Random Matrix Ensemble for the Level Statistics of Many-Body Localization

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The GOE, GUE, and GSE are covered by to the Gaussian $\beta$ ensemble. These statistics generalize Wigner-Dyson level statistics from the discrete set of Dyson indices $\beta = 1, 2, 4$ to the continuous range $0 < \beta < \infty$. The Gaussian $\beta$ ensemble covers Poissonian level statistics for $\beta \to 0$, and provides a smooth interpolation between Poissonian and Wigner-Dyson level statistics. We establish the physical relevance of the level statistics of the Gaussian $\beta$ ensemble by showing near-perfect agreement with the level statistics of a paradigmatic model in studies on many-body localization over the entire crossover range from the thermal to the many-body localized phase. In addition, we show similar agreement for a related Hamiltonian with broken time-reversal symmetry.

We numerically study the level statistics of the Gaussian $\beta$ ensemble. These statistics generalize Wigner-Dyson level statistics from the discrete set of Dyson indices $\beta = 1, 2, 4$ to the continuous range $0 < \beta < \infty$. The Gaussian $\beta$ ensemble covers Poissonian level statistics for $\beta \to 0$, and provides a smooth interpolation between Poissonian and Wigner-Dyson level statistics. We establish the physical relevance of the level statistics of the Gaussian $\beta$ ensemble by showing near-perfect agreement with the level statistics of a paradigmatic model in studies on many-body localization over the entire crossover range from the thermal to the many-body localized phase. In addition, we show similar agreement for a related Hamiltonian with broken time-reversal symmetry.

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The eigenvalues of the tridiagonal matrix ensemble, for which the probability density is given by

\[ P(a_i) = \frac{1}{\sqrt{2\pi}} e^{-a_i^2/2}, \]

and \( b_i \) distributed according to the \( \chi \) distribution with the shape parameter given by \( i\beta \), for which the probability density is given by

\[ P(b_i) = \begin{cases} \frac{1}{\Gamma(i\beta/2)} b_i^{i\beta-1} e^{-b_i^2/2} & \text{if } b_i \leq 0, \\ \frac{2}{\Gamma(i\beta/2)} b_i^{i\beta-1} e^{-b_i^2/2} & \text{if } b_i > 0, \end{cases} \]

are distributed according to Eq. (2). This matrix ensemble has the property that the eigendistribution factorizes into separate terms for the eigenvalues and the eigenvectors. Equation (3) allows one to sample from the Gaussian \( \beta \) ensemble at low computational costs, and thus to generalize Wigner-Dyson level statistics beyond \( \beta = 1, 2, 4 \). Various aspects of the Gaussian \( \beta \) ensemble have been studied in mathematical [18,28] and physical [29–31] contexts.

First, we study the eigenvalue statistics of the Gaussian \( \beta \) ensemble for \( \beta \in [0,1] \) by focusing on two common statistical measures: the distribution of the ratios of consecutive level spacings [7,32] and the level spacing distribution [1]. For a set of eigenvalues \( \{s_i\} \) sorted in ascending order, the level spacings \( \{s_i\} \) are given by

\[ s_i = e_{i+1} - e_i, \]

and the ratios \( \{r_i\} \) of consecutive level spacings are given by

\[ r_i = \min \left( \frac{s_{i+1}}{s_i}, \frac{s_i}{s_{i+1}} \right). \]

For Poissonian level statistics (\( \beta = 0 \)), the level spacing distribution is given by \( P(s) = \exp(-s) \), where the spacings have been rescaled such that \( \langle s \rangle = 1 \). Correspondingly, the distribution of \( r \in [0,1] \) is given by \( P(r) = 2/(1+r)^2 \), with \( \langle r \rangle = 2 \ln(2) - 1 \approx 0.386 \). For \( \beta > 0 \), we obtain data by numerically diagonalizing matrices \( T \) as given in Eq. (3) of dimension \( n = 10^5 \). We determine the 100 eigenvalues closest to zero for each realization, accumulating at least \( 10^6 \) eigenvalues. Aiming to maximize the accuracy of the results, we unfold [33] data before analysis. For \( n \to \infty \), the density of states is given by a semicircle with radius \( 2\sqrt{n} \). This asymptotic result, which we use here to unfold data sampled from the Gaussian \( \beta \) ensemble, serves as a good approximation at finite (\( n \gtrsim 100 \)) values of \( n \) [28].

Figure 1 shows the distributions of \( r \) and \( s \) for the Gaussian \( \beta \) ensemble at various values of \( \beta \in [0,1] \), indicating how the Gaussian \( \beta \) ensemble interpolates between Poissonian and Wigner-Dyson level statistics. Table I shows the average \( \langle r \rangle \) as a function of \( \beta \), which will be used as the fitting parameter when comparing the eigenvalue statistics of the Gaussian \( \beta \) ensemble with the level statistics of a physical system.
Comparison with spectral statistics.—Here, we compare the level statistics of a standard model in studies on MBL with the eigenvalue statistics of the Gaussian $\beta$ ensemble. We consider a disordered spin-$1/2$ XXZ chain, for which the Hamiltonian is given by

$$H = \sum_{i=1}^{L} \left( S_{ix}^i S_{ix}^{i+1} + S_{iy}^i S_{iy}^{i+1} + \Delta S_{iz}^i S_{iz}^{i+1} \right) + \sum_{i=1}^{L} h_{ix}^i, \quad (7)$$

with $S_{i}^{\alpha} = \frac{1}{2} \sigma_{i}^{\alpha}$, where $\sigma_{i}^{\alpha}$ are Pauli matrices ($\alpha = x, y, z$) acting on site $i$. During the last decade, the level statistics of this Hamiltonian have been studied extensively in, e.g., Refs. [7,8,34–38]. In particular, the intermediate level statistics between the thermal and the MBL phase have been studied by means of a two-stage flow picture in Ref. [8]. Following these references, we impose periodic boundary conditions $\sigma_{i+L}^\alpha = \sigma_{i}^\alpha$, sample $h_{ix}^i$ from the uniform distribution ranging over $[-W, W]$, set $\Delta = 1$ (unless stated otherwise), and restrict the focus to the symmetry sector $\sum_{i} S_{ix}^i = 0$. We set $L = 16$, for which $\dim(H) = 12870$. We consider at least 1000 disorder realizations for each value of $W$. For each value of $W$ separately, we restrict the focus to the energy window containing the middle 10% of the union of all sampled spectra. The system exhibits a smooth crossover from Poissonian to Wigner-Dyson level statistics in the region $1.7 \lesssim W \lesssim 4.0$.

Figure 2 shows the distributions of $r$ and $s$ for the spectra of the Hamiltonian compared with the corresponding distributions for the Gaussian $\beta$ ensemble, where $\beta$ is estimated from $\langle r \rangle$. Note that, since $r$ is independent of the average level spacing, no unfolding [33] is required for drawing the distribution of this quantity. Before drawing the histograms of $s$ for the Hamiltonian, the spectra are unfolded by numerically estimating the smooth part of the density of states [39]. Remarkably, we observe near-perfect agreement between the spectral statistics of the Hamiltonian and the corresponding eigenvalue statistics of the Gaussian $\beta$ ensemble at all disorder strengths. Similar agreement can be found for $\Delta = 2$, which is illustrated in the lower right panel.

In Fig. 3, we study the sensitivity to finite-size effects. The top panels show that the agreement between the level
statistics of the Hamiltonian and the Gaussian $\beta$ is near perfect also at $L = 12, 14$. The bottom panels show a flow towards Wigner-Dyson (Poissonian) level statistics for $W \lesssim 3\, (W \gtrsim 3)$ with increasing system size. For $L \to \infty$, the system is believed to display an MBL transition at $W \approx 3.6\ [34]$. As there is a one-to-one relation between $\langle r \rangle$ and $\beta$, these results can in principle be appended with previous results from, e.g., Ref. [35]. Studying $\langle r \rangle$ as a function of the matrix dimension $n$ for the Gaussian $\beta$ ensemble indicates a difference of less than 1% between the value for $n = 500$ and $n = 10^5$ at all values $\beta \in [0, 1]$. 

Breaking time-reversal symmetry.—Ergodic systems with broken time-reversal symmetry are characterized by Wigner-Dyson level statistics for $\beta = 2\ [1]$. For the Hamiltonian given in Eq. (7), time-reversal symmetry can be broken in an experimentally relevant way by adding the three-body term

$$H' = \sum_{i=1}^{L} \vec{S}_i \cdot (\vec{S}_{i+1} \times \vec{S}_{i+2}),$$

where $\vec{S}_i = [S_i^x, S_i^y, S_i^z]^T\ [40]$. For $\beta \gtrsim 1$, the distribution of the ratio of consecutive level spacings for the Gaussian $\beta$ ensemble can be approximated with high precision [32] from Eq. (2) with $n = 3$, giving

$$P(r) \sim \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+3\beta/2}}.$$  

In what follows, estimates of $\beta \gtrsim 1$ from $\langle r \rangle$ are obtained by using Eq. (9). Figure 4 shows the distribution of $r$ for the spectra of $H + H'$ at several values of $W$ compared with the corresponding distributions for the Gaussian $\beta$ ensemble, where $\beta$ is estimated from $\langle r \rangle$. Again, we observe near-perfect agreement between corresponding curves at all disorder strengths.

Higher order spacing ratios.—Going beyond the study of the distribution of the ratios of consecutive level spacings and the level spacing distribution, we here study the higher order ratios $r^{(n)} \in [0, 1]$ of level spacings, for a spectrum $\{E_i\}$ sorted in ascending order defined as

$$r^{(n)}_i = \min \left( \frac{E_{i+2n} - E_{i+n}}{E_{i+n} - E_i}, \frac{E_{i+2n} - E_i}{E_{i+n} - E_{i+n}} \right).$$

Note that $r^{(1)} = r$. For the Gaussian $\beta$ ensemble, it can be shown rigorously that the distribution of $r^{(n)}$ for $\beta = 2/(n+1)$ is equivalent to the distribution of $r^{(1)}$ for $\beta = 2(n+1)/\beta\ [41]$. Evidence for a broader class of inter-relations involving $\beta = 1, 2, 4$ has been provided recently in Ref. [42].

Figure 5 shows the distributions of $r^{(n)}$ for the Hamiltonian compared with the corresponding distributions for the Gaussian $\beta$ ensemble, where the value of $\beta$ is estimated from $\langle r \rangle$. No unfolding is applied to the spectra of the Hamiltonian. We observe qualitative agreement up to $n = 3$ (i.e., up to 6 level spacings) for all values of $W$. The algorithm used to unfold the spectra can be suboptimal for the system under consideration. Attempts to compare the spectra of the Hamiltonian and the Gaussian $\beta$ ensemble on longer ranges by other measures such as the spectral rigidity [43], density-density correlation function [44], and the spectral form factor [45] did not provide conclusive results, presumably due to this effect.

Discussion and conclusions.—We have proposed a generalization of Wigner-Dyson level statistics from the discrete taxonomy $\beta = 1, 2, 4$ to the continuous one $\beta \in (0, \infty)$. Using the matrix model for the Gaussian $\beta$ ensemble introduced in Ref. [21], we have shown how the Gaussian $\beta$ ensemble provides a smooth interpolation between Poissonian and Wigner-Dyson level statistics.
We have studied the level statistics of a paradigmatic model in studies on MBL, and found near-perfect agreement with the corresponding statistics of the Gaussian $\beta$ ensemble over the full crossover range between the thermal (corresponding to $\beta \approx 1$) and many-body localized (corresponding to $\beta \approx 0$) phase, where $\beta$ is a single fitting parameter. We have shown that similar agreement holds for a related Hamiltonian with broken time-reversal symmetry.

We expect that this work paves a way for further investigations in various fields. Primarily, we believe it would be of significant interest to explore how universal the Gaussian $\beta$ ensemble describes the spectral statistics of quantum systems that show intermediate level statistics between Poissonian and Wigner-Dyson. In view of this, we note that there are several known physical and mathematical models supporting intermediate level statistics, studied mostly in the context of either single-particle models of quantum chaos [46–49] or the Anderson localization transition for noninteracting systems [50–53]. A crossover between Poissonian and Wigner-Dyson level statistics for $\beta = 2$ has also been found recently in a generalized SYK model [54].

Next, we expect that our results are of relevance in the field of MBL. In this field, level statistics are a key ingredient in both numerical [7,35,37] and analytical [55] studies. The detailed quantitative characterization of the level statistics of the Hamiltonian provided in this work might be valuable in, e.g., the finite-size scaling analysis of the MBL transition [34,35] and studies on the intermediate phase separating the thermal from the MBL phase [26] at finite system sizes. Finally, we hope that this Letter can contribute to the ongoing studies [45,56,57] on the fundamental correspondence between classical and quantum chaos.

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PHYSICAL REVIEW LETTERS 122, 180601 (2019)


[39] We estimate the (scaled) smooth part of the density of states for each spectrum separately by using the Wolfram Research, Inc. Mathematica version 10.0 function SmoothKernelDistribution with bandwidth specifications {"Adaptive", Automatic, 1}, "Biweight" (\( W = 2, 3 \)) or {"Adaptive", Automatic, 1} (\( W = 4, 5 \)).


