Supplemental Material

S1. A MOST GENERAL RATIONAL SOLUTION FOR THE YANG-BAXTER EQUATION: ALGEBRAIC BETHE ANSATZ AND RICHARDSON-GAUDIN LIMIT

Algebraic Bethe ansatz

In order to be self-contained, we first derive the general form of Bethe equations for the model under study and show how to derive Richardson-Gaudin integrability starting from a solution to the Yang-Baxter Eq. [S49]. Given such an R-matrix parametrized as

$$R_{12}(u,v) = \frac{I_{12} + \eta F(u,v) P_{12}}{1+\eta F(u,v)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u,v) & c(u,v) & 0 \\ 0 & c(u,v) & b(u,v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{S1}$$

with $b(u,v) = 1/(1+\eta F(u,v))$ and $c(u,v) = \eta F(u,v)/(1+\eta(u,v))$, the building block of any integrable model is a Lax operator satisfying

$$R_{12}(u,v) L_{j,1}(u) L_{j,2}(v) = L_{j,2}(v) L_{j,1}(u) R_{12}(u,v), \tag{S2}$$

from which a monodromy matrix $T_a(u)$ and a transfer matrix $\tau(u)$ can be constructed as

$$T_a(u) = L_{L,a}(u) \dots L_{2,a}(u) L_{1,a}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a, \qquad \tau(u) = \mathrm{Tr}_a \left[T_a(u) \right] = A(u) + D(u), \tag{S3}$$

where the indices $i = 1 \dots L$ denote the physical Hilbert spaces, and a is an auxiliary space which is traced over (this construction corresponds to periodic boundary conditions). This gives rise to a continuous set of commuting operators

$$[\tau(u), \tau(v)] = 0, \qquad \forall u, v \in \mathbb{C}. \tag{S4}$$

Following the usual techniques of the Algebraic Bethe ansatz, the parametrization of the R-matrix in terms of $b(u,v)$ and $c(u,v)$ allows results to be directly transferred to the present situation [S49]. The eigenstates of $\tau(u)$ are given by Bethe states

$$\left| \{\lambda\}_N \right\rangle = \prod_{j=1}^{N} B(\lambda_j) \left| 0 \right\rangle, \tag{S5}$$

acting on a vacuum state satisfying $C(u)\left|0\right\rangle = 0$, $A(u)\left|0\right\rangle = a(u)\left|0\right\rangle$ and $D(u)\left|0\right\rangle = d(u)\left|0\right\rangle$, leading to eigenvalues

$$\tau(u|\{\lambda\}_N) = a(u) \prod_{k=1}^{N} \frac{1}{b(\lambda_k, u)} + du) \prod_{k=1}^{N} \frac{1}{b(u, \lambda_k)}, \tag{S6}$$

provided the Bethe equations are satisfied

$$\frac{a(\lambda_l)}{d(\lambda_l)} \prod_{k \neq l}^{N} \frac{b(\lambda_l, \lambda_k)}{b(\lambda_k, \lambda_l)} = 1, \qquad \forall l = 1 \dots N. \tag{S7}$$

Richardson-Gaudin integrability

The limit $\eta \to 0$ is known as the quasi-classical limit, leading to the class of integrable Richardson-Gaudin models [S50, S51]. This can be seen as a linearization of the usual construction, where a series expansion for the R-matrix in η results in

$$R_{12}(u,v) = I_{12} - \eta \mathcal{R}_{12}(u,v) + \mathcal{O}(\eta^2), \tag{S8}$$

defining the quasi-classical $\mathcal{R}$-matrix as

$$\mathcal{R}_{12}(u,v) = F(u,v)(I_{12} - P_{12}) = -2F(u,v)\left(\vec{S}_1 \cdot \vec{S}_2 - \frac{1}{4}\right). \tag{S9}$$

A Lax operator can be obtained from the R-matrix as $L_{j,a}(u) = g_a R_{aj}(u,\epsilon_j)$ with $\epsilon_j \in \mathbb{R}$ and $g_a = e^{-2\eta S_a^z/L}$ acting solely on the auxiliary space, satisfying the RLL-relation (S2) by construction. This operator can similarly be expanded in η as

$$L_{j,a}(u) = I_{aj} - 2\eta\left[\frac{S_a^z}{L} - F(u,\epsilon_j)\left(\vec{S}_a \cdot \vec{S}_j - \frac{1}{4}\right)\right] + \mathcal{O}(\eta^2). \tag{S10}$$

Taking the quasi-classical limit, the only nontrivial contributions to the transfer matrix will be at values $u = \epsilon_j$, where

$$\begin{aligned}\tau(\epsilon_j) =& \mathrm{Tr}_a\left[L_{L,a}(\epsilon_j)\dots L_{j+1,a}(\epsilon_j) g_a \mathbb{P}_{aj} L_{j-1,a}(\epsilon_j)\dots L_{1,a}(\epsilon_j)\right] \\ =& L_{j-1,j}(\epsilon_j)\dots L_{1,j}(\epsilon_j) L_{L,j}(\epsilon_j)\dots L_{j+1,j}(\epsilon_j) g_j,\end{aligned} \tag{S11}$$

due to the cyclic invariance of the trace. Retaining all first-order terms in η in the quasi-classical limit then results in

$$\tau(\epsilon_j) = 1 - 2\eta\left[S_j^z - \sum_{k\neq j}^{L} F(\epsilon_j,\epsilon_k)\left(\vec{S}_j \cdot \vec{S}_k - \frac{1}{4}\right)\right] + \mathcal{O}(\eta^2). \tag{S12}$$

It follows that the operators

$$Q_j = S_j^z - \sum_{k\neq j}^{L} F(\epsilon_j,\epsilon_k)\left(\vec{S}_j \cdot \vec{S}_k - \frac{1}{4}\right), \qquad j = 1\dots L, \tag{S13}$$

construe a set of conserved operators $[Q_i, Q_j] = 0, \forall i,j = 1\dots L$, which are exactly the conserved charges defining a class of Richardson-Gaudin models.

Taking the $\eta \to 0$ limit of the Bethe states and Bethe equations, these conserved charges can again be immediately diagonalized with Bethe states

$$|\{\lambda\}_N\rangle = \prod_{j=1}^{N} \mathcal{B}(\lambda_j)|0\rangle, \qquad \mathcal{B}(\lambda) = \sum_{j=1}^{L} F(\epsilon_j,\lambda) S_j^-, \qquad |0\rangle = |\uparrow\dots\uparrow\rangle, \tag{S14}$$

where the eigenvalues follow as

$$Q_j|\{\lambda\}_N\rangle = \frac{1}{2}\left(1 - \sum_{k=1}^{N} F(\epsilon_j,\lambda_k)\right)|\{\lambda\}_M\rangle, \tag{S15}$$

provided the rapidities satisfy the Bethe equations

$$-1 + \frac{1}{2}\sum_{j=1}^{L} F(\epsilon_j,\lambda_k) - \sum_{l\neq k}^{N} F(\lambda_l,\lambda_k) = 0, \qquad k = 1\dots N. \tag{S16}$$

S2. DERIVING THE RICHARDSON-GAUDIN HAMILTONIAN

Starting from the conserved charges (where a constant has been subtracted)

$$Q_i = \left(S_i^z + \frac{1}{2}\right) - \sum_{j\neq i}^{L} F_{ij}\left[\frac{1}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right) + S_i^z S_j^z\right], \tag{S17}$$

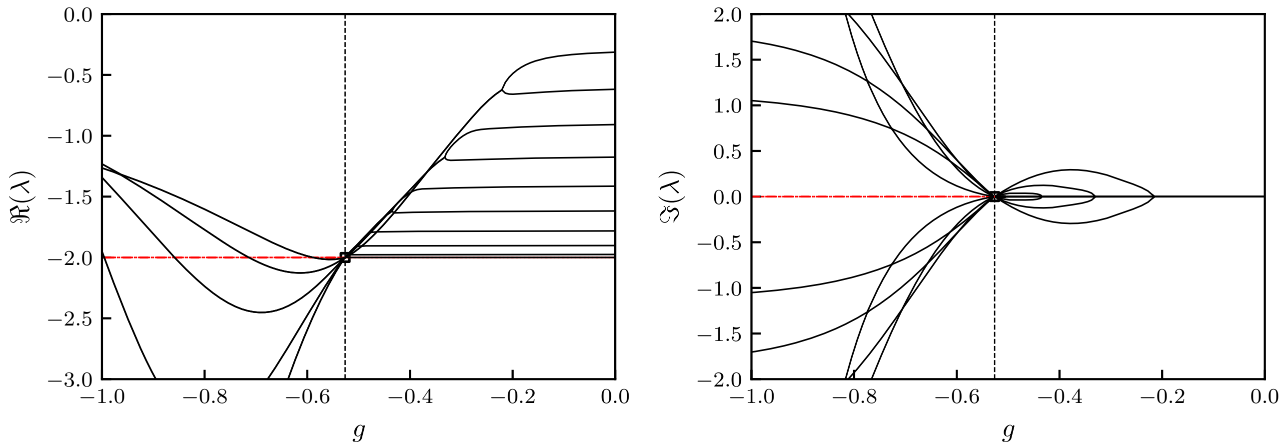

FIG. S1: Real and imaginary parts of the rapidities $\{\lambda_1 \ldots \lambda_N\}$ for a system with the parameters of Figure 1 at half-filling with $L = 20$, $N = 10$ and varying $g = GL$. The vertical line marks $g_c = G_c L$, while the horizontal red line marks $\lim_{k\to 0} \epsilon_k = -2(t_1 + t_2)$.

with $F_{ij} = F(\epsilon_i, \epsilon_j) = (c_0 + c_1\epsilon_i)(c_0 + c_1\epsilon_j)/(\epsilon_i - \epsilon_j) = G\eta_i\eta_j/(\epsilon_i - \epsilon_j)$ (with $\sqrt{G}\eta_i = (c_0 + c_1\epsilon_i)$), taking $H = \sum_i \epsilon_i Q_i$ results in

$$H = \sum_{i=1}^{L} \left(S_i^z + \frac{1}{2}\right) - \frac{G}{2}\sum_{i,j\neq i}^{L} \eta_i\eta_j \left[\frac{1}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right) + S_i^z S_j^z\right]. \tag{S18}$$

The Hamiltonian can now be rewritten by associating a one-dimensional (positive) momentum k with each level and mapping the $su(2)$-algebra to a quasispin algebra, introducing the fermion pair operators $S_k^z = \frac{1}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1\right) \equiv \frac{1}{2}\left(n_k + n_{-k} - 1\right)$, $S_k^+ = c_k^\dagger c_{-k}^\dagger \equiv b_k^\dagger$ and $S_k^- = c_{-k}c_k \equiv b_k$. This leads to a Hamiltonian

$$\sum_k \epsilon_k Q_k = \frac{1}{2}\sum_k \epsilon_k\left(n_k + n_{-k}\right) - \frac{G}{4}\sum_{k,k'}\eta_k\eta_{k'}\left(b_k^\dagger b_{k'} + b_k b_{k'}^\dagger\right) - \frac{G}{8}\sum_{k,k'}\eta_k\eta_{k'}\left(n_k + n_{-k}\right)\left(n_{k'} + n_{-k'}\right)$$
$$+ \frac{G}{8}\sum_{k,k'}\eta_k\eta_{k'}\left(n_k + n_{-k} + n_{k'} + n_{-k'}\right) + Cst. \tag{S19}$$

The sum over $k' \neq k$ has been extended to include $k = k'$ by adding and subtracting the Casimir operators of the spin algebras from the Hamiltonian, since these only lead to a global shift in the energy and can be absorbed in the constant term. By grouping terms together, we can get

$$\sum_k \epsilon_k Q_k = \frac{1}{2}\sum_k \epsilon_k\left(n_k + n_{-k}\right)\left[1 + \frac{c_1}{2}\sum_{k'}(c_0 + c_1\epsilon_{k'})\right] - \frac{G}{4}\sum_{k,k}\eta_k\eta_{k'}\left(b_k^\dagger b_{k'} + b_k b_{k'}^\dagger\right)$$
$$- \frac{G}{8}\sum_{k,k'}\eta_k\eta_{k'}\left(n_k + n_{-k}\right)\left(n_{k'} + n_{-k'}\right) + \frac{1}{4}\sum_{k'}\left(c_0 + c_1\epsilon_{k'}\right)\sum_k\left(n_k + n_{-k}\right) + Cst. \tag{S20}$$

Since the total number of fermions $\sum_k(n_k + n_{-k})$ is a symmetry of the system, this term also reduces to a constant, and this total operator can be divided by $\frac{1}{2}\left[1 + \frac{c_1}{2}\sum_k(c_0 + c_1\epsilon_k)\right]$, resulting in the proposed Hamiltonian with $\epsilon_k = -2t_1\cos k - 2t_2\cos 2k$, $2(t_1 + t_2) = c_0/c_1$ and $G^{-1} = 2c_1^{-2} - \sum_k \eta_k$. At the phase transition $c_1^2 \to \infty$ with c_0/c_1 fixed, which is reflected in $G_c^{-1} = -\sum_k \eta_k$.

S3. NUMERICAL SOLUTIONS TO THE RICHARDSON-GAUDIN EQUATIONS

The solutions to the Richardson-Gaudin equations for the ground state of the presented Hamiltonian are given in Fig. S1. These always arise as either real variables or in complex pairs, and it can be seen that for $G = G_c$, all solutions collapse to a single point $\lambda_\alpha = -c_0/c_1 = -2(t_1+t_2)$ corresponding to the noninteracting mode. Similar behaviour has been observed in p-wave [S52–S54] and extended d-wave superconductors [S45]. For $g > g_c$ a single variable $\lambda = -2(t_1+t_2) = \lim_{k\to 0}\epsilon_k$, unlike for $g < g_c$, indicating a change in the topological invariant $\lim_{k\to 0} n_k$ from 1 to 0.

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