasi-cardinality that emerged from educational mathematics and is due to Griesel [12] (see also Padberg [14, p.30]). The aspect of quasi-cardinality for addition of fracpairs, which can also be called the quasi-cardinality law, is expressed by equation (17). So we find that the quasi-cardinality law, which features as a central fact in many textbooks on elementary arithmetic, follows from the equations for fracpairs and the definition of addition on fracpairs.
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References


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5Bourbaki group, officially known as the *Association des collaborateurs de Nicolas Bourbaki*.
