Supplementary Materials for
“Liquid Helix: How Capillary Jets Adhere to Vertical Cylinders”

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(Dated: April 15, 2019)

In this supplementary document, we provide technical details on the experiment and the modelling presented in the main text. The latter involves the detailed derivation and analysis of the equations for the helix shapes, and tests of some of the assumptions.

EXPERIMENTAL DETAILS

We use Schlick nozzles (model 629) with bore diameters $D_j = 0.3, 0.4, 0.5, 0.8, 1.0, 1.5$ mm and tap water (whose interfacial tension was checked with pendant-drop tensiometry, Kruss EasyDrop) to produce our jets. The flow rates are measured with a Bio-Tech mini turbine tensiometry, Kruss EasyDrop) to produce our jets. The flow rates are measured with a Bio-Tech mini turbine flowmeter that we calibrated. We rinsed the cylinders with ethanol or acetone and then water before each experiment. All our glass cylinders are straight with a constant diameter ($\pm 1\%$). For teflon surfaces we used a full cylinder ($D_c = 4.20$ mm) and wrapped teflon tape around cylinders of glass or steel. We measured the advancing $\theta_a$ and receding $\theta_r$ contact angle of cylinder from a water bath. For the teflon tape, we measured the same angles on sessile drops after injection or withdrawal of liquid (Kruss EasyDrop). We show in Table S1 the average contact angle ($\theta_a + \theta_r$)/2 and the half hysteresis ($\theta_a - \theta_r$)/2 for our cylinders. The value and significant hysteresis observed are in agreement with previous studies using these substrates [1–4]. This uncertainty on the contact angle results in the horizontal error bars on Fig. 4(e).

MODELING

Mass and momentum conservation

As the rivulet flows down, it is subjected to three forces: basal viscous friction, gravity and the inertial-capillary depression in the stream. The steady momentum balance on an immobile control volume (C.V) bounded by a control surface (C.S) is:

$$\iint_{C.S} \rho \mathbf{u} \cdot \hat{n}_{C.S} \, dS =$$

$$\iiint_{C.V} \rho \mathbf{g} dV + \iint_{C.S} (\mathbf{r}^* - P \hat{n}_{C.S}) \, dS,$$

with $\rho$ the fluid density, $\mathbf{u}$ its velocity, $\mathbf{r}^*$ the shear stress, $P$ the pressure and $\hat{n}_{C.S}$ the C.S normal unit vector. If we apply it to an infinitesimal section of the rivulet between $s$ and $s+ds$ [as shown in Fig 2(a)], since the shear stress is non-zero only on the solid surface and $\mathbf{u}$ is aligned with the inlet and outlet normal (which is the rivulet centerline tangent unit vector $\hat{t}$), it simplifies to

$$\rho \hat{t} \left( s + ds \right) \int_{A(s+ds)} u^2 dS - \rho \hat{t} (s) \int_{A(s)} u^2 dS =$$

$$ds \left[ \rho g A(s) + \hat{t}(s) \int_{W(s)} \mathbf{r}^* d\ell - \int_{C} P \hat{n}_{C.S} d\ell \right].$$

Here $A$ is the area of the rivulet cross section, $C$ its perimeter and $W$ the length of the solid-liquid contact (which is the width of the rivulet for wetting solids). The perimeter of the pressure term $C$ can be decomposed in the solid-liquid portion $W$ and the free surface (F.S). Since the rivulet is symmetric the resultant will be along the cylinder normal $\hat{n}$ and we rewrite it

$$\int_{C} P \hat{n}_{C.S} d\ell = \hat{n} \left[ \int_{W(s)} P d\ell - \int_{F.S} P d\ell \right]$$

$$= W \Delta P \hat{n}.$$

Now if we assume a constant speed $u$ equals to its average over the cross section $U = \frac{1}{A} \int_{A} u dS$ and average the shear stress $\mathbf{r} = \frac{1}{\pi \sigma} \int_{W(s)} \mathbf{r}^* d\ell$ we get

$$\rho \frac{d ( A U^2 \hat{t} )}{ds} = W ( \mathbf{r} \hat{t} - \Delta P \hat{n} ) + \rho g.$$

Applying mass conservation $Q = A(s) U(s)$ finally reduces the momentum balance to its form in the main text [Eq. (1)].

Application to the helix

In the helical rivulet regime, as the jet completely sticks to the cylinder the position vector of the trajectory
can be written \( \mathbf{r}(s) = R_h \cos \phi(s) \hat{e}_x + R_h \sin \phi(s) \hat{e}_y + z(s) \hat{e}_z \) with \( R_h \) the (constant) helix radius, \( \phi \) the azimuthal angle and \( s \) the arc-length. The tangent vector is then

\[
\mathbf{t}(s) \equiv \frac{\mathbf{dr}(s)}{ds} = R_h \frac{d\phi}{ds} (-\sin \phi(s) \hat{e}_x + \cos \phi(s) \hat{e}_y) + \frac{dz}{ds} \hat{e}_z
\]

The unwrapped 2D coordinate system \((X,z)\) presented in Fig. 2 naturally appears by rewriting the tangent vector as \( \mathbf{t}(s) = \frac{dX}{ds} \hat{e}_X + \frac{dz}{ds} \hat{e}_z \) with \( X(s) = R_h \phi(s) \) and \( \hat{e}_X = \hat{e}_\phi = -\sin \phi(s) \hat{e}_x + \cos \phi(s) \hat{e}_y \). The later can then be fully expressed with only the local angle \( \psi(s) \) of the tangent vector with respect to the vertical as \( \frac{dX}{ds} = \cos \psi(s) \) and \( \frac{dz}{ds} = \sin \psi(s) \).

If we now look at the derivative of the speed in the full momentum balance Eq. (1), it gives in this geometry

\[
\frac{d}{ds}(U \hat{t}) = \frac{dU}{ds} \hat{t}(s) + U(s) \frac{d\hat{t}(s)}{ds} = \left( \frac{dU}{ds} \sin \psi(s) + U(s) \frac{d\phi}{ds} \cos \psi(s) \right) \hat{e}_X + \left( \frac{dU}{ds} \cos \psi(s) - U(s) \frac{d\phi}{ds} \sin \psi(s) \right) \hat{e}_z - U(s) \frac{d\phi}{ds} \sin \psi(s) \hat{e}_r.
\]

The last term is purely radial and thus in the normal direction as \( \hat{e}_r = \mathbf{n} \) in this geometry. Therefore, it can be interpreted as a centrifugal force balancing the pressure term in Eq. (1). Balancing the two other terms in the \( X \) and \( z \) direction with the friction force and gravity using the assumption made in the main text \( \tau_W = -3C \eta D_j^2 U^2/Q \) yields

\[
\frac{dU}{ds} \sin \psi + U \frac{d\psi}{ds} \cos \psi = -\frac{3\eta C D_j^2 U^2}{\rho Q^2} \sin \psi,
\]

\[
\frac{dU}{ds} \cos \psi - U \frac{d\psi}{ds} \sin \psi = -\frac{3\eta C D_j^2 U^2}{\rho Q^2} \cos \psi + \frac{g}{U}.
\]

The final equations [Eqs. (2)(3)] are then obtained by isolating the rivulet speed and angle and by replacing the flowrate with the initial jet speed \( Q = U_0 \pi D_j^2/4 \).

**Application to the jet bending**

**Two dimensional reduction.**— In the jet bending regime, as the jet velocity is higher, we neglect gravitational and frictional forces and the flow is assumed steady, laminar, irrotational and inviscid. A more detailed sketch of the projected problem is shown in Fig. 1. We call \( \alpha \) the angle (with respect to the horizontal \( x \)) at which the jet separate from the cylinder and balance the angle that the wetted portion of the cylinder makes. At the separation point a capillary meniscus, assumed circular with a radius \( r_m \), is formed to accommodate the solid-liquid-air contact angle \( \theta \) as shown by Duez et al. [5] (see Figs. 4(a) and [S1]). Only the pressure terms remains in the momentum balance:

\[
\rho Q \frac{d(U \hat{t})}{ds} = -\mathbf{n} \cdot \nabla P.
\]

Since only the pressure force (which is purely normal and thus have no \( z \) component) remains the jet angle is constant \( \psi(s) = \psi_0 \) and \( U(s) = U_0 \). It is then convenient to integrate the momentum balance in the control volume whose projection is shown in Fig. 1.

\[
\int_{s_{out}}^{s_{out}} \rho QU_0 \frac{d\mathbf{t}}{ds} ds' = \rho QU_0 \mathbf{t}(s_{out}) - \mathbf{t}(s_{in}) = \rho QU_0 \sin \psi_0 [(\cos \alpha - 1) \hat{e}_x + \sin \alpha \hat{e}_y] = -\int ds' \Delta P W \mathbf{n}
\]

If we now assume for simplicity a constant square cross section \( A(s) = W(s)^2 = D_j^2 \), the pressure is uniform along the width and we can reduce the integral over the whole control surface (C.S) to a line integral over the top \( \ell_{top} \) and bottom \( \ell_{bot} \) section of the C.S.

\[
\int \Delta P dW \mathbf{n} = \int_{\ell_{top}} \rho s' P_{top} \mathbf{n} + \int_{\ell_{bot}} ds' P_{bot} \mathbf{n}
\]

Since the inclination angle \( \psi_0 \) is constant, this line integral can then be projected in the \((x,y)\) plane as \( ds' = ds'_y / \sin \psi_0 \), with \( ds'_y = \sqrt{dx^2 + dy^2} \). We finally obtain

\[
\rho U_0^2 D_j [(\cos \alpha - 1) \hat{e}_x + \sin \alpha \hat{e}_y] = \int_{\ell_{top} + \ell_{bot}} P \mathbf{n} ds'_y
\]

with \( U_{0,x} = U_0 \sin \psi_0 \) the projected initial speed in the \( x \) direction. The full 3D momentum equation can therefore be projected in 2D and becomes equivalent to the problem presented in Fig. [S1] which was also considered in [6].

**TABLE S1.** Diameters and air-water contact angles of our cylinders.

<table>
<thead>
<tr>
<th>Material</th>
<th>Glass</th>
<th>Teflon</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_c ) (mm)</td>
<td>1.05</td>
<td>2.20</td>
</tr>
<tr>
<td>( \theta_0 ) (deg)</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>( \theta_0 ) (deg)</td>
<td>13</td>
<td>15</td>
</tr>
</tbody>
</table>
Pressure calculation. The pressure differs from the atmospheric pressure in the curved region of the line integrals only (all pressure terms are expressed relative to the atmospheric pressure). In the curved part of \( \ell_{\text{top}} \) and in the meniscus region of \( \ell_{\text{bot}} \) there is a capillary pressure jump which creates forces of capillary origin \( F_{\text{top}} \) and \( F_{\text{men}} \). In addition the bending of the streamlines in the fluid create a depression which creates the hydrodynamic force \( F_{\text{hyd}} \) on the portion of \( \ell_{\text{bot}} \) in contact with the cylinder. The momentum balance projected in both \( x \) and \( y \) directions can be written:

\[
\begin{align*}
\rho U_0^2 x D_j (\cos \alpha - 1) &= F_{\text{hyd},x} + F_{\text{top},x} + F_{\text{men},x}, \\
-\rho U_0^2 D_j \sin \alpha &= F_{\text{hyd},y} + F_{\text{top},y} + F_{\text{men},y}.
\end{align*}
\]

We now calculate these forces separately by integrating the fluid pressure over the relevant line integrals (boundaries of the control volume).

At the top interface in the curved circular region there is a first overpressure of capillary origin \( P_{\text{top}} = \gamma / (D_j/2 + D_j) \). The top capillary force is then:

\[
F_{\text{top},x} = -\int_0^\alpha P_{\text{top}} \left( \frac{D_j}{2} + D_j \right) \sin \beta^* d\beta^* = \gamma (\cos \alpha - 1),
\]

\[
F_{\text{top},y} = -\int_0^\alpha P_{\text{top}} \left( \frac{D_j}{2} + D_j \right) \cos \beta^* d\beta^* = -\gamma \sin \alpha.
\]

Similarly within the meniscus region, \( P_{\text{men}} = -\gamma / r_m \) and knowing the meniscus angle \( \pi - \beta - \theta + \alpha \) from geometry

\[
F_{\text{men},x} = \int_{-(\pi-\beta-\theta)}^\alpha P_{\text{men}} r_m \sin \beta^* d\beta^*
= \gamma (\cos (\beta + \theta) + \cos \alpha),
\]

\[
F_{\text{men},y} = \int_{-(\pi-\beta-\theta)}^\alpha P_{\text{men}} r_m \cos \beta^* d\beta^*
= -\gamma (\sin (\beta + \theta) + \sin \alpha).
\]

On the wetted solid surface there is an additional depression of hydrodynamic origin. To compute the pressure distribution in the liquid we must know the velocity profile \( u(r) \) inside the flowing liquid (with \( r \) the radial coordinate), for a given geometry characterized by \( \tilde{R} \). As explained in the main text, we focus on the case \( \tilde{R} \gg 1 \) for which we expect from potential flow theory \( u \sim 1/r \). The prefactor is then determined by mass conservation in the jet:

\[
U_0 x D_j = \int_{D_j/2}^{D_j/2+D_j} u(r) dr \Rightarrow u = \frac{U_0 x D_j}{r \ln \left(1 + 1/\tilde{R}\right)}.
\]

The depression on the cylinder surface \( P_c \) is thus:

\[
P_c = P_{\text{top}} - \rho \int_{D_j/2}^{D_j/2+D_j} u(r)^2 dr
= \frac{\gamma / D_j}{1 + 1/\tilde{R}} - \rho U_0^2 D_j \mathcal{G}(\tilde{R}) / \tilde{R}
\]

with \( \mathcal{G}(\tilde{R}) = \left( \frac{1}{\tilde{R}} + \tilde{R} \right) \left( \tilde{R} \left( 1 + \tilde{R} \right)^2 \left[ \ln(1 + \tilde{R}) \right]^2 \right)^{-1} \) encoding the information about the velocity profile: assuming a different velocity profile simply modifies \( \mathcal{G}(\tilde{R}) \). We finally obtain the hydrodynamic force by integration of the pressure over the wetted area:

\[
F_{\text{hyd},x} = \int_0^\beta P_c \frac{D_j}{2} \sin \beta^* d\beta^*
= \left[ \rho U_0^2 x D_j \mathcal{G}(\tilde{R}) - \gamma \left( \frac{\tilde{R}}{1 + \tilde{R}} \right) \right] (\cos \beta - 1)
\]

\[
F_{\text{hyd},y} = \int_0^\beta P_c \frac{D_j}{2} \cos \beta^* d\beta^*
= -\left[ \rho U_0^2 D_j \mathcal{G}(\tilde{R}) - \gamma \left( \frac{\tilde{R}}{1 + \tilde{R}} \right) \right] \sin \beta.
\]

It is instructive to note that if there is no flow within the meniscus, by continuity \( P_c = P_{\text{men}} \) and the flow sets the meniscus size

\[
\frac{r_m}{D_j} = \frac{1 + \tilde{R}}{\text{We}_\parallel \left(1 + \frac{1}{\tilde{R}}\right) \mathcal{G}(\tilde{R}) - 1}.
\]
Finally, replacing the forces given by equations (S.3) to (S.8) into the momentum balance (S.1) and solving the momentum balance (S.1)(S.2) and switching to dimensionless numbers finally leads to the final equations (see also (b)):

\[
\begin{align*}
\text{We}_{\parallel} - \text{We}_{\parallel} G(R) - \frac{1}{(1 + R)} \right) \cos \beta + \cos(\beta + \theta) &= 0, \\
\sin \beta + \sin(\beta + \theta) &= 0.
\end{align*}
\] (S.9) (S.10)

**Analysis of the model.** We solve the momentum balance (S.9)(S.10) numerically and plot in Fig. S2(a) the separation angle \( \alpha \) as a function of \( \text{We}_{\parallel} \) for a typical cylinder radius \( R = 4 \) at various wettabilities. Solutions only exist above a critical value of the Weber number, \( \text{We}_{\parallel,c} \), which we identify as the threshold for the sticking transition. The critical point is found to coincide with \( \alpha = 180^\circ \). Above \( \text{We}_{\parallel,c} \), the momentum balance admits two possible solutions. However, solutions for \( \alpha \) larger than \( 180^\circ \) are only observed around the critical point [see Fig. S2(b)] and are experimentally unstable, thus suggesting a saddle node bifurcation with only the lower branch as stable. Now focusing on the lower branch, as expected, the deflection angle \( \alpha \) increases when the jet speed is reduced, with much more rapid variations around the critical point following the saddle node scaling \( \alpha_c - \alpha \sim (\text{We}_{\parallel} - \text{We}_{\parallel,c})^{1/2} \). The inset of Fig. S2(a) shows a zoom around the critical point for both angles, \( \alpha \) and \( \beta \). The two angles always take similar values, with a maximum difference of about \( 20^\circ \). The global minimum of \( \beta \) is also reached at \( \text{We}_{\parallel,c} \), but has a value slightly below \( 180^\circ \).

The critical Weber number depends on both the wettability of the solid and the curvature of the cylinder. Fig. S3(a) illustrates the wettability dependence: \( \text{We}_{\parallel,c} \) varies linearly with \( 1 + \cos \theta \) as in Duez et al. [5] scaling analysis, except in the superhydrophobic limit (\( \theta \approx 180^\circ \)) where \( \text{We}_{\parallel,c} \rightarrow 2 \) for all cylinder radii. Fig. S3(b) illustrates the solid edge curvature dependence: for large \( R \), the trend is also linear as in our experiments, a result which differs notably from the previous scaling analysis which found a quadratic dependence [5]. Note that the model is only valid for \( R > 1 \), owing to the assumptions of the velocity field. The model predicts an unphysical divergence for \( R \ll 1 \) that appears when \( G \rightarrow 1 \). This divergence disappears when considering the more realistic corner solution \( u(r) \sim r^{-\Omega} \) with \( 0 < \Omega < 1/2 \) (see Fig. S4).

**Asymptotic expansion.** Since the sticking transition coincides with \( \alpha = 180^\circ \) and \( \beta = 180^\circ - \epsilon \) with \( \epsilon \) a small parameter (< 20°), we can recover the critical Weber number through an asymptotic expansion around the critical point. Hence, we expand Eqs. (S.9)(S.10) around the critical point and linearize it up to the first order in \( \epsilon \), i.e. replacing \( \sin \epsilon \sim \epsilon \) and \( \cos \epsilon \sim 1 \). We find a quadratic
FIG. S4. Influence of the shape of the velocity profile on the critical Weber number \( \text{We}_{\|c} \). Blue circles are the same data as Fig. 4(e), the solid curves are the full numerical solution of the momentum balance with \( \theta = 30^\circ \) for the different profiles shown in Table S2.

\[
\begin{align*}
    u(r) & \sim 1/r & \sim 1/\sqrt{r} & \sim 1 & \sim r \\
    G(\tilde{R}) & \sim \left( \frac{1}{2} + \tilde{R} \right) / \left( \tilde{R} (1 + \tilde{R})^2 (\ln(1 + \frac{1}{\tilde{R}}))^2 \right) & 1/ \left( 4(1 + \tilde{R}) (\sqrt{\tilde{R}} - \sqrt{1 + \tilde{R}})^2 \right) & \tilde{R} \ln \left( 1 + \frac{1}{\tilde{R}} \right) & \tilde{R} / \left( 1 + \frac{\tilde{R}}{2} \right)
\end{align*}
\]

TABLE S2. Velocity profile encoding function \( G(\tilde{R}) \) for different power law velocity profiles \( u(r) \sim r^\alpha \) (the prefactor is determined from mass conservation).

Equation for \( \text{We}_{\|c} \) such that:

\[
\text{We}_{\|c} = -A + \sqrt{A^2 - B},
\]

\[
\epsilon = \frac{\sin \theta}{G(\tilde{R}) \text{We}_{\|c} + \cos \theta - \frac{\tilde{R}}{1 + \tilde{R}}},
\]

with

\[
\begin{align*}
    A &= \frac{2 \tilde{R} + (3G(\tilde{R}) - 2) \cos \theta + \tilde{G}(\tilde{R})(3 - \frac{\tilde{R}}{1 + \tilde{R}})}{4G(\tilde{R})G(\tilde{R}) - 1} \\
    B &= \frac{1 - \tilde{R}}{1 + \tilde{R}} + \frac{3}{1 + \tilde{R}} \cos \theta \frac{1}{2G(\tilde{R})G(\tilde{R}) - 1}.
\end{align*}
\]

The analytical solution \( \text{S.11} \) is superimposed as dashed lines in Fig. S3 and gives a very good description of the full numerical solutions. The difference is largest for \( \theta \sim 90^\circ \), where the largest values for \( \epsilon \) are encountered. We can even further simplify \( \text{S.11} \) when \( \tilde{R} \gg 1 \) as \( G(\tilde{R}) = 1 - \frac{1}{2\tilde{R}} + o \left( \frac{1}{\tilde{R}^2} \right) \) which yields

\[
\text{We}_{\|c} \approx 4 + \tilde{R} (1 + \cos \theta),
\]

predicting a linear behavior of the critical Weber number with respect to both the aspect ratio \( \tilde{R} \) and to \( 1 + \cos \theta \).

**Discussion of the assumptions.**—The first assumption is about the flow profile, the \( u \sim 1/r \) profile is only valid for inviscid, irrotational flows around large cylinders. In reality, there must be some viscous effects with a boundary layer around the surface and we expect a slightly different flow structure [7]. Besides, for small cylinders even in the inviscid case we expect a different solution: the corner flow. We thus calculated the hydrodynamic force for different power law profiles using the same procedure, which yielded different \( G(\tilde{R}) \) (see Table S2). We replot in Fig. S4 the experimental \( \text{We}_{\|c} \) as a function of \( \tilde{R} \) for our glass cylinders and compare it to the result of the model with the different velocity profiles of Table S2 (still with \( \theta = 30^\circ \)). The choice of velocity profile has a minor influence on the critical Weber number except for \( \tilde{R} \ll 1 \) where the unphysical behavior \( \text{We}_{\|c} \to \infty \) as \( \tilde{R} \to 0 \) only appears for \( u \sim 1/r \) and cannot explain the discrepancy with experiments.

The remaining assumptions are mostly geometric. Although in the model the jet is assumed coherent and square with a constant cross-section this is not the case in experiments. The jet is initially circular, then it makes a small hydraulic jump upon impact [8] (it widens with two rims and a thinner part in the center) that recoils and turn circular again after the separation point. Moreover, at very high speed, the jet leaving the cylinder breaks up in a stream of droplets with a well defined angle \( \alpha \) [the one reported, see Fig. S5(a)] while at intermediates velocities although the jet does not break it forms a chain like structure similar to the one observed in pouring flows or colliding jets [9, 10] [see Fig. S5(b) and the leftmost picture of Fig. 1(b)]. Finally, the meniscus is also often more complex than the simple circular picture (see Fig.
We thus believe that the discrepancies between the model and the experiments come from our oversimplified geometry. Though, we expect the exact jet geometry to be only accessible through computational fluid dynamic simulations [8] and therefore these assumptions to be hard to improve.