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Quadratic transformations for orthogonal polynomials in one and two variables

Tom H. Koornwinder

Dedicated to Masatoshi Noumi on the occasion of his sixtieth birthday

Abstract

We discuss quadratic transformations for orthogonal polynomials in one and two variables. In the one-variable case we list many (or all) quadratic transformations between families in the Askey scheme or q -Askey scheme. In the two-variable case we focus, after some generalities, on the polynomials associated with root system BC_2 , i.e., BC_2 -type Jacobi polynomials if $q = 1$ and Koornwinder polynomials in two variables in the q -case.

1 Introduction

Whenever we have a system of orthogonal polynomials $\{p_n\}$ in one variable with respect to an even orthogonality measure μ on \mathbb{R} , then we can write $p_{2n}(x) = q_n(x^2)$, $p_{2n+1}(x) = x r_n(x^2)$ with $\{q_n\}$ and $\{r_n\}$ systems of orthogonal polynomials on $\mathbb{R}_{\geq 0}$ with respect to orthogonality measures which are immediately obtained from μ . These mappings from $\{p_n\}$ to $\{q_n\}$ and $\{r_n\}$ are called quadratic transformations. For quite some multi-parameter families of orthogonal polynomials in the Askey scheme and the q -Askey scheme such quadratic transformations can be given explicitly. Very well-known are the quadratic transformations for Jacobi polynomials connecting $\{P_n^{(\alpha, \alpha)}\}$ with $\{P_n^{(\alpha, \pm \frac{1}{2})}\}$. Since all such polynomials can be expressed as (q -)hypergeometric functions, their quadratic transformations are equivalent to certain quadratic transformations for terminating (q -)hypergeometric functions.

The first aim of this paper, in Section 2, is to survey many (maybe all) instances of quadratic transformations in the (q -)Askey scheme, and how they are related by the limit arrows in those schemes. While the quadratic transformations for Askey-Wilson polynomials were already given in the Memoir [1] by Askey & Wilson, some of the other quadratic transformations given below may occur here for the first time, in particular the ones on the discrete side of the (q -)Askey scheme.

Quadratic transformations occur also for orthogonal polynomials in several variables as soon as the orthogonality measure is invariant under the transformation $x_1 \mapsto -x_1$ of the first variable x_1 . This sounds like a trivial generalization of the one-variable case, but this reflection map already takes some unexpected form when we look for quadratic transformations within multi-parameter families of special orthogonal polynomials in two variables. For the systems

associated with root system BC_2 the deeper explanation for the existence of the quadratic transformations is the isomorphism between the root systems B_2 and C_2 , both of which are contained in BC_2 .

These quadratic transformations in the two-variable case will be discussed in Section 3. For BC_2 -type Jacobi polynomials they go back to Sprinkhuizen-Kuyper [18], while they may be new for Koornwinder polynomials. We will also argue that quadratic transformations for orthogonal polynomials associated with BC_n cannot occur if $n > 2$, at least not in the simple form as for $n = 1$ and 2.

The paper concludes in Section 4 with a discussion how quadratic transformations can be helpful as heuristics for extending results to a larger realm of parameters, and with mentioning some possible work which would be a natural follow-up of this paper.

Conventions For definition and notation of hypergeometric and q -hypergeometric series see [4]. Throughout we will assume that $0 < q < 1$.

2 The one-variable case

2.1 Ordinary polynomials

Let $\{p_n(x)\}$ be a system of monic orthogonal polynomials on \mathbb{R} which are orthogonal with respect to an even (nonnegative) weight function $w(x) = w(-x) = v(x^2)$. Then $p_n(-x) = (-1)^n p_n(x)$. Put

$$q_n(x^2) := p_{2n}(x), \quad r_n(x^2) := x^{-1} p_{2n+1}(x). \quad (2.1)$$

Then (see [3, Ch. 1, §8]) $\{q_n(x)\}$ and $\{r_n(x)\}$ are systems of monic orthogonal polynomials on $[0, \infty)$:

- the q_n with respect to weight function $x^{-\frac{1}{2}}v(x)$,
- the r_n with respect to weight function $x^{\frac{1}{2}}v(x)$.

Note that from (2.1) we have, for any $x_0 \in \mathbb{R}$ on which the p_n do not vanish, that

$$\frac{q_n(x^2)}{q_n(x_0^2)} = \frac{p_{2n}(x)}{p_{2n}(x_0)}, \quad \frac{r_n(x^2)}{r_n(x_0^2)} = \frac{x_0 p_{2n+1}(x)}{x p_{2n+1}(x_0)}. \quad (2.2)$$

The identities (2.2) remain valid for arbitrary normalizations of the p_n , q_n , r_n .

As a slight variant of the above, let $\{p_n(x)\}$ be a system of orthogonal polynomials on $[-1, 1]$ which are orthogonal with respect to an even weight function $w(x) = w(-x) = v(2x^2 - 1)$. Let $x_0 \in \mathbb{R}$ such that $p_n(x_0) \neq 0$ for all n . Let $q_n(x)$ and $r_n(x)$ be polynomials of degree n such that

$$\frac{q_n(2x^2 - 1)}{q_n(2x_0^2 - 1)} = \frac{p_{2n}(x)}{p_{2n}(x_0)}, \quad \frac{r_n(2x^2 - 1)}{r_n(2x_0^2 - 1)} = \frac{x_0 p_{2n+1}(x)}{x p_{2n+1}(x_0)}.$$

Then $\{q_n(x)\}$ and $\{r_n(x)\}$ are systems of orthogonal polynomials on $[-1, 1]$:

- the q_n with respect to weight function $(1+x)^{-\frac{1}{2}}v(x)$,
- the r_n with respect to weight function $(1+x)^{\frac{1}{2}}v(x)$.

Example 2.1. Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) := \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1}{2}(1-x)\right)$$

are orthogonal on $[-1, 1]$ with weight function $(1-x)^\alpha(1+x)^\beta$ ($\alpha, \beta > -1$). So we have quadratic transformations (see [20, Theorem 4.1])

$$\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)}, \quad \frac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)} = \frac{xP_n^{(\alpha,\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,\frac{1}{2})}(1)}. \quad (2.3)$$

Example 2.2. Laguerre polynomials

$$L_n^\alpha(x) := \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}; x\right)$$

are orthogonal on $[0, \infty)$ with weight function $x^\alpha e^{-x}$ ($\alpha > -1$), while Hermite polynomials

$$H_n(x) := (2x)^n {}_2F_0\left(\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}(n-1) \\ - \end{matrix}; -x^{-2}\right)$$

are orthogonal on $(-\infty, \infty)$ with weight function e^{-x^2} . So we have quadratic transformations (see [20, (5.6.1)])

$$\frac{H_{2n}(x)}{H_{2n}(0)} = \frac{L_n^{-\frac{1}{2}}(x^2)}{L_n^{-\frac{1}{2}}(0)}, \quad \frac{H_{2n+1}(x)}{H'_{2n+1}(0)} = \frac{xL_n^{\frac{1}{2}}(x^2)}{L_n^{\frac{1}{2}}(0)}. \quad (2.4)$$

These are limit cases of (2.3) by the limits [6, (9.8.16), (9.8.18)].

Remark 2.3. In connection with (2.1) and (2.2) we had weight functions $w(x) = w(-x) = v(x^2)$. Then $d\mu(x) := w(x)dx$ is an even measure on \mathbb{R} and $d\nu(x) := 2x^{-\frac{1}{2}}v(x)dx$ is the pushforward measure $\nu = \phi_*\mu$ on $\mathbb{R}_{\geq 0}$ with $\phi: x \mapsto x^2: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$. In general, the quadratic transformations (2.1), (2.2) remain true if the p_n are orthogonal polynomials with respect to a (positive) even measure on \mathbb{R} , the q_n are orthogonal with respect to the measure $\nu = \phi_*\mu$ on $\mathbb{R}_{\geq 0}$, i.e.,

$$\int_{\mathbb{R}_{\geq 0}} p(y) d\nu(y) = \int_{\mathbb{R}} p(x^2) d\mu(x) \quad \text{for all polynomials } p,$$

and the r_n are orthogonal with respect to the measure $x d\nu(x)$ on $\mathbb{R}_{\geq 0}$. Similar remarks will apply to other quadratic transformations. This becomes in particular relevant in examples involving discrete mass points or q -integrals.

2.2 Symmetric Laurent polynomials

As a further variant of the above, with $w(x)$ a weight function on $[-1, 1]$, we substitute $x = \frac{1}{2}(z + z^{-1})$ so that z runs from -1 to 1 on the upper half unit circle if x runs from -1 to 1 on

the interval $[-1, 1]$. Let $\Delta(z)$ be a real-valued weight function on the upper half unit circle such that

$$w(x) = w\left(\frac{1}{2}(z + z^{-1})\right) = \frac{2i\Delta(z)}{z - z^{-1}}.$$

Then

$$\int_{-1}^1 f(x) w(x) dx = i^{-1} \int_C f\left(\frac{1}{2}(z + z^{-1})\right) \Delta(z) \frac{dz}{z},$$

where the contour C is the upper half unit circle starting at 1 and ending at -1 . Now suppose that $\Delta(z) = \Delta(-z^{-1})$ and put

$$\tilde{\Delta}(z^2) := \Delta(z) = \Delta(-z^{-1}) \quad (|z| = 1, 0 \leq \arg z \leq \pi/2). \quad (2.5)$$

Equivalently, $w(x) = w(-x)$. As before, put

$$v(2x^2 - 1) := w(x) = w(-x).$$

Then

$$w\left(\frac{1}{2}(z + z^{-1})\right) = v\left(\frac{1}{2}(z^2 + z^{-2})\right) = \frac{2i}{z - z^{-1}} \tilde{\Delta}(z^2).$$

Hence

$$\begin{aligned} (1 + \frac{1}{2}(z^2 + z^{-2}))^{-\frac{1}{2}} v\left(\frac{1}{2}(z^2 + z^{-2})\right) &= 2^{\frac{1}{2}} \frac{2i}{z^2 - z^{-2}} \tilde{\Delta}(z^2), \\ (1 + \frac{1}{2}(z^2 + z^{-2}))^{\frac{1}{2}} v\left(\frac{1}{2}(z^2 + z^{-2})\right) &= 2^{-\frac{1}{2}} (1 + z^2)(1 + z^{-2}) \frac{2i}{z^2 - z^{-2}} \tilde{\Delta}(z^2). \end{aligned}$$

Thus, with $x = \frac{1}{2}(z + z^{-1})$,

$$\begin{aligned} (1 + x)^{-\frac{1}{2}} v(x) &= 2^{\frac{1}{2}} \frac{2i}{z - z^{-1}} \tilde{\Delta}(z), \\ (1 + x)^{\frac{1}{2}} v(x) &= 2^{-\frac{1}{2}} (1 + z)(1 + z^{-1}) \frac{2i}{z - z^{-1}} \tilde{\Delta}(z). \end{aligned}$$

We arrive at the following result. Let $\{\tilde{p}_n(z)\}$ be a system of symmetric (i.e., invariant under $z \rightarrow z^{-1}$) Laurent polynomials which are orthogonal on C with respect to the measure $\Delta(z)z^{-1}dz$, where Δ satisfies (2.5). Let $z_0 \in \mathbb{C}$ such that $p_n(z_0) \neq 0$ for all n . Let $\tilde{q}_n(z)$ and $\tilde{r}_n(z)$ be symmetric Laurent polynomials of degree n such that

$$\frac{\tilde{q}_n(z^2)}{\tilde{q}_n(z_0^2)} = \frac{\tilde{p}_{2n}(z)}{\tilde{p}_{2n}(z_0)}, \quad \frac{\tilde{r}_n(z^2)}{\tilde{r}_n(z_0^2)} = \frac{(z_0 + z_0^{-1})\tilde{p}_{2n+1}(z)}{(z + z^{-1})\tilde{p}_{2n+1}(z_0)}.$$

Then $\{\tilde{q}_n(z)\}$ and $\{\tilde{r}_n(z)\}$ are systems of symmetric orthogonal Laurent polynomials on C :

- the \tilde{q}_n with orthogonality measure $\tilde{\Delta}(z)z^{-1}dz$,
- the \tilde{r}_n with orthogonality measure $(1 + z)(1 + z^{-1})\tilde{\Delta}(z)z^{-1}dz$.

If we go back to Example 2.1 then, with the above notation and up to constant factors,

$$\begin{aligned} \tilde{p}_n(z) &= P_n^{(\alpha, \alpha)}\left(\frac{1}{2}(z + z^{-1})\right), \quad \tilde{q}_n(z) = P_n^{(\alpha, -\frac{1}{2})}\left(\frac{1}{2}(z + z^{-1})\right), \quad \tilde{r}_n(z) = P_n^{(\alpha, \frac{1}{2})}\left(\frac{1}{2}(z + z^{-1})\right), \\ \Delta(z) &= (2 - z^2 - z^{-2})^{\alpha + \frac{1}{2}}, \quad \tilde{\Delta}(z) = (2 - z - z^{-1})^{\alpha + \frac{1}{2}}. \end{aligned}$$

Example 2.4. Recall Askey-Wilson polynomials [1], [6, §14.1], which we write as monic symmetric Laurent polynomials:

$$\begin{aligned} P_n(z) = P_n(z; a, b, c, d | q) &:= \frac{1}{(abcdq^{n-1}; q)_n} p_n\left(\frac{1}{2}(z + z^{-1}); a, b, c, d | q\right) \\ &= \frac{(ab, ac, ad; q)_n}{a^n(abcdq^{n-1}; q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right). \end{aligned} \quad (2.6)$$

Here $P_n(z)$ is invariant under permutations of the parameters a, b, c, d . Observe that

$$\begin{aligned} P_n(a; a, b, c, d | q) &= \frac{(ab, ac, ad; q)_n}{a^n(abcdq^{n-1}; q)_n}, \quad p_n\left(\frac{1}{2}(a + a^{-1}); a, b, c, d | q\right) = \frac{1}{(abcdq^{n-1}; q)_n}, \\ \frac{p_n\left(\frac{1}{2}(z + z^{-1})\right)}{p_n\left(\frac{1}{2}(a + a^{-1})\right)} &= {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right). \end{aligned} \quad (2.7)$$

Assume that a, b, c, d have absolute value ≤ 1 but do not have pairwise products equal to 1, and that non-real parameters occur in complex conjugate pairs. The polynomials $P_n(z)$ are orthogonal on the upper half unit circle C with respect to the orthogonality measure $\Delta(z) z^{-1} dz$, where

$$\Delta(z) = \Delta_+(z)\Delta_+(z^{-1}), \quad \Delta_+(z) = \Delta_+(z; a, b, c, d | q) := \frac{(z^2; q)_\infty}{(az, bz, cz, dz | q)_\infty}.$$

Since

$$\Delta(z; a, b, -a, -b | q) = \Delta(z^2; a^2, b^2, -1, -q | q^2) = \frac{\Delta(z; a^2, b^2, -q, -q^2 | q^2)}{(1 + z^2)(1 + z^{-2})},$$

we have:

$$P_{2n}(z; a, b, -a, -b | q) = P_n(z^2; a^2, b^2, -1, -q | q^2), \quad (2.8)$$

$$P_{2n+1}(z; a, b, -a, -b; q) = (z + z^{-1})P_n(z^2; a^2, b^2, -q, -q^2 | q^2), \quad (2.9)$$

or, in the normalization (2.7),

$$\frac{p_{2n}(x; a, b, -a, -b | q)}{p_{2n}\left(\frac{1}{2}(a + a^{-1}); a, b, -a, -b | q\right)} = \frac{p_n(2x^2 - 1; a^2, b^2, -1, -q | q^2)}{p_n\left(\frac{1}{2}(a^2 + a^{-2}); a^2, b^2, -1, -q | q^2\right)}, \quad (2.10)$$

$$\frac{p_{2n+1}(x; a, b, -a, -b; q)}{p_{2n+1}\left(\frac{1}{2}(a + a^{-1}); a, b, -a, -b; q\right)} = \frac{2x p_n(2x^2 - 1; a^2, b^2, -q, -q^2 | q^2)}{(a + a^{-1}) p_n\left(\frac{1}{2}(a^2 + a^{-2}); a^2, b^2, -q, -q^2 | q^2\right)}. \quad (2.11)$$

Formula (2.10) is given by Askey & Wilson [1, Section 3.1] in terms of q -hypergeometric functions, but similarly derived as above (a, b below different from a, b above):

$${}_4\phi_3\left(\begin{matrix} a^2, qb^2, c, -d \\ qab, -qab, cd \end{matrix}; q, q\right) = {}_4\phi_3\left(\begin{matrix} a^2, qb^2, c^2, d^2 \\ q^2 a^2 b^2, cd, qcd \end{matrix}; q^2, q^2\right) \quad (2.12)$$

when both sides terminate. The identity (2.12) can also be obtained from Singh [17, (22)] (see also [4, (3.10.11)]) by applying Sears' transformation [4, (2.10.4)].

While we arrived at (2.12) in the terminating case $a = q^{-n}$, the identity holds also in the terminating case $c = q^{-n}$. Then a resulting identity for Askey-Wilson polynomials is

$$P_n(z; a, b, q^{\frac{1}{2}}, -q^{\frac{1}{2}} | q) = P_n(z; a, qa, b, qb | q^2). \quad (2.13)$$

This relates two different ways of writing continuous q -Jacobi polynomials as Askey-Wilson polynomials, see [1, (4.20)] or [4, (7.5.26)]. Formula (2.13) also follows by observing that

$$\Delta(z; a, b, q^{\frac{1}{2}}, -q^{\frac{1}{2}} | q) = \Delta(z; a, qa, b, qb | q^2).$$

The quadratic transformation (2.11) can be written in terms of q -hypergeometric functions as

$${}_4\phi_3\left(\begin{matrix} a^2, qb^2, c, -d \\ qab, -qab, cd \end{matrix}; q, q\right) = \frac{c-d}{1-cd} {}_4\phi_3\left(\begin{matrix} qa^2, q^2b^2, c^2, d^2 \\ q^2a^2b^2, q^2cd, qcd \end{matrix}; q^2, q^2\right) \quad (2.14)$$

when both series terminate. For $c = q^{-n}$ the resulting identity for Askey-Wilson polynomials is again (2.13), with a and qa interchanged in the parameter list on the right-hand side. Hence, if $c = q^{-n}$ then (2.14) follows from (2.12) by applying Sears' transformation to the right-hand side of (2.12).

With $ab = q^{\alpha+1}$ formulas (2.10), (2.11) give a two-parameter q -analogue of (2.3). Indeed if $a = a_q, b = b_q$ in (2.10), (2.11) such that $a_q b_q = q^{\alpha+1}$ and $a_q \rightarrow 1$ as $q \uparrow 1$ then the quadratic transformations (2.10), (2.11) have the quadratic transformations (2.3) as limits for $q \uparrow 1$.

The quadratic transformations (2.10), (2.11) remain valid for less constrained parameter values by analytic continuation. In the case of orthogonality involving additionally a finite number of mass points (see [6, (14.1.3)]) we may still give a proof of (2.10), (2.11) by orthogonality in view of Remark 2.3.

There are various noteworthy special cases of the quadratic transformations (2.10), (2.11). For $b = q^{\frac{1}{2}}a$ we get continuous q -Jacobi polynomials on the left-hand sides and continuous q -ultraspherical polynomials on the right-hand sides. For $b = 0$ we get Al-Salam-Chihara polynomials on the left-hand sides and continuous dual q -Hahn polynomials on the right-hand sides. For $a = b = 0$ we get continuous q -Hermite polynomials on the left-hand sides and Al-Salam-Chihara polynomials (in this context also called continuous q -Laguerre polynomials) on the right-hand sides. See [6, Ch. 14] for details about the mentioned families of orthogonal polynomials.

2.3 Further examples of quadratic transformations in the q -Askey scheme

First we discuss some limit cases of the quadratic transformations (2.10), (2.11) for Askey-Wilson polynomials, where we stay in the continuous part of the q -Askey scheme.

Example 2.5. For big q -Jacobi polynomials [6, §14.5]

$$P_n(x; a, b, c, d; q) = P_n(qac^{-1}x; a, b, -ac^{-1}d; q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, qac^{-1}x \\ qa, -qac^{-1}d \end{matrix}; q, q\right).$$

and little q -Jacobi polynomials [6, §14.12], [12, §2.4]

$$p_n(x; a, b; q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1}ab \\ qa \end{matrix}; q, qx\right), \quad p_n(q^{-1}b^{-1}; a, b; q) = \frac{(-1)^n (qb; q)_n}{q^{\frac{1}{2}n(n+1)} b^n (qa; q)_n} \quad (2.15)$$

there are the quadratic transformations

$$P_{2n}(x; a, a, 1, 1; q) = \frac{p_n(x^2; q^{-1}, a^2; q^2)}{p_n((qa)^{-2}; q^{-1}, a^2; q^2)}, \quad (2.16)$$

$$P_{2n+1}(x; a, a, 1, 1; q) = \frac{qa x p_n(x^2; q, a^2; q^2)}{p_n((qa)^{-2}; q, a^2; q^2)}. \quad (2.17)$$

These were earlier given in [12, (2.48), (2.49)]. They are limit cases of (2.10), (2.11) by the limit formulas [9, (6.2), (6.4)].

The orthogonality relations for big and little q -Jacobi polynomials are given by q -integrals. In view of Remark 2.3 the quadratic transformations (2.16) and (2.17) can be obtained in a straightforward way by comparing the q -weights for the polynomials involved. The relevant observation is that, with $w(x) = v(x^2)$ and polynomials p , we have

$$\begin{aligned} \int_0^1 p(x) x^{-\frac{1}{2}} v(x) d_{q^2} x &= (1 - q^2) \sum_{k=0}^{\infty} p(q^{2k}) v(q^{2k}) q^k \\ &= (1 - q^2) \sum_{k=0}^{\infty} p((q^k)^2) w(q^k) q^k = (1 + q) \int_0^1 p(x^2) w(x) d_q x. \end{aligned}$$

Example 2.6. For discrete q -Hermite I polynomials [6, §14.28]

$$h_n(x; q) := q^{\frac{1}{2}n(n-1)} {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}; q, -qx\right)$$

and the little q -Laguerre polynomials (or Wall polynomials) $p_n(x; a; q) = p_n(x; a, 0; q)$ [6, §14.20] ((2.15) with $b = 0$) there are the quadratic transformations

$$h_{2n}(x; q) = (-1)^n q^{n(n-1)} (q; q^2)_n p_n(x^2; q^{-1}; q^2), \quad (2.18)$$

$$h_{2n+1}(x; q) = (-1)^n q^{n(n-1)} (q^3; q^2)_n x p_n(x^2; q; q^2). \quad (2.19)$$

These are limit cases of (2.16), (2.17) by the limit formula [13, §14.5]

$$\lim_{a \rightarrow 0} a^{-n} P_n(x; a, a, 1, 1; q) = q^n h_n(x; q).$$

The quadratic transformations (2.18), (2.19) immediately imply quadratic transformations [13, §14.21] connecting discrete q -Hermite II polynomials [6, §14.29] and q -Laguerre polynomials [6, §14.21] because these two orthogonal polynomials can be expressed as $i^{-n} h_n(ix; q^{-1})$ and $\text{const. } p_n(-x; q^{-\alpha}; q^{-1})$ in terms of discrete q -Hermite I polynomials and little q -Laguerre polynomials, respectively. Note that both families of orthogonal polynomials have non-unique orthogonality measures, see for instance [2]. Quite probably these last quadratic transformations are limit cases of rewritings of (2.16), (2.17) which can be interpreted as quadratic transformations for pseudo big q -Jacobi polynomials [5, Prop. 2.2].

Next we turn to the discrete part of the q -Askey scheme.

Example 2.7. On top there is a quadratic transformation between q -Racah polynomials (see [6, §14.2])

$$R_n(q^{-x} + \gamma\delta q^{x+1}; \alpha, \beta, \gamma, \delta | q) := {}_4\phi_3 \left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix}; q, q \right) \quad (n = 0, 1, \dots, N), \quad (2.20)$$

where αq or $\beta\delta q$ or γq is equal to q^{-N} . It reads, with $N \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$,

$$\begin{aligned} & R_{2n}(q^{-x-N-\frac{1}{2}} - q^{x-N-\frac{1}{2}}; \alpha, \alpha, q^{-2N-2}, -1 | q) \\ &= R_n(q^{-2x-2N-1} + q^{2x-2N-1}; \alpha^2, q^{-1}, q^{-2N-2}, q^{-2N-2} | q^2) \quad (n = 0, 1, \dots, [N + \frac{1}{2}]). \end{aligned} \quad (2.21)$$

Indeed, as a function of $q^{-x-N-\frac{1}{2}} - q^{x-N-\frac{1}{2}}$ the polynomials on the left-hand side of (2.21) are orthogonal on the points $q^{-x-N-\frac{1}{2}} - q^{x-N-\frac{1}{2}}$ ($x = -N - \frac{1}{2}, -N + \frac{1}{2}, \dots, N + \frac{1}{2}$) with respect to the weights

$$(q^x + q^{-x}) \frac{(\alpha^2 q^2; q^2)_{x+N+\frac{1}{2}} (\alpha^2 q^2; q^2)_{-x+N+\frac{1}{2}}}{(q^2; q^2)_{x+N+\frac{1}{2}} (q^2; q^2)_{-x+N+\frac{1}{2}}}$$

while the polynomials on the right-hand side are orthogonal on the points $(q^{-x-N-\frac{1}{2}} - q^{x-N-\frac{1}{2}})^2$ (x running over $-N - \frac{1}{2}, -N + \frac{1}{2}, \dots, -\frac{1}{2}$ or 0) with respect to the same weights. These weights are positive if $-1 < q\alpha < 1$.

In terms of q -hypergeometric functions (2.21) can be written as

$$\begin{aligned} & {}_4\phi_3 \left(\begin{matrix} q^{-2n}, \alpha^2 q^{2n+1}, q^{-x-N-\frac{1}{2}}, -q^{x-N-\frac{1}{2}} \\ q\alpha, -q\alpha, q^{-2N-1} \end{matrix}; q, q \right) \\ &= {}_4\phi_3 \left(\begin{matrix} q^{-2n}, \alpha^2 q^{2n+1}, q^{-2x-2N-1}, q^{2x-2N-1} \\ q^2 \alpha^2, q^{-2N-1}, q^{-2N} \end{matrix}; q^2, q^2 \right), \end{aligned}$$

which is the case $a = q^{-n}$, $b = q^n \alpha$, $c = q^{-x-N-\frac{1}{2}}$, $d = q^{x-N-\frac{1}{2}}$ of (2.12).

Similarly, from (2.14), we have the quadratic transformation

$$\begin{aligned} & R_{2n+1}(q^{-x-N-\frac{1}{2}} - q^{x-N-\frac{1}{2}}; \alpha, \alpha, q^{-2N-2}, -1 | q) \\ &= \frac{q^{-x-N-\frac{1}{2}} - q^{x-N-\frac{1}{2}}}{1 - q^{-2N-1}} R_n(q^{-2x-2N-1} + q^{2x-2N-1}; \alpha^2, q, q^{-2N-2}, q^{-2N-2} | q^2), \end{aligned} \quad (2.22)$$

where $N \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$ and $n = 0, 1, \dots, [N]$. Formula (2.22) can also be proved by orthogonality.

The special case $\alpha = 0$ of (2.21) and (2.22) gives quadratic transformations involving dual q -Krawtchouk polynomials [6, §14.17] and dual q -Hahn polynomials [6, §14.7].

Remark 2.8. The quadratic transformations (2.16), (2.17) involving big and little q -Jacobi polynomials can be obtained as limit cases of (2.21) and (2.22). For this we need the following special case of the limit formula [11, (2.2)] from q -Racah polynomials to big q -Jacobi polynomials:

$$\lim_{N \rightarrow \infty} R_n(q^{-2N-1}x; a, a, q^{-2N-2}, -1 | q) = \frac{P_n(x; a, a, 1, 1; q)}{P_n(-1; a, a, 1, 1; q)}.$$

We need also a limit formula from q -Racah polynomials to little q -Jacobi polynomials, not yet observed in [11]:

$$\lim_{N \rightarrow \infty} R_n(q^{-2N}x; a, b, q^{-N-1}, \delta q^{-N} | q) = \frac{p_n(\delta^{-1}x; b, a; q)}{p_n(1; b, a; q)}. \quad (2.23)$$

This is obtained from the limit formula (straightforward from (2.20))

$$\lim_{N \rightarrow \infty} R_n(q^{-2N}x; a, b, q^{-N-1}, \delta q^{-N} | q) = {}_3\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1}, \delta x^{-1} \\ qa \end{matrix}; q, b^{-1}\delta^{-1}x \right)$$

combined with [4, (III.8)] and (2.15).

Furthermore, the quadratic transformations (2.18), (2.19) can be obtained as limits of the cases $\alpha = 0$ of (2.21) and (2.22).

Example 2.9. Rather non-standard quadratic transformations for q -Racah polynomials can be obtained by another specialization of (2.12) and (2.14):

$${}_4\phi_3 \left(\begin{matrix} q^{-2n}, q^{-2(N-n)-1}, q^{-x}, -\gamma q^{x+1} \\ q^{-N}, -q^{-N}, \gamma q \end{matrix}; q, q \right) = {}_4\phi_3 \left(\begin{matrix} q^{-2n}, q^{2n-2N-1}, q^{-2x}, \gamma^2 q^{2x+2} \\ q^{-2N}, \gamma q, \gamma q^2 \end{matrix}; q^2, q^2 \right), \quad (2.24)$$

$$\begin{aligned} {}_4\phi_3 \left(\begin{matrix} q^{-2n-1}, q^{-2(N-n)}, q^{-x}, -\gamma q^{x+1} \\ q^{-N}, -q^{-N}, \gamma q \end{matrix}; q, q \right) &= \frac{q^{-x} - \gamma q^{x+1}}{1 - \gamma q} \\ &\times {}_4\phi_3 \left(\begin{matrix} q^{-2n}, q^{2n-2N+1}, q^{-2x}, \gamma^2 q^{2x+2} \\ q^{-2N}, \gamma q^2, \gamma q^3 \end{matrix}; q^2, q^2 \right). \end{aligned} \quad (2.25)$$

Here N is a positive integer and $n = 0, 1, \dots, N$. For $2n \leq N$ (2.24) is valid for all $x \in \mathbb{C}$. However, by the subtlety of passing to a lower parameter q^{-N} in (2.12) or (2.14), formula (2.24) is only valid for $x = 0, 1, \dots, N$ if $2n > N$. Similarly, (2.25) is valid for all $x \in \mathbb{C}$ if $2n + 1 \leq N$, but only valid for $x = 0, 1, \dots, N$ if $2n + 1 > N$.

By substitution of (2.20) in (2.24) and (2.25) we obtain quadratic transformations for q -Racah polynomials:

$$\begin{aligned} R_n(q^{-2x} + \gamma^2 q^{2x+2}; q^{-2N-2}, q^{-1}, \gamma, \gamma | q^2) \\ = \begin{cases} R_{2n}(q^{-x} - \gamma q^{x+1}; q^{-N-1}, q^{-N-1}, \gamma, -1 | q) & (2n \leq N), \\ R_{2N-2n+1}(q^{-x} - \gamma q^{x+1}; q^{-N-1}, q^{-N-1}, \gamma, -1 | q) & (2n > N), \end{cases} \end{aligned} \quad (2.26)$$

$$\begin{aligned} & \frac{q^{-x} - \gamma q^{x+1}}{1 - \gamma q} R_n(q^{-2x} + \gamma^2 q^{2x+2}; q^{-2N-2}, q, \gamma, \gamma | q^2) \\ &= \begin{cases} R_{2n+1}(q^{-x} - \gamma q^{x+1}; q^{-N-1}, q^{-N-1}, \gamma, -1 | q) & (2n+1 \leq N), \\ R_{2N-2n}(q^{-x} - \gamma q^{x+1}; q^{-N-1}, q^{-N-1}, \gamma, -1 | q) & (2n+1 > N). \end{cases} \end{aligned} \quad (2.27)$$

Both in (2.26) and (2.27) the identities corresponding to the first case of the right-hand side are valid for all complex $y := q^{-x} - \gamma q^{x+1}$ (then $q^{-2x} + \gamma^2 q^{2x+2} = y^2 + 2\gamma q$). But the identities corresponding to the second case of the right-hand side are only valid for $x = 0, 1, \dots, N$.

By [6, (14.2.2)] the q -Racah polynomials on the left-hand side of (2.26) are orthogonal on the set of points $q^{-2x} + \gamma^2 q^{2x+2}$ ($x = 0, 1, \dots, N$) with respect to the weights

$$w_x = q^{(2N+1)x} \frac{1 + q^{2x+1}\gamma}{1 + q\gamma} \frac{(q^{-2N}, q^2\gamma^2; q^2)_x}{(q^2, q^{2N+4}\gamma^2; q^2)_x}. \quad (2.28)$$

These weights are positive if $q^{-N} < \gamma < q^{-N-2}$. Inspection of the positivity of the coefficient of $p_{n-1}(x)$ in [6, (14.2.4)] for $n = 1, \dots, N$ gives the same constraint on γ . Again by [6, (14.2.2)], the q -Racah polynomials on the right-hand side of (2.26) are orthogonal on the set of points $q^{-x} - \gamma q^{x+1}$ ($x = 0, 1, \dots, N$) with respect to the weights w_x given by (2.28). This is compatible with (2.26), but on the other hand (2.26) can be proved from this equality of weights only if $2n \leq N$. Similar remarks can be made about (2.27).

If we put for $n = 0, 1, \dots, 2N + 1$

$$p_n(y) := \begin{cases} R_{\frac{1}{2}n}(y^2 + 2q\gamma; q^{-2N-2}, q^{-1}, \gamma, \gamma | q^2) & (n \text{ even}), \\ (1 - \gamma q)^{-1} y^{-1} R_{\frac{1}{2}(n-1)}(y^2 + 2q\gamma; q^{-2N-2}, q, \gamma, \gamma | q^2) & (n \text{ odd}) \end{cases}$$

then $p_n(-y) = (-1)^n p_n(y)$ and the p_n are orthogonal on the set of points $\pm(q^{-x} - \gamma q^{x+1})$ (for $x = 0, 1, \dots, N$) with respect to the weights w_x given by (2.28). For $n \leq N$ the explicit expressions for the p_n as polynomials of general argument are given by the first cases of the right-hand sides of (2.26), (2.27), but the expressions for $n > N$ will be more complicated.

2.4 Further examples of quadratic transformations in the Askey scheme

First we discuss limit cases for $q \uparrow 1$ of the quadratic transformations in the continuous part of the Askey scheme.

Example 2.10. Between Wilson polynomials [6, §9.1]

$$\frac{W_n(x^2; a, b, c, d)}{W_n(-a^2; a, b, c, d)} = \frac{W_n(x^2; a, b, c, d)}{(a+b)_n(a+c)_n(a+d)_n} := {}_4F_3 \left(\begin{matrix} -n, a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1 \right)$$

and continuous Hahn polynomials [6, §9.4]

$$\frac{p_n(x; a, b, \bar{a}, \bar{b})}{p_n(ia; a, b, \bar{a}, \bar{b})} = \frac{n! p_n(x; a, b, \bar{a}, \bar{b})}{i^n (a + \bar{a})_n (a + \bar{b})_n} := {}_3F_2 \left(\begin{matrix} -n, n + 2 \operatorname{Re}(a+b) - 1, a + ix \\ a + \bar{a}, a + \bar{b} \end{matrix}; 1 \right)$$

there are the quadratic transformations

$$\frac{p_{2n}(x; a, b, \bar{a}, \bar{b})}{p_{2n}(ia; a, b, \bar{a}, \bar{b})} = \frac{W_n(x^2; a, b, \frac{1}{2}, 0)}{W_n(-a^2; a, b, \frac{1}{2}, 0)}, \quad (2.29)$$

$$\frac{p_{2n+1}(x; a, b, \bar{a}, \bar{b})}{p_{2n+1}(ia; a, b, \bar{a}, \bar{b})} = \frac{xW_n(x^2; a, b, \frac{1}{2}, 1)}{iaW_n(-a^2; a, b, \frac{1}{2}, 1)}, \quad (2.30)$$

where $a, b \in \mathbb{R}$ or $b = \bar{a}$. This follows by comparing the orthogonality relations [6, (9.1,2), (9.4.2)] with each other.

In fact, (2.29) and (2.30) are limit cases of the quadratic transformations (2.10), (2.11) for Askey-Wilson polynomials by the limits

$$\lim_{q \uparrow 1} \frac{p_n(1 - \frac{1}{2}x(1-q)^2; q^a, q^b, q^c, q^d | q)}{(1-q)^{3n}} = W_n(x; a, b, c, d)$$

and

$$\begin{aligned} \lim_{q \uparrow 1} \frac{p_n(\cos \phi - x(1-q)\sin \phi; q^a e^{i\phi}, q^b e^{i\phi}, q^{\bar{a}} e^{-i\phi}, q^{\bar{b}} e^{-i\phi} | q)}{(1-q)^{2n}} \\ = (-2 \sin \phi)^n n! p_n(x; a, b, \bar{a}, \bar{b}) \quad (0 < \phi < \pi). \end{aligned}$$

There are corresponding limit cases of (2.12) and (2.14):

$${}_3F_2 \left(\begin{matrix} 2a, 2b+1, c \\ a+b+1, c+d \end{matrix}; 1 \right) = {}_4F_3 \left(\begin{matrix} a, b + \frac{1}{2}, c, d \\ a+b+1, \frac{1}{2}(c+d), \frac{1}{2}(c+d+1) \end{matrix}; 1 \right), \quad (2.31)$$

$${}_3F_2 \left(\begin{matrix} 2a, 2b+1, c \\ a+b+1, c+d \end{matrix}; 1 \right) = \frac{d-c}{d+c} {}_4F_3 \left(\begin{matrix} a + \frac{1}{2}, b+1, c, d \\ a+b+1, \frac{1}{2}(c+d)+1, \frac{1}{2}(c+d+1) \end{matrix}; 1 \right), \quad (2.32)$$

which are valid whenever both sides terminate.

Also note that (2.29) and (2.30) have the quadratic transformations (2.3) as limit cases. This follows by [6, (9.4.15)] and the limit (extension of [6, (9.1.18)])

$$\lim_{t \rightarrow \infty} \frac{W_n(\frac{1}{2}(1-x)t^2; a, \alpha + 1 - a, c + it, \beta + 1 - c - it)}{t^{2n} n!} = P_n^{(\alpha, \beta)}(x).$$

Example 2.11. Between continuous dual Hahn polynomials [6, §9.3]

$$\frac{S_n(x^2; a, b, c)}{S_n(-a^2; a, b, c)} := {}_3F_2 \left(\begin{matrix} -n, a + ix, a - ix \\ a + b, a + c \end{matrix}; 1 \right), \quad S_n(-a^2; a, b, c) = (a+b)_n (a+c)_n,$$

and Meixner-Pollaczek polynomials [6, §9.7]

$$\frac{P_n^{(\lambda)}(x; \phi)}{P_n^{(\lambda)}(i\lambda; \phi)} := {}_2F_1 \left(\begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix}; 1 - e^{-2i\phi} \right), \quad P_n^{(\lambda)}(i\lambda; \phi) = \frac{(2\lambda)_n}{n!} e^{in\phi},$$

there are the quadratic transformations

$$\frac{P_{2n}^{(a)}(x; \frac{1}{2}\pi)}{P_{2n}^{(a)}(ia; \frac{1}{2}\pi)} = \frac{S_n(x^2; a, \frac{1}{2}, 0)}{S_n(-a^2; a, \frac{1}{2}, 0)}, \quad (2.33)$$

$$\frac{P_{2n+1}^{(a)}(x; \frac{1}{2}\pi)}{P_{2n+1}^{(a)}(ia; \frac{1}{2}\pi)} = \frac{xS_n(x^2; a, \frac{1}{2}, 1)}{iaS_n(-a^2; a, \frac{1}{2}, 1)}. \quad (2.34)$$

These are limit cases of (2.29) and (2.30) by the limits [6, (9.1.16), (9.4.14)] Furthermore, (2.33) and (2.34) have the quadratic transformations (2.4) as limit cases by [6, (9.7.15)] and the limit

$$\lim_{a \rightarrow \infty} \frac{S_n(ax; a, b, c)}{a^n n!} = L_n^{b+c-1}(x). \quad (2.35)$$

For the proof of (2.35) compare the recurrence relations [6, (9.3.5), (9.12.4)] with each other.

Next we turn to the discrete part of the Askey scheme.

Example 2.12. On top there is a quadratic transformation between Racah polynomials [6, §9.2]

$$R_n(x(x + \gamma + \delta + 1); \alpha, \beta, \gamma, \delta) := {}_4F_3 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1 \\ \alpha + 1, \beta + \delta + 1, \gamma + 1 \end{matrix}; 1 \right) \\ (\alpha + 1 \text{ or } \beta + \delta + 1 \text{ or } \gamma + 1 = -N; n = 0, 1, \dots, N)$$

and Hahn polynomials [6, §9.5]

$$Q_n(x; \alpha, \beta, N) := {}_3F_2 \left(\begin{matrix} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{matrix}; 1 \right) \quad (n = 0, 1, \dots, N).$$

It reads

$$Q_{2n}(x + N + \frac{1}{2}; \alpha, \alpha, 2N + 1) = R_n(x^2 - (N + \frac{1}{2})^2; \alpha, -\frac{1}{2}, -N - 1, -N - 1) \\ (N \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}, n = 0, 1, \dots, [N + \frac{1}{2}]). \quad (2.36)$$

Indeed, as a function of x the polynomials on the left-hand side of (2.36) are orthogonal on the points $x = -N - \frac{1}{2}, -N + \frac{1}{2}, \dots, N + \frac{1}{2}$ with respect to the weights

$$\frac{(\alpha + 1)_{N + \frac{1}{2} + x} (\alpha + 1)_{N + \frac{1}{2} - x}}{(N + \frac{1}{2} + x)! (N + \frac{1}{2} - x)!},$$

while the polynomials on the right-hand side are orthogonal on the points x^2 (x running over $-N - \frac{1}{2}, -N + \frac{1}{2}, \dots, -\frac{1}{2}$ or 0) with respect to the same weights.

The quadratic transformation (2.36) is the case $a = -n$, $b = n + \alpha$, $c = -x - N - \frac{1}{2}$, $d = x - N - \frac{1}{2}$ of formula (2.31). By specialization of (2.32), also as a limit case for $q \uparrow 1$ of (2.22), we have the quadratic transformation

$$Q_{2n+1}(x + N + \frac{1}{2}; \alpha, \alpha, 2N + 1) = \frac{2N + 1 - 2x}{2N + 1} R_n(x^2 - (N + \frac{1}{2})^2; \alpha + 1, \frac{1}{2}, -N - 1, -N - 1) \\ (N \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}, n = 0, 1, \dots, [N]). \quad (2.37)$$

The quadratic transformations (2.3) for Jacobi polynomials can be obtained as limit cases of (2.36) and (2.37).

Example 2.13. Quadratic transformations involving Krawtchouk polynomials [6, §9.11]

$$K_n(x; p, N) := {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix}; p^{-1}\right)$$

and dual Hahn polynomials [6, §9.6]

$$R_n(x(x + \gamma + \delta + 1); \gamma, \delta, N) := {}_3F_2\left(\begin{matrix} -n, -x, x + \gamma + \delta + 1 \\ \gamma + 1, -N \end{matrix}; 1\right)$$

are given by

$$K_{2m}(x + N; \frac{1}{2}, 2N) = \frac{(\frac{1}{2})_m}{(-N + \frac{1}{2})_m} R_m(x^2; -\frac{1}{2}, -\frac{1}{2}, N), \quad (2.38)$$

$$K_{2m+1}(x + N; \frac{1}{2}, 2N) = -\frac{(\frac{3}{2})_m}{N(-N + \frac{1}{2})_m} x R_m(x^2 - 1; \frac{1}{2}, \frac{1}{2}, N - 1), \quad (2.39)$$

$$K_{2m}(x + N + 1; \frac{1}{2}, 2N + 1) = \frac{(\frac{1}{2})_m}{(-N - \frac{1}{2})_m} R_m(x(x + 1); -\frac{1}{2}, \frac{1}{2}, N), \quad (2.40)$$

$$K_{2m+1}(x + N + 1; \frac{1}{2}, 2N + 1) = \frac{(\frac{3}{2})_m}{(-N - \frac{1}{2})_{m+1}} (x + \frac{1}{2}) R_m(x(x + 1); \frac{1}{2}, -\frac{1}{2}, N). \quad (2.41)$$

They can be proved by orthogonality, they are limit cases of (2.36) and (2.37), and they have the quadratic transformations (2.4) involving Hermite and Laguerre polynomials as limit cases.

2.5 The (q -)Askey scheme of quadratic transformations

Let us summarize the quadratic transformations for families in the (q -)Askey scheme. In the q -case we have:

1a Askey-Wilson (2.10), (2.11)

1b q -Racah (2.21), (2.22)

2 big q -Jacobi to little q -Jacobi (2.16), (2.17)

Let $p_{n,k}(x, y)$ be polynomials of the form

$$p_{n,k}(x, y) = \sum_{(m,l) \leq (n,k)} c_{m,l} x^{m-l} y^l, \quad c_{n,k} \neq 0, \quad (3.1)$$

such that

$$\int_{\Omega} p_{n,k}(x, y) x^{m-l} y^l w(x, y) dx dy = 0 \quad \text{if } (m, l) < (n, k). \quad (3.2)$$

We call the polynomials $p_{n,k}(x, y)$ *dominance orthogonal* polynomials. For convenience we assume that they are monic, i.e., $c_{n,k} = 1$ in (3.1). Thus $p_{n,k}(x, y)$ and $p_{m,l}(x, y)$ with $(n, k) \neq (m, l)$ are orthogonal on Ω with weight function $w(x, y)$ if (n, k) and (m, l) are related in the partial ordering \leq , but the orthogonality will usually fail if (n, k) and (m, l) are not related in this partial ordering, except for very special Ω and $w(x, y)$, as will occur for cases related to root systems.

Now suppose that Ω is invariant under $(x, y) \rightarrow (-x, y)$, and also $w(x, y) = w(-x, y)$. Then, by (3.2), $p_{n,k}(-x, y) = (-1)^{n-k} p_{n,k}(x, y)$, and in (3.1) $c_{m,l} = 0$ if $n - k$ and $m - l$ do not have the same parity.

Put

$$\Omega' := \{(y, x^2) \mid (x, y) \in \Omega\} \quad \text{and} \quad v(x, y) := w(y^{\frac{1}{2}}, x) \quad ((x, y) \in \Omega'). \quad (3.3)$$

Proposition 3.1. *Let $q_{n,k}(x, y)$ and $r_{n,k}(x, y)$ be dominance orthogonal polynomials on Ω' with respect to weight functions $y^{-\frac{1}{2}}v(x, y)$ and $y^{\frac{1}{2}}v(x, y)$, respectively. Then*

$$q_{n,k}(y, x^2) = p_{n+k, n-k}(x, y), \quad x r_{n,k}(y, x^2) = p_{n+k+1, n-k}(x, y). \quad (3.4)$$

Proof We have

$$p_{n+k, n-k}(x, y) = \sum_{(i,j) \leq (n+k, n-k)} c_{i,j} x^{i-j} y^j,$$

where only terms with $i - j$ even occur. So we can substitute $i - j = 2l$ and $i + j = 2m$, $c_{i,j} = c'_{m,l}$. Then $(i, j) \leq (n+k, n-k)$ iff $(m, l) \leq (n, k)$. Hence

$$p_{n+k, n-k}(x, y) = \sum_{(m,l) \leq (n,k)} c'_{m,l} y^{m-l} x^{2l},$$

while from (3.2) we have

$$\int_{\Omega'} p_{n+k, n-k}(y^{\frac{1}{2}}, x) x^{m-l} y^l v(x, y) y^{-\frac{1}{2}} dx dy = 0 \quad \text{if } (m, l) < (n, k).$$

This settles (3.4) for $q_{n,k}$. A similar proof can be given for $r_{n,k}$. □

In particular, let Ω be the region

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid 1 - x + y, 1 + x + y, x^2 - 4y > 0\}. \quad (3.5)$$

Then (3.3) and (3.5) yield that

$$\Omega' = \{(x, y) \in \mathbb{R}^2 \mid y, y - 4x, (1 + x)^2 - y > 0\} = \{(\frac{1}{2}x, 1 + x + y) \mid (x, y) \in \Omega\}. \quad (3.6)$$

So if Ω is given by (3.5) then, by (3.6), an affine transformation respecting the dominance partial order of monomials maps Ω' onto Ω . Thus we can formulate a variant of Proposition 3.1 which again generalizes the proof of Theorem 10.1 in Sprinkhuizen [18]:

Proposition 3.2. *Let Ω be given by (3.5). Let the $p_{n,k}(x, y)$ be monic dominance orthogonal polynomials on Ω with respect to a weight function $w(x, y) = w(-x, y)$. Define $v(x, y)$ on Ω by*

$$v(2y, x^2 - 2y - 1) = w(x, y).$$

Let $q_{n,k}(x, y)$ and $r_{n,k}(x, y)$ be dominance orthogonal polynomials on Ω with respect to weight functions $(1 + x + y)^{-\frac{1}{2}}v(x, y)$ and $(1 + x + y)^{\frac{1}{2}}v(x, y)$, respectively. Then

$$2^{-n+k} q_{n,k}(2y, x^2 - 2y - 1) = p_{n+k, n-k}(x, y), \quad (3.7)$$

$$2^{-n+k} x r_{n,k}(2y, x^2 - 2y - 1) = p_{n+k+1, n-k}(x, y). \quad (3.8)$$

If the $p_{n,k}(x, y)$ in the above Proposition are not monic but satisfy $p_{n,k}(2, 1) \neq 0$ (which probably is implied by the dominance orthogonality) then we can replace (3.7), (3.8) by

$$\frac{q_{n,k}(2y, x^2 - 2y - 1)}{q_{n,k}(2, 1)} = \frac{p_{n+k, n-k}(x, y)}{p_{n+k, n-k}(2, 1)}, \quad (3.9)$$

$$\frac{x r_{n,k}(2y, x^2 - 2y - 1)}{2r_{n,k}(2, 1)} = \frac{p_{n+k+1, n-k}(x, y)}{p_{n+k+1, n-k}(2, 1)}. \quad (3.10)$$

In further variants of these results, to be discussed below, we will formulate results in a normalization as in (3.9), (3.10). If the assumption corresponding to $p_{n,k}(2, 1) \neq 0$ would fail then formulations in terms of monic polynomials would still be true.

3.2 Symmetric polynomials

In Proposition 3.2 replace x, y by ξ, η , and next put $\xi = x + y$, $\eta = xy$. Then we can rephrase this proposition in terms of symmetric polynomials in x, y . For this purpose make the following observations.

- The map $(x, y) \rightarrow (\xi, \eta)$ is a diffeomorphism from

$$\Lambda := \{(x, y) \mid -1 < y < x < 1\} \quad (3.11)$$

onto Ω given by (3.5). Furthermore $d\xi d\eta = (x - y) dx dy$.

- Let $n > k$. Then, for certain a_i, b_i with $a_0 = b_0 = 1$ we have

$$(x + y)^{n-k} (xy)^k = \sum_{i=0}^{\lfloor \frac{1}{2}(n-k) \rfloor} a_i (x^{n-i} y^{k+i} + x^{k+i} y^{n-i}),$$

$$x^n y^k + x^k y^n = \sum_{i=0}^{\lfloor \frac{1}{2}(n-k) \rfloor} b_i (x + y)^{n-k-2i} (xy)^{k+i}.$$

- If

$$p(\xi, \eta) = \sum_{(m,l) \leq (n,k)} a_{m,l} \xi^{m-l} \eta^l \quad (3.12)$$

for certain $a_{m,l}$ with $a_{n,k} \neq 0$ then

$$p(x+y, xy) = \sum_{(m,l) \leq (n,k)} b_{m,l} (x^m y^l + x^l y^m) \quad (3.13)$$

for certain $b_{m,l}$ with $b_{n,k} \neq 0$. Conversely, any symmetric polynomial given by the right-hand side of (3.13) can be written as $p(x+y, xy)$ for some polynomial $p(\xi, \eta)$ of the form (3.12).

Now let $W(x, y)$ be a weight function on Λ and let $P_{n,k}(x, y)$ be symmetric polynomials of the form of the right-hand side of (3.13) with $b_{n,k} \neq 0$ such that

$$\int_{\Lambda} P_{n,k}(x, y) (x^m y^l + x^l y^m) W(x, y) (x - y) dx dy = 0 \quad \text{if } (m, l) < (n, k).$$

We call the polynomials $P_{n,k}(x, y)$ dominance orthogonal symmetric polynomials. Observe that the polynomials $p_{n,k}(\xi, \eta)$ are dominance orthogonal on Ω with weight function $w(\xi, \eta)$ iff the polynomials $P_{n,k}(x, y) := p_{n,k}(x+y, xy)$ are dominance orthogonal on Λ with orthogonality measure $W(x, y)(x-y) dx dy$, where $W(x, y) := w(x+y, xy)$.

Now we can rephrase Proposition 3.2 as follows.

Proposition 3.3. *Let the $P_{n,k}(x, y)$ be dominance orthogonal symmetric polynomials on Λ with respect to a measure $W(x, y)(x-y) dx dy$, where $W(x, y) = W(-y, -x)$. Define a weight function V on Λ by*

$$V(xy + (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}, xy - (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}) = W(x, y). \quad (3.14)$$

Let $Q_{n,k}(x, y)$ and $R_{n,k}(x, y)$ be dominance orthogonal symmetric polynomials on Λ with respect to measures $(1+x)^{-\frac{1}{2}}(1+y)^{-\frac{1}{2}}V(x, y)(x-y) dx dy$ and $(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}V(x, y)(x-y) dx dy$, respectively. Then

$$\frac{Q_{n,k}(xy + (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}, xy - (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}})}{Q_{n,k}(1, 1)} = \frac{P_{n+k, n-k}(x, y)}{P_{n+k, n-k}(1, 1)}, \quad (3.15)$$

$$\frac{(x+y)R_{n,k}(xy + (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}, xy - (1-x^2)^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}})}{2R_{n,k}(1, 1)} = \frac{P_{n+k+1, n-k}(x, y)}{P_{n+k+1, n-k}(1, 1)}. \quad (3.16)$$

if $P_{n,k}(1, 1) \neq 0$ for all n, k , or the same identities without denominators for monic polynomials.

On passing to trigonometric coordinates (3.15) and (3.16) can be rewritten as

$$\frac{Q_{n,k}(\cos(\theta_1 - \theta_2), \cos(\theta_1 + \theta_2))}{Q_{n,k}(1, 1)} = \frac{P_{n+k, n-k}(\cos \theta_1, \cos \theta_2)}{P_{n+k, n-k}(1, 1)}, \quad (3.17)$$

$$\frac{(\cos \theta_1 + \cos \theta_2)R_{n,k}(\cos(\theta_1 - \theta_2), \cos(\theta_1 + \theta_2))}{2R_{n,k}(1, 1)} = \frac{P_{n+k+1, n-k}(\cos \theta_1, \cos \theta_2)}{P_{n+k+1, n-k}(1, 1)}. \quad (3.18)$$

Example 3.4. In the notation of Proposition 3.2 let

$$w(\xi, \eta) = w_{\alpha, \beta, \gamma}(\xi, \eta) := (1 - \xi + \eta)^\alpha (1 + \xi + \eta)^\beta (\xi^2 - 4\eta)^\gamma \quad (\alpha, \beta, \gamma > -1, \alpha + \gamma, \beta + \gamma > -\frac{3}{2}),$$

and put $p_{n,k}(\xi, \eta) = p_{n,k}^{\alpha, \beta, \gamma}(\xi, \eta)$ for the corresponding dominance orthogonal polynomials on the region Ω defined by (3.5). These polynomials, nowadays known as Jacobi polynomials for root system BC_2 , were first studied in [7] and subsequently in [18]. It follows from [7, (3.14)] and [18, Theorem 8.1] that these polynomials, even if they are defined as dominance orthogonal polynomials, still satisfy full orthogonality, and that they are nonzero at $(2, 1)$ by the explicit value [18, (7.3)].

Since

$$w_{\gamma, 0, \alpha}(2\eta, \xi^2 - 2\eta - 1) = 4^\alpha w_{\alpha, \alpha, \gamma}(\xi, \eta),$$

we have

$$\frac{p_{n,k}^{\gamma, -\frac{1}{2}, \alpha}(2\eta, \xi^2 - 2\eta - 1)}{p_{n,k}^{\gamma, -\frac{1}{2}, \alpha}(2, 1)} = \frac{p_{n+k, n-k}^{\alpha, \alpha, \gamma}(\xi, \eta)}{p_{n+k, n-k}^{\alpha, \alpha, \gamma}(2, 1)}, \quad \frac{\xi p_{n,k}^{\gamma, \frac{1}{2}, \alpha}(2\eta, \xi^2 - 2\eta - 1)}{2p_{n,k}^{\gamma, \frac{1}{2}, \alpha}(2, 1)} = \frac{p_{n+k+1, n-k}^{\alpha, \alpha, \gamma}(\xi, \eta)}{p_{n+k+1, n-k}^{\alpha, \alpha, \gamma}(2, 1)}. \quad (3.19)$$

These quadratic transformations were first given by Sprinkhuizen [18, Theorem 10.1]. They can be conceptually explained by the fact that B_2 and C_2 , while special cases of BC_2 , are isomorphic root systems.

Equivalently, in the notation of Proposition 3.3, let

$$W(x, y) = W_{\alpha, \beta, \gamma}(x, y) := (1 - x)^\alpha (1 - y)^\alpha (1 + x)^\beta (1 + y)^\beta (x - y)^{2\gamma}$$

and put $P_{n,k}(x, y) = P_{n,k}^{\alpha, \beta, \gamma}(x, y)$ for the corresponding dominance orthogonal symmetric polynomials on the region Λ . Then

$$\frac{P_{n,k}^{\gamma, -\frac{1}{2}, \alpha}(\cos(\theta_1 - \theta_2), \cos(\theta_1 + \theta_2))}{P_{n,k}^{\gamma, -\frac{1}{2}, \alpha}(1, 1)} = \frac{P_{n+k, n-k}^{\alpha, \alpha, \gamma}(\cos \theta_1, \cos \theta_2)}{P_{n+k, n-k}^{\alpha, \alpha, \gamma}(1, 1)},$$

$$\frac{(\cos \theta_1 + \cos \theta_2) P_{n,k}^{\gamma, \frac{1}{2}, \alpha}(\cos(\theta_1 - \theta_2), \cos(\theta_1 + \theta_2))}{2P_{n,k}^{\gamma, \frac{1}{2}, \alpha}(1, 1)} = \frac{P_{n+k+1, n-k}^{\alpha, \alpha, \gamma}(\cos \theta_1, \cos \theta_2)}{P_{n+k+1, n-k}^{\alpha, \alpha, \gamma}(1, 1)}.$$

3.3 Symmetric Laurent polynomials

Let S_2 be the symmetric group in 2 letters and $W_2 := S_2 \ltimes (\mathbb{Z}_2)^2$ (the Weyl group of BC_2). These groups naturally act on \mathbb{Z}^2 . For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ and $x = (x_1, x_2) \in \mathbb{C}^2$ put $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2}$. Put

$$m_\lambda(x) := \sum_{\mu \in S_2 \lambda} x^\mu, \quad \tilde{m}_\lambda(z) := \sum_{\mu \in W_2 \lambda} z^\mu \quad (\lambda_1 \geq \lambda_2 \geq 0).$$

For certain a_μ, b_μ with $a_\lambda, b_\lambda \neq 0$ we have

$$\begin{aligned} m_\lambda\left(\frac{1}{2}(z_1 + z_1^{-1}), \frac{1}{2}(z_2 + z_2^{-1})\right) &= \sum_{\mu \leq \lambda} a_\mu \tilde{m}_\mu(z_1, z_2), \\ \tilde{m}_\lambda(z_1, z_2) &= \sum_{\mu \leq \lambda} b_\mu m_\mu\left(\frac{1}{2}(z_1 + z_1^{-1}), \frac{1}{2}(z_2 + z_2^{-1})\right). \end{aligned}$$

Let $W(x, y)$ be a weight function on the region Λ given by (3.11). Let

$$\Gamma := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| = |z_2| = 1, 0 < \arg z_1 < \arg z_2 < \pi\}. \quad (3.20)$$

Then $(z_1, z_2) \mapsto \left(\frac{1}{2}(z_1 + z_1^{-1}), \frac{1}{2}(z_2 + z_2^{-1})\right)$ is a diffeomorphism from Γ onto Λ . On Γ define a weight function $\Delta(z_1, z_2)$ such that

$$W\left(\frac{1}{2}(z_1 + z_1^{-1}), \frac{1}{2}(z_2 + z_2^{-1})\right) = \frac{8\Delta(z_1, z_2)}{(z_1 - z_1^{-1})(z_2 - z_2^{-1})(z_1 + z_1^{-1} - z_2 - z_2^{-1})}.$$

Then

$$\int_{\Delta} f(x, y) W(x, y) (x - y) dx dy = \int_{\Gamma} f\left(\frac{1}{2}(z_1 + z_1^{-1}), \frac{1}{2}(z_2 + z_2^{-1})\right) \Delta(z_1, z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2}.$$

Hence, if the $P_{n,k}(x, y)$ are dominance orthogonal symmetric polynomials on Λ with orthogonality measure $W(x, y)(x - y) dx dy$ and if

$$p_{n,k}(z_1, z_2) := P_{n,k}\left(\frac{1}{2}(z_1 + z_1^{-1}), \frac{1}{2}(z_2 + z_2^{-1})\right)$$

then the $p_{n,k}(z_1, z_2)$ are dominance orthogonal W_2 -invariant Laurent polynomials on Γ with weight function $\Delta(z_1, z_2)$, i.e., we have for certain $c_{m,l}$ with $c_{n,k} \neq 0$ that

$$p_{n,k}(z_1, z_2) = \sum_{(m,l) \leq (n,k)} c_{m,l} \tilde{m}_{m,l}(z_1, z_2) \quad (3.21)$$

such that

$$\int_{\Delta} p_{n,k}(z_1, z_2) \tilde{m}_{m,l}(z_1, z_2) \Delta(z_1, z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2} = 0 \quad \text{if } (m, l) < (n, k).$$

Call the polynomials $p_{n,k}(z_1, z_2)$ monic if $c_{n,k} = 1$ in (3.21).

Now we can rephrase Proposition 3.3 as follows.

Proposition 3.5. *Let the $p_{n,k}(z_1, z_2)$ be dominance orthogonal W_2 -invariant polynomials on Γ with respect to a weight function $\Delta(z_1, z_2)$, where $\Delta(z_1, z_2) = \Delta(-z_2^{-1}, -z_1^{-1})$. Define a weight function $\tilde{\Delta}$ on Γ by*

$$\tilde{\Delta}(z_1 z_2, z_1 z_2^{-1}) = \Delta(z_1, z_2).$$

Let $q_{n,k}(x, y)$ and $r_{n,k}(x, y)$ be dominance orthogonal W_2 -invariant polynomials on Γ with respect to weight functions $\tilde{\Delta}(z_1, z_2)$ and $(1 + z_1)(1 + z_1^{-1})(1 + z_2)(1 + z_2^{-1})\tilde{\Delta}(z_1, z_2)$, respectively. Then

$$\frac{q_{n,k}(z_1 z_2, z_1 z_2^{-1})}{q_{n,k}(1, 1)} = \frac{p_{n+k, n-k}(z_1, z_2)}{p_{n,k}(1, 1)}, \quad (3.22)$$

$$\frac{(z_1 + z_1^{-1} + z_2 + z_2^{-1})r_{n,k}(z_1 z_2, z_1 z_2^{-1})}{4r_{n,k}(1, 1)} = \frac{p_{n+k+1, n-k}(z_1, z_2)}{p_{n+k+1, n-k}(1, 1)}. \quad (3.23)$$

if $p_{n,k}(1, 1) \neq 0$ for all n, k , or the same identities without denominators for monic polynomials.

Example 3.6. In the notation of Proposition 3.2 let

$$\Delta(z_1, z_2) = \Delta(z_1, z_2; q, t; a, b, c, d) = \Delta_+(z_1, z_2)\Delta_+(z_1^{-1}, z_2^{-1}),$$

where

$$\Delta_+(z_1, z_2) := \frac{(z_1^2; q)_\infty}{(az_1, bz_1, cz_1, dz_1; q)_\infty} \frac{(z_2^2; q)_\infty}{(az_2, bz_2, cz_2, dz_2; q)_\infty} \frac{(z_1 z_2, z_1 z_2^{-1}; q)_\infty}{(tz_1 z_2, tz_1 z_2^{-1}; q)_\infty},$$

and put $p_{n,k}(z_1, z_2) = p_{n,k}(z_1, z_2; q, t; a, b, c, d)$ for the corresponding dominance orthogonal W_2 -invariant monic Laurent polynomials on the region Γ defined by (3.20). These polynomials are the two-variable case of the n -variable Koornwinder polynomials [8], [14], which are associated with root system BC_n . These polynomials are fully orthogonal [8]. Now observe that

$$\Delta(z_1, z_2; q, t; a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = \Delta(z_1 z_2, z_1 z_2^{-1}; q^2, a^2; t, qt, -1, -q).$$

Hence, by Proposition 3.2, we have for $n + k$ even that

$$P_{n,k}(z_1, z_2; q, t; a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) = P_{\frac{1}{2}(n+k), \frac{1}{2}(n-k)}(z_1 z_2, z_1 z_2^{-1}; q^2, a^2; t, qt, -1, -q), \quad (3.24)$$

$$\begin{aligned} P_{n+1,k}(z_1, z_2; q, t; a, -a, q^{\frac{1}{2}}, -q^{\frac{1}{2}}) &= (z_1 + z_2 + z_1^{-1} + z_2^{-1}) \\ &\times P_{\frac{1}{2}(n+k), \frac{1}{2}(n-k)}(z_1 z_2, z_1 z_2^{-1}; q^2, a^2; t, qt, -q, -q^2). \end{aligned} \quad (3.25)$$

3.4 Failure of quadratic transformations in the n -variable case if $n > 2$

There are no straightforward analogues in $n > 2$ variables of Propositions 3.2 and 3.3. Indeed, symmetric polynomials in x_1, \dots, x_n invariant under $x_i \rightarrow -x_i$ ($i = 1, \dots, n$) correspond to polynomials in e_1, \dots, e_n (the elementary symmetric polynomials in x_1, \dots, x_n) which are invariant under $e_{2i-1} \rightarrow -e_{2i-1}$ ($i = 1, \dots, [\frac{1}{2}(n+1)]$). If $n > 2$ then this last involutive linear transformation has more than one eigenvalue unequal to 1. Therefore, by Stanley [19, Theorem 4.1] (a theorem going back to Shephard & Todd [16]), there do not exist n algebraically independent invariants for this involution if $n > 2$.

4 Discussion of results and further perspective

4.1 New results suggested by extrapolation from very few data

In all examples of quadratic transformations within multi-parameter families of special orthogonal polynomials in one or two variables we start with a subfamily depending on less than the full number of parameters, and then there is an even degree case and an odd degree case giving rise to two systems of orthogonal polynomials for which one of the parameters takes two special values, say $-\frac{1}{2}$ in the even case and $\frac{1}{2}$ in the odd case. Thus formulas and other results already known for the system with which we started give results for these parameter values $\pm\frac{1}{2}$ which can be tentatively extrapolated for more general values of the parameter.

Example 4.1. Consider the quadratic transformations (2.3) for Jacobi polynomials. They map from the Gegenbauer case of parameters (α, α) to the Jacobi cases $(\alpha, \pm\frac{1}{2})$. The Gegenbauer case is easier than the general Jacobi case, so (2.3) may be helpful as a start to derive from known results in the Gegenbauer case yet unknown results in the general Jacobi case. For instance, for a system of orthogonal polynomials $\{p_n\}$ it is remarkable to have a lowering formula of the form

$$\frac{d}{dx} \left(\psi(x)^n p_n(\phi(x)) \right) = \lambda_n \psi(x)^{n-1} p_{n-1}(\phi(x)).$$

For Gegenbauer polynomials such a formula does exist (see [10, (3.3)]):

$$\frac{d}{dx} \left((1+x^2)^{\frac{1}{2}n} P_n^{(\alpha, \alpha)} \left(\frac{x}{\sqrt{1+x^2}} \right) \right) = (n+\alpha) (1+x^2)^{\frac{1}{2}(n-1)} P_{n-1}^{(\alpha, \alpha)} \left(\frac{x}{\sqrt{1+x^2}} \right), \quad (4.1)$$

but probably not for general Jacobi polynomials. But let us see what we get for $(\alpha, \pm\frac{1}{2})$ by quadratic transformation of (4.1). First we have to iterate (4.1) once. Then apply (2.3). We obtain

$$\begin{aligned} \frac{d^2}{dx^2} \left((1+x^2)^n P_n^{(\alpha, -\frac{1}{2})} \left(\frac{x^2-1}{x^2+1} \right) \right) &= 4(n+\alpha)(n-\frac{1}{2}) (1+x^2)^{n-1} P_{n-1}^{(\alpha, -\frac{1}{2})} \left(\frac{x^2-1}{x^2+1} \right), \\ \left(\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} \right) \left((1+x^2)^n P_n^{(\alpha, \frac{1}{2})} \left(\frac{x^2-1}{x^2+1} \right) \right) &= 4(n+\alpha)(n+\frac{1}{2}) (1+x^2)^{n-1} P_{n-1}^{(\alpha, \frac{1}{2})} \left(\frac{x^2-1}{x^2+1} \right). \end{aligned}$$

Then the straightforward extrrapolation

$$\left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x} \frac{d}{dx} \right) \left((1+x^2)^n P_n^{(\alpha, \beta)} \left(\frac{x^2-1}{x^2+1} \right) \right) = 4(n+\alpha)(n+\beta) (1+x^2)^{n-1} P_{n-1}^{(\alpha, \beta)} \left(\frac{x^2-1}{x^2+1} \right)$$

can indeed be proved, see [10, (4.4)].

Example 4.2. In this example we again have a result obtained by quadratic transformation, now valid on a two-dimensional subdomain of a three-dimensional parameter space, but still can make a meaningful guess how to extrapolate.

The BC_2 -type Jacobi polynomials $p_{n,k}^{\alpha,\beta,\gamma}(\xi, \eta)$ given in Example 3.4 have an explicit expansion [15, (6.11)] (with different notation following [15, (3.3)]) in terms of polynomials

$$(1 - \xi + \eta)^{\frac{1}{2}(m+l)} P_{m-l}^{(\gamma,\gamma)} \left(\frac{1 - \frac{1}{2}\xi}{(1 - \xi + \eta)^{\frac{1}{2}}} \right) \quad (0 \leq l \leq m \leq n, l \leq k). \quad (4.2)$$

The polynomials (4.2) can be recognized as Jack polynomials in two variables and the mentioned expansion was seen in [14, Section 11.2] as a limit case of Okounkov's binomial formula for Koornwinder polynomials in two variables. Now by (3.19) and parity we have a quadratic transformation

$$p_{n+k,n-k}^{\alpha,\alpha,\gamma}(\xi, \eta) = \text{const.} \cdot p_{n,k}^{-\frac{1}{2},\gamma,\alpha}(-2\eta, \xi^2 - 2\eta - 1). \quad (4.3)$$

We can explicitly expand the right-hand side of (4.3) in terms of polynomials

$$\xi^{m+l} P_{m-l}^{(\alpha,\alpha)} \left(\frac{1 + \eta}{\xi} \right), \quad (4.4)$$

and thus this is also an explicit expansion for the left-hand side of (4.3). The other quadratic transformation in (3.19) gives a similar result for $p_{n+k+1,n-k}^{\alpha,\alpha,\gamma}(\xi, \eta)$ (with x^{m+l} in (4.4) replaced by x^{m+l+1}). This suggests, and is indeed confirmed in [15, Section 7], that $p_{n,k}^{\alpha,\beta,\gamma}(\xi, \eta)$ has a nice expansion in terms of the polynomials

$$\xi^m P_l^{(\alpha,\beta)} \left(\frac{1 + \eta}{\xi} \right) \quad (0 \leq m - l \leq n - k, m + l \leq n + k), \quad (4.5)$$

which can be considered as a two-parameter extension of the Jack polynomials in two variables.

In fact, by [15, Theorem 7.7], the polynomials (4.2) and (4.5) are limit cases of $p_{n,k}^{\alpha,\beta,\gamma}(\xi, \eta)$ for $\beta \rightarrow \infty$ and $\gamma \rightarrow \infty$, respectively.

4.2 Further perspective

In the one-variable part of this paper we gave a quite extensive treatment of quadratic transformations between families in the Askey and q -Askey scheme. Similar treatments should be given in the two-variable case. On the one hand we have orthogonal polynomials in two variables which are products of two polynomials from the (q -)Askey scheme and an elementary function, of which the orthogonal polynomials on the triangle involving products of two Jacobi polynomials are a well-known example. Quadratic transformations for such polynomials can be derived by suitable substitutions of quadratic transformations for polynomials in one variable. On the other hand there are the orthogonal polynomials associated with root system BC_2 . By work of various authors a large part of the (q -)Askey scheme has now been realized for BC_2 . It can be expected that corresponding schemes of quadratic transformations can also be given in the BC_2 case.

Finally it would be interesting to do further explicit work for Koornwinder polynomials in two variables analogous to the $q = 1$ case treated in (3.2) and extending [14, Section 11.1]. Analogous to Example 4.2 for $q = 1$, the quadratic transformations (3.24), (3.25) may be helpful for making a start in such work.

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