Revisable Justified Belief

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Revisable Justified Belief: Preliminary Report

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Abstract

The theory CDL of *Conditional Doxastic Logic* is the single-agent version of Board's multi-agent theory BRSIC of conditional belief. CDL may be viewed as a version of AGM belief revision theory in which Boolean combinations of revisions are expressible in the language. We introduce a theory JCDL of *Justified Conditional Doxastic Logic* that replaces conditional belief formulas $B^{\psi}\varphi$ by expressions $t{:}^{\psi}\varphi$ made up of a term t whose syntactic structure suggests a derivation of the belief φ after revision by ψ. This allows us to think of terms t as reasons justifying a belief in various formulas after a revision takes place. We show that JCDL-theorems are the exact analogs of CDL-theorems, and that this result holds the other way around as well. This allows us to think of JCDL as a theory of revisable justified belief.

1 Introduction

Conditional Doxastic Logic is Baltag and Smets' [4] name for a single-agent version of Board's [5] multi-agent theory of conditional belief BRSIC. CDL has formulas $B^{\psi}\varphi$ to express that the agent believes φ conditional on ψ. As we will see, CDL has a certain relationship with the "AGM theory" of belief revision due to due to Alchourrón, Gärdenfors, and Makinson [1]. So we may also think of $B^{\psi}\varphi$ as say that the agent will believe φ after revising her belief state by successfully incorporating the information that ψ is true. As with AGM theory, CDL assumes that conditionalization (i.e., revision) is always successful: the agent is to assume that the incoming information ψ is completely trustworthy and therefore update her belief state by consistently incorporating this incoming information. If before the revision she holds beliefs that imply $\neg\psi$, then she must give up these beliefs so that she will come to believe ψ after the revision takes place. The question then is how to do this in general.

The models of CDL are "plausibility models." These are very close to Grove's "system of spheres" for AGM theory [6]. As we will see, we can view the conditionalization/revision process either from the semantic perspective, as a definite operation on plausibility models, or from the syntactic perspective, as an axiomatic formulation analogous to the postulate-based approach of AGM. However, it is perhaps simplest to start with the semantic perspective.

2 Plausibility models

Definition 2.1 (Plausibility models)**.** Let $\mathscr{P}$ be a set of propositional letters. A *plausibility model* is a structure $M = (W, \leq, V)$ consisting of a nonempty set W of "worlds," a preorder (i.e., a reflexive and transitive binary relation) $\leq$ on W, and a propositional valuation $V : W \to \wp(\mathscr{P})$ mapping each world w to the set $V(w)$ of propositional letters true at w. We call $\leq$ a *plausibility relation* on W. In terms of $\leq$, we

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define the converse relation $\geq$, the strict version $<$, the strict converse relation $>$, and the various negations of these (denoted by writing a slash through the symbol to be negated) as usual. $\simeq$ denotes the relation $\simeq := (\geq \cap \leq)$ of *equi-plausibility*. A *pointed plausibility model* is a pair (M, w) consisting of a plausibility model M and the *point* w, itself a world in M. Notation: for each $w \in W$, we define the set

$$w^{\downarrow} := \{x \in W \mid x \leq w\} \ .$$

$x \leq y$ read, "x is no less plausible than y." According to this reading, if we think of $\leq$ as a "less than or equal to" relation, then it is the "lesser" elements that are *more* plausible. Therefore, if $\leq$ is a well-order and S is a nonempty set of worlds, $\min S$ is the set of worlds that are the most plausble in S. While it may at first seem counterintuitive to the uninitiated, this convention of "lesser is more plausible" is nevertheless standard in Belief Revision Theory.[1]

We think of the plausibility relation as describing the judgments of an unnamed agent: for each pair of worlds (x, y), she either judges one world to be more plausible than the other or the two to be of equal plausibility. Plausibility models for multiple agents have a number of plausibility relations, one for each agent. For present purposes, we restrict attention to the single-agent case, though we say more about the multi-agent situation later.

Definition 2.2 (Plausibility model terminology)**.** Let $M = (W, \leq, V)$ be a plausibility model.

- To say M is *finite* means W is finite.
- To say M is *connected* means that for each $x \in W$, we have $\mathsf{cc}(x) = W$, where

  $$\mathsf{cc}(x) := \{y \in x \mid x(\geq \cup \leq)^{+} y\}$$

  is the *connected component of* x and $(\geq \cup \leq)^{+}$ is the transitive closure of $\geq \cup \leq$.[2] A *connected component* is a subset $S \subseteq W$ for which there exists an $x \in W$ such that $\mathsf{cc}(x) = S$.
- To say M is *well-founded* means $\leq$ is well-founded: for each nonempty $S \subseteq W$, the set

  $$\min S := \{x \in S \mid \forall y \in S : y \not< x\}$$

  of *minimal elements of* S is nonempty.
- To say that a set $S \subseteq W$ of worlds is *smooth* in M means that for each world $x \in S$, either $x \in \min(S)$ or there exists $y \in \min(S)$ such that $y < x$. Given a collection $\Gamma \subseteq \wp(W)$ of sets of worlds, to say that M is *smooth with respect to* Γ means that every $S \in \Gamma$ is smooth in M. To say that M is *smooth* means that M is smooth with respect to $\wp(W)$.
- To say that M is *total* means that $\leq$ is total on W: for each $(x, y) \in W \times W$, we have $x \leq y$ or $y \leq x$.
- To say M is *well-ordered* (equivalently, that M is a *well-order*) means that $\leq$ is well-ordered (i.e., it is total and well-founded).
- To say that M is *locally total* means that $\leq$ is *total on each connected component*: for each $w \in W$ and $(x, y) \in \mathsf{cc}(w) \times \mathsf{cc}(w)$, we have $x \leq y$ or $y \leq x$.

[1] This convention stems from the notion of "Grove spheres" [6]: given a well-order, worlds are arranged so that they sit on the surface of a number of concentric spheres. Worlds of strictly greater plausibility are assigned to spheres with strictly shorter radii, and equi-plausible worlds are assigned to the same sphere. In this way, the most plausible worlds sit on the surface of the innermost sphere, which has the *minimum* radius. Similarly, if we restrict attention to a nonempty set S of worlds, then we "recenter" the sphere around S. By this we mean that we create a new system of spheres consisting of just those worlds in S. After doing so, the most plausible worlds again sit on the surface of the innermost sphere, which again has the minimum radius.

[2] The *transitive closure* of a binary relation R is the smallest extension $R^{+} \supseteq R$ satisfying the property that $xR^{+}y$ and $yR^{+}z$ together imply $xR^{+}z$.

- To say M is *locally well-ordered* (equivalently, that M is a *local well-order*) means that $\leq$ is locally well-ordered (i.e., it is well-founded and total on each connected component).

To say that a pointed plausibility model (M, w) satisfies one of the model-applicable adjectives above means that M itself satisfies the adjective in question.

Theorem 2.3 (Relationships between terminology)**.** Let $M = (W, \leq, V)$ be a plausibility model and $S \subseteq W$.

1. If M is finite, then M is well-founded.
2. If M is locally well-ordered, and S is a connected component, then
$$\min S = \{x \in S \mid \forall y \in S : x \leq y\} \ .$$
3. If M is well-ordered, then $\min S = \{x \in S \mid \forall y \in S : x \leq y\}$.
4. M is well-founded iff M is smooth.
5. M is well-ordered iff M is smooth and total.
6. M is locally well-ordered iff M is smooth and locally total.

Proof. See the appendix. □

From now on, in this paper we will restrict ourselves to locally well-ordered plausibility models, unless otherwise specified.

3 Conditional Doxastic Logic

3.1 Language and semantics

Definition 3.1 ($\mathscr{L}_{\mathsf{CDL}}$)**.** Let $\mathscr{P}$ be a fixed set of propositional letters. The language of *Conditional Doxastic Logic* consists of the set of $\mathscr{L}_{\mathsf{CDL}}$ formulas φ formed by the following grammar:

$$\varphi ::= \bot \mid p \mid (\varphi \to \varphi) \mid B^{\varphi}\varphi \qquad p \in \mathscr{P}$$

The logical constant $\top$ (for truth) and the various familiar Boolean connectives are defined by the usual abbreviations. Other important abbreviations: $B(\varphi|\psi)$ denotes $B^{\psi}\varphi$, and $B\varphi$ denotes $B^{\top}\varphi$.

The formula $B^{\psi}\varphi$ is read, "Conditional on ψ, the agent believes φ." Intuitively, this means that each of the most plausible ψ-worlds satisfies φ. The forthcoming semantics will clarify this further. The basic idea is that a belief conditional on ψ is a belief the agent would hold were she to minimally revise her beliefs so that she comes to believe ψ.

Definition 3.2 ($\mathscr{L}_{\mathsf{CDL}}$-truth)**.** Let $M = (W, \leq, V)$ be a locally well-ordered plausibility model. We define a binary satisfaction relation $\models$ between locally well-ordered pointed plausibility models (M, w) (written without surrounding parentheses) and $\mathscr{L}_{\mathsf{CDL}}$-formulas and we define a function $[\![-]\!] : \mathscr{L}_{\mathsf{CDL}} \to \wp(W)$ as follows.

- $[\![\varphi]\!]_M \coloneqq \{v \in W \mid M, v \models \varphi\}$. The subscript M may be suppressed.
- $M, w \not\models \bot$.
- $M, w \models p$ iff $p \in V(w)$ for $p \in \mathscr{P}$.

- $M, w \models \varphi \to \psi$ iff $M, w \not\models \varphi$ or $M, w \models \psi$.
- $M, w \models B^\psi \varphi$ iff for all $x \in \mathsf{cc}(w)$, we have

$$x^\downarrow \cap [\![\psi]\!] = \emptyset \quad \text{or} \quad \exists y \in x^\downarrow \cap [\![\psi]\!] : y^\downarrow \cap [\![\psi]\!] \subseteq [\![\varphi]\!] \ .$$

  $B^\psi \varphi$ holds at w iff for every world connected to w that has an equally or more plausible ψ-world y, the ψ-worlds that are equally or more plausible than y satisfy φ.

We extend the above so that we may have sets $S \subseteq \mathscr{L}_{\mathsf{CDL}}$ of formulas on the right-hand side:

$$M, w \models S \quad \text{means} \quad M, w \models \varphi \text{ for each } \varphi \in S \ .$$

Also, we will have occasion to use the following notion of *local consequence*: given a set $S \cup \{\varphi\} \subseteq \mathscr{L}_{\mathsf{CDL}}$ of formulas and writing $\mathfrak{P}_*$ to denote the class of pointed plausibility models,

$$S \models_\ell \varphi \quad \text{means} \quad \forall (M, w) \in \mathfrak{P}_* : M, w \models S \text{ implies } M, w \models \varphi \ .$$

Finally, we write $M \models \varphi$ to mean that $M, v \models \varphi$ for each world v in M ("φ is valid within M").

Remark 3.3 (Knowledge)**.** Baltag and Smets [4] read the abbreviation

$$K\varphi \quad := \quad B^{\neg\varphi}\bot$$

as "the agent knows φ." This notion of "knowledge" is based on the rejection of a proposed belief revision. In particular, $K\varphi = B^{\neg\varphi}\bot$ says that the most plausible $\neg\varphi$-worlds are $\bot$-worlds. The propositional constant $\bot$ for falsehood is true nowhere, so this amounts to us saying that the agent does not consider any $\neg\varphi$-worlds possible. Hence all the worlds she considers possible are φ-worlds. It is in this sense that we say she "knows" that φ is true: she will not revise her beliefs by $\neg\varphi$ (on pain of contradiction). It is easy to see that the semantics ensures that K so-defined is an $\mathsf{S5}$ modal operator: knowledge is closed under classical implication, what is known is true, it is known what is known, it is known what is not known, and all validities are known.

In well-founded plausibility models, belief in φ conditional on ψ is equivalent to having φ true at the most plausible ψ-worlds that are within the connected component of the actual world. And if the models are well-ordered, then we can omit mention of the connected component.

Theorem 3.4 ($\mathscr{L}_{\mathsf{CDL}}$-truth in well-founded models)**.** Let $M = (W, \leq, V)$ be a plausibility model.

(a) If M is well-founded: $M, w \models B^\psi \varphi \quad \Leftrightarrow \quad \min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$.

(b) If M is well-ordered: $M, w \models B^\psi \varphi \quad \Leftrightarrow \quad \min[\![\psi]\!] \subseteq [\![\varphi]\!]$.

Proof. See the appendix for the proof of (a). For (b), if M is well-ordered, then $\leq$ is total and therefore $\mathsf{cc}(w) = W$ for each $w \in W$. Apply (a). $\square$

The intended models for CDL are the well-ordered (and hence well-founded) plausibility models.

Definition 3.5 ($\mathscr{L}_{\mathsf{CDL}}$-validity)**.** To say that a $\mathscr{L}_{\mathsf{CDL}}$-formula φ is *valid*, written $\models \varphi$, means that we have $M \models \varphi$ for each well-ordered plausibility model M.

As per Theorem 3.4, restricting validity to the well-orders allows us to read $B^\psi \varphi$ as follows: "the most plausible ψ-worlds satisfy φ." While the intended models for CDL are well-ordered, and validity is defined accordingly (as per Definition 3.5), the following theorem shows that locally well-ordered plausibility models would suffice.

Axiom Schemes	
(CL)	Schemes for Classical Propositional Logic
(K)	$B^{\psi}(\varphi_1 \to \varphi_2) \to (B^{\psi}\varphi_1 \to B^{\psi}\varphi_2)$
(Succ)	$B^{\psi}\psi$
(IEa)	$B^{\psi}\varphi \to (B^{\psi\wedge\varphi}\chi \leftrightarrow B^{\psi}\chi)$
(IEb)	$\neg B^{\psi}\neg\varphi \to (B^{\psi\wedge\varphi}\chi \leftrightarrow B^{\psi}(\varphi \to \chi))$
(PI)	$B^{\psi}\chi \to B^{\varphi}B^{\psi}\chi$
(NI)	$\neg B^{\psi}\chi \to B^{\varphi}\neg B^{\psi}\chi$
(WCon)	$B^{\psi}\bot \to \neg\psi$

Rules

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}\,(\text{MP}) \qquad \frac{\varphi}{B^{\psi}\varphi}\,(\text{MN}) \qquad \frac{\psi \leftrightarrow \psi'}{B^{\psi}\varphi \leftrightarrow B^{\psi'}\varphi}\,(\text{LE})$$

Table 1. The theory CDL_0, a single-agent variant of Board's theory BRSIC [5]

Theorem 3.6 ($\mathscr{L}_{\mathsf{CDL}}$-validity with respect to local well-orders)**.** Let $\mathfrak{P}_L$ be the class of locally well-ordered plausibility models. For each $\varphi \in \mathscr{L}_{\mathsf{CDL}}$, we have:

$$\models \varphi \qquad \text{iff} \qquad \forall M \in \mathfrak{P}_L,\ M \models \varphi\ .$$

Proof. Right to left ("if"): obvious. Left to right ("only if"): assume $\models \varphi$ and take $M = (W, \leq, V) \in \mathfrak{P}_L$ and a world $w \in W$. Let M' be the sub-model of $M = (W', \leq', V')$ obtained by restricting to $\mathsf{cc}(w)$:

$$W' = \mathsf{cc}(w), \quad \leq' = \leq \cap (W' \times W'), \quad V'(v) = V(v) \text{ for } v \in W'.$$

Since $M \in \mathfrak{P}_L$, it follows that M' is well-ordered and therefore, since $\models \varphi$, we have $M', w \models \varphi$. It follows by a straightforward induction on the construction of $\mathscr{L}_{\mathsf{CDL}}$-formulas θ that $M', w \models \theta$ iff $M, w \models \theta$. Hence $M, w \models \varphi$. Since $w \in W$ and $M \in \mathfrak{P}_L$ were chosen arbitrarily, we conclude that $M \models \varphi$ for each $M \in \mathfrak{P}_L$. □

3.2 Board's theory BRSIC and CDL_0

The Hilbert theory of Conditional Doxastic Logic was first studied by Board [5] under the name BRSIC. Baltag and Smets [4] subsequently developed various alternative axiomatizations and extensions and introduced the name *Conditional Doxastic Logic*. The single-agent version of Board's theory BRSIC is equivalent to what we call CDL_0.

Definition 3.7 (CDL_0 theory)**.** CDL_0 is defined in Table 1.

CDL_0 is actually a simplification of BRSIC. In particular, BRSIC is a multi-agent theory for a nonempty set A of agents using a language similar to $\mathscr{L}_{\mathsf{CDL}}$ except that it has as primitives both conditional belief $B_a^{\psi}\varphi$ for each agent $a \in A$ and unconditional belief $B_a\varphi$ for each agent $a \in A$. Since the BRSIC axiom $B_a\varphi \leftrightarrow B_a^{\top}\varphi$ ("Triv") requires that unconditional belief be equivalent to conditional belief based on a tautological conditional, we have decided upon a streamlined language that contains conditional belief only. This allowed us to define away Board's axiom Triv in the following way (in Definition 3.1): let $B_a\varphi$ abbreviate $B_a^{\top}\varphi$. We have also renamed some of Board's axioms and rules: his Taut is now called (CL), his Dist is now called (K), his IE(a) is now called (IEa), his IE(b) is now called (IEb), his TPI is now called (PI), his

NPI is now called (NI), his RE is now called (MN), and all of his other axiom names have been enclosed in parenthesis. Finally, what we call (WCon) is the contrapositive of what Board called WCon. Restricting to a single-agent setting and thereby dropping subscripted agent names, we obtain the theory CDL_0.

Remark 3.8 (Multi-agent CDL_0)**.** A multi-agent version of CDL_0 is obtained by making trivial modifications to the language, axiomatization, and semantics of CDL_0. In particular, for a nonempty set A of agents, the multi-agent language $\mathscr{L}^A_{\mathsf{CDL}}$ is like the single-agent language $\mathscr{L}_{\mathsf{CDL}}$ except that each conditional belief operator B^ψ is replaced by a number of operators B^ψ_a, one for each agent $a \in A$. The multi-agent theory CDL^A_0 is obtained by adding a metavariable agent subscript a to each of the belief operators in Table 1. The models of CDL^A_0 are *multi-agent plausibility models*: these are like single-agent plausibility models presented above (in Definition 2.1) except that the preorder $\leq$ is replaced by a preorder $\leq_a$ for each agent $a \in A$. The definition of truth for $\mathscr{L}^A_{\mathsf{CDL}}$ on these models is like that in Definition 3.2 except that the meaning of $M, w \models B^\psi_a \varphi$ is changed so as to refer to the preorder $\leq_a$. Validity is defined with respect to the class of multi-agent plausibility models satisfying the property that each $\leq_a$ is locally well-ordered.

So, in essence, the multi-agent version consists of multiple single agent versions such that each agent's conditional beliefs are always restricted to the worlds connected (for that agent) to the given world w currently under consideration. It is clear that restricting to one agent a yields a framework that is equivalent to the version of CDL_0 we have presented here.

We note that the multi-agent version allows us to describe what one agent conditionally believes about what another agent conditionally believes. This is feature of interest in a wide variety of applications. However, from the technical perspective, the difference between the single- and multi-agent frameworks does not amount to too much in the way of mathematical shenanigans. It therefore suffices to indicate, as we have here, how the multi-agent version is obtained from the single-agent version and then restrict our study to the single-agent version. Of course, one may consult Board [5] for the fully specified account of the multi-agent theory BRSIC.

3.3 The theory CDL

It will be our task in this paper to develop a version of Conditional Doxastic Logic with justifications in the tradition of Justification Logic [2]. We will say more about this later, but for now it suffices to say that justifications in this tradition are meant to encode the individual reasoning steps that the agent uses to support her belief in one statement based on justifications she has for beliefs in other statements. In this way, justifications are supposed to present a stepwise explanation for how the agent derives complex beliefs from more basic ones. It is in this sense that justifications are "proof-like."

In order to make this precise, we require an axiomatization of Conditional Doxastic Logic that is more perspicious than is CDL_0 with regard to the ways in which conditional beliefs obtain. In particular, (IEa), (IEb), and (LE) are powerful principles that in fact encode a number of more basic principles and, as such, these powerful principles compress a number of reasoning steps into a small number of postulates. This is especially obvious with (LE): a belief conditional ψ may be replaced by a provably equivalent conditional ψ' in one step, which does not reflect the complexity of the derivation that was used to prove the equivalence $\psi \leftrightarrow \psi'$. From the point of view of the Justification Logic tradition, wherein justifications should explain in a stepwise fashion how one conditional belief follows from another, this is undesirable. Intuitively, if the agent believes φ conditional on ψ, then the reason she believes φ conditional on a provably equivalent ψ' depends crucially on the reasoning as to why ψ' is in fact equivalent to ψ. If π_1 and a more complex π_2 are proofs of this equivalence, then an agent who bases her belief on the more complex π_2 should have a correspondingly more complex justification witnessing her belief. We therefore require an alternative but equivalent axiomatization of the theory CDL_0 that makes such stepwise reasoning operations more explicit. The exact criteria we seek for such a theory are not precisely defined but are based on the authors' experience in working in the Justification Logic tradition. We call the theory we have settled upon CDL, and later we

Axiom Schemes	
(CL)	Schemes for Classical Propositional Logic
(K)	$B^{\psi}(\varphi_1 \to \varphi_2) \to (B^{\psi}\varphi_1 \to B^{\psi}\varphi_2)$
(Succ)	$B^{\psi}\psi$
(KM)	$B^{\psi}\bot \to B^{\psi\wedge\varphi}\bot$
(RM)	$\neg B^{\psi}\neg\varphi \to (B^{\psi}\chi \to B^{\psi\wedge\varphi}\chi)$
(Inc)	$B^{\psi\wedge\varphi}\chi \to B^{\psi}(\varphi \to \chi)$
(Comm)	$B^{\psi\wedge\varphi}\chi \to B^{\varphi\wedge\psi}\chi$
(PI)	$B^{\psi}\chi \to B^{\varphi}B^{\psi}\chi$
(NI)	$\neg B^{\psi}\chi \to B^{\varphi}\neg B^{\psi}\chi$
(WCon)	$B^{\psi}\bot \to \neg\psi$
Rules	
$\dfrac{\varphi \to \psi \quad \varphi}{\psi}$ (MP)	$\dfrac{\varphi}{B^{\psi}\varphi}$ (MN)

Table 2. The theory CDL

will explain how this theory gives rise to a theory of Conditional Doxastic Logic with justifications.

Definition 3.9 (CDL theory)**.** CDL is defined in Table 2.

The scheme (CL) of *Classical Logic* and the rule (MP) of *Modus Ponens* tell us that CDL is an extension of Classical Propositional Logic. The rule (MN) of *Modal Necessitation* tells us that derivable formulas hold in all conditional belief states.

Scheme (K) is just Kripke's axiom for our conditional belief operator B^{ψ}. The scheme (Succ) of *Success* says that every belief revision is always successful: if the agent revises her belief based on the information that ψ, then she will always arrive in a belief state in which ψ is one of her beliefs.

Making use of the definition of knowledge $K\varphi := B^{\neg\varphi}\bot$ from Remark 3.3, we can look at the following special case of the scheme (KM) of *Knowledge Monotonicity*:

$$B^{\neg\psi}\bot \to B^{\neg\psi\wedge\neg\varphi}\bot \ .$$

Since $\neg\neg\psi$ is equivalent to ψ and $\neg(\neg\psi \wedge \neg\varphi)$ is equivalent to $\psi \vee \varphi$, we may interpret the above instance of (KM) as telling us that knowledge is closed under disjunction: if ψ is known, then so is $\psi \vee \varphi$. But another interpretation perhaps better explains the word "Monotonicity" in the name of this scheme. Returning now to the official formulation

$$B^{\psi}\bot \to B^{\psi\wedge\varphi}\bot$$

of (KM), this scheme tells us that if we can conclude that a belief state conditional on ψ is contradictory, then conjunctively adding any further information φ yields a belief state conditional on $\psi \wedge \varphi$ that is still contradictory. Accordingly, the belief state is unchanged by the conjunctive addition of any further conditional information, and so the belief state is trivially "monotonic" in the conjunctive addition of conditional information.

The scheme (RM) of *Rational Monotonicity* permits a more subtle kind of conjunctive addition. This scheme,

$$\neg B^{\psi}\neg\varphi \to (B^{\psi}\chi \to B^{\psi\wedge\varphi}\chi) \ ,$$

says that if φ is *consistent* with the belief state conditional on ψ, then we may conjunctively add φ to our conditional without losing any beliefs from the original belief state. This is a non-trivial monotonicity: incorporating the information φ by forming the conditional $\psi \wedge \varphi$ yields a belief state that includes all the beliefs from the belief state conditional on ψ, but it may also include more.

The scheme (Inc) of *Inclusion* says that a belief state conditional on ψ includes every χ implied by φ whenever the belief state conditional on the conjunction $\psi \wedge \varphi$ includes χ. The scheme (Comm) of *Commutativity* says that the belief state conditional on a conjunction is invariant to the ordering of the conjuncts. The schemes (PI) of *Positive Introspection* and (NI) or *Negative Introspection* tell us that conditional beliefs are identical in all belief states. The scheme (WCon) of *Weak Consistency* tells us that belief revision is consistent with the actual state of affairs: if a revision by ψ yields a contradictory belief state, then ψ cannot be true.

Remark 3.10 (Classical reasoning (CR), modal reasoning (MR))**.** When discussing derivation in CDL, we will often suppress elementary reasoning steps familiar from the study of normal modal logics. Toward this end, "classical reasoning," which may be denoted by (CR), refers to a derivation with one or more steps that makes use solely of (CL) and (MP). "Modal reasoning," which may be denoted by (MR), refers to a derivation with one or more steps that makes use solely of (CL), (K), (MP), and (MN).

Theorem 3.11 (CDL-theorems)**.** The following schemes of (Cut), *Cautious Monotonicity* (CM), (Taut), (And), (Or), *Positive Reduction* (PR), and *Negative Reduction* (NR) are all derivable in CDL:

$$\begin{array}{rl}
\text{(Cut)} & B^{\psi}\varphi \to (B^{\psi\wedge\varphi}\chi \to B^{\psi}\chi) \\
\text{(CM)} & B^{\psi}\varphi \to (B^{\psi}\chi \to B^{\psi\wedge\varphi}\chi) \\
\text{(Taut)} & B\varphi \leftrightarrow B^{\top}\varphi \\
\text{(And)} & B^{\psi}\varphi_1 \to (B^{\psi}\varphi_2 \to B^{\psi}(\varphi_1 \wedge \varphi_2)) \\
\text{(Or)} & B^{\psi_1}\varphi \to (B^{\psi_2}\varphi \to B^{\psi_1\vee\psi_2}\varphi) \\
\text{(PR)} & B^{\varphi}B^{\psi}\chi \leftrightarrow (B^{\varphi}\bot \vee B^{\psi}\chi) \\
\text{(NR)} & B^{\varphi}\neg B^{\psi}\chi \leftrightarrow (B^{\varphi}\bot \vee \neg B^{\psi}\chi)
\end{array}$$

Also, the following rules of *(Left) Logical Equivalence* (LE), *Right Weakening* (RW), and *Supraclassicality* (SC) are all derivable in CDL:

$$\frac{\psi \leftrightarrow \psi'}{B^{\psi}\chi \leftrightarrow B^{\psi'}\chi}\,(\text{LE}) \qquad \frac{\chi \to \chi'}{B^{\psi}\chi \to B^{\psi}\chi'}\,(\text{RW}) \qquad \frac{\psi \to \chi}{B^{\psi}\chi}\,(\text{SC})$$

Proof. See the appendix. □

The following result shows that CDL and CDL_0 derive the same theorems and therefore that these theories are identical.

Theorem 3.12 (CDL-CDL_0 equivalence)**.** For each $\varphi \in \mathscr{L}_{\mathsf{CDL}}$:

$$\vdash_{\mathsf{CDL}} \varphi \quad \text{iff} \quad \vdash_{\mathsf{CDL}_0} \varphi \ .$$

Proof. See the appendix. □

That CDL is sound and complete with respect to the class of well-ordered plausibility follows by Theorem 3.12 and the results in Board's work [5]. However, we provide a full proof of this in the appendix because the details will be useful when we consider a justified version of CDL.

Theorem 3.13 (CDL soundness and completeness; [5])**.** For each $\varphi \in \mathscr{L}_{\mathsf{CDL}}$:

$$\vdash_{\mathsf{CDL}} \varphi \quad \text{iff} \quad \models \varphi \ .$$

Proof. See the appendix. □

4 AGM Belief Revision

The most influential theory of belief change is due to Alchourrón, Gärdenfors, and Makinson [1]. Their theory, commonly called to the "AGM theory," takes the view that an agent's belief state (or "database") is represented by a deductively closed set of sentences T called a "belief set." The agent is understood to believe exactly those sentences in her belief set T, and various operators on T are used to describe various kinds of changes in her belief state. Of particular interest is the *revision* operator, now often denoted using the symbol "$*$". This operator takes new information in the form of a sentence ψ and produces another belief set $T * \psi$ that contains ψ. Intuitively, the revision operation assumes that the incoming information ψ is completely trustworthy and so it should be incorporated into the database. However, simply adding ψ and taking the deductive closure, forming the *expansion*

$$T + \psi \coloneqq \mathsf{Cn}(T \cup \{\psi\}) \tag{1}$$

using an assumed consequence-closure operator $\mathsf{Cn}(-)$ underlying the setting, might lead to an inconsistent belief set. By this it is meant that $T + \psi$ might be logically inconsistent according to the logic governing $\mathsf{Cn}(-)$. Therefore, we cannot simply equate revision with expansion but must do something more clever so that the revised belief set $T * \psi$ not only contains ψ but is also consistent whenever ψ is consistent.

Instead of providing an exact procedure for computing revision, the AGM approach is "postulate based": a number of axiomatic postulates are provided, some intuitive justification is given as to why a revision operator should satisfy each of the postulates, and any operation on belief sets that satisfies all of the postulates is said to be an *AGM revision operator*. So in principle, there are many revision operators, and each is to be studied from an axiomatic point of view using the AGM revision postulates.

Following the exposition of AGM theory from [7] (but with some minor modifications), we begin with a set $\mathscr{P}$ of propositional letters (usually countable). The set $\mathscr{L}_{\mathsf{CPL}}$ of formulas (the "propositional formulas" or "formulas of Classical Propositional Logic") consists of those expressions that can be built up from the propositional letters and the Boolean constants $\bot$ (falsehood) and $\top$ (truth) using the usual Boolean connectives. A deductive theory is assumed, and this theory is specified in terms of a Tarskian consequence operator: for any set S of formulas, $\mathsf{Cn}(S)$ is the set of logical consequences of S. It is assumed that $\mathsf{Cn}(-)$ satisfies the following conditions:

- Inclusion: $S \subseteq \mathsf{Cn}(S)$,
- Monotony (also sometimes called "Monotonicity"): $S \subseteq S'$ implies $\mathsf{Cn}(S) \subseteq \mathsf{Cn}(S')$,
- Iteration: $\mathsf{Cn}(S) = \mathsf{Cn}(\mathsf{Cn}(S))$, and
- Supraclassicality: $\mathsf{Cn}(S)$ contains each classical tautology in $\mathscr{L}_{\mathsf{CPL}}$.

It is usually assumed that $\mathsf{Cn}(-)$ also satisfies the following conditions:

- Deductive Consistency: $\bot \notin \mathsf{Cn}(\emptyset)$; and
- Compactness: $\varphi \in \mathsf{Cn}(S)$ iff there exists a finite $S' \subseteq S$ such that $\varphi \in \mathsf{Cn}(S')$.

The alternative notation $S \vdash \varphi$ is used to express $\varphi \in \mathsf{Cn}(S)$. A *belief base* is a set of formulas in the language, and a *belief set* is a deductively closed belief base (i.e., $\mathsf{Cn}(S) = S$). Note that the phrase "belief state" (a.k.a., "database") is an intuitive notion meant to describe the agent's situation with regard to her beliefs. This intuitive notion is formalized either by a belief base (not necessarily deductively closed) or a belief set (necessarily deductively closed). A belief base T' gives rise to a belief set T by applying the consequence operator: $T \coloneqq \mathsf{Cn}(T')$.

A revision operator is meant to take an existing belief set T and some incoming information ψ and produce a new belief set $T * \psi$ that incorporates the incoming information ψ (i.e., $\psi \in T * \psi$), is consistent

Postulates of AGM Belief Revision	
1. Closure:	$T * \psi = \mathsf{Cn}(T * \psi)$
2. Success:	$\psi \in T * \psi$
3. Inclusion:	$T * \psi \subseteq T + \psi$
4. Vacuity:	if $\neg\psi \notin T$, then $T * \psi = T + \psi$
5. Consistency:	if $\neg\psi \notin \mathsf{Cn}(\emptyset)$, then $\bot \notin \mathsf{Cn}(T * \psi)$
6. Extensionality:	if $(\psi \leftrightarrow \psi') \in \mathsf{Cn}(\emptyset)$, then $T * \psi = T * \psi'$
7. Superexpansion:	$T * (\psi \wedge \varphi) \subseteq (T * \psi) + \varphi$
8. Subexpansion:	if $\neg\varphi \notin \mathsf{Cn}(T * \psi)$, then $(T * \psi) + \varphi \subseteq T * (\psi \wedge \varphi)$

Notes: $S + \psi \coloneqq \mathsf{Cn}(S \cup \{\psi\})$;
$\mathsf{Cn}(-)$ satisfies Inclusion, Monotony, Iteration, and Supraclassicality;
Vacuity may be called "Preservation."

Table 3. The AGM revision postulates (as presented in [7])

whenever the incoming information ψ is consistent (i.e., $T * \psi \nvdash \bot$ if ψ is consistent), and is obtained from T by way of a "minimal change." The latter is an intuitive (and non-formalized) guiding principle that is used to persuade the reader that certain proposed postulates are desirable. From a formal perspective, it can be safely ignored.

The AGM revision postulates are reproduced in Table 3. Postulates 1–6 are called the *Gärdenfors postulates* (or, more elaborately, the "basic Gärdenfors postulates for revision"). Postulates 7–8 are called the *supplementary postulates.*

Grove [6] proposed a possible worlds modeling of the AGM postulates. Modulo certain details we gloss over, his proposal essentially amounts to this: represent the agent's belief set using the minimal worlds of a well-ordered plausibility model and define revision in terms of belief conditionalization.

Theorem 4.1 (AGM revision and Grove spheres; adapted from [6])**.** Let $\mathsf{Cn}(-)$ be the consequence function $\mathsf{CPL}(-)$ of Classical Propositional Logic over the language $\mathscr{L}_{\mathsf{CPL}}$. For each well-ordered plausibility model M, each belief set $T \subseteq \mathscr{L}_{\mathsf{CPL}}$, and each propositional formula $\psi \in \mathscr{L}_{\mathsf{CPL}}$, define

$$\begin{aligned} M^{\downarrow} &\coloneqq \{\varphi \in \mathscr{L}_{\mathsf{CPL}} \mid \min(W) \subseteq [\![\varphi]\!]_M\} \ , \\ T *_M \psi &\coloneqq \{\varphi \in \mathscr{L}_{\mathsf{CPL}} \mid \min[\![\psi]\!]_M \subseteq [\![\varphi]\!]_M\} \ , \\ \mathfrak{B}_{\mathsf{CPL}} &\coloneqq \{L \subseteq \mathscr{L}_{\mathsf{CPL}} \mid \mathsf{CPL}(L) = L\} \ . \end{aligned}$$

Note that $\mathfrak{B}_{\mathsf{CPL}}$ is the collection of all propositional belief sets. For a propositional formula or set of propositional formulas $X \in \mathscr{L}_{\mathsf{CPL}} \cup \wp(\mathscr{L}_{\mathsf{CPL}})$, to say X is *consistent* means $\bot \notin \mathsf{CPL}(X)$ (or equivalently, that $\mathsf{CPL}(X) \neq \mathscr{L}_{\mathsf{CPL}}$), and to say X is *inconsistent* means it is not consistent. To say that a plausibility model M is a *system of spheres* means that M is well-ordered and the function

$$M^{\downarrow} *_M (-) : \mathscr{L}_{\mathsf{CPL}} \to \mathfrak{B}_{\mathsf{CPL}}$$

mapping propositional formulas $\psi \in \mathscr{L}_{\mathsf{CPL}}$ to belief sets $M^{\downarrow} *_M \psi \in \mathfrak{B}_{\mathsf{CPL}}$ satisfies the AGM revision postulates. To say that M is a *Grove system* for a belief set $T \in \mathfrak{B}_{\mathsf{CPL}}$ means that M is a system of spheres and $M^{\downarrow} = T$.

(a) Each consistent belief set $T \in \mathfrak{B}_{\mathsf{CPL}}$ has a Grove system.

(b) Suppose for a function $* : \mathfrak{B}_{\mathsf{CPL}} \times \mathscr{L}_{\mathsf{CPL}} \to \mathfrak{B}_{\mathsf{CPL}}$ and each $(T, \psi) \in \mathfrak{B}_{\mathsf{CPL}} \times \mathscr{L}_{\mathsf{CPL}}$ we have:

(i) if T is inconsistent, then $T * \psi = M_*^{\downarrow} *_{M_*} \psi$ for some fixed system of spheres M_*; and

(ii) if T is consistent, then $T * \psi = T *_{M_T} \psi$ for some fixed Grove system M_T for T.

It follows that $*$ is an AGM revision operator.

Proof. See the appendix. □

This suggests that we may view CDL as a version of AGM belief revision in which the revision process itself can be described in the language [5, 4]. In particular, for propositional formulas φ and ψ, the formula $B\varphi$, which is our abbreviation for $B^{\top}\varphi$, says that the agent believes φ before the revision takes place; and the formula $B^{\psi}\varphi$ says that the agent believes φ after revision by ψ. So by restricting to propositional φ and ψ, we can use conditional belief formulas $B^{\psi}\varphi$ to describe a version of the AGM revision process directly in the language of $\mathscr{L}_{\mathsf{CDL}}$. This leads us to the following overview of what the axiomatic theory CDL (Table 2) has to say about the CDL-based version of AGM revision.

- (CL) and (MP) indicate that we use a classical meta-theory.
- The principles we have grouped together under the name "modal reasoning"—(CL), (MP), (K), and (MN)—together correspond to AGM Closure and AGM Extensionality. In addition, it follows by modal reasoning that our underlying "consequence operator" satisfies Inclusion, Monotony, Iteration, and Supraclassicality.
- (Succ) corresponds to AGM Success.
- Under the assumption of (Succ), scheme (KM) corresponds to consequence of AGM Consistency: inconsistency of revision by ψ implies inconsistency of ψ and therefore of $\psi \wedge \varphi$, and so inconsistency of revision by $\psi \wedge \varphi$ follows via (Succ).
- Under the assumption of (Succ), scheme (RM) corresponds to AGM Subexpansion. And if $\varphi = \top$, then (Succ) and (RM) together correspond to AGM Vacuity.
- (Inc) corresponds to AGM Subexpansion. And, if $\varphi = \top$, then (Inc) corresponds to AGM Inclusion.
- (Comm) corresponds to a special case of AGM Extensionality (i.e., commutativity of conjunction).
- (PI) and (NI) do not corresponds to principles in the AGM setting (belief sets are subset of $\mathscr{L}_{\mathsf{CPL}}$).
- (WCon) corresponds to AGM Consistency.

5 Justified Conditional Doxastic Logic

Though CDL may be viewed as a version of AGM revision in which Boolean combinations of revisions are expressible in the language, one key aspect that is missing: the *reasons* as to why revisions result in a state in which certain formulas are believed. This is a deficit present also in AGM revision: while a revision by ψ may lead to a belief set that includes φ, it is not immediately clear why it is that φ ought to obtain. What is missing is some language-describable reason that explains how it is φ came about as a result of the revision. It shall be our task in this section to study how we might "fill in" these reasons in a theory based on CDL. Our approach follows the general paradigm of Justification Logic [2], where modal operators are replaced by syntactically structured objects called *terms*. Terms are meant to suggest "reasons" in the sense that the syntactic structure of a term accords with certain derivational principles in the underlying logic. Our goal is to adapt this methodology to CDL. In particular, for a term t, we introduce new formulas

$$t :^{\psi} \varphi$$

with the intended meaning that, whenever the agent revises her belief state by incorporating the formula ψ, then t will be a reason justifying her belief of φ in the resulting belief state. Intuitively, the formula $t :^{\psi} \varphi$ tells us two things. First, it tells us that the agent believes φ conditional on ψ, which was the information conveyed in CDL by the formula $B^{\psi}\varphi$. Second, something new: the formula $t :^{\psi} \varphi$ tells us that the reason encoded by t supports φ. Taken together, $t :^{\psi} \varphi$ tells us that the agent has a reason-based belief of φ after revising by ψ.

Terms will be built up using a simple grammar. At the base of this grammar are the "certificates," which are terms of the form c_{φ} for some formula φ. Intuitively, whenever we have one or more reasons in support of φ, the certificate c_{φ} picks out the "best" one. So whenever φ has any support at all, we are always guaranteed that c_{φ} names a particular reason in support of φ. As such, the theory we eventually define will derive the principle

$$t :^{\psi} \varphi \to c_{\varphi} :^{\psi} \varphi \ ,$$

which says that the certificate c_{φ} supports φ after a revision by ψ whenever there is some reason t that support φ after the same revision. In essence, certificates allow us to "forget" the details of a complex argument in support of some assertion, remembering only that we at some point found such an argument.

Though a certificate c_{φ} must support the formula φ it certifies, as per the above-mentioned derivable principle, we do not prevent c_{φ} from supporting other formulas as well. For example, it is consistent with the theory we will develop for us to have $c_{\varphi} :^{\psi} \chi$ for some $\chi \neq \varphi$. As such, though we require certificates to provide "best evidence" for the formulas they certify, we do not require that this be the only evidence that they provide.

Other terms are formed from certificates using one of two operators. The first is the *Application operator* "$\cdot$" from Justification Logic. This operator is used to indicate that terms are to be combined using a single step of the rule of *Modus Ponens*. In particular, the logic we will develop will derive the following principle:

$$t :^{\psi} (\varphi_1 \to \varphi_2) \to (s :^{\psi} \varphi_1 \to (t \cdot s) :^{\psi} \varphi_2) \ .$$

This says that if t supports an implication after the revision by ψ and s supports the antecedent after the same revision, then the combination $t \cdot s$ supports the consequence after that revision. This is the reason-explicit version of the principle (K) of CDL. The difference is that the present version tells us something about how we obtained the consequent: the form of $t \cdot s$, with t to the left and s to the right, indicates that t supports an implication and s supports the antecedent, and hence we were able to derive the consequent via one step of Modus Ponens in virtue of the fact that we use a single instance of the Application operator "$\cdot$" to combine t with s to form $t \cdot s$.

The second term-combining operator we introduce is the *Sum operator* "+" from Justification Logic. This operator allows us to combine to reasons in a way that preserves support. In particular, the logic we develop will derive the following principle:

$$(t :^{\psi} \varphi \vee s :^{\psi} \varphi) \to (t + s) :^{\psi} \varphi \ .$$

This says that $t + s$ supports φ whenever at least one of t or s does so. As such, the sum $t + s$ combines the supported statements of t and of s without performing logical inference.

Formulas of the language will be built up from the language of Classical Propositional Logic (based on propositional letters, the constant $\bot$ for falsehood, and material implication) by adding formulas of the form $t :^{\psi} \varphi$, where t is a term and φ and ψ are other formulas. Intuitively, $t :^{\psi} \varphi$ says that t supports the agent's belief of φ after she revises her beliefs by incorporating ψ.

The theory we shall define is called *Justified Conditional Doxastic Logic* or JCDL. In addition to the derivable principles mentioned above, JCDL has a number of additional principles that make explicit the revision-based principles of CDL. In particular, we will see that every CDL-principle gives rise to a corresponding JCDL-principle, and the other way around as well. We explain this in more detail after the main definitions are in place.

5.1 Language and axiomatics

Definition 5.1 ($\mathscr{L}_{\mathsf{JCDL}}$)**.** Let $\mathscr{P}$ be a fixed set of propositional letters. The language of *Justified Conditional Doxastic Logic* consists of the set $\mathscr{L}_{\mathsf{JCDL}}$ of formulas φ and set $\mathscr{T}_{\mathsf{JCDL}}$ of terms t formed by the following grammar:

$$\varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid t :^{\varphi} \varphi \qquad p \in \mathscr{P}$$

$$t ::= c_{\varphi} \mid (t \cdot t) \mid (t + t)$$

Standard abbreviations for Boolean constants and connectives are used, and parentheses are dropped when doing so will cause no confusion. We adopt the following key abbreviation:

$$\dot{B}^{\psi}\varphi \quad \text{denotes} \quad c_{\varphi} :^{\psi} \varphi \ .$$

We may write $c(\varphi)$ as an abbreviation for c_{φ} when convenient. Also, we let $t : \varphi$ abbreviate $t :^{\emptyset} \varphi$, and we let $\dot{B}\varphi$ abbreviate $\dot{B}^{\emptyset}\varphi$.

Roughly speaking, it will be useful to think of $\dot{B}^{\psi}\varphi$ as the JCDL-analog of the CDL-expression $B^{\psi}\varphi$. We will see that every CDL-principle gives rise to a JCDL-principle obtained by replacing modal operators "B^{ψ}" by reason-based operators "$t :^{\psi}$". Accordingly, if $B^{\psi}\varphi$ is CDL-derivable, then a corresponding $t :^{\psi'} \varphi'$ will be derivable, where ψ' corresponds to ψ and φ' corresponds to φ). So using the certificate $c_{\varphi'}$ for φ', it will follow that $\dot{B}^{\psi'}\varphi'$ is derivable as well. It is in this sense that $\dot{B}^{\psi'}\varphi'$ may be thought of as the JCDL-analog of the CDL-principle $B^{\psi}\varphi$.

Definition 5.2 (JCDL theory)**.** The theory JCDL is defined in Table 4.

(CL) and (MP) tell us that JCDL is an extension of Classical Propositional Logic.

(eK) is our reason-explicit analog of the CDL-scheme (K); it tells us that reason support is closed under the rule of Modus Ponens using the Application operator "$\cdot$". (eSucc) is our reason-explicit analog of CDL (Succ); it tells us that certificates are used to certify the success of belief revisions. (eKM) is our reason-explicit version of (KM); it tells us that a reason supporting a contradiction conditional on some ψ continues to do so no matter what additional information we conjunctively add to the conditional.

(eRM) is our reason-explicit version of (RM). The antecedent $\neg\dot{B}^{\psi}\neg\varphi$ is just $lnotc_{\neg\varphi} :^{\psi} \neg\varphi$. Since the certificate $c_{\neg\varphi}$ will always stand in for some argument relevant to $\neg\varphi$, the antecedent $\neg\dot{B}^{\psi}\neg\varphi$ tells us that φ is consistent with the belief state obtained after revision by ψ. And (eRM) tells us that if this is so and t is a reason to believe χ after revision by ψ, then it follows that t is still a reason to believe χ even after we revise by the conjunction $\psi \wedge \varphi$. Notice that the reason t for χ does not change; it is only the revision formula itself that changes.

(eInc) is our reason-explicit version of (Inc); it tells us that that a belief of $\varphi \to \psi$ is certified after revising by ψ so long as there is a reason t for believing ψ after revising by $\psi \wedge \varphi$. (eComm) tells us that reasons are invariant to the order of conjuncts in revisions; this is the reason-explicit analog of (Comm). (ePI) and (eNI) are the reason-explicit versions of (PI) and (NI), respectively; these tell us that all support and negated support statements are certified. (eWCon) is the reason-explicit analog of (WCon); it tells us that if t is a reason supporting a belief in $\bot$ after revision by ψ, then $\neg\psi$ must have been true.

(eA) is particular to JCDL. This scheme tells us that if t supports a belief in φ conditional on ψ and φ is also certified conditional on some other χ, then t must itself support φ conditional on χ as well. This

Axiom Schemes

(CL)	Schemes for Classical Propositional Logic
(eCert)	$t:^{\psi}\varphi \to \dot{B}^{\psi}\varphi$
(eK)	$t:^{\psi}(\varphi_1 \to \varphi_2) \to (s:^{\psi}\varphi_1 \to (t\cdot s):^{\psi}\varphi_2)$
(eSum)	$(t:^{\psi}\varphi \vee s:^{\psi}\varphi) \to (t+s):^{\psi}\varphi$
(eSucc)	$\dot{B}^{\psi}\psi$
(eKM)	$t:^{\psi}\bot \to t:^{\psi\wedge\varphi}\bot$
(eRM)	$\neg\dot{B}^{\psi}\neg\varphi \to (t:^{\psi}\chi \to t:^{\psi\wedge\varphi}\chi)$
(eInc)	$t:^{\psi\wedge\varphi}\chi \to \dot{B}^{\psi}(\varphi \to \chi)$
(eComm)	$t:^{\psi\wedge\varphi}\chi \to t:^{\varphi\wedge\psi}\chi$
(ePI)	$t:^{\psi}\chi \to \dot{B}^{\varphi}(t:^{\psi}\chi)$
(eNI)	$\neg t:^{\psi}\chi \to \dot{B}^{\varphi}(\neg t:^{\psi}\chi)$
(eWCon)	$t:^{\psi}\bot \to \neg\psi$
(eA)	$t:^{\psi}\varphi \to (\dot{B}^{\chi}\varphi \to t:^{\chi}\varphi)$

Rules

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}\,(\mathrm{MP}) \qquad\qquad \frac{\varphi}{\dot{B}^{\psi}\varphi}\,(\mathrm{eMN})$$

Note: $\dot{B}^{\psi}\varphi$ abbreviates $c_{\varphi}:^{\psi}\varphi$.

Table 4. The theory JCDL

tells us that if a reason supports a belief of φ after some revision, it does so no matter the particulars of the revision. Said another way, reason support depends only on the statement supported (and not on the revision).

(eMN) says that every derivable formula is "certified" (i.e., supported by its certificate). This corresponds to the CDL-principle (MN). But while all derivable principles are certified, intuitively such certification omits a great deal of information; in particular, it is not clear from which axioms a given principle follows and how it is that it follows by way of the rules of the theory. Toward this end, it will be useful to consider a restriction of (eMN) in which we apply this rule only to axioms, possibly multiple times in a row. An axiom to which we apply (eMN) zero or more times in a row will be called a "possibly necessitated axiom." We will see that if we remove (eMN) from the theory, then all JCDL-theorems can be derived from the possibly necessitated axioms using (MP) as the only rule. The trick to this will be to eliminate from JCDL-derivations uses of (eMN) that are "troublesome necessitations": these are the derivable consequences of (eMN) that are not themselves possibly necessitated axioms. If we can show that all such "troublesome necessitations" can be eliminated, then the result follows. This will be our task now.

The terms that can be formed from certificates of possibly necessitated axioms using the Application operator "$\cdot$" will be called the "logical terms." These terms play a special role: every JCDL-theorem φ gives rise to a logical term t that supports φ (i.e., $t :^{\psi} \varphi$ is derivable for each ψ).

Definition 5.3 (Necessitations, logical terms)**.** A *necessitation* is a $\mathscr{L}_{\mathsf{JCDL}}$-formula of the form

$$\underbrace{\dot{B}^{\psi_n} \dot{B}^{\psi_{n-1}} \dot{B}^{\psi_{n-2}} \cdots \dot{B}^{\psi_1}}_{\text{zero or more of these}} \varphi \tag{2}$$

for some integer $n \geq 0$. A *possibly necessitated axiom* is a $\mathscr{L}_{\mathsf{JCDL}}$-formula of the form (2) for which $n \geq 0$ and φ is a JCDL-axiom. The set $\mathscr{T}^{L}_{\mathsf{JCDL}}$ of *logical terms* is the smallest set that contains certificates c_φ for each possibly necessitated axiom φ and is closed under the term-forming operation $t, s \mapsto t \cdot s$.

Definition 5.4 (Troublesome necessitations, notation $\pi \vdash^{n}_{\mathsf{JCDL}} \varphi$)**.** A JCDL-*derivation* is finite nonempty sequence of $\mathscr{L}_{\mathsf{JCDL}}$-formulas, each of which is either a JCDL-axiom or follows by a JCDL-rule from formulas occurring earlier in the sequence. A *line* of a JCDL-derivation π is an element of the sequence π. A *troublesome necessitation* is a line φ of a JCDL-derivation that neither is a possibly necessitated axiom nor follows from earlier lines by (MP). Clearly, a troublesome necessitation must follow by applying (eMN) to an earlier line that is not itself a possibly necessitated axiom. For $n \in \mathbb{N}$, we write $\pi \vdash^{n}_{\mathsf{JCDL}} \varphi$ to mean that π is a JCDL-derivation that contains at most n troublesome necessitations and whose last line is φ. For $n \in \mathbb{N}$, we write $\vdash^{n}_{\mathsf{JCDL}} \varphi$ to mean that there exists a JCDL-derivation π such that $\pi \vdash^{n}_{\mathsf{JCDL}} \varphi$. Obviously, $\vdash_{\mathsf{JCDL}} \varphi$ implies $\vdash^{n}_{\mathsf{JCDL}} \varphi$ for some $n \in \mathbb{N}$.

The following lemma shows that every JCDL-theorem can be derived from the possibly necessitated axioms using (MP) as the only rule of inference.

Lemma 5.5 (Elimination of troublesome necessitations)**.** For each $\varphi \in \mathscr{L}_{\mathsf{JCDL}}$, we have:

$$\pi \vdash^{n}_{\mathsf{JCDL}} \varphi \quad \Rightarrow \quad \exists \pi_* \supseteq \pi, \;\; \pi_* \vdash^{0}_{\mathsf{JCDL}} \varphi \ . \tag{3}$$

Proof. See the appendix. □

"Theorem Internalization" is a property of Justification Logics whereby every theorem φ of the logic is witnessed by supporting term. Since certificates trivially support JCDL-theorems by (eMN), the usual formulation of Theorem Internalization is trivialized in our setting. However, we can prove a stronger variant: every JCDL-theorem φ is witnessed by a *logical* supporting term. This stronger version tells us that all JCDL-theorems are witnessed by terms that refer only to possibly necessitated axioms and combinations of these using rule (MP).

Theorem 5.6 (JCDL Theorem Internalization)**.** For each $\psi \in \mathscr{L}_{\mathsf{JCDL}}$, we have:

$$\vdash_{\mathsf{JCDL}} \varphi \quad \Rightarrow \quad \exists t \in \mathscr{T}^L_{\mathsf{JCDL}},\ \vdash_{\mathsf{JCDL}} t{:}^{\psi}\varphi\ .$$

Proof. See the appendix. □

5.2 Relationship to CDL

In our motivation of JCDL, we have described the formula $t{:}^{\psi}\varphi$ as an analog of a corresponding $\mathscr{L}_{\mathsf{CDL}}$-formula $B^{\psi}\varphi$. Up to this point, the idea was mere intuition. We now make this intuition precise by defining two mappings. The first, called "forgetful projection," maps $\mathscr{L}_{\mathsf{JCDL}}$-formulas to $\mathscr{L}_{\mathsf{CDL}}$-formulas by replacing each "$t{:}^{\psi}$" prefix by the prefix "B^{ψ}". The second, called "trivial realization," maps $\mathscr{L}_{\mathsf{CDL}}$-formulas to $\mathscr{L}_{\mathsf{JCDL}}$-formulas by replacing each prefix "B^{ψ}" by the prefix "$\dot{B}^{\psi}$". We will see that these operations preserve derivability of schemes.

Definition 5.7 (The forgetful projection)**.** The *forgetful projection* is the function

$$(-)^{\circ} : \mathscr{L}_{\mathsf{JCDL}} \to \mathscr{L}_{\mathsf{CDL}}$$

defined by:

$$\begin{aligned} q^{\circ} &\coloneqq q \quad \text{for } q \in \mathscr{P} \cup \{\bot\} \\ (\varphi \to \psi)^{\circ} &\coloneqq \varphi^{\circ} \to \psi^{\circ} \\ (t{:}^{\psi}\varphi)^{\circ} &\coloneqq B^{\psi^{\circ}}\varphi^{\circ} \end{aligned}$$

The *forgetful projection* of $\psi \in \mathscr{L}_{\mathsf{JCDL}}$ is the $\psi^{\circ} \in \mathscr{L}_{\mathsf{CDL}}$. Extend the function $(-)^{\circ}$ to sets $\Gamma \subseteq \mathscr{L}_{\mathsf{JCDL}}$ by $\Gamma^{\circ} \coloneqq \{\varphi^{\circ} \mid \varphi \in \Gamma\}$. We further extend the function $(-)^{\circ}$ to schemes. In particular, let $\mathcal{S}$ be a fixed set of schematic variables (i.e., "metavariables" or placeholders for formulas) that includes all schematic variables used in this paper and that has cardinality $\min\{|\mathscr{P}|, \omega\}$. For each of our languages $\mathscr{L} \in \{\mathscr{L}_{\mathsf{CDL}}, \mathscr{L}_{\mathsf{JCDL}}\}$, let $\mathscr{L}(\mathcal{S})$ be the set of formula schemes that can be formed using the formula formation grammar of $\mathscr{L}$ but with schematic variables in $\mathcal{S}$ used in place of propositional letters in $\mathscr{P}$. Define $\mathscr{T}_{\mathsf{JCDL}}(\mathcal{S})$ similarly. Using Φ and Ψ as metavariables ranging over members of $\mathscr{L}_{\mathsf{JCDL}}(\mathcal{S})$ and T as a metavariable ranging over members of $\mathscr{T}_{\mathsf{JCDL}}(\mathcal{S})$, let

$$\begin{aligned} X^{\circ} &\coloneqq X \quad \text{for } X \in \mathcal{S} \cup \{\bot\} \\ (\Phi \to \Psi)^{\circ} &\coloneqq \Phi^{\circ} \to \Psi^{\circ} \\ (T{:}^{\Psi}\Phi)^{\circ} &\coloneqq B^{\Psi^{\circ}}\Phi^{\circ} \end{aligned}$$

Extend the function $(-)^{\circ}$ to sets $\Gamma \subseteq \mathscr{L}_{\mathsf{JCDL}}(\mathcal{S})$ by $\Gamma^{\circ} \coloneqq \{\Phi^{\circ} \mid \Phi \in \Gamma\}$.

Definition 5.8 (Realizations and the trivial realization)**.** A *realization* of a formula $\varphi \in \mathscr{L}_{\mathsf{CDL}}$ is a formula $\psi \in \mathscr{L}_{\mathsf{JCDL}}$ for which $\psi^{\circ} = \varphi$ and $\vdash_{\mathsf{JCDL}} \psi$. The *trivial realization* is the function $(-)^t : \mathscr{L}_{\mathsf{CDL}} \to \mathscr{L}_{\mathsf{JCDL}}$ defined by:

$$\begin{aligned} q^{t} &\coloneqq q \quad \text{for } q \in \mathscr{P} \cup \{\bot\} \\ (\varphi \to \psi)^{t} &\coloneqq \varphi^{t} \to \psi^{t} \\ (B^{\psi}\varphi)^{t} &\coloneqq \dot{B}^{\psi^{t}}\varphi^{t} \end{aligned}$$

We extend the function $(-)^t$ to sets $\Gamma \subseteq \mathscr{L}_{\mathsf{CDL}}$ by $\Gamma^t \coloneqq \{\varphi^t \mid \varphi \in \Gamma\}$. As in Definition 5.7 and using the notation from that definition, we extend the function $(-)^t$ to schemes:

$$X^{t} \coloneqq X \quad \text{for } X \in \mathcal{S} \cup \{\bot\}$$

$$(\Phi \to \Psi)^t := \Phi^t \to \Psi^t$$
$$(B^\Psi \Phi)^t := \dot{B}^{\Psi^t} \Phi^t$$

Finally, we apply $(-)^t$ to sets $\Gamma \subseteq \mathscr{L}_{\mathsf{CDL}}(\mathcal{S})$ by defining $\Gamma^t := \{\Phi^t \mid \Phi \in \Gamma\}$. For some object Z in the domain of $(-)^t$ we say that Z^t is the *trivial realization* of Z.

The trivial realization of a formula or scheme of CDL is obtained by replacing each "B" with "$\dot{B}$". Note that while the word "realization" in the phrase "trivial realization" suggests that the trivial realization of a formula or scheme is indeed a realization (i.e., we obtain something derivable in JCDL), this does not come automatically (i.e., by definition) because the trivial realization is a mere syntactic translation and so it must be proved that this translation satisfies the requisite property before the conclusion can be drawn. However, the following theorem guarantees that the trivial realization is indeed a realization. The theorem also tells us that the forgetful projection maps JCDL-theorems to CDL-theorems.

Theorem 5.9 (Projection and realization for schemes)**.** We use the notation from Definitions 5.7 and 5.8. For each $\Gamma \cup \{\varphi\} \subseteq \mathscr{L}_{\mathsf{JCDL}}(\mathcal{S})$ and each $\Delta \cup \{\psi\} \subseteq \mathscr{L}_{\mathsf{CDL}}(\mathcal{S})$, we have:

1. $\Gamma \vdash_{\mathsf{JCDL}} \varphi$ implies $\Gamma^\circ \vdash_{\mathsf{CDL}} \varphi^\circ$, and
2. $\Delta \vdash_{\mathsf{CDL}} \psi$ implies $\Delta^t \vdash_{\mathsf{JCDL}} \psi^t$.

Proof. See the appendix. □

Theorem 5.9 tells us that JCDL really is an explicit analog of CDL: every JCDL-derivable statement gives rise to a CDL-derivable statement (by forgetful projection), and every CDL-derivable statement gives rise to a JCDL-derivable statement (by trivial realization). This link makes precise our intuition that the $\mathscr{L}_{\mathsf{CDL}}$-formula $B^\psi\varphi$ should correspond to the $\mathscr{L}_{\mathsf{JCDL}}$-formula $\dot{B}^\psi\varphi$.

Though Theorem 5.9 is stated with respect to schemes, we have the analogous result for formulas as well.

Theorem 5.10 (Projection and realization for formulas)**.** We use the notation from Definitions 5.7 and 5.8. For each $\Gamma \cup \{\varphi\} \subseteq \mathscr{L}_{\mathsf{JCDL}}$ and each $\Delta \cup \{\psi\} \subseteq \mathscr{L}_{\mathsf{CDL}}$, we have:

1. $\Gamma \vdash_{\mathsf{JCDL}} \varphi$ implies $\Gamma^\circ \vdash_{\mathsf{CDL}} \varphi^\circ$, and
2. $\Delta \vdash_{\mathsf{CDL}} \psi$ implies $\Delta^t \vdash_{\mathsf{JCDL}} \psi^t$.

Proof. Replace each distinct propositional letter with a distinct schematic variable, apply Theorem 5.9, and take the instances of the resulting derivable schemes obtained by substituting the original propositional variables back into their corresponding positions. □

Using Theorem 5.9, the CDL-principles from Theorem 3.11 translate into JCDL-principles.

Theorem 5.11 (JCDL-theorems)**.** The following schemes of (eCut), *Explicit Cautious Monotonicity* (eCM), (eTaut), (eAnd), (eOr), *Explicit Positive Reduction* (ePR), and *Explicit Negative Reduction* (eNR) are all derivable in JCDL:

$$\begin{array}{rl}
\text{(eCut)} & \dot{B}^\psi\varphi \to (\dot{B}^{\psi\wedge\varphi}\chi \to \dot{B}^\psi\chi) \\
\text{(eCM)} & \dot{B}^\psi\varphi \to (\dot{B}^\psi\chi \to \dot{B}^{\psi\wedge\varphi}\chi) \\
\text{(eTaut)} & \dot{B}\varphi \leftrightarrow \dot{B}^\top\varphi \\
\text{(eAnd)} & \dot{B}^\psi\varphi_1 \to (\dot{B}^\psi\varphi_2 \to \dot{B}^\psi(\varphi_1 \wedge \varphi_2)) \\
\text{(eOr)} & \dot{B}^{\psi_1}\varphi \to (\dot{B}^{\psi_2}\varphi \to \dot{B}^{\psi_1\vee\psi_2}\varphi) \\
\text{(ePR)} & \dot{B}^\varphi\dot{B}^\psi\chi \leftrightarrow (\dot{B}^\psi\bot \vee \dot{B}^\psi\chi)
\end{array}$$

$$\text{(eNR)} \quad \dot{B}^{\varphi}\neg\dot{B}^{\psi}\chi \leftrightarrow (\dot{B}^{\varphi}\bot \vee \neg\dot{B}^{\psi}\chi)$$

Also, the following rules of *Explicit (Left) Logical Equivalence* (eLE), *Explicit Right Weakening* (eRW), and *Explicit Supraclassicality* (eSC) are all derivable in JCDL:

$$\frac{\psi \leftrightarrow \psi'}{\dot{B}^{\psi}\chi \leftrightarrow \dot{B}^{\psi'}\chi}\text{(eLE)} \qquad \frac{\chi \to \chi'}{\dot{B}^{\psi}\chi \to \dot{B}^{\psi}\chi'}\text{(eRW)} \qquad \frac{\psi \to \chi}{\dot{B}^{\psi}\chi}\text{(eSC)}$$

Proof. Apply Theorems 3.11 and 5.9(2). □

5.3 Semantics

One of the main possible worlds semantics for Justification Logic is the semantics due to Fitting (see [2] for details). Here we adapt the traditional Fitting semantics for use in our language $\mathscr{L}_{\mathsf{JCDL}}$.

Definition 5.12 (Fitting models)**.** A *Fitting model* is a structure $M = (W, \leq, V, A)$ for which $(W, \leq, V)$ is a locally well-ordered plausibility model and A is an *admissibility function*: a function

$$A : (\mathscr{T}_{\mathsf{JCDL}} \times \mathscr{L}_{\mathsf{JCDL}}) \to \wp(W)$$

that maps each term-formula pair (t, φ) to a set $A(t, \varphi) \subseteq W$ of worlds subject to the following restrictions:

- Certification: $A(c_{\varphi}, \varphi) = W$,

  which says formulas are certified by their certificates;

- Application: $A(t, \varphi_1 \to \varphi_2) \cap A(s, \varphi_1) \subseteq A(t \cdot s, \varphi_2)$,

  which says the Application operator encodes instances of (MP);

- Sum: $A(t, \varphi) \cup A(s, \varphi) \subseteq A(t + s, \varphi)$,

  which says the Sum operator encodes support aggregation without logical inference; and

- Admissibility Indefeasibility: if $x \in A(t, \varphi)$ and $y \in \mathsf{cc}(x)$, then $y \in A(t, \varphi)$,

  which says admissibility is constant within each connected component. Using the notion of "knowledge" from Remark 3.3, this tells us that the agent knows her admissibility function.

The notion of *pointed Fitting model* is similar to the corresponding definition found in Definition 2.2; we also apply the terminology from that definition to Fitting models in the obvious way.

If A is an admissibility function, then $w \in A(t, \varphi)$ says that, from the perspective of world w, term t has the proper "syntactic shape" to be a reason in support of φ. This does not, however, guarantee that t does indeed support φ. For this we shall require something more.

Definition 5.13 ($\mathscr{L}_{\mathsf{JCDL}}$-truth)**.** Let $M = (W, \leq, V, A)$ be a Fitting model. We extend the binary satisfaction relation $\models$ from Definition 3.2 to one between pointed Fitting models (M, w) (written without surrounding parentheses) and $\mathscr{L}_{\mathsf{JCDL}}$-formulas and we extend the function $[\![-]\!]$ from Definition 3.2 to include a function $[\![-]\!] : \mathscr{L}_{\mathsf{JCDL}} \to \wp(W)$ as follows.

- $[\![\varphi]\!]_M \coloneqq \{v \in W \mid M, v \models \varphi\}$. The subscript M may be suppressed.
- $M, w \not\models \bot$.
- $M, w \models p$ iff $p \in V(w)$ for $p \in \mathscr{P}$.
- $M, w \models \varphi \to \psi$ iff $M, w \not\models \varphi$ or $M, w \models \psi$.

- $M, w \models t:^{\psi}\varphi$ iff $w \in A(t,\varphi)$ and

$$\forall x \in \mathsf{cc}(w): \quad x^{\downarrow} \cap [\![\psi]\!] = \emptyset \quad \text{or} \quad \exists y \in x^{\downarrow} \cap [\![\psi]\!] : y^{\downarrow} \cap [\![\psi]\!] \subseteq [\![\varphi]\!] \ . \tag{4}$$

So to have $t:^{\psi}\varphi$ true at a world w, we must have two things. First, from the perspective of w, term t must have the correct "syntactic shape" for an argument in support of φ; that is, we must have $w \in A(t,\varphi)$. Second, we must satisfy the condition (4), which is the same condition we had for truth of a $\mathscr{L}_{\mathsf{CDL}}$-formula $B^{\psi}\varphi$. So, taken together, to have $t:^{\psi}\varphi$ true at world w means that t has the "shape" of an argument for φ and the agent believes φ after revising her beliefs by ψ. The following theorem states this precisely.

Theorem 5.14 (JCDL-truth in terms of belief formulas)**.** Let $M = (W, \leq, V, A)$ be a Fitting model.

$$M, w \models t:^{\psi}\varphi \qquad \text{iff} \qquad w \in A(t,\varphi) \text{ and } M, w \models \dot{B}^{\psi}\varphi \ .$$

Proof. Left to right ("only if"): assume $M, w \models t:^{\psi}\varphi$. By Definition 5.13, we have $w \in A(t,\varphi)$ and (4). Since we have $w \in A(c_{\varphi},\varphi)$ by the Certification property of admissibility functions and we have $\dot{B}^{\psi}\varphi = c_{\varphi}:^{\psi}\varphi$ by definition, it follows from by (4) and $w \in A(c_{\varphi},\varphi)$ by Definition 5.13 that $M, w \models \dot{B}^{\psi}\varphi$.

Right to left ("if"): assume $w \in A(t,\varphi)$ and $M, w \models \dot{B}^{\psi}\varphi$. Applying Definition 5.13 and the definition $\dot{B}^{\psi}\varphi = c_{\varphi}:^{\psi}\varphi$, it follows that (4). So since we have $w \in A(t,\varphi)$ and (4), it follows by Definition 5.13 that $M, w \models t:^{\psi}\varphi$. □

In well-founded Fitting models, belief in φ conditional on ψ is equivalent to having φ true at the most plausible ψ-worlds that are within the connected component of the actual world.

Theorem 5.15 (JCDL-truth in well-founded models)**.** Let $M = (W, \leq, V, A)$ be a Fitting model.

(a) If M is well-founded: $M, w \models t:^{\psi}\varphi \;\;\Leftrightarrow\;\; w \in A(t,\varphi)$ and $\min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$.

(b) If M is well-ordered: $M, w \models t:^{\psi}\varphi \;\;\Leftrightarrow\;\; w \in A(t,\varphi)$ and $\min[\![\psi]\!] \subseteq [\![\varphi]\!]$.

Proof. (a), left to right ("only if"): assume $M, w \models t:^{\psi}\varphi$. By Definition 5.13, we have $w \in A(t,\varphi)$ and (4). Use the argument in the proof of Theorem 3.4 to conclude that $\min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$. (a), right to left ("if"): assume $w \in A(t,\varphi)$ and $\min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$. Use the argument in the proof of Theorem 3.4 to conclude that (4). Applying Definition 5.13, it follows that $M, w \models t:^{\psi}\varphi$.

(b): M is well-ordered, then $\mathsf{cc}(w) = W$ and M is well-founded. Apply (a). □

And so in well-founded Fitting models, we can see that formulas $\dot{B}^{\psi}\varphi$ really do play the semantic analog of $\mathscr{L}_{\mathsf{CDL}}$-belief formulas.

Theorem 5.16 (JCDL-truth in well-founded models in terms of belief formulas)**.** Let M be a Fitting model.

(a) If M is well-founded: $M, w \models \dot{B}^{\psi}\varphi \;\;\Leftrightarrow\;\; \min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$.

(b) If M is well-ordered: $M, w \models \dot{B}^{\psi}\varphi \;\;\Leftrightarrow\;\; \min[\![\psi]\!] \subseteq [\![\varphi]\!]$.

Proof. By the Certification (Definition 5.12) and Theorems 5.14 and 5.15. □

Similar to $\mathscr{L}_{\mathsf{CDL}}$, the intended semantic objects for $\mathscr{L}_{\mathsf{JCDL}}$ are the well-ordered models of the appropriate type (in this case Fitting models, as opposed to simple plausibility models).

Definition 5.17 ($\mathscr{L}_{\mathsf{JCDL}}$-validity)**.** To say that a $\mathscr{L}_{\mathsf{JCDL}}$-formula φ is *valid*, written $\models \varphi$, means that $M \models \varphi$ for each well-ordered Fitting model M. Though we use the same symbol "$\models$" for $\mathscr{L}_{\mathsf{JCDL}}$-validity as we did for $\mathscr{L}_{\mathsf{CDL}}$-validity, it will be clear from context which notion is meant.

And as for CDL, locally well-ordered plausibility models would suffice.

Theorem 5.18 ($\mathscr{L}_{\mathsf{JCDL}}$-validity with respect to local well-orders)**.** Let $\mathfrak{F}_L$ be the class of locally well-ordered Fitting models. For each $\varphi \in \mathscr{L}_{\mathsf{JCDL}}$, we have:

$$\models \varphi \qquad \text{iff} \qquad \forall M \in \mathfrak{F}_L,\ M \models \varphi \ .$$

Proof. As in the proof of Theorem 3.6. □

Soundness and completeness of JCDL with respect to its intended semantics (i.e., well-ordered Fitting models) makes use of many components of the proof of Theorem 3.13, itself essentially due to [5].

Theorem 5.19 (JCDL soundness and completeness)**.** For each $\varphi \in \mathscr{L}_{\mathsf{JCDL}}$:

$$\vdash_{\mathsf{JCDL}} \varphi \quad \text{iff} \quad \models \varphi \ .$$

Proof. See the appendix. □

6 Conclusion

We saw earlier that CDL is a version of AGM revision in which Boolean combinations of revisions are expressible in the language. Since JCDL is a reason-explicit analog of CDL (as per Theorem 5.9), we are led to the following suggestion: JCDL is a version of AGM revision in which Boolean combinations of *reason-explicit revisions* are expressible in the language. In essence, a formula φ that is part of the belief state after revision by ψ may be witnessed by a specific reason t whose syntactic structure tracks the genesis of φ stepwise from basic principles. This suggests we think of JCDL as a theory of *revisable justified belief.* It would be interesting to see if there is some explicit version of the AGM revision principles that matches up with JCDL in the way that standard AGM matches up with CDL. However, we leave this issue for future work.

A Technical results

A.1 Results for plausibility models

Proof of Theorem 2.3. Item 1 is obvious. Item 2 is a well-known result from order theory, but we reprove it anyway for completeness purposes. So assume M is locally well-ordered and $\emptyset \neq S = \mathsf{cc}(w) \subseteq W$ for some $w \in W$. Define

$$S' := \{x \in S \mid \forall y \in S : x \leq y\} \ .$$

We wish to prove that $\min S = S'$. Proceeding, take $x \in \min S$. If $y \in S$ as well, then it follows from $x \in \min S$ by the definition of $\min S$ that $y \not< x$, from which we obtain $x \leq y$ because S is a connected component and $\leq$ is total on each connected component. Since $y \in S$ was chosen arbitrarily, it follows that $x \in S'$. Hence $\min S \subseteq S'$. To show the inclusion holds in the other direction, take $x \in S'$. If $y \in S$, then it follows from $x \in S'$ by the definition of S' that $x \leq y$, from which we obtain $y \not< x$ by the definition of $\not<$. Since $y \in S$ was chosen arbitrarily, it follows that $x \in \min S$. Hence $S' \subseteq \min S$.

For Item 3, it follows from the fact that $\leq$ is well-ordered that W is a connected component. Further, since $\leq$ is well-ordered, it is also locally well-ordered. The result therefore follows by Item 2.

For Item 4, let us first assume that M is well-founded. We wish to prove that each $S \in \wp(W)$ is smooth in M. So take $S \in \wp(W)$. Since $\emptyset$ is smooth in M, let us assume further that $S \neq \emptyset$. Now take $x \in S$. Since $x \in x^{\downarrow} \cap S$ by the reflexivity of $\leq$, it follows that $x^{\downarrow} \cap S \neq \emptyset$. Therefore, since M is well-founded, it follows that $\min(x^{\downarrow} \cap S) \neq \emptyset$. That is, there exists $y \in \min(x^{\downarrow} \cap S)$. But then $y \leq x$ and $x \not< y$, from which it follows that $y \simeq x$ or $y < x$. And if $y \simeq x$, then it follows from $y \in \min(x^{\downarrow} \cap S)$ by the transitivity of $\leq$ that $x \in \min(x^{\downarrow} \cap S)$. So either we have $x \in \min(x^{\downarrow} \cap S)$ or we have $y \in \min(x^{\downarrow} \cap S)$ and $y < x$. Further, for each $m \in \min(x^{\downarrow} \cap S)$ and $z \in S - (x^{\downarrow} \cap S)$, we have by the transitivity of $\leq$ that $z \not< m$. And for each $m \in \min(x^{\downarrow} \cap S)$ and $z \in (x^{\downarrow} \cap S)$, we have by the definition of $\min(x^{\downarrow} \cap S)$ that $z \not< m$. But then $m \in \min(x^{\downarrow} \cap S)$ implies $m \in \min S$. Taken together, we have shown that for each $x \in S$, either $x \in \min S$ or there exists $y \in \min S$ such that $y < x$. It follows that S is smooth in M. Since we have shown that every $S \in \wp(W)$ is smooth in M, it follows that M is smooth.

For the converse of Item 4, we assume that M is smooth. We wish to prove that M is well-founded. So take a nonempty $S \subseteq W$. Since S is nonempty, we have $x \in S$. But M is smooth and so S is smooth in M, and so it follows that $x \in \min S$ or there exists $y \in \min S$ such that $y < x$. In either case, we have $\min S \neq \emptyset$. So M is well-founded.

Items 5 and 6 follow from Item 4 by Definition 2.2. □

A.2 Results for CDL

Proof of Theorem 3.4(a). Assume M is well-founded and $M, w \models B^{\psi}\varphi$. The latter means

$$\forall x \in \mathsf{cc}(w): \quad x^{\downarrow} \cap [\![\psi]\!] = \emptyset \quad \text{or} \quad \exists y \in x^{\downarrow} \cap [\![\psi]\!] : y^{\downarrow} \cap [\![\psi]\!] \subseteq [\![\varphi]\!] \ . \tag{5}$$

If $\min[\![\psi]\!] \cap \mathsf{cc}(w) = \emptyset$, then $\min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$. So let us assume further that $\min[\![\psi]\!] \cap \mathsf{cc}(w) \neq \emptyset$. Take $z \in \min[\![\psi]\!] \cap \mathsf{cc}(w)$. Since we then have $z \in z^{\downarrow} \cap [\![\psi]\!]$ by the reflexivity of $\leq$ and the definition of $\min[\![\psi]\!]$, it follows by (5) that

$$\exists y \in z^{\downarrow} \cap [\![\psi]\!] : y^{\downarrow} \cap [\![\psi]\!] \subseteq [\![\varphi]\!] \ .$$

Now $z \in \mathsf{cc}(w)$ and $y \in z^{\downarrow} \cap [\![\psi]\!]$, so it follows that $y \in \mathsf{cc}(w)$ and therefore that $y \in [\![\psi]\!] \cap \mathsf{cc}(w)$. From this we obtain by $z \in \min[\![\psi]\!] \cap \mathsf{cc}(w)$ that $y \not< z$. But $y \in z^{\downarrow}$, and therefore we have $y \leq z$ and $z \leq y$. As a result, $z \in y^{\downarrow} \cap [\![\psi]\!] \subseteq [\![\varphi]\!]$. Since $z \in \min[\![\psi]\!] \cap \mathsf{cc}(w)$ was chosen arbitrarily, we have proved that $\min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$.

Conversely, assume M is well-founded and $\min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$. To show that we have $M, w \models B^{\psi}\varphi$, we must show that (5) obtains. For this it suffices for us to take $x \in \mathsf{cc}(w)$ satisfying $x^{\downarrow} \cap [\![\psi]\!] \neq \emptyset$ and prove that

$$\exists y \in x^{\downarrow} \cap [\![\psi]\!] : y^{\downarrow} \cap [\![\psi]\!] \subseteq [\![\varphi]\!] \ . \tag{6}$$

Proceeding, since $x^{\downarrow} \cap [\![\psi]\!] \neq \emptyset$ and M is well-founded, it follows that there exists $y \in \min(x^{\downarrow} \cap [\![\psi]\!])$. Since $\min[\![\psi]\!] \cap \mathsf{cc}(w) \subseteq [\![\varphi]\!]$, if we can show that for every $z \in y^{\downarrow} \cap [\![\psi]\!]$ we have $z \in \min[\![\psi]\!] \cap \mathsf{cc}(w)$, then it would follow that $y^{\downarrow} \cap [\![\psi]\!] \subseteq [\![\varphi]\!]$ and therefore that (6), completing the argument. So take $z \in y^{\downarrow} \cap [\![\psi]\!]$. It follows by $z \in y^{\downarrow}$, $y \in x^{\downarrow}$, and $x \in \mathsf{cc}(w)$ that $z \in \mathsf{cc}(w)$. So to show that $z \in \min[\![\psi]\!] \cap \mathsf{cc}(w)$, all that remains is to prove that $z \in \min[\![\psi]\!]$, and for this it suffices to prove that $u \in z^{\downarrow} \cap [\![\psi]\!]$ implies $u \not< z$. So take $u \in z^{\downarrow} \cap [\![\psi]\!]$. Since $u \in z^{\downarrow}$, $z \in y^{\downarrow}$, and $y \in x^{\downarrow}$, we have

$$u \leq z \leq y \leq x \ .$$

By the transitivity of $\leq$, it follows that $u \leq y$ and $u \leq x$. Hence $u \in x^{\downarrow} \cap [\![\psi]\!]$. Since $y \in \min(x^{\downarrow} \cap [\![\psi]\!])$, it follows that $u \not< y$. But from $u \not< y$ and $u \leq y$, it follows by the definition of $\not<$ that $y \leq u$. And from $z \leq y \leq u$, it follows by the transitivity of $\leq$ that $z \leq u$. Applying the definition of $\not<$, we obtain $u \not< z$. □

Proof of Theorem 3.11. We reason in CDL. For (Cut), assume $B^{\psi}\varphi$ and $B^{\psi\wedge\varphi}\chi$. It follows from $B^{\psi\wedge\varphi}\chi$ by (Inc) that $B^{\psi}(\varphi \to \chi)$. But from $B^{\psi}(\varphi \to \chi)$ and $B^{\psi}\varphi$ we obtain by (MR) that $B^{\psi}\chi$.

For (CM), we reason by cases under the assumption $B^{\psi}\varphi$. First: if $\neg B^{\psi}\neg\varphi$, then we obtain $B^{\psi}\chi \to B^{\psi\wedge\varphi}\chi$ by (RM) and (CR). Second: if $B^{\psi}\neg\varphi$, then it follows by our assumption $B^{\psi}\varphi$ and (MR) that $B^{\psi}\bot$; applying (KM) and (CR) yields $B^{\psi\wedge\varphi}\bot$, from which we obtain $B^{\psi\wedge\varphi}\chi$ by (MR).

(Taut) obtains by (CR) since $B\varphi = B^{\top}\varphi$. (And) obtains by (MR).

For (Or), we assume $B^{\psi_1}\varphi$ and $B^{\psi_2}\varphi$. Take $i \in \{1, 2\}$. By (Succ) and (MR) we obtain $B^{\psi_i}(\psi_1 \vee \psi_2)$. From $B^{\psi_i}(\psi_1 \vee \psi_2)$ and our assumption $B^{\psi_i}\varphi$, we obtain by (CM) and (Comm) that $B^{(\psi_1\vee\psi_2)\wedge\psi_i}\varphi$. From this it follows by (Inc) that $B^{\psi_1\vee\psi_2}(\psi_i\to\varphi)$. Since we have this for each $i\in\{1,2\}$, it follows by (MR) that $B^{\psi_1\vee\psi_2}((\psi_1\vee\psi_2)\to\varphi)$. Since $B^{\psi_1\vee\psi_2}(\psi_1\vee\psi_2)$ by (Succ), we obtain the result $B^{\psi_1\vee\psi_2}\varphi$ by (MR).

For (PR), we have the following:

1. $B^\varphi\neg B^\psi\chi\to(B^\varphi B^\psi\chi\to B^\varphi\bot)$ (MR)
2. $(B^\varphi B^\psi\chi\wedge\neg B^\varphi\bot)\to\neg B^\varphi\neg B^\psi\chi$ (CR), 1
3. $\neg B^\psi\chi\to B^\varphi\neg B^\psi\chi$ (NI)
4. $\neg B^\varphi\neg B^\psi\chi\to B^\psi\chi$ (CR), 3
5. $(B^\varphi B^\psi\chi\wedge\neg B^\varphi\bot)\to B^\psi\chi$ (CR), 2, 4
6. $B^\varphi B^\psi\chi\to(B^\varphi\bot\vee B^\psi\chi)$ (CR), 5
7. $B^\psi\chi\to B^\varphi B^\psi\chi$ (PI)
8. $B^\varphi\bot\to B^\varphi B^\psi\chi$ (MR)
9. $(B^\varphi\bot\vee B^\psi\chi)\to B^\varphi B^\psi\chi$ (CR), 7, 8
10. $B^\varphi B^\psi\chi\leftrightarrow(B^\varphi\bot\vee B^\psi\chi)$ (CR), 6, 9

To obtain the proof for (NR), replace each occurrence of $B^\psi\chi$ in the above proof with $\neg B^\psi\chi$ and change the reason for line 7 from (PI) to (NI). It is straightforward to verify that this operation yields a derivation of (NR).

For (LE), assume $\psi\leftrightarrow\psi'$. We have $B^\psi\psi$ by (Succ). Applying (MR) to our assumption, we obtain $B^\psi(\psi\to\psi')$. Hence $B^\psi\psi'$ by (MR). By similar reasoning, we obtain $B^{\psi'}\psi$. Now by (CM), $B^\psi\psi'$, (Comm), and (CR), we obtain $B^\psi\chi\to B^{\psi'\wedge\psi}\chi$. By $B^{\psi'}\psi$, (Cut), and (CR), we obtain $B^{\psi'\wedge\psi}\chi\to B^{\psi'}\chi$. But then it follows by (CR) that $B^\psi\chi\to B^{\psi'}\chi$. A similar argument shows that $B^{\psi'}\chi\to B^\psi\chi$. By (CR), we conclude that $B^\psi\chi\leftrightarrow B^{\psi'}\chi$.

(RW) follows by (MR). For (SC), from $\psi\to\varphi$ we obtain $B^\psi(\psi\to\varphi)$ by (MN); however, we have $B^\psi\psi$ by (Succ), and so it follows by (MR) that $B^\psi\varphi$. □

Proof of Theorem 3.12. Left to right: it suffices to show that CDL derives (IEa), (IEb), and (LE).

- (IEa): $\vdash_{\mathsf{CDL}} B^\psi\varphi\to(B^{\psi\wedge\varphi}\chi\leftrightarrow B^\psi\chi)$.

  By (CM), (Cut), and (CR).

- (IEb): $\vdash_{\mathsf{CDL}} \neg B^\psi\neg\varphi\to(B^{\psi\wedge\varphi}\chi\leftrightarrow B^\psi(\varphi\to\chi))$.

  Reasoning in CDL, by (Inc) and (CR) we have

  $$\neg B^\psi\neg\varphi\to(B^{\psi\wedge\varphi}\chi\to B^\psi(\varphi\to\chi))\ ,$$

  and so it suffices by (CR) to prove that

  $$\neg B^\psi\neg\varphi\to(B^\psi(\varphi\to\chi)\to B^{\psi\wedge\varphi}\chi)\ . \tag{7}$$

  Proceeding, we have

  $$\neg B^\psi\neg\varphi\to(B^\psi(\varphi\to\chi)\to B^{\psi\wedge\varphi}(\varphi\to\chi)) \tag{8}$$

  by (RM). We also have $B^{\psi\wedge\varphi}(\psi\wedge\varphi)$ by (Succ) and therefore $B^{\psi\wedge\varphi}\varphi$ by (MR). But then we obtain (7) from (8) and $B^{\psi\wedge\varphi}\varphi$ by (MR). The result follows.

- (LE): if $\vdash_{\mathsf{CDL}}\psi\leftrightarrow\psi'$, then $\vdash_{\mathsf{CDL}} B^\psi\chi\leftrightarrow B^{\psi'}\chi$.

  By Theorem 3.11.

This completes the left-to-right direction. Right to left: it suffices to show that CDL_0 derives (KM), (RM), (Inc), and (Comm).

- (KM): $\vdash_{\mathsf{CDL}_0} B^{\psi}\bot \to B^{\psi\wedge\varphi}\bot$.

  We reason in CDL_0. We have $B^{\psi}\bot \to (B^{\psi}\varphi \wedge B^{\psi}\bot)$ by (MR). Applying (IEa) and (CR), we obtain $B^{\varphi}\bot \to B^{\psi\wedge\varphi}\bot$.

- (RM): $\vdash_{\mathsf{CDL}_0} \neg B^{\psi}\neg\varphi \to (B^{\psi}\chi \to B^{\psi\wedge\varphi}\chi)$.

  Reasoning in CDL_0, we assume $\neg B^{\psi}\neg\varphi$ and $B^{\psi}\chi$. It follows from $B^{\psi}\chi$ by (MR) that $B^{\psi}(\varphi \to \chi)$. From this and our assumption $\neg B^{\psi}\neg\varphi$, we obtain $B^{\psi\wedge\varphi}\chi$ by (IEb) and (CR).

- (Inc): $\vdash_{\mathsf{CDL}_0} B^{\psi\wedge\varphi}\chi \to B^{\psi}(\varphi \to \chi)$.

  We reason in CDL_0 by cases. First: if $\neg B^{\psi}\neg\varphi$, then we obtain $B^{\psi\wedge\varphi}\chi \to B^{\psi}(\varphi \to \chi)$ by (IEb). Second: if $B^{\psi}\neg\varphi$, then we obtain $B^{\psi}(\varphi \to \chi)$ by (MR) and therefore $B^{\psi\wedge\varphi}\chi \to B^{\psi}(\varphi \to \chi)$ by (CR).

- (Comm): $\vdash_{\mathsf{CDL}_0} B^{\psi\wedge\varphi}\chi \to B^{\varphi\wedge\psi}\chi$.

  By (LE) and (CR). □

Proof of Theorem 3.13. We use the notation and concepts from Remark 3.8. Let

$$\mathscr{L}_{\mathsf{CDL}_a} := \mathscr{L}_{\mathsf{CDL}}^{\{a\}} \quad \text{and} \quad \mathsf{CDL}_a := \mathsf{CDL}_0^{\{a\}} \ .$$

We write χ^a for the $\mathscr{L}_{\mathsf{CDL}_a}$-formula obtained from the $\mathscr{L}_{\mathsf{CDL}_0}$-formula χ by recursively replacing each occurrence of a modal operator B^{θ} in χ by B^{θ}_a. Obviously, $(\chi^a)' = \chi$ and $(\theta')^a = \theta$.

It was shown by Board [5] that we have $\vdash_{\mathsf{CDL}_a} \chi$ iff $\models^{\{a\}}_{\mathsf{CDL}_a} \chi$. By Remark 3.8, this is equivalent to the statement that

$$\forall \chi \in \mathscr{L}_{\mathsf{CDL}_a} : \ \vdash_{\mathsf{CDL}_a} \chi \quad \text{iff} \quad \models \chi' \ . \tag{9}$$

By induction on derivation length, it is easy to see that the operation $\chi \mapsto \chi^a$ maps CDL_0-theorems to CDL_a-theorems and the operation $\chi \mapsto \chi'$ maps CDL_a-theorems to CDL_0-theorems. So $\vdash_{\mathsf{CDL}_0} \varphi$ iff $\vdash_{\mathsf{CDL}_0} \varphi^a$. Applying (9) and the fact that $(\varphi^a)' = \varphi$, we obtain $\vdash_{\mathsf{CDL}_0} \varphi$ iff $\models \varphi$. Applying Theorem 3.12, we obtain $\vdash_{\mathsf{CDL}} \varphi$ iff $\models \varphi$. □

Proof of Theorem 3.13. The argument can be obtained by combining the ideas from the various proofs in [5]. However, this requires restricting to the single-agent case and combining multiple arguments, so it is not so transparent how the argument should go. In the interest of making the argument clear and so that we have some constructions available for us later when we turn to the JCDL case, we provide a full proof here. However, the argument is truly due to [5].

For soundness, we proceed by induction on the length of derivation. In the induction base, we must show that each axiom scheme is wf-valid. (CL) is straightforward, so we proceed with the remaining schemes. Let M be an arbitrary well-founded plausibility model. We make tacit use of Theorem 3.4.

- (K) is valid: $\models B^{\psi}(\varphi_1 \to \varphi_2) \to (B^{\psi}\varphi_1 \to B^{\psi}\varphi_2)$.

  Assume (M, w) satisfies $B^{\psi}(\varphi_1 \to \varphi_2)$ and $B^{\psi}\varphi_1$. Then $\min[\![\psi]\!] \subseteq [\![\varphi_1 \to \varphi_2]\!]$ and $\min[\![\psi]\!] \subseteq [\![\varphi_1]\!]$. Hence $\min[\![\psi]\!] \subseteq [\![\varphi_2]\!]$. So (M, w) satisfies $B^{\psi}\varphi_2$.

- (Succ) is valid: $\models B^{\psi}\psi$.

  We have $\min[\![\psi]\!] \subseteq [\![\psi]\!]$ by the definition of $\min[\![\psi]\!]$. So (M, w) satisfies $B^{\psi}\psi$.

- (KM) is valid: $\models B^{\psi}\bot \to B^{\psi\wedge\varphi}\bot$.

  Suppose (M,w) satisfies $B^{\psi}\bot$. Then $\min[\![\psi]\!] \subseteq [\![\bot]\!] = \emptyset$. Since M is well-founded, it follows that $[\![\psi]\!] = \emptyset$. But then $[\![\psi \wedge \varphi]\!] = \emptyset$, from which it follows that $\min[\![\psi \wedge \varphi]\!] = \emptyset \subseteq [\![\bot]\!]$. So (M,w) satisfies $B^{\psi\wedge\varphi}\bot$.

- (RM) is valid: $\models \neg B^{\psi}\neg\varphi \to (B^{\psi}\chi \to B^{\psi\wedge\varphi}\chi)$.

  Suppose (M,w) satisfies $\neg B^{\psi}\neg\varphi$ and $B^{\psi}\chi$. It follows that $\min[\![\psi]\!] \nsubseteq [\![\neg\varphi]\!]$ and $\min[\![\psi]\!] \subseteq [\![\chi]\!]$. Hence $\min[\![\psi]\!] \cap [\![\varphi]\!] \neq \emptyset$. We prove that $\min[\![\psi \wedge \varphi]\!] \subseteq \min[\![\psi]\!] \cap [\![\varphi]\!]$. Proceeding, take $x \in \min[\![\psi \wedge \varphi]\!]$. Since $\min[\![\psi]\!] \cap [\![\varphi]\!] \neq \emptyset$, there exists $y \in \min[\![\psi]\!] \cap [\![\varphi]\!]$. Hence $y \in [\![\psi \wedge \varphi]\!]$. Since $x \in \min[\![\psi \wedge \varphi]\!]$ and $\leq$ is total, we have $x \leq y$. But $x \in [\![\psi]\!]$ and $y \in \min[\![\psi]\!]$, and therefore it follows from $x \leq y$ that $x \in \min[\![\psi]\!]$ as well. Since $x \in [\![\varphi]\!]$, we have $x \in \min[\![\psi]\!] \cap [\![\varphi]\!]$. Conclusion: $\min[\![\psi \wedge \varphi]\!] \subseteq \min[\![\psi]\!] \cap [\![\varphi]\!]$. So since $\min[\![\psi]\!] \subseteq [\![\chi]\!]$, it follows that

  $$\min[\![\psi \wedge \varphi]\!] \subseteq \min[\![\psi]\!] \cap [\![\varphi]\!] \subseteq [\![\chi]\!] \cap [\![\varphi]\!] \subseteq [\![\chi]\!] \ .$$

  That is, $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\chi]\!]$. So (M,w) satisfies $B^{\psi\wedge\varphi}\chi$.

- (Inc) is valid: $\models B^{\psi\wedge\varphi}\chi \to B^{\psi}(\varphi \to \chi)$.

  Suppose (M,w) satisfies $B^{\psi\wedge\varphi}\chi$. Then $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\chi]\!]$. We prove that $\min[\![\psi]\!] \subseteq [\![\varphi \to \chi]\!]$. Proceeding, take $x \in \min[\![\psi]\!]$. If $x \notin [\![\varphi]\!]$, then $x \in [\![\varphi \to \chi]\!]$. So let us assume further that $x \in [\![\varphi]\!]$ and therefore that $x \in [\![\psi \wedge \varphi]\!]$. Now take any $y \in [\![\psi \wedge \varphi]\!]$. Since $y \in [\![\psi]\!]$, if we had $y < x$, then it would follow that $x \notin \min[\![\psi]\!]$, contradicting our choice of x. Hence $y \in [\![\psi \wedge \varphi]\!]$ implies $y \not< x$, from which it follows by $x \in [\![\psi \wedge \varphi]\!]$ that $x \in \min[\![\psi \wedge \varphi]\!]$. But we have $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\chi]\!]$ and hence $x \in [\![\chi]\!]$, from which we obtain $x \in [\![\varphi \to \chi]\!]$. Conclusion: (M,w) satisfies $B^{\psi}(\varphi \to \chi)$.

- (Comm) is valid: $\models B^{\psi\wedge\varphi}\chi \to B^{\varphi\wedge\psi}\chi$.

  Suppose (M,w) satisfies $B^{\psi\wedge\varphi}\chi$. Then $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\chi]\!]$. Since $[\![\psi \wedge \varphi]\!] = [\![\varphi \wedge \psi]\!]$, it follows that $\min[\![\varphi \wedge \psi]\!] \subseteq [\![\chi]\!]$. But then (M,w) satisfies $B^{\varphi\wedge\psi}\chi$.

- (PI) is valid: $\models B^{\psi}\chi \to B^{\varphi}B^{\psi}\chi$.

  Suppose (M,w) satisfies $B^{\psi}\chi$. Hence $\min[\![\psi]\!] \subseteq [\![\chi]\!]$, which implies that (M,v) satisfies $B^{\psi}\chi$ for any given $v \in \mathsf{cc}(w) = W$. That is, $[\![B^{\psi}\chi]\!] = W$. Therefore, $\min[\![\varphi]\!] \subseteq [\![B^{\psi}\chi]\!]$. Conclusion: (M,w) satisfies $B^{\varphi}B^{\psi}\chi$.

- (NI) is valid: $\models \neg B^{\psi}\chi \to B^{\varphi}\neg B^{\psi}\chi$.

  Suppose (M,w) satisfies $\neg B^{\psi}\chi$. Then $\min[\![\psi]\!] \nsubseteq [\![\chi]\!]$. It follows that $M,v \models \neg B^{\psi}\chi$ for each $v \in \mathsf{cc}(w) = W$. Therefore, $[\![\neg B^{\psi}\chi]\!] = W$, from which it follows that $\min[\![\varphi]\!] \subseteq [\![\neg B^{\psi}\chi]\!]$. Conclusion: (M,w) satisfies $B^{\varphi}\neg B^{\psi}\chi$.

- (WCon) is valid: $\models B^{\psi}\bot \to \neg\psi$.

  Suppose (M,w) satisfies $B^{\psi}\bot$. It follows that $\min[\![\psi]\!] \subseteq [\![\bot]\!] = \emptyset$. Since $\leq$ is well-founded, it follows that $[\![\psi]\!] = \emptyset$. But this implies $[\![\neg\psi]\!] = W$. So (M,w) satisfies $\neg\psi$.

This completes the induction base. For the induction step, we must show that validity is preserved under the rules of (MP) and (MN). The argument for (MP) is standard, so let us focus on (MN). We assume $\models \varphi$ for the CDL-derivable φ (this is the "induction hypothesis"), and we prove that $\models B^{\psi}\varphi$. Proceeding, since we have $\models \varphi$ by the induction hypothesis, it follows that $[\![\varphi]\!] = W$ and hence that $\min[\![\psi]\!] \subseteq [\![\varphi]\!]$. But then (M,w) satisfies $B^{\psi}\varphi$. Soundness has been proved.

Since CDL is sound with respect to the class of well-ordered plausibility models we note that CDL is consistent (i.e., $\nvdash_{\mathsf{CDL}} \bot$). In particular, take any pointed plausibility model (M,w) containing only the single world w. Since there is only one world, M is well-ordered. Further, by soundness, we have that $\vdash_{\mathsf{CDL}} \varphi$ to $M, w \models \varphi$. Therefore, since $M, w \not\models \bot$ by Definition 3.2, it follows that $\nvdash_{\mathsf{CDL}} \bot$. That is, CDL is consistent. We make use of this fact tacitly in what follows.

For completeness, we write $\vdash$ without any subscript in the remainder of this proof as an abbreviation for $\vdash_{\mathsf{CDL}}$. Take θ such that $\nvdash \neg\theta$. We shall prove that θ is satisfiable at a locally well-ordered pointed plausibility model and then apply Theorem 3.6 to draw the desired conclusion. Our construction is based on the completeness constructions given in [5]. Proceeding, provability will always be taken with respect to CDL, the language is assumed to be $\mathscr{L}_{\mathsf{CDL}}$, and we make tacit use of Theorem 3.11. To say that a set of formulas is *consistent* means that for no finite subset does the conjunction provably imply $\bot$. For sets S and S' of formulas, to say that S is *maximal consistent in* (or "maxcons in") S' means that $S \subseteq S'$, S is consistent, and extending S by adding any formula in S' not already present would yield a set that is *inconsistent* (i.e., not consistent). Given a formula φ, we write $\mathsf{sub}(\varphi)$ to denote the set of sub-formulas of φ, including φ itself. We extend this definition to sets of formulas: for a set S of formulas, $\mathsf{sub}(S) := \bigcup_{\varphi \in S} \mathsf{sub}(\varphi)$. Given a set S of formulas, we write $\oplus S$ to denote the *Boolean closure* of S (with respect to the language): this is the smallest extension of S that contains all Boolean constants that are primitive to the language (i.e., $\bot$) and is closed under all Boolean operations that are primitive to the language (i.e., implication). Since our language is Boolean complete (i.e., every Boolean constant and every Boolean connective is definable in terms of the Boolean constants and Boolean connectives primitive to the language), it follows that the Boolean closure $\oplus S$ of S is the smallest extension of S that contains all definable Boolean constants (i.e., $\bot$ and $\top$) and is closed under all definable Boolean connectives (e.g., implication, conjunction, disjunction, and negation). For a set S of formulas, we define:

$$
\begin{aligned}
\pm S &:= S \cup \{\neg\varphi \mid \varphi \in S\} \ , \\
B_0 S &:= S \ , \\
B_{i+1} S &:= \pm\{B^{\psi}\varphi \mid \psi \in S \text{ and } \varphi \in B_i S\} \ , \\
B_\omega S &:= \textstyle\bigcup_{0<i<\omega} B_i S \ , \\
C_0 &:= \pm\,\mathsf{sub}(\{\theta, \bot, \top\}) \ , \\
C_1 &:= \oplus C_0 \ , \\
B &:= B_\omega C_1 \ , \\
C &:= C_1 \cup B \ .
\end{aligned}
$$

Notice that 0 is excluded in the definition of $B_\omega S$. Further, C_0 is finite. Since $\nvdash \neg\theta$, we may extend $\{\neg\theta\}$ to a set w_θ that is maxcons in C. We then define:

$$
\begin{aligned}
W &:= \{x \subseteq C \mid x \text{ is maxcons in } C\} \ , \\
\bar{x} &:= \textstyle\bigwedge(x \cap C_0) \text{ for } x \in W \ , \\
x^{\psi} &:= \{\varphi \mid B^{\psi}\varphi \in x\} \text{ for } x \in W \text{ and } \psi \in C_1 \ , \\
\leq &:= \{(x, y) \in W \times W \mid \exists \psi \in (x \cap y \cap C_1),\, y^{\psi} \subseteq x\} \ , \\
V(x) &:= \mathscr{P} \cap x \text{ for } x \in W \ , \\
M &:= (W, \leq, V) \ .
\end{aligned}
$$

Notice that for $x \in W$, we have $\bar{x} \in C_1$, from which it follows by the maximal consistency of x in $C \supseteq C_1$ that $\bar{x} \in x$. Further, it follows by the definition $\bar{x} = \bigwedge(x \cap C_0)$ and the fact that x is maximal consistent in $C = C_1 \cup B$ that for each $\psi \in C_1 = \oplus C_0$, we have $\vdash \bar{x} \to \psi$ or $\vdash \bar{x} \to \neg\psi$. Finally, if $x \leq y$ and $x \neq y$, then it follows by the maximal consistency of x and of y in C that $\bar{x} \notin y$ and $\bar{y} \notin x$ and therefore that $\vdash \bar{x} \to \neg\bar{y}$ and $\vdash \bar{y} \to \neg\bar{x}$. We make tacit use of the facts mentioned in this paragraph in what follows.

We prove that W is finite. First a definition due to [8]: to say a set S of formulas is *logically finite* means that S has a *finite basis*, which is a finite $S' \subseteq S$ satisfying the property that for every $\chi \in S$, there exists

$\chi' \in S'$ such that $\vdash \chi \leftrightarrow \chi'$. It can be shown by a normal form argument that if S is logically finite, then there can be only finitely many sets that are maximal consistent in S. So to prove that W is finite, it suffices to prove that C is logically finite. Proceeding, since C_1 was obtained as the Boolean closure of the finite set C_0, it follows by a normal form argument that C_1 has a finite basis C_1'. So for $B^\psi\varphi \in B_1C_1$, since we have that $\psi \in C_1$ and $\varphi \in B_0C_1 = C_1$, there exists $\psi' \in C_1'$ and $\varphi' \in C_1'$ such that

$$\vdash \psi \leftrightarrow \psi' \quad \text{and} \quad \vdash \varphi \leftrightarrow \varphi' \ .$$

Applying (LE) and modal reasoning, it follows that

$$\vdash B^\psi\varphi \leftrightarrow B^{\psi'}\varphi' \ .$$

Since $B^{\psi'}\varphi' \in B_1C_1'$ and $\neg B^{\psi'}\varphi' \in B_1C_1'$, it follows by classical reasoning that B_1C_1' is a finite basis for B_1C_1. Therefore, there exists a finite basis B_1' for $\oplus B_1C_1'$. We prove by induction on positive $i < \omega$ that the set B_1' is also a finite basis for B_iC_1. The induction base case $i = 1$ (for B_1C_1) has already been handled. So let us proceed with the induction step: we assume B_1' is a finite basis for B_jC_1 for each non-negative integer j that does not exceed some fixed $i \geq 1$ (this is the "induction hypothesis"), and we prove that B_1' is a finite basis for $B_{i+1}C_1$. Proceeding, take $\chi \in B_{i+1}C_1$. Since $i \geq 1$, we have $i + 1 \geq 2$ and therefore χ has one of the forms $B^\psi B^\delta\varphi$, $B^\psi\neg B^\delta\varphi$, $\neg B^\psi B^\delta\varphi$, or $B^\psi\neg B^\delta\varphi$ for some $\varphi \in B_{i-1}C_1$. We have by (PR), (NR), and classical reasoning that the following "reductive equivalences" obtain:

$$\begin{aligned}
&\vdash B^\psi B^\delta\varphi \leftrightarrow (B^\psi\bot \vee B^\delta\varphi) \ , \\
&\vdash \neg B^\psi B^\delta\varphi \leftrightarrow (\neg B^\psi\bot \wedge \neg B^\delta\varphi) \ , \\
&\vdash B^\psi\neg B^\delta\varphi \leftrightarrow (B^\psi\bot \vee \neg B^\delta\varphi) \ , \\
&\vdash \neg B^\psi\neg B^\delta\varphi \leftrightarrow (\neg B^\psi\bot \wedge B^\delta\varphi) \ .
\end{aligned}$$

$B^\psi\bot$ and $\neg B^\psi\bot$ are members of B_1C_1, and $B^\delta\varphi$ and $\neg B^\delta\varphi$ are members of B_iC_1. Since $i + 1 > 1$ and $i + 1 > i$, we may apply the induction hypothesis: there exist members $(B^\psi\bot)'$, $(\neg B^\psi\bot)'$, $(B^\delta\varphi)'$, and $(\neg B^\delta\varphi)'$ of B_1' such that the following "inductive equivalences" obtain:

$$\begin{aligned}
&\vdash B^\psi\bot \leftrightarrow (B^\psi\bot)' \ , & \qquad &\vdash \neg B^\psi\bot \leftrightarrow (\neg B^\psi\bot)' \ , \\
&\vdash B^\delta\varphi \leftrightarrow (B^\delta\varphi)' \ , & &\vdash \neg B^\delta\varphi \leftrightarrow (\neg B^\delta\varphi)' \ .
\end{aligned}$$

Let us call the four formulas appearing on the right sides of the inductive equivalences the "reduced formulas." Each reduced formula is a member of the finite basis B_1' for $\oplus B_1C_1$, and therefore each reduced formula is also a member of $\oplus B_1C_1$. Since $\oplus B_1C_1$ is closed under all definable Boolean operations and B_1' is a finite basis for $\oplus B_1C_1$, it follows that any Boolean combination of the reduced formulas is also a member of $\oplus B_1C_1$ and therefore that any such Boolean combination is provably equivalent to a formula in B_1'. But then it follows by the inductive equivalences and classical reasoning that the right side of each reductive equivalence is provably equivalent to a Boolean combination of reduced formulas, and the latter combination is itself provably equivalent to a formula in B_1'. It follows that our original formula $\chi \in B_{i+1}C$ must be provably equivalent to a formula in B_1' as well. Therefore, B_1' is indeed a finite basis for $B_{i+1}C_1$. This completes the induction step. We conclude that B_1' is a finite basis for B_iC_1 for each $i \geq 1$. As a result, it follows that B_1' is a finite basis for $B = B_\omega C_1 = \bigcup_{0<i<\omega} B_iC_1$. But then $C_1' \cup B_1'$ is a finite basis for $C = C_1 \cup B$. Conclusion: W is finite.

Suppose we are given $x \in W$ and $\varphi \in x$. If $\vdash \varphi \to \psi$ and $\psi \in C$, then it follows by the maximal consistency of x in C that $\psi \in C$. It is tedious to repeatedly verify membership assertions in C and state that the reason the result follows is by the fact that x is maximal consistent in C. Therefore, we adopt the convention that we shall generally only write that $\vdash \varphi \to \psi$ and $\varphi \in x$ together imply $\psi \in x$. In so doing, we tacitly indicate (and the reader should verify) that $\psi \in C$, x is maximal consistent in C, and so the result follows by the maximal consistency of x in C. The reader will always know when such tacit use takes place (and requires verification), since this use occurs every time it is stated that a membership assertion $\psi \in x$ obtains as a logical consequence of some collection of assumptions that does not include the assumption $\psi \in x$ itself. Finally, for convenience in the remainder of the proof, we shall say that a set is "maximal consistent" to mean that it is maximal consistent in C.

Agreement Lemma: for each $\{x, y\} \subseteq W$, if $\psi \in C_1$ and $x^\psi \subseteq y$, then $x \cap B = y \cap B$. We prove this now. Proceeding, assume $\psi \in C_1$ and $x^\psi \subseteq y$. If $B^\chi\varphi \in x$, then $B^\psi B^\chi\varphi \in x$ by (PI) and therefore $B^\chi\varphi \in y$ by $x^\psi \subseteq y$. So $B^\chi\varphi \in x$ implies $B^\chi\varphi \in y$. Now suppose $B^\chi\varphi \in y$. If we had $B^\chi\varphi \notin x$, it would follow by maximal consistency that $\neg B^\chi\varphi \in x$, hence $B^\psi\neg B^\chi\varphi \in x$ by (NI), and hence $\neg B^\chi\varphi \in y$, contradicting the consistency of y because $B^\chi\varphi \in y$. So $B^\chi\varphi \in y$ implies $B^\chi\varphi \in x$. Conclusion: the Agreement Lemma obtains. Note that we obtain from this lemma by the definition of $\leq$ that that $x \leq y$ implies $x \cap B = y \cap B$.

We prove that $\leq$ is reflexive. By the definition of $\leq$, it suffices to prove that $x^{\bar{x}} \subseteq x$. Proceeding, since $B^{\bar{x}}\varphi \in x$ implies $B^{\bar{x}}\varphi \in B = \bigcup_{0<i<\omega} B_iC_1$, all we need do is prove by induction on $i \geq 1$ that $B^{\bar{x}}\varphi \in x \cap B_iC_1$ implies $\varphi \in x$.

- Induction base: $B^{\bar{x}}\varphi \in x \cap B_1C_1$.

  We have $\varphi \in C_1$ and hence $\vdash \bar{x} \to \varphi$ or $\vdash \bar{x} \to \neg\varphi$. If $\vdash \bar{x} \to \neg\varphi$, then it follows by (SC) that $B^{\bar{x}}\neg\varphi \in x$. Since we also have $B^{\bar{x}}\varphi \in x$, we obtain by modal reasoning that $B^{\bar{x}}\bot \in x$, from which it follows by (WCon) that $\neg\bar{x} \in x$, a contradiction. Therefore it follows that $\vdash \bar{x} \to \varphi$ and hence $\varphi \in x$.

- Induction step: we assume the result for $i = 1, \ldots, k$ and we prove the result for $i = k + 1$.

  Assume $B^{\bar{x}}\varphi \in x \cap B_{k+1}C_1$. Since $k \geq 2$, it follows that $\varphi = B^\psi\chi$ or $\varphi = \neg B^\psi\chi$. Therefore, by (PR) or (NR) we have $B^{\bar{x}}\bot \vee \varphi \in x$. As in the induction base, it follows by (WCon) that $B^{\bar{x}}\bot \notin x$. Therefore, $\varphi \in x$.

Conclusion: $\leq$ is reflexive.

We prove that $\leq$ is transitive. Proceeding, assume $x \leq y \leq z$. This means there exists $a \in (x \cap y \cap C_1)$ and $b \in (y \cap z \cap C_1)$ such that $y^a \subseteq x$ and $z^b \subseteq x$. It follows that $a \vee b \in (x \cap z \cap C_1)$, and so to conclude $x \leq z$, it suffices for us to prove that $z^{a\vee b} \subseteq x$. Proceeding, we take an arbitrary $B^{a\vee b}\varphi \in x$ and we seek to prove that $\varphi \in z$. We consider two cases.

- Case: $B^{a\vee b}\neg a \in z$.

  Since $B^{a\vee b}(a \vee b) \in z$ by (Succ), it follows by the assumption of this case and modal reasoning that $B^{a\vee b}b \in z$. Applying the latter and the assumption of the case again, we obtain by (CM) that $B^{(a\vee b)\wedge b}\neg a \in z$, from which it follows by (LE) that $B^b\neg a \in z$. But $z^b \subseteq y$ and therefore $\neg a \in y$. Since $a \in y$, we have reached a contradiction. So this case cannot obtain, and so there is nothing more to prove.

- Case: $\neg B^{a\vee b}\neg a \in z$.

  From the assumption of this case and $B^{a\vee b}\varphi \in z$ we obtain by (RM) that $B^{(a\vee b)\wedge a}\varphi \in z$. Applying (LE), it follows that $B^a\varphi \in z$. Hence $B^bB^a\varphi \in z$ by (PI). Since $z^b \subseteq y$, it follows that $B^a\varphi \in y$. Since $y^a \subseteq x$, we obtain $\varphi \in x$.

Conclusion: $\leq$ is transitive.

We prove that $\leq$ is total on each connected component. Proceeding, suppose we have $w \in W$ and $(x, y) \in \mathsf{cc}(w) \times \mathsf{cc}(w)$. Since $x \in \mathsf{cc}(w)$ and $y \in \mathsf{cc}(w)$, it follows that $v \leq w$ or $w \leq v$ for each $v \in \{x, y\}$. Applying the definition of $\leq$ and the Agreement Lemma, we obtain $x \cap B = y \cap B$. We wish to prove that $x \leq y$ or $y \leq x$. We consider two cases.

- Case: $B^{\bar{x}\vee\bar{y}}\neg\bar{x} \in x \cap y$.

  By (Succ), we have $B^{\bar{x}\vee\bar{y}}(\bar{x} \vee \bar{y}) \in x \cap y$. Applying the assumption of this case and modal reasoning, we obtain $B^{\bar{x}\vee\bar{y}}\bar{y} \in x \cap y$. We shall now prove that $y \leq x$. Proceeding, take an arbitrary $B^{\bar{x}\vee y}\varphi \in x$. It follows from this by $B^{\bar{x}\vee\bar{y}}\bar{y} \in x$ and (CM) that $B^{(\bar{x}\vee\bar{y})\wedge\bar{y}}\varphi \in x$. Applying (LE), we obtain $B^{\bar{y}}\varphi \in x$. Since $x \cap B = y \cap B$, we have $B^{\bar{y}}\varphi \in y$. But we saw in the argument for reflexivity that $y^{\bar{y}} \subseteq y$ and therefore $\varphi \in y$. So we have proved that for an arbitrary $B^{\bar{y}}\varphi \in y$, we obtain $\varphi \in y$. That is, we have shown that $x^{\bar{x}\vee\bar{y}} \subseteq y$. Since $\bar{x} \vee \bar{y} \in (x \cap y \cap C_1)$, it follows that $y \leq x$.

- Case: $\neg B^{\bar{x}\vee\bar{y}}\neg\bar{x} \in x \cap y$.

  We prove that $x \leq y$. Proceeding, take an arbitrary $B^{\bar{x}\vee\bar{y}}\varphi \in y$. It follows from this by the assumption of this case and (RM) that $B^{(\bar{x}\vee\bar{y})\wedge\bar{x}}\varphi \in y$. Applying (LE), $B^{\bar{x}}\varphi \in y$. Since $y \cap B = x \cap B$, we obtain $B^{\bar{x}}\varphi \in x$. But we saw in the argument for reflexivity that $x^{\bar{x}} \subseteq x$ and therefore $\varphi \in x$. So we have proved that for an arbitrary $B^{\bar{x}\vee\bar{y}}\varphi \in y$, we obtain $\varphi \in x$. That is, $y^{\bar{x}\vee\bar{y}} \subseteq x$. Since $\bar{x}\vee\bar{y} \in (x \cap y \cap C_1)$, it follows that $x \leq y$.

Conclusion: $\leq$ is total on each connected component.

Since $w_\theta \in W$, it follows that W is nonempty. Therefore, $\leq$ is a reflexive and transitive binary relation over the nonempty finite set W, and $\leq$ is total on each connected component. It follows from the finiteness of W that $\leq$ is well-founded. Therefore, M is a locally well-ordered plausibility model. We now prove a few lemmas that will be of assistance.

Consistency Lemma: for each $x \in W$, if $\psi \in x$, then x^ψ is consistent. We prove this now. Proceeding, suppose x^ψ is not consistent. It follows that there exists a nonempty $\{\chi_1, \ldots, \chi_n\} \subseteq x^\psi$ such that $\vdash (\bigwedge_{i\leq n} \chi_i) \to \bot$. By modal reasoning, $\vdash (\bigwedge_{i\leq n} B^\psi\chi_i) \to B^\psi\bot$. Since $\chi_i \in x^\psi$ and hence $B^\psi\chi_i \in x$ for each $i \leq n$, it follows that $B^\psi\bot \in x$. Applying (WCon), $\neg\psi \in x$, which contradicts the consistency of x because we assumed $\psi \in x$. Conclusion: x^ψ is consistent.

Minimality Lemma: for each $\psi \in C_1$ and $x \in [\psi]$, where

$$[\psi] := \{x \in W \mid \psi \in x\} \ ,$$

we have $x \in \min[\psi]$ iff $\neg B^\psi\neg\bar{x} \in x$.

- Left to right: for $\psi \in C_1$ and $x \in [\psi]$, we prove $x \in \min[\psi]$ implies $\neg B^\psi\neg\bar{x} \in x$.

  Assume $\psi \in C_1$ and $x \in \min[\psi]$. Toward a contradiction, suppose $B^\psi\neg\bar{x} \in x$. Applying the Consistency Lemma, x^ψ is consistent and so may be extended to a maximal consistent $y \in W$. Since $B^\psi\psi \in x$ by (Succ), it follows that $\psi \in y$. And since $B^\psi\neg\bar{x} \in x$, it follows that $\neg\bar{x} \in y$ and therefore that $y \neq x$. But then $\psi \in (x \cap y \cap C_1)$ and $y^\psi \subseteq x$, from which it follows that $y \leq x$. Since $y \in [\psi]$, $y \leq x$, and $x \in \min[\psi]$, it follows that $x \leq y$. That is, there exists $\delta \in (x \cap y \cap C_1)$ such that $y^\delta \subseteq x$. Now if we had $B^\psi\neg\delta \in x$, then it would follow that $\neg\delta \in y$, contradicting the fact that $\delta \in y$. Hence $\neg B^\psi\neg\delta \in x$. Since $B^\psi\neg\bar{x} \in x$ as well, it follows by (RM) and (Comm) that $B^{\delta\wedge\psi}\neg\bar{x} \in x$. Applying (Inc), $B^\delta(\psi \to \neg\bar{x}) \in x$. Since $x \leq y$ implies $x \cap B = y \cap B$, it follows that $B^\delta(\psi \to \neg\bar{x}) \in y$, from which it follows by $y^\delta \subseteq x$ that $\psi \to \neg\bar{x} \in x$. Since $\psi \in x$, we obtain $\neg\bar{x} \in x$, a contradiction. Our assumption $B^\psi\neg\bar{x} \in x$ must have been incorrect, and so we must have $\neg B^\psi\neg\bar{x} \in x$ after all.

- Right to left: for $\psi \in C_1$ and $x \in [\psi]$, we prove $\neg B^\psi\neg\bar{x} \in x$ implies $x \in \min[\psi]$.

  Assume $\psi \in C_1$, $x \in [\psi]$ and, $\neg B^\psi\neg\bar{x} \in x$. It suffices to show that for each $y \in [\psi] \cap \mathsf{cc}(x)$, we have $x \leq y$. Proceeding, take an arbitrary $y \in [\psi] \cap \mathsf{cc}(x)$. It follows that $y \cap B = x \cap B$ and that $\psi \in (x \cap y \cap C_1)$. So from $\neg B^\psi\neg\bar{x} \in x$ we obtain $\neg B^\psi\neg\bar{x} \in y$. Now take an arbitrary $B^\psi\varphi \in y$. It follows from this and $\neg B^\psi\neg\bar{x} \in y$ by (RM) that $B^{\psi\wedge\bar{x}}\varphi \in y$. Since $\psi \in x \cap C_1$, we have $\vdash \bar{x} \to \psi$ and therefore it follows by (LE) that $B^{\bar{x}}\varphi \in y$. Since $x \cap B = y \cap B$, we obtain $B^{\bar{x}}\varphi \in x$. However, we saw in the argument for reflexivity that $x^{\bar{x}} \subseteq x$, so it follows that $\varphi \in x$. That is, we have shown that $\psi \in (x \cap y \cap C_1)$ and $y^\psi \subseteq x$. Hence $x \leq y$. Since $y \in [\psi] \cap \mathsf{cc}(x)$ was chosen arbitrarily and $x \in [\psi]$, it follows that $x \in \min[\psi]$.

This completes the proof of the Minimality Lemma.

Truth Lemma: for each $\varphi \in C_1$, we have $[\varphi] = [\![\varphi]\!]_M$. We prove this now. The proof is by induction on the construction of formulas in C_1. The induction base case and Boolean induction step cases are standard, so we only consider the induction step case for formulas $B^\psi\varphi \in C_1$. Note: by the definition of C_1, we have $B^\psi\varphi \in C_1$ iff $B^\psi\varphi \in C_0$. Further, from $B^\psi\varphi \in C_0$, it follows by the definition of C_0 that $\psi \in C_0$ and $\varphi \in C_0$.

- Induction step $B^\psi\varphi$ (left to right): if $x \in W$ and $B^\psi\varphi \in x \cap C_1$, then $M, x \models B^\psi\varphi$.

  Assume $B^\psi\varphi \in x \cap C_1$. If $\min[\![\psi]\!] \cap \mathsf{cc}(x) = \emptyset$, then, since M is well-founded, the result follows immediately by Theorem 3.4. So assume $\min[\![\psi]\!] \cap \mathsf{cc}(x) \neq \emptyset$ and take an arbitrary $y \in \min[\![\psi]\!] \cap \mathsf{cc}(x)$. Applying the induction hypothesis, $y \in \min[\psi]$, from which it follows by the Minimality Lemma that $\neg B^\psi\neg\bar{y} \in y$. Since $y \in \mathsf{cc}(x)$ implies $x \cap B = y \cap B$, it follows that $\neg B^\psi\neg\bar{y} \in x$. It follows from this by the assumption and (RM) that $B^{\psi\wedge\bar{y}}\varphi \in x$. Since $\psi \in y \cap C_1$, we have $\vdash \bar{y} \to \psi$ and therefore it follows from $B^{\psi\wedge\bar{y}}\varphi \in x$ by (LE) that $B^{\bar{y}}\varphi \in x$. Since $x \cap B = y \cap B$, we have $B^{\bar{y}}\varphi \in y$. But as we saw that $y^{\bar{y}} \subseteq y$ in the argument for reflexivity, it follows that $\varphi \in y$ and hence that $y \in [\varphi]$. Applying the induction hypothesis, $y \in [\![\varphi]\!]$. Since $y \in \min[\![\psi]\!] \cap \mathsf{cc}(x)$ was chosen arbitrarily, we have shown that $\min[\![\psi]\!] \cap \mathsf{cc}(x) \subseteq [\![\varphi]\!]$. Since M is well-founded, it follows by Theorem 3.4 that $M, w \models B^\psi\varphi$.

- Induction step $B^\psi\varphi$ (right to left): if $x \in W$, $B^\psi\varphi \in C_1$, and $M, x \models B^\psi\varphi$, then $B^\psi\varphi \in x$.

  Assume $B^\psi\varphi \in C_1$ and $M, x \models B^\psi\varphi$. Since M is well-founded, it follows by Theorem 3.4 that $\min[\![\psi]\!] \cap \mathsf{cc}(x) \subseteq [\![\varphi]\!]$. By the induction hypothesis, $\min[\psi] \cap \mathsf{cc}(x) \subseteq [\varphi]$. We wish to prove that $B^\psi\varphi \in x$. We consider two cases.

  Case: $\min[\psi] \cap \mathsf{cc}(x) = \emptyset$. Since M is well-founded, it follows that $[\psi] \cap \mathsf{cc}(x) = \emptyset$. Toward a contradiction, assume $B^\psi\bot \notin x$. Now if x^ψ were not consistent, then there would exist $\{\chi_1, \dots, \chi_n\} \subseteq x^\psi$ such that $\vdash (\bigwedge_{i \leq n} \chi_i) \to \bot$, from which it would follow by modal reasoning and the fact that $B^\psi\chi_i \in x$ for each $i \leq n$ that $B^\psi\bot \in x$, contradicting our assumption that $B^\psi\bot \notin x$. So x^ψ is consistent after all, and we may extend this set to some $y \in W$. Since $B^\psi\psi \in x$ by (Succ), it follows that $\psi \in y$ and therefore that $y \in [\psi]$. But $[\psi] \cap \mathsf{cc}(x) = \emptyset$, and so we must have that $y \notin \mathsf{cc}(x)$. However, $x^\psi \subseteq y$, so it follows by the Agreement Lemma that $x \cap B = y \cap B$ and hence $x^{\bar{x}\vee\bar{y}} = y^{\bar{x}\vee\bar{y}}$. If $x^{\bar{x}\vee\bar{y}}$ were not consistent, then there would exist $\{\chi_1, \dots, \chi_n\} \subseteq x^{\bar{x}\vee\bar{y}}$ such that $\vdash (\bigwedge_{i \leq n} \chi_i) \to \bot$, from which it would follow by modal reasoning and the fact that $B^{\bar{x}\vee\bar{y}}\chi_i \in x$ for each $i \leq n$ that $B^{\bar{x}\vee\bar{y}}\bot \in x$, from which we would obtain by (WCon) that $\neg(\bar{x} \vee \bar{y}) \in x$ and hence that $\neg\bar{x} \in x$, a contradiction. So $x^{\bar{x}\vee\bar{y}} = y^{\bar{x}\vee\bar{y}}$ is consistent and so may be extended to some $z \in W$. By (Succ), $B^{\bar{x}\vee\bar{y}}(\bar{x} \vee \bar{y}) \in x \cap y$ and therefore $\bar{x} \vee \bar{y} \in z$. But then $\bar{x} \vee \bar{y} \in (x \cap y \cap z \cap C_1)$, $x^{\bar{x}\vee\bar{y}} \subseteq z$, and $y^{\bar{x}\vee\bar{y}} \subseteq z$; that is, $z \leq x$ and $z \leq y$, from which it follows that $y \in \mathsf{cc}(x)$, a contradiction. Therefore our original assumption that $B^\psi\bot \notin x$ must have been incorrect, and so we must have $B^\psi\bot \in x$ after all. Applying modal reasoning, we obtain $B^\psi\varphi \in x$, as desired.

  Case: $\min[\psi] \cap \mathsf{cc}(x) \neq \emptyset$. It follows that there exists $y \in \min[\psi] \cap \mathsf{cc}(x)$. Toward a contradiction, assume $B^\psi\varphi \notin y$. If $y^\psi \cup \{\neg\varphi\}$ were not consistent, then we would have a finite $\{\chi_1, \dots, \chi_n\} \subseteq y^\psi$ such that $\vdash (\bigwedge_{i \leq n} \chi_i) \to \varphi$, from which it would follow by modal reasoning and the fact that $B^\psi\chi_i \in y$ for each $i \leq n$ that $B^\psi\varphi \in y$, contradicting the consistency of y by our assumption that $B^\psi\varphi \notin y$. So $y^\psi \cup \{\neg\varphi\}$ is indeed consistent and may be extended to some $z \in W$. Since we have $B^\psi\psi \in y$ by (Succ), it follows that $\psi \in z \cap y \cap C_1$. So since $y^\psi \subseteq z$, it follows that $z \leq y$. But then $z \leq y$, $\psi \in z$, and $y \in \min[\psi] \cap \mathsf{cc}(x)$, so it follows that $z \in \min[\psi] \cap \mathsf{cc}(x)$. Since $\min[\psi] \cap \mathsf{cc}(x) \subseteq [\varphi]$, it follows that $\varphi \in z$. However, by the construction of z as a maximal consistent extension of $y^\psi \cup \{\neg\varphi\}$, we also have

$\neg\varphi \in z$. So z is inconsistent, a contradiction. It follows that our assumption $\neg B^\psi\varphi \in y$ must have been incorrect and therefore we must have $B^\psi\varphi \in y$ after all. Since $y \in \mathsf{cc}(x)$, we have $x \cap B = y \cap B$ and therefore that $B^\psi\varphi \in x$, as desired.

This completes the proof of the Truth Lemma. Since $\neg\theta \in w_\theta \cap C_1$, it follows by the Truth Lemma that $M, w_\theta \not\models \theta$. But then we have shown that $\nvdash \theta$ implies $M \not\models \theta$ for our locally well-founded model $M \in \mathfrak{P}_L$. Applying Theorem 3.6, $\nvdash \theta$ implies $\not\models \varphi$. By contraposition, we have that $\models \theta$ implies $\vdash \theta$. So completeness obtains. □

A.3 Results for AGM revision

Proof of Theorem 4.1. For (a), assume $T \in \mathfrak{B}_{\mathsf{CPL}}$ is consistent. To say a S is *maximal* CPL*-consistent* means that $S \subseteq \mathscr{L}_{\mathsf{CPL}}$, S is consistent, and adding to S any $\psi \in \mathscr{L}_{\mathsf{CPL}}$ not already present would result in a set that is inconsistent. We make tacit use of various well-known facts about maximal consistent sets. Let $\mathfrak{S}$ be the collection of all maximal CPL-consistent sets. Let $\{\psi_i\}_{0<i<\omega}$ be an enumeration of $\mathscr{L}_{\mathsf{CPL}}$. Define $\omega^+ := \omega - \{0\}$ and $S + \psi := \mathsf{CPL}(S \cup \{\psi\})$. Take

$$\begin{aligned}
W &:= \{(S,0) \in \mathfrak{S} \times \{0\} \mid T \subseteq S\} \cup \\
&\qquad \{(S,i) \in \mathfrak{S} \times \omega^+ \mid \neg\psi_i \in T - \mathsf{CPL}(\emptyset) \text{ and } \mathsf{CPL}(\psi_i) \subseteq S\} \ , \\
\leq &:= \{((S,i),(S',i')) \in W \times W \mid i \leq i'\} \ , \\
V &:= \{((S,i),P) \in W \times \wp(\mathscr{P}) \mid P = S \cap \mathscr{P}\} \ , \\
M &:= (W, \leq, V) \ .
\end{aligned}$$

Since T is consistent, it follows that there exists $S \in \mathfrak{S}$ such that $S \supseteq T$, which implies $(S,0) \in W$. Therefore $W \neq \emptyset$. Similarly, if $i \in \omega^+$ and $\neg\psi_i \in \mathsf{CPL}(T) - \mathsf{CPL}(\emptyset)$, then it follows that ψ_i is consistent and therefore there exists $S \in \mathfrak{S}$ such that $S \supseteq \mathsf{CPL}(\psi_i)$, which implies $(S,i) \in W$. The relation $\leq$ on W is a well-order because the relation $\leq$ on ω is a well-order. So M is a well-ordered plausibility model.

Truth Lemma: for each $\varphi \in \mathscr{L}_{\mathsf{CPL}}$ and $(S,i) \in W$, we have $M,(S,i) \models \varphi$ iff $\varphi \in S$. The proof is by induction on the construction of $\varphi \in \mathscr{L}_{\mathsf{CPL}}$. Induction base: if $\varphi = \bot$, then $\bot \notin S$ since S is consistent and $M,(S,i) \not\models \bot$ by Definition 3.2. Induction base: if $\varphi = p \in \mathscr{P}$, then the result follows by the definition of V and Definition 3.2. Induction step: if $\varphi = \varphi_1 \to \varphi_2$, then the result follows by the induction hypothesis and Definition 3.2. The lemma follows.

Theory Lemma: for each $\psi \in \mathscr{L}_{\mathsf{CPL}}$, we have

$$M^\downarrow *_M \psi = \begin{cases} T + \psi & \text{if } \neg\psi \notin T, \\ \mathsf{CPL}(\psi) & \text{if } \neg\psi \in T - \mathsf{CPL}(\emptyset), \\ \mathscr{L}_{\mathsf{CPL}} & \text{if } \neg\psi \in \mathsf{CPL}(\emptyset). \end{cases}$$

We prove this now by a case distinction. Proceeding, for each $W' \subseteq W$, define

$$T(W') := \{\varphi \in \mathscr{L}_{\mathsf{CDL}} \mid \forall w \in W' : M, w \models \varphi\} \ .$$

- Case: $\neg\psi \notin T$.

  Let $\mathfrak{S}_\psi := \{S \in \mathfrak{S} \mid S \supseteq T \cup \{\psi\}\}$. It follows by the assumption of this case that $\mathfrak{S}_\psi \neq \emptyset$ and hence $\mathfrak{S}_\psi \times \{0\} \subseteq W$. By the Truth Lemma and the definition of $\leq$, it follows that $\min[\![\psi]\!]_M = \mathfrak{S}_\psi \times \{0\}$. Now notice that since $S \in \mathfrak{S}_\psi$ is maximal CPL-consistent, it follows that for each $S' \subseteq S$ we have $\mathsf{CPL}(S') \subseteq S$. Therefore, since $T \cup \{\psi\} \subseteq S$ for each $S \in \mathfrak{S}_\psi$, it follows that $T + \psi \subseteq S$ for each $S \in \mathfrak{S}_\psi$. So $T + \psi \subseteq \bigcap \mathfrak{S}_\psi$. Now if we had some $\varphi \in \bigcap \mathfrak{S}_\psi$ such that $\varphi \notin T + \psi$, then it would follow that $\varphi \notin \mathsf{CPL}(T \cup \{\psi\})$ and therefore we could extend $T \cup \{\psi, \neg\varphi\}$ to a maximal CPL-consistent

$S \in \mathfrak{S}_\psi$ having $\varphi \notin S$, contradicting the fact that $\varphi \in \bigcap \mathfrak{S}_\psi$. Therefore it must be the case that $\bigcap \mathfrak{S}_\psi \subseteq T + \psi$. Hence $T + \psi = \bigcap \mathfrak{S}_\psi$. But then we have

$$T(\mathfrak{S}_\psi \times \{0\}) = \bigcap \mathfrak{S}_\psi = T + \psi \ ,$$

where the leftmost equality follows by the Truth Lemma. Since $\min[\![\psi]\!]_M = \mathfrak{S}_\psi \times \{0\}$, it follows by the definition of $*_M$ that $M^\downarrow *_M \psi = T + \psi$.

- Case: $\neg\psi \in T - \mathsf{CPL}(\emptyset)$.

  Define $\mathfrak{S}_\psi := \{S \in \mathfrak{S} \mid \psi \in S\}$. It follows by the definition of W that there exists $j \in \omega^+$ and $S \in \mathfrak{S}_\psi$ such that $\psi_j = \psi$ and $(S, j) \in W$. So let $i \in \omega^+$ be the least positive integer satisfying the property that there exists $(S, i) \in W$ with $\psi \in S$. By the Truth Lemma, the definition of $\leq$, and our choice of i, it follows that $\min[\![\psi]\!]_M = \mathfrak{S}_\psi \times \{i\}$. Since $S \in \mathfrak{S}_\psi$ is maximal CPL-consistent, it follows that for each $S' \subseteq S$ we have $\mathsf{CPL}(S') \subseteq S$. Therefore, since $\psi \in S$ for each $S \in \mathfrak{S}_\psi$, it follows that $\mathsf{CPL}(\psi) \subseteq S$ for each $S \in \mathfrak{S}_\psi$. So $\mathsf{CPL}(\psi) \subseteq \bigcap \mathfrak{S}_\psi$. If we had some $\varphi \in \bigcap \mathfrak{S}_\psi$ such that $\varphi \notin \mathsf{CPL}(\psi)$, then we could extend $\{\psi, \neg\varphi\}$ to a maximal CPL-consistent $S \in \mathfrak{S}_\psi$ having $\varphi \notin S$, contradicting the fact that $\varphi \in \bigcap \mathfrak{S}_\psi$. Therefore it must be the case that $\bigcap \mathfrak{S}_\psi \subseteq \mathsf{CPL}(\psi)$. Hence $\mathsf{CPL}(\psi) = \bigcap \mathfrak{S}_\psi$. But then we have

  $$T(\mathfrak{S}_\psi \times \{i\}) = \bigcap \mathfrak{S}_\psi = \mathsf{CPL}(\psi) \ ,$$

  where the leftmost equality follows by the Truth Lemma. Since $\min[\![\psi]\!]_M = \mathfrak{S}_\psi \times \{i\}$, it follows by the definition of $*_M$ that $M^\downarrow *_M \psi = \mathsf{CPL}(\psi)$.

- Case: $\neg\psi \in \mathsf{CPL}(\emptyset)$.

  It follows from $\neg\psi \in \mathsf{CPL}(\emptyset)$ by the soundness of Classical Propositional Logic that $\not\models \psi$ and therefore $\min[\![\psi]\!]_M = \emptyset$. Hence $\min[\![\psi]\!]_M \subseteq [\![\varphi]\!]_M$ for each $\varphi \in \mathscr{L}_{\mathsf{CPL}}$. Applying the definition of $*_M$, it follows that $M^\downarrow *_M \psi = \mathscr{L}_{\mathsf{CPL}}$.

The lemma follows.

We now prove that M is a Grove system for T. First, since $M^\downarrow = L_0$, it follows by the Theory Lemma that $M^\downarrow = T$. Second, we have already seen that M is a well-ordered plausibility model. So all that remains is to prove that $M^\downarrow *_M (-)$ satisfies the AGM revision postulates. So given an arbitrary $\psi \in \mathscr{L}_{\mathsf{CPL}}$, we check each postulate in turn.

- Closure: $M^\downarrow *_M \psi = \mathsf{CPL}(M^\downarrow *_M \psi)$.

  By the Theory Lemma, $M^\downarrow *_M \psi$ is either $T + \psi$, $\mathsf{CPL}(\psi)$, or $\mathscr{L}_{\mathsf{CPL}}$. However, for each of these sets S, we have $\mathsf{CPL}(S) = S$.

- Success: $\psi \in M^\downarrow *_M \psi$.

  By the Theory Lemma, $M^\downarrow *_M \psi$ is either $T + \psi$, $\mathsf{CPL}(\psi)$, or $\mathscr{L}_{\mathsf{CPL}}$. However, each of these sets contains ψ.

- Inclusion: $M^\downarrow *_M \psi \subseteq M^\downarrow + \psi$.

  By the Theory Lemma, $M^\downarrow *_M \psi$ is $T + \psi$ if $\neg\psi \notin T$, is $\mathsf{CPL}(\psi)$ if $\neg\psi \in T - \mathsf{CPL}(\emptyset)$, and is $\mathscr{L}_{\mathsf{CPL}}$ if $\neg\psi \in \mathsf{CPL}(\emptyset)$. Further $M^\downarrow = T$. Inclusion obviously follows for the case $\neg\psi \notin T$. for the case $\neg\psi \in T - \mathsf{CPL}(\emptyset)$, Inclusion follows because $\mathsf{CPL}(\psi) \subseteq T + \psi = \mathsf{CPL}(T \cup \{\psi\})$. For the case $\neg\psi \in \mathsf{CPL}(\emptyset)$, Inclusion follows because $T + \psi = \mathsf{CPL}(T \cup \{\psi\}) = \mathscr{L}_{\mathsf{CPL}}$.

- Vacuity: if $\neg\psi \notin M^\downarrow$, then $M^\downarrow *_M \psi = M^\downarrow + \psi$.

  Since $M^\downarrow = T$, the result follows by the Theory Lemma.

- Consistency: if $\neg\psi \notin \mathsf{CPL}(\emptyset)$, then $\bot \notin \mathsf{CPL}(M^{\downarrow} *_M \psi)$.

  If $\neg\psi \notin \mathsf{CPL}(\emptyset)$, then either $\neg\psi \notin T$ or $\neg\psi \in T - \mathsf{CPL}(\emptyset)$. Applying the Theory Lemma, $M^{\downarrow} *_M \psi$ is either $T + \psi$ or $\mathsf{CPL}(\psi)$. Since T is consistent and we assumed ψ is consistent, each of $T + \psi$ and $\mathsf{CPL}(\psi)$ is consistent. Hence $\bot \notin \mathsf{CPL}(M^{\downarrow} *_M \psi)$.

- Extensionality: if $(\psi \leftrightarrow \psi') \in \mathsf{CPL}(\emptyset)$, then $M^{\downarrow} *_M \psi = M^{\downarrow} *_M \psi'$.

  Assume $(\psi \leftrightarrow \psi') \in \mathsf{CPL}(\emptyset)$. It follows that $T + \psi = T + \psi'$ and $\mathsf{CPL}(\psi) = \mathsf{CPL}(\psi')$. The result therefore follows by the Theory Lemma.

- Superexpansion: $M^{\downarrow} *_M (\psi \wedge \varphi) \subseteq (M^{\downarrow} *_M \psi) + \varphi$.

  Case: $\neg(\psi \wedge \varphi) \notin T$. By the Theory Lemma, $M^{\downarrow} *_M (\psi \wedge \varphi) = T + (\psi \wedge \varphi)$. Since T is a theory, it follows from the assumption of this case that $\neg\psi \notin T$. Applying the Theory Lemma, $M^{\downarrow} *_M \psi = T + \psi$. The result follows because

  $$T + (\psi \wedge \varphi) = \mathsf{CPL}(T \cup \{\psi \wedge \varphi\}) = \mathsf{CPL}(\mathsf{CPL}(T \cup \{\psi\}) \cup \{\varphi\}) = (T + \psi) + \varphi \ .$$

  Case: $\neg(\psi \wedge \varphi) \in T - \mathsf{CPL}(\emptyset)$. By the Theory Lemma, $M^{\downarrow} *_M (\psi \wedge \varphi) = \mathsf{CPL}(\psi \wedge \varphi)$. If $\neg\psi \notin T$, then it follows by the Theory Lemma that $M^{\downarrow} *_M \psi = T + \psi$ and hence

  $$\mathsf{CPL}(\psi \wedge \varphi) \subseteq \mathsf{CPL}(T \cup \{\psi, \varphi\}) = \mathsf{CPL}(\mathsf{CPL}(T \cup \{\psi\}) \cup \{\varphi\}) = (T + \psi) + \varphi \ .$$

  And if $\neg\psi \in T - \mathsf{CPL}(\emptyset)$, then it follows by the Theory Lemma that $M^{\downarrow} *_M \psi = \mathsf{CPL}(\psi)$ and hence $\mathsf{CPL}(\psi \wedge \varphi) = \mathsf{CPL}(\{\psi\} \cup \{\varphi\}) = \mathsf{CPL}(\psi) + \varphi$. Finally, if $\neg\psi \in \mathsf{CPL}(\emptyset)$, then it follows by the Theory Lemma that $M^{\downarrow} *_M \psi = \mathscr{L}_{\mathsf{CPL}}$, from which we obtain

  $$\mathsf{CPL}(\psi \wedge \varphi) = \mathscr{L}_{\mathsf{CPL}} = \mathscr{L}_{\mathsf{CPL}} + \varphi \ .$$

  Case: $\neg(\psi \wedge \varphi) \in \mathsf{CPL}(\emptyset)$. By the Theory Lemma, $M^{\downarrow} *_M (\psi \wedge \varphi) = \mathscr{L}_{\mathsf{CPL}}$. If $\neg\psi \notin T$, then it follows by the Theory Lemma that $M^{\downarrow} *_M \psi = T + \psi$; however, since $\varphi \wedge \psi \in (T + \psi) + \varphi$, it follows from the assumption of the case that $(T + \psi) + \varphi = \mathscr{L}_{\mathsf{CPL}}$, which implies the result. And if $\neg\psi \in T - \mathsf{CPL}(\emptyset)$, then it follows by the Theory Lemma that $M^{\downarrow} *_M \psi = \mathsf{CPL}(\psi)$; however, since $\varphi \wedge \psi \in \mathsf{CPL}(\psi) + \varphi$, it follows from the assumption of the case that $\mathsf{CPL}(\psi) + \varphi = \mathscr{L}_{\mathsf{CPL}}$, which implies the result. Finally, if $\neg\psi \in \mathsf{CPL}(\emptyset)$, then it follows by the Theory Lemma that $M^{\downarrow} *_M \psi = \mathscr{L}_{\mathsf{CPL}}$ and the result follows because $\mathscr{L}_{\mathsf{CPL}} + \varphi = \mathscr{L}_{\mathsf{CPL}}$.

- Subexpansion: if $\neg\varphi \notin \mathsf{CPL}(M^{\downarrow} *_M \psi)$, then $(M^{\downarrow} *_M \psi) + \varphi \subseteq M^{\downarrow} *_M (\psi \wedge \varphi)$.

  Suppose $\neg\varphi \notin \mathsf{CPL}(M^{\downarrow} *_M \psi)$. We consider a few cases.

  Case: $\neg\psi \notin T$. By the Theory Lemma, $\mathsf{CPL}(M^{\downarrow} *_M \psi) = \mathsf{CPL}(T + \psi) = T + \psi$. If $\neg(\psi \wedge \varphi) \notin T$, then it follows by the Theory Lemma that $M^{\downarrow} *_M (\psi \wedge \varphi) = T + (\psi \wedge \varphi)$, from which we obtain the result because

  $$(T + \psi) + \varphi = \mathsf{CPL}(\mathsf{CPL}(T \cup \{\psi\}) \cup \{\varphi\}) = \mathsf{CPL}(T \cup \{\psi\} \cup \{\varphi\}) = \mathsf{CPL}(T \cup \{\psi \wedge \varphi\}) = T + (\psi \wedge \varphi) \ .$$

  And if $\neg(\psi \wedge \varphi) \in T - \mathsf{CPL}(\emptyset)$, then we have $(T + \psi) + \varphi = \mathscr{L}_{\mathsf{CPL}}$ by the definition of $+$; however, we originally assumed that $\neg\varphi \notin \mathsf{CPL}(M^{\downarrow} *_M \psi) = T + \psi$, which implies $(T + \psi) + \varphi \neq \mathscr{L}_{\mathsf{CPL}}$, a contradiction that allows us to conclude that the hypothesis $\neg(\psi \wedge \varphi) \in T - \mathsf{CPL}(\emptyset)$ cannot obtain under the assumption of this case. Finally, if $\neg(\psi \wedge \varphi) \in \mathsf{CPL}(\emptyset)$, then it follows by the Theory Lemma that $M^{\downarrow} *_M (\psi \wedge \varphi) = \mathscr{L}_{\mathsf{CDL}}$, which trivially implies the result.

Case: $\neg\psi \in T - \mathsf{CPL}(\emptyset)$. By the Theory Lemma, $\mathsf{CPL}(M^{\downarrow} *_M \psi) = \mathsf{CPL}(\mathsf{CPL}(\psi)) = \mathsf{CPL}(\psi)$. Since T is a theory, it follows from the assumption of this case that $\neg(\psi \wedge \varphi) \in T - \mathsf{CPL}(\emptyset)$. Applying the Theory Lemma, $M^{\downarrow} *_M (\psi \wedge \varphi) = \mathsf{CPL}(\psi \wedge \varphi)$. But then the result follows because

$$\mathsf{CPL}(\psi) + \varphi = \mathsf{CPL}(\mathsf{CPL}(\psi) \cup \{\varphi\}) = \mathsf{CPL}(\{\psi\} \cup \{\varphi\}) = \mathsf{CPL}(\psi \wedge \varphi) \ .$$

Case: $\neg\psi \in \mathsf{CPL}(\emptyset)$. By the Theory Lemma $\mathsf{CPL}(M^{\downarrow} *_M \psi) = \mathsf{CPL}(\mathscr{L}_{\mathsf{CPL}}) = \mathscr{L}_{\mathsf{CPL}}$. It follows from the assumption of this case that $\neg(\psi \wedge \varphi) \in \mathsf{CPL}(\emptyset)$. Applying the Theory Lemma, $M^{\downarrow} *_M \psi = \mathscr{L}_{\mathsf{CPL}}$. The result follows.

Conclusion: M is a Grove system for T.

For (b), if T is inconsistent, then since M_* is a system of spheres, it follows that $T * (-) = M_*^{\downarrow} *_M (-)$ satisfies the AGM revision postulates. And if T is consistent, then since $T * \psi = T *_{M_T} \psi$ and M_T is a Grove system for T, we have $M^{\downarrow} = T$ and that M is a system of spheres; therefore, $T *_{M_T} (-) = T * (-)$ satisfies the AGM revision postulates. Conclusion: $*$ is an AGM revision operator. □

A.4 Results for JCDL

Proof of Lemma 5.5. For this argument, a *derivation* is a JCDL-derivation. We wish to establish the following result:

$$\pi \vdash^n_{\mathsf{JCDL}} \varphi \quad \Rightarrow \quad \exists \pi_* \supseteq \pi, \ \ \pi_* \vdash^0_{\mathsf{JCDL}} \varphi \ . \tag{10}$$

The proof of (10), an adaptation of [3, Lemma 4.6] to the present setting, is by induction on n with a sub-induction on the length $|\pi|$ of π.

- *Induction base:* $n = 0$. Take $\pi_* = \pi$.
- *Induction step:* assume the result holds for all $m < n$ (this is the "induction hypothesis"), and prove it holds for n by a sub-induction on $|\pi|$.

  Sub-induction base: $|\pi| = 1$. Since π contains a single line, φ is an axiom and therefore $\pi \vdash^0_{\mathsf{JCDL}} \varphi$. Take $\pi_* = \pi$.

  Sub-induction step: assume the result holds for all derivations π' having $|\pi'| < |\pi|$ (this is the "sub-induction hypothesis"), and prove it holds for π. Proceeding, assume $\pi \vdash^n_{\mathsf{JCDL}} \varphi$. If $\pi \vdash^0_{\mathsf{JCDL}} \varphi$, we are done. So let us assume further that $\pi \nvdash^0_{\mathsf{JCDL}} \varphi$. Therefore

  $$\pi = \theta_1, \dots, \theta_{|\pi|-1}, \varphi$$

  contains at least one troublesome necessitation. It follows that there exists a shortest prefix π' of π whose last line is a troublesome necessitation: we have

  $$\pi' = \theta_1, \dots, \theta_{|\pi'|-1}, \dot{B}^{\delta}\theta_m$$

  with $|\pi'|$ the minimum value such that $\dot{B}^{\delta}\theta_m$ neither is not a possibly necessitated axiom nor follows from a previous line by (MP). We consider two cases.

  Case: $|\pi'| < |\pi|$. By the sub-induction hypothesis, there exists $\pi'_* \supseteq \pi'$ such that $\pi'_* \vdash^0_{\mathsf{JCDL}} \dot{B}^{\delta}\theta_m$. Let σ be the suffix of π such that $\pi'\sigma = \pi$, where we have denoted sequence concatenation by juxtaposition. Since π'_* is a derivation and $\pi'_* \supseteq \pi'$, it follows that $\pi'_*\sigma$ is a derivation and this derivation has the same last line as π. Therefore, since we chose π' as the shortest prefix π' of π whose last line is a troublesome necessitation, we have $\pi'_*\sigma \vdash^{n-1}_{\mathsf{JCDL}} \varphi$. Applying the induction hypothesis, it follows that there exists a derivation $\pi_* \supseteq \pi'_*\sigma \supseteq \pi'\sigma = \pi$ such that $\pi_* \vdash^0_{\mathsf{JCDL}} \varphi$.

Case: $\pi' = \pi$. Hence $\varphi = \dot{B}^\delta\theta_m$. Since the shortest prefix of π whose last line is a troublesome necessitation is π itself, it follows that $\pi \vdash^1_{\mathsf{JCDL}} \dot{B}^\delta\theta_m$. But then θ_m is not a troublesome necessitation. Moreover, θ_m is not a possibly necessitated axiom (since if it were, $\dot{B}^\delta\theta_m$ would not be a troublesome necessitation, contrary to our assumption implying that it is). So θ_m must follow by way of (MN) from lines $\theta_n \to \theta_m$ and θ_n appearing earlier in π than line m. That is,

$$\begin{aligned}\pi = \;& \sigma, \theta_m, \tau, X\theta_m \\ & \text{with both } \theta_n \to \theta_m \text{ and } \theta_n \text{ in } \sigma\end{aligned}$$

where σ and τ denote sequences of formulas (and we note that τ may be empty). By (eMN), we have derivations

$$\begin{aligned}&\pi^1 = \sigma, \dot{B}^\delta\theta_n && \text{with } |\pi^1| \leq m < |\pi| \\ &\pi^2 = \sigma, \dot{B}^\delta(\theta_n \to \theta_m) && \text{with } |\pi^2| \leq m < |\pi|\end{aligned}$$

such that $\pi^1 \vdash^1_{\mathsf{JCDL}} \dot{B}^\delta\theta_n$ and $\pi^2 \vdash^1_{\mathsf{JCDL}} \dot{B}^\delta(\theta_n \to \theta_m)$. Applying the sub-induction hypothesis to π^1 and to π^2, there exist $\pi^1_* \supseteq \pi^1$ and $\pi^2_* \supseteq \pi^2$ such that $\pi^1_* \vdash^0_{\mathsf{JCDL}} \dot{B}^\delta\theta_n$ and $\pi^2_* \vdash^0_{\mathsf{JCDL}} \dot{B}^\delta(\theta_n \to \theta_m)$. Recalling the abbreviation $\dot{B}^\delta\chi = c_\chi :^\delta \chi$ and making use of an instance of (eK), the sequence

$$\begin{aligned}\pi_* = \;& \pi^1_*, \pi^2_*, \theta_m, \tau, \\ & \dot{B}^\delta(\theta_n \to \theta_m) \to (\dot{B}^\delta\theta_n \to (c_{\theta_n\to\theta_m} \cdot c_{\theta_n}) :^\delta \theta_m), \\ & \dot{B}^\delta\theta_n \to (c_{\theta_n\to\theta_m} \cdot c_{\theta_n}) :^\delta \theta_m, \\ & (c_{\theta_n\to\theta_m} \cdot c_{\theta_n}) :^\delta \theta_m\end{aligned}$$

is a derivation satisfying $\pi_* \supseteq \pi$ and $\pi_* \vdash^0_{\mathsf{JCDL}} \varphi$.

This completes the proof that (10) holds. □

Proof of Theorem 5.6. In light of Lemma 5.5, to prove the statement of the present theorem, it suffices to prove the following: if $\vdash^0_{\mathsf{JCDL}} \varphi$, then there exists a logical term t such that $\vdash^0_{\mathsf{JCDL}} t :^\delta \varphi$. This we prove by induction on the length of derivation.

- Induction base and induction step for (eMN): φ is a possibly necessitated axiom. It follows that $\vdash^0_{\mathsf{JCDL}} c_\varphi :^\delta \varphi$ for the logical term c_φ.
- Induction step for (MP): we assume $\vdash^0_{\mathsf{JCDL}} \varphi \to \psi$ and $\vdash^0_{\mathsf{JCDL}} \varphi$ along with the following "induction hypothesis": there exist logical terms t and s such that $\vdash^0_{\mathsf{JCDL}} t :^\delta (\varphi \to \psi)$ and $\vdash^0_{\mathsf{JCDL}} s :^\delta \varphi$. But then it follows by the induction hypothesis, (eK), and two applications of (MP) that $\vdash^0_{\mathsf{JCDL}} (t \cdot s) :^\delta \psi$ for the logical term $t \cdot s$.

The result follows. □

Proof of Theorem 5.9. For Item 1, we prove by induction on the length of derivation in JCDL from hypotheses Γ that

$$\Gamma \vdash_{\mathsf{JCDL}} \varphi \quad \text{implies} \quad \Gamma^\circ \vdash_{\mathsf{CDL}} \varphi^\circ \ .$$

- Induction base for hypotheses: if $\varphi \in \Gamma$, then $\varphi^\circ \in \Gamma^\circ$. So $\Gamma^\circ \vdash_{\mathsf{CDL}} \varphi^\circ$.
- Induction base for axioms: φ is an axiom. But for each axiom scheme (eX) of JCDL for which there is a "matching" axiom scheme (X) of CDL, the forgetful projection of an instance of the JCDL axiom is an instance of the matching CDL scheme. Regarding the three remaining JCDL schemes (eSum), (eCert), and (eA) that have no matching CDL scheme: each of (eSum) and (eCert) is mapped to an instance of the CDL-theorem $p \to p$, and (eA) is mapped to an instance of the CDL-theorem $q \to (p \to p)$. Conclusion: $\Gamma^\circ \vdash_{\mathsf{CDL}} \varphi^\circ$.

- Induction step for (MP): we assume the result holds for JCDL-theorems $\varphi \to \psi$ and φ derived from hypotheses Γ, and we prove that $\Gamma^\circ \vdash_{\mathsf{CDL}} \psi^\circ$. By our assumption, $\varphi^\circ \to \psi^\circ$ and φ° are CDL-theorems derived from hypotheses Γ°. Applying (MP), ψ° as a CDL-theorem derived from hypotheses Γ°.

- Induction step for (eMN): we assume the result holds for the JCDL-theorem φ derived from hypotheses Γ, and we prove that the result holds for the JCDL-theorem $\dot{B}^{\psi}\varphi$ derived from hypotheses Γ; that is, we prove that $\Gamma^\circ \vdash_{\mathsf{CDL}} (\dot{B}^{\psi}\varphi)^\circ$. By our assumption, φ° is a CDL-theorem derived from hypotheses Γ°. Applying (MN) and the definition of the forgetful projection, $B^{\psi^\circ}\varphi^\circ = (\dot{B}^{\psi}\varphi)^\circ$ is a CDL-theorem derived from hypotheses in Γ°.

This completes the proof of Item 1. For Item 2, we by induction on the length of derivation in CDL from hypotheses Δ that

$$\Delta \vdash_{\mathsf{CDL}} \psi \quad \text{implies} \quad \Delta^t \vdash_{\mathsf{JCDL}} \psi^t \ .$$

- Induction base for hypotheses: If $\psi \in \Delta$, then $\psi^t \in \Delta^t$. So $\Delta^t \vdash_{\mathsf{CDL}} \psi^t$.

- Induction base for axiom scheme: ψ is an axiom scheme of CDL, so we consider each possibility.
  - $(\mathrm{CL})^t$ is an instance of (CL) and therefore a JCDL-derivable from hypotheses Δ^t.
  - $(\mathrm{K})^t$ is the scheme $\dot{B}^{\gamma^t}(\varphi_1^t \to \varphi_2^t) \to (\dot{B}^{\gamma^t}\varphi_1^t \to \dot{B}^{\gamma^t}\varphi_2^t)$. This has the form

    $$\dot{B}^{\delta}(\chi_1 \to \chi_2) \to (\dot{B}^{\delta}\chi_1 \to \dot{B}^{\delta}\chi_2) \ .$$

    Applying (eK), we obtain

    $$\Delta^t \vdash_{\mathsf{JCDL}} \dot{B}^{\delta}(\chi_1 \to \chi_2) \to (\dot{B}^{\delta}\chi_1 \to (c_{\chi_1 \to \chi_2} \cdot c_{\chi_1}) :^{\delta} \chi_2) \ .$$

    Applying (eCert) and classical reasoning,

    $$\Delta^t \vdash_{\mathsf{JCDL}} \dot{B}^{\delta}(\chi_1 \to \chi_2) \to (\dot{B}^{\delta}\chi_1 \to \dot{B}^{\delta}\chi_2) \ .$$

    That is, $\Delta^t \vdash_{\mathsf{JCDL}} (\mathrm{K})^t$.
  - For $X \in \{\mathrm{Succ}, \mathrm{RM}, \mathrm{Inc}, \mathrm{Comm}, \mathrm{PI}, \mathrm{NI}\}$: $(X)^t$ is a schematic instance of (eX).
  - Similar to the argument for $(\mathrm{K})^t$:
    * $(\mathrm{KM})^t$ follows by (eKM) and (eCert), and
    * $(\mathrm{WCon})^t$ follows by (eWCon) and (eCert).

- Induction step for (MP): we assume the result holds for the CDL-theorems $\varphi_1 \to \varphi_2$ and φ_1 derived from hypotheses Δ, and we prove the result holds for the CDL-theorem φ_2 derived from hypothesis Δ. By our assumption,

  $$(\varphi_1 \to \varphi_2)^t = \varphi_1^t \to \varphi_2^t \quad \text{and} \quad \varphi_1^t$$

  is a JCDL-theorem derivable from hypotheses Δ^t. So it follows by (MN) that φ_2^t is a JCDL-theorem derivable from hypotheses Δ^t as well.

- Induction step for (MN): we assume the result holds for the CDL-theorem φ derived from hypotheses Δ, and we prove the result holds for the CDL-theorem $B^{\gamma}\varphi$ derived from hypotheses Δ; that is, we prove that $\Delta^t \vdash_{\mathsf{JCDL}} (B^{\gamma}\varphi)^t$. By our assumption, the formula φ^t is a JCDL-theorem derived from hypotheses Δ^t, but then it follows by (eMN) that $\dot{B}^{\gamma^t}\varphi^t = (\dot{B}^{\gamma}\varphi)^t$. is a JCDL-theorem derived from hypotheses Δ^t.

This completes the proof of Item 2. □

Proof of Theorem 5.19. This proof uses much of the work from the proof of Theorem 3.13 (the "old proof"). In utilizing portions of the argument from the old proof in the present argument (the "new proof"), we adopt the following conventions (the "JCDL-conventions"):

- the language (and set of formulas) is assumed to be $\mathscr{L}_{\mathsf{JCDL}}$;
- occurrences of "B" from the old proof are replaced by occurrences of "$\dot{B}$";
- truth or validity for CDL is replaced by truth or validity for JCDL;
- use of Theorem 3.4 or 3.4 is replaced by use of Theorem 5.16;
- use of Theorem 3.6 is replaced by Theorem 5.18;
- derivability in CDL is replaced by derivability in JCDL
- use of the non-subscripted turnstile $\vdash$ denotes $\vdash_{\mathsf{JCDL}}$;
- use of Theorem 3.11 is replaced by Theorem 5.11; and
- use of a derivable principle (X) of CDL (perhaps by tacit use of Theorem 3.11) is replaced by use of the corresponding derivable principle (eX) of JCDL (with corresponding tacit use of Theorem 5.11 when appropriate).

Having established the above JCDL-conventions and the "old/new proof" terminology, we proceed.

For soundness, we proceed by induction on the length of derivation. In the induction base, we must show that each axiom scheme is valid. (CL) is straightforward, so we proceed with the remaining schemes. Let (M, w) be an arbitrary well-ordered pointed Fitting model. We make tacit use of Theorems 2.3(3) and 5.15.

(eCert) is valid $\models t:^{\psi}\varphi \to \dot{B}^{\psi}\varphi$.

Assume (M, w) satisfies $t:^{\psi}\varphi$. Then $w \in A(t, \varphi)$ and $\min[\![\psi]\!] \subseteq [\![\varphi]\!]$. Applying Theorem 5.16, (M, w) satisfies $B^{\psi}\varphi$.

(eK) is valid: $\models t:^{\psi}(\varphi_1 \to \varphi_2) \to (s:^{\psi}\varphi_1 \to (t \cdot s):^{\psi}\varphi_2)$.

Assume (M, w) satisfies $t:^{\psi}(\varphi_1 \to \varphi_2)$ and $s:^{\psi}\varphi_1$. Then $w \in A(t, \varphi_1 \to \varphi_2) \cap A(s, \varphi_1)$, $\min[\![\psi]\!] \subseteq [\![\varphi_1 \to \varphi_2]\!]$, and $\min[\![\psi]\!] \subseteq [\![\varphi_1]\!]$. Hence $\min[\![\psi]\!] \subseteq [\![\varphi_2]\!]$ and, by Application (Definition 5.12), $w \in A(t \cdot s, \varphi_2)$. Conclusion: (M, w) satisfies $(t \cdot s):^{\psi}\varphi_2$.

(eSum) is valid: $\models (t:^{\psi}\varphi \vee s:^{\psi}\varphi) \to (t + s):^{\psi}\varphi$.

Assume (M, w) satisfies $t:^{\psi}\varphi \vee s:^{\psi}\varphi$. Then $w \in A(t, \varphi) \cup A(s, \varphi)$ and $\min[\![\psi]\!] \subseteq [\![\varphi]\!]$. By Sum (Definition 5.12), $w \in A(t + s, \varphi)$. Conclusion: (M, w) satisfies $(t + s):^{\psi}\varphi$.

(eSucc) is valid: $\models \dot{B}^{\psi}\psi$.

We have $\min[\![\psi]\!] \subseteq [\![\psi]\!]$. Applying Theorem 5.16, (M, w) satisfies $\dot{B}^{\psi}\psi$.

(eKM) is valid: $\models t:^{\psi}\bot \to t:^{\psi\wedge\varphi}\bot$.

Assume (M, w) satisfies $t:^{\psi}\bot$. Then $w \in A(t, \bot)$ and $\min[\![\psi]\!] \subseteq [\![\bot]\!]$. It follows from the latter by the old proof that $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\bot]\!]$. Conclusion: (M, w) satisfies $t:^{\psi\wedge\varphi}\bot$.

(eRM) is valid: $\models \neg\dot{B}^{\psi}\neg\varphi \to (t:^{\psi}\chi \to t:^{\psi\wedge\varphi}\chi)$.

Assume (M, w) satisfies $\neg\dot{B}^{\psi}\neg\varphi$ and $t :^{\psi} \chi$.

Since $\neg\dot{B}^{\psi}\neg\varphi = \neg c_{\neg\varphi} :^{\psi} \neg\varphi$, it follows by Certification (Definition 5.12) that $\min[\![\psi]\!] \not\subseteq [\![\varphi]\!]$. And it follows from $t :^{\psi} \chi$ that $w \in A(t, \chi)$ and $\min[\![\psi]\!] \subseteq [\![\chi]\!]$. By the old proof, we have by $\min[\![\psi]\!] \not\subseteq [\![\varphi]\!]$ and $\min[\![\psi]\!] \subseteq [\![\chi]\!]$ that $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\chi]\!]$. Since $w \in A(t, \chi)$, we conclude that (M, w) satisfies $t :^{\psi\wedge\varphi} \chi$.

(eInc) is valid: $\models t :^{\psi\wedge\varphi} \chi \to \dot{B}^{\psi}(\varphi \to \chi)$.

Suppose (M, w) satisfies $t :^{\psi\wedge\varphi} \chi$. Then $w \in A(t, \chi)$ and $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\chi]\!]$. It follows from the latter by the old proof that $\min[\![\psi]\!] \subseteq [\![\varphi \to \chi]\!]$. Conclusion: (M, w) satisfies $\dot{B}^{\psi}(\varphi \to \chi)$.

(eComm) is valid: $\models t :^{\psi\wedge\varphi} \chi \to t :^{\varphi\wedge\psi} \chi$.

Suppose (M, w) satisfies $t :^{\psi\wedge\varphi} \chi$. Then $w \in A(t, \chi)$ and $\min[\![\psi \wedge \varphi]\!] \subseteq [\![\chi]\!]$. Hence $\min[\![\varphi \wedge \psi]\!] \subseteq [\![\chi]\!]$. Conclusion: (M, w) satisfies $t :^{\varphi\wedge\psi} \chi$.

(ePI) is valid: $\models t :^{\psi} \chi \to \dot{B}^{\varphi}(t :^{\psi} \chi)$.

Suppose (M, w) satisfies $t :^{\psi} \chi$. Then $w \in A(t, \chi)$ and $\min[\![\psi]\!] \subseteq [\![\chi]\!]$. The latter implies that (M, v) satisfies $\dot{B}^{\psi}\chi$ for any given $v \in W$. Since M is well-ordered, we have $v \in \mathsf{cc}(w)$ for each $v \in W$, and therefore it follows from $w \in A(t, \chi)$ by Admissibility Indefeasibility (Definition 5.12) that $A(t, \chi) = W$. But then for each $v \in W$, we have $v \in A(t, \chi)$ and $M, v \models \dot{B}^{\psi}\chi$. Applying Theorem 5.14, we have for each $v \in W$ that $M, v \models t :^{\psi} \chi$. Therefore, $[\![t :^{\psi} \chi]\!] = W$, from which it follows that $\min[\![\varphi]\!] \subseteq [\![t :^{\psi} \chi]\!]$. Applying Theorem 5.16, we conclude that (M, w) satisfies $\dot{B}^{\varphi}(t :^{\psi} \chi)$.

(eNI) is valid: $\models \neg t :^{\psi} \chi \to \dot{B}^{\varphi}(\neg t :^{\psi} \chi)$.

Suppose (M, w) satisfies $\neg t :^{\psi} \chi$. Then $w \notin A(t, \varphi)$ or $\min[\![\psi]\!] \not\subseteq [\![\chi]\!]$.

Case: $w \notin A(t, \varphi)$. Since M is well-ordered, we have $v \in \mathsf{cc}(w)$ for each $v \in W$, and so it follows from $w \notin A(t, \varphi)$ by Admissibility Indefeasibility (Definition 5.12) that $A(t, \varphi) = \emptyset$. Therefore, $[\![\neg t :^{\psi} \chi]\!] = W$, from which it follows that $\min[\![\varphi]\!] \subseteq [\![\neg t :^{\psi} \chi]\!]$. Applying Theorem 5.16, we conclude that (M, w) satisfies $\dot{B}^{\varphi}(\neg t :^{\psi} \chi)$.

Case: $\min[\![\psi]\!] \not\subseteq [\![\chi]\!]$. It follows that $M, v \models \neg t :^{\psi} \chi$ for each $v \in \mathsf{cc}(w) = W$. Therefore, $[\![\neg t :^{\psi} \chi]\!] = W$, from which it follows that $\min[\![\varphi]\!] \subseteq [\![\neg t :^{\psi} \chi]\!]$. Applying Theorem 5.16, we conclude that (M, w) satisfies $\dot{B}^{\varphi}(\neg t :^{\psi} \chi)$.

(eWCon) is valid: $\models t :^{\psi} \bot \to \neg\psi$.

Suppose (M, w) satisfies $t :^{\psi} \bot$. By the old proof, $[\![\neg\psi]\!] = W$. So (M, w) satisfies $\neg\psi$.

(eA) is valid: $\models t :^{\psi} \varphi \to (\dot{B}^{\chi}\varphi \to t :^{\chi} \varphi)$.

Suppose (M, w) satisfies $t :^{\psi} \varphi$ and $\dot{B}^{\chi}\varphi$. Then $w \in A(t, \varphi)$ and $\min[\![\chi]\!] \subseteq [\![\varphi]\!]$. Conclusion: (M, w) satisfies $t :^{\chi} \varphi$.

This completes the induction base. For the induction step, we must show that validity is preserved under the rules of (MP) and (eMN). The argument for (MP) is standard, so let us focus on (eMN). We assume $\models \varphi$ for the JCDL-derivable φ (this is the "induction hypothesis"), and we prove that $\models \dot{B}^{\psi}\varphi$. Proceeding, since $\models \varphi$, it follows that $[\![\varphi]\!] = W$ and hence that $\min[\![\psi]\!] \subseteq [\![\varphi]\!]$. By Certification (Definition 5.12, $A(c_{\varphi}, \varphi) = W$. But then (M, w) satisfies $c_{\varphi} :^{\psi} \varphi = \dot{B}^{\psi}\varphi$. Soundness has been proved.

Since JCDL is sound with respect to the class of well-ordered Fitting models we note that JCDL is consistent (i.e., $\nvdash_{\mathsf{JCDL}} \bot$). In particular, take any pointed Fitting model (M, w) containing only the single world w. It is simple to construct such a model: take

$$M := (\{w\}, \{(w, w)\}, \{(w, \emptyset)\}, A) \ ,$$

where A is the "total" admissibility function defined by setting $A(t,\varphi) := W$ for all $(t,\varphi) \in \mathscr{T}_{\mathsf{JCDL}} \times \mathscr{L}_{\mathsf{JCDL}}$. It is obvious that the requisite properties from Definition 5.12 obtain. Since there is only one world, M is well-ordered. Further, by soundness, we have that $\vdash_{\mathsf{JCDL}} \varphi$ implies $M,w \models \varphi$. Therefore, since $M,w \not\models \bot$ by Definition 5.13, it follows that $\nvdash_{\mathsf{JCDL}} \bot$. That is, JCDL is consistent. We make use of this fact tacitly in what follows.

For completeness, take a formula θ such that $\nvdash \neg\theta$. Define consistency, inconsistency, maximal consistency in ("maxcons in") a set of formulas, the set $\mathsf{sub}(\varphi)$ of subformulas of φ (including φ itself), the set $\mathsf{sub}(S)$ containing all subformulas of each formula in the set S (including the formulas themselves), and the Boolean closure $\oplus S$ as in the old proof (but of course using the JCDL-conventions). For a set S of formulas, we define:

$$
\begin{aligned}
\vec{B}S &:= S \cup \{\dot{B}^{\psi}\varphi \mid t\!:^{\psi}\!\varphi \in S\} \ , \\
\pm S &:= S \cup \{\neg\varphi \mid \varphi \in S\} \ , \\
\dot{B}_0 S &:= S \ , \\
\dot{B}_{i+1} S &:= \pm\{\dot{B}^{\psi}\varphi \mid \psi \in S \text{ and } \varphi \in \dot{B}_i S\} \ , \\
\dot{B}_{\omega} S &:= \textstyle\bigcup_{0<i<\omega} \dot{B}_i S \ , \\
C_0 &:= \pm\vec{B}\,\mathsf{sub}(\{\theta,\bot,\top\}) \ , \\
C_1 &:= \oplus C_0 \ , \\
\dot{B} &:= \dot{B}_{\omega} C_1 \ , \\
C &:= C_1 \cup \dot{B} \ , \\
T_0 &:= \pm\{t\!:^{\psi}\!\varphi \in \mathscr{L}_{\mathsf{JCDL}} \mid t\!:^{\psi}\!\varphi \in C_0\} \ .
\end{aligned}
$$

Key differences from the old proof:

- the new operator $\vec{B}S$ adds the "certified version" $\dot{B}^{\psi}\varphi = c_{\varphi}\!:^{\psi}\!\varphi$ of each formula $t\!:^{\psi}\!\varphi$ in S,
- C_0 has been changed by adding the operator $\vec{B}$,
- every "B" in the old proof has been replaced by "$\dot{B}$",
- we use the JCDL-conventions (e.g., we are working in the language of JCDL), and
- the set T_0 (used later) is new.

We then define:

$$
\begin{aligned}
W &:= \{x \subseteq C \mid x \text{ is maxcons in } C\} \ , \\
\bar{x} &:= \textstyle\bigwedge(x \cap C_0) \text{ for } x \in W \ , \\
x^{\psi} &:= \{\varphi \mid \dot{B}^{\psi}\varphi \in x\} \text{ for } x \in W \text{ and } \psi \in C_1 \ , \\
\leq &:= \{(x,y) \in W \times W \mid \exists\psi \in (x \cap y \cap C_1),\, y^{\psi} \subseteq x\} \ , \\
V(x) &:= \mathscr{P} \cap x \text{ for } x \in W \ , \\
\hat{x} &:= \textstyle\bigwedge(x \cap T_0) \text{ for } x \in W \ , \\
A(t,\varphi) &:= \{x \in W \mid \exists\psi \in \mathscr{L}_{\mathsf{JCDL}},\ \vdash \hat{x} \to t\!:^{\psi}\!\varphi\} \ , \\
M &:= (W, \leq, V, A) \ .
\end{aligned}
$$

Key differences from the old proof:

- every "B" in the old proof has been replaced by "$\dot{B}$",

- the mapping taking a world $x \in W$ to a formula $\hat{x}$ is new,
- the function A of type $(\mathscr{T}_{\mathsf{JCDL}} \times \mathscr{L}_{\mathsf{JCDL}}) \to \wp(W)$ is new, and
- M has been expanded to contain A.

The arguments in the old proof (modulo the JCDL-conventions) are used to prove W is finite and nonempty, $\leq$ is reflexive and transitive, $\leq$ is total on each connected component, and $\leq$ is well-founded. We prove the following result particular to the new proof:

$$y \in \mathsf{cc}(x) \quad \Rightarrow \quad (\forall t{:}^{\psi}\varphi \in C_1,\ \ t{:}^{\psi}\varphi \in x \text{ iff } t{:}^{\psi}\varphi \in y)\ . \tag{11}$$

Proceeding, suppose $y \in \mathsf{cc}(x)$. Since $\leq$ is total on each connected component, we may assume without loss of generality that $x \leq y$. (The case where $y \leq x$ is argued similarly.) Now $x \leq y$ implies there exists $\delta \in (x \cap y \cap C_1)$ such that $y^{\delta} \subseteq x$. For the left-to-right direction: suppose $t{:}^{\psi}\varphi \in x \cap C_1$. If we had $t{:}^{\psi}\varphi \notin y$, then it would follow by maximal consistency that $\neg t{:}^{\psi}\varphi \in y$, hence $\dot{B}^{\delta}(\neg t{:}^{\psi}\varphi) \in y$ by (eNI), hence $\neg t{:}^{\psi}\varphi \in x$ by $y^{\delta} \subseteq x$, thereby contradicting the consistency of x. So it must be the case that $t{:}^{\psi}\varphi \in y$ after all. Now for the right-to-left direction: suppose $t{:}^{\psi}\varphi \in y \cap C_1$. It follows by (ePI) that $\dot{B}^{\delta}(t{:}^{\psi}\varphi) \in y$. Since $y^{\delta} \subseteq x$, we obtain $t{:}^{\psi}\varphi \in x$. So (11) indeed obtains.

In order to conclude that M is a Fitting model, we must prove that A satisfies the properties required of an admissibility function (Definition 5.12). We state and prove these in turn.

- Certification: $A(c_{\varphi}, \varphi) = W$.

  Take an arbitrary $x \in W$. We have $\vdash \hat{x} \to \dot{B}^{\varphi}\varphi$ by (Succ). Since $\dot{B}^{\varphi}\varphi = c_{\varphi}{:}^{\varphi}\varphi$, it follows that $\vdash \hat{x} \to c_{\varphi}{:}^{\varphi}\varphi$. Applying the definition of A, we obtain $x \in A(c_{\varphi}, \varphi)$. Since $x \in W$ was chosen arbitrarily, we conclude that $A(c_{\varphi}, \varphi) = W$.

- Application: $A(t, \varphi_1 \to \varphi_2) \cap A(s, \varphi_1) \subseteq A(t \cdot s, \varphi_2)$.

  Assume $x \in A(t, \varphi_1 \to \varphi_2) \cap A(s, \varphi_1)$. Applying the definition of A, this means there exists $\psi_1 \in \mathscr{L}_{\mathsf{JCDL}}$ and $\psi_2 \in \mathscr{L}_{\mathsf{JCDL}}$ such that $\vdash \hat{x} \to t{:}^{\psi_1}(\varphi_1 \to \varphi_2)$ and $\vdash \hat{x} \to s{:}^{\psi_2}\varphi_1$. Now we have each of $\vdash B^{\varphi_1 \wedge \varphi_2}(\varphi_1 \to \varphi_2)$ and $\vdash B^{\varphi_1 \wedge \varphi_2}\varphi_1$ by (Succ) and modal reasoning. So it follows that

  $$\vdash \hat{x} \to t{:}^{\psi_1}(\varphi_1 \to \varphi_2) \wedge B^{\varphi_1 \wedge \varphi_2}(\varphi_1 \to \varphi_2) \wedge s{:}^{\psi_2}\varphi_1 \wedge B^{\varphi_1 \wedge \varphi_2}\varphi_1\ .$$

  Applying (eA), we obtain

  $$\vdash \hat{x} \to t{:}^{\varphi_1 \wedge \varphi_2}(\varphi_1 \to \varphi_2) \wedge s{:}^{\varphi_1 \wedge \varphi_2}\varphi_1\ ,$$

  from which it follows by (eK) that $\vdash \hat{x} \to (t \cdot s){:}^{\varphi_1 \wedge \varphi_2}\varphi_2$. Applying the definition of A, it follows that $x \in A(t \cdot s, \varphi_2)$.

- Sum: $A(t, \varphi) \cup A(s, \varphi) \subseteq A(t + s, \varphi)$.

  Suppose $x \in A(t, \varphi) \cup A(s, \varphi)$. Applying the definition of A, this means there exists $\psi_t \in \mathscr{L}_{\mathsf{JCDL}}$ such that $\vdash \hat{x} \to t{:}^{\psi_t}\varphi$ or there exists $\psi_s \in \mathscr{L}_{\mathsf{JCDL}}$ such that $\vdash \hat{x} \to s{:}^{\psi_s}\varphi$. Applying (eSum), it follows that $\vdash \hat{x} \to (t+s){:}^{\psi}\varphi$ for some $\psi \in \{\psi_t, \psi_s\}$. But then we obtain by the definition of A that $x \in A(t+s, \varphi)$.

- Admissibility Indefeasibility: if $x \in A(t, \varphi)$ and $y \in \mathsf{cc}(x)$, then $y \in A(t, \varphi)$.

  Suppose $x \in A(t, \varphi)$ and $y \in \mathsf{cc}(x)$. It follows from $x \in A(t, \varphi)$ by the definition of A that there exists $\psi \in \mathscr{L}_{\mathsf{JCDL}}$ such that $\vdash \hat{x} \to t{:}^{\psi}\varphi$. But $y \in \mathsf{cc}(x)$, and so it follows by (11) that $\hat{y} = \hat{x}$. Therefore $\vdash \hat{y} \to t{:}^{\psi}\varphi$, from which it follows by the definition of A that $y \in A(t, \varphi)$.

So A satisfies the properties required of an admissibility function.

Since W is nonempty, A satisfies the properties of an admissibility function, and $\leq$ is a locally well-ordered, it follows that M is a locally well-ordered Fitting model. The arguments in the old proof (modulo the JCDL-conventions) are then used to prove that the Consistency Lemma holds and that the Minimality Lemma holds.

By the old proof (modulo the JCDL-conventions), all of the induction base and step cases of the Truth Lemma hold, except for the induction step case for formulas $t:^{\psi}\varphi$. What remains is to check this remaining case. Before we do this, note: by the definition of C_1 as the Boolean closure of C_0, it follows from $t:^{\psi}\varphi \in C_1$ that $t:^{\psi}\varphi \in C_0$ and therefore $t:^{\psi}\varphi \in T_0$. We now proceed with the argument.

- Induction step $t:^{\psi}\varphi$ (left to right): if $x \in W$ and $t:^{\psi}\varphi \in x \cap C_1$, then $M, x \models t:^{\psi}\varphi$.

  Assume $t:^{\psi}\varphi \in x \cap C_1$. By (eCert) and the definition of C_1, it follows that $\dot{B}^{\psi}\varphi \in x \cap C_1$. Using the old proof (modulo the JCDL-conventions and using Theorem 5.16 in place of Theorem 3.4), we obtain $M, x \models \dot{B}^{\psi}\varphi$. Also, since $t:^{\psi}\varphi \in x \cap C_1$ implies $t:^{\psi}\varphi \in x \cap T_0$, it follows that $\vdash \hat{x} \to t:^{\psi}\varphi$, from which we obtain $x \in A(t, \varphi)$ by the definition of A. So since $x \in A(t, \varphi)$ and $M, x \models \dot{B}^{\psi}\varphi$, it follows by Theorem 5.14 that $M, x \models t:^{\psi}\varphi$.

- Induction step $t:^{\psi}\varphi$ (right to left): if $x \in W$, $t:^{\psi}\varphi \in C_1$, and $M, x \models t:^{\psi}\varphi$, then $t:^{\psi}\varphi \in x$.

  Assume $t:^{\psi}\varphi \in C_1$ and $M, x \models t:^{\psi}\varphi$. By Theorem 5.14, it follows that $M, x \models \dot{B}^{\psi}\varphi$ and $x \in A(t, \varphi)$. Using the old proof (modulo the JCDL-conventions and using Theorem 5.16 in place of Theorem 3.4), it follows from $M, x \models \dot{B}^{\psi}\varphi$ that $\dot{B}^{\psi}\varphi \in x$. Applying the definition of A, it follows from $x \in A(t, \varphi)$ that there exists $\chi \in \mathscr{L}_{\mathsf{JCDL}}$ such that $\vdash \hat{x} \to t:^{\chi}\varphi$. But then we have $\vdash \hat{x} \to t:^{\chi}\varphi \wedge \dot{B}^{\psi}\varphi$. Applying (eA), we obtain $\vdash \hat{x} \to t:^{\psi}\varphi$. Since $t:^{\psi}\varphi \in C_1$ implies $t:^{\psi}\varphi \in T_0$, it follows from $\vdash \hat{x} \to t:^{\psi}\varphi$ by the maximal consistency of x in $C \supseteq T_0$ that $t:^{\psi}\varphi \in x$.

This completes the proof of the Truth Lemma. We then apply the argument as in the old proof (modulo the JCDL-conventions and using Theorem 5.18 in place of Theorem 3.6) to conclude that JCDL is complete with respect to the class of well-ordered Fitting models. □

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