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AN EXAMPLE CONCERNING SADULLAEV’S BOUNDARY RELATIVE
EXTREMAL FUNCTIONS

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In memory of Józef Siciak

ABSTRACT. We exhibit a smoothly bounded domain Ω with the property that for suitable
K ⊂ ∂Ω and z ∈ Ω the Sadullaev boundary relative extremal functions satisfy the inequality
ω₁(z, K, Ω) < ω₂(z, K, Ω) ≤ ω(z, K, Ω).

1. Introduction

In [5] Sadullaev introduced several so-called boundary relative extremal functions
for compact sets K in the boundary of domains D ⊂ C^n, and asked whether their regularizations are perhaps
always equal. Recently Djire and the author [1, 2] gave a positive answer in certain cases where
D and K are particularly nice.

In this note we show that in general equality does not hold. The example is formed by a suitable
compact set in the boundary of the domain Ω that was constructed by Fornæss and the author
[3] as an example of a domain D where bounded plurisubharmonic functions that are continuous
on D cannot be approximated by plurisubharmonic functions that are continuous on D. We start
by briefly recalling the definitions of boundary relative extremal functions and the construction of
the domain Ω.

1.1. Boundary relative extremal functions. We follow Sadullaev [5, Section 27]. Let D be
a domain with smooth boundary in C^n, ξ ∈ ∂D, and A_α(ξ) = {z ∈ D; |z − ξ| < αδ_ξ(z)}, where
α ≥ 1 and δ_ξ(z) is the distance from z to the tangent plane at ξ to ∂D. For a function u defined
on D, put

\[ \tilde{u}(ξ) = \sup_{α > 1} \limsup_{z \to ξ} u(z), \quad ξ \in ∂D. \]

Definition 1.1. Let PSH(D) denote the plurisubharmonic functions on D and let K ⊂ ∂D be compact. We define the following boundary relative extremal functions

1. \[ \omega(z, K, D) = \sup\{u(z) : u ∈ PSH(D), u ≤ 0, \tilde{u} |_K ≤ -1\}; \]
2. \[ \omega₁(z, K, D) = \sup\{u(z) : u ∈ PSH(D) \cap C(\overline{D}), u ≤ 0, u |_K ≤ -1\}; \]
3. \[ \omega₂(z, K, D) = \sup\{u(z) : u ∈ PSH(D), u ≤ 0, \limsup_{z \to ξ} u \leq -1, \text{ for all } ξ \in K\}. \]

The upper semi-continuous regularization u* of a function u on a domain D is defined as

\[ u^*(z) = \limsup_{z \to w} u(w). \]

The functions ω*, ω₁*, ω₂* are plurisubharmonic. Observing that \( \omega₁(z, K, D) ≤ \omega₂(z, K, D) ≤ \omega(z, K, D) \), Sadullaev’s question is for what j is \( \omega^*(z, K, D) \equiv \omega_j^*(z, K, D) \)?

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1.2. The domain $\Omega$. We briefly recall the construction and properties of the domain $\Omega$ from $[3]$. 
\begin{equation}
\Omega = \{(z,w) \in \mathbb{C}^2; |w - e^{i\varphi(|z|)}|^2 < r(|z|)\}.
\end{equation}
Here $r$ and $\varphi$ are in $\mathcal{C}^\infty(\mathbb{R})$ with the following properties: $-1 \leq r \leq 2$; $r(t) = 0$ for $t \leq 1$ and for $t \geq 17$; $r(t) \equiv 1$ for $3 \leq t \leq 8$ and for $10 \leq t \leq 15$; $r(t)$ takes its maximum value $= 2$ precisely at $t = 2, 9, 16$. Moreover, $r'(t) > 0$ on $1 \leq t \leq 8$, $8 < t < 15$ and $15 < t < 16$, while $f'(t) < 0$ on $2 < t < 3$, $9 < t < 10$, and $16 < t < 17$. Next $\varphi$ satisfies $\varphi(t) < -\pi/2$ for $t \leq 4$ and for $t \geq 14$; $\varphi(t) > \pi/2 + 100$ for $5 \leq t \leq 6$ and for $12 \leq t \leq 13$ and $\varphi(t) < -\pi/2 + 100$ for $7 < t < 10$, and we demand in addition that $\varphi < 108$.

From $[3]$ we recall that $\Omega$ is a Hartogs domain with smooth boundary, and that the annulus 
\begin{equation}
A = \{(z,w); w = 0, 2 \leq |z| \leq 15\}
\end{equation}
is contained in $\overline{\Omega}$.

2. Negative answer to Sadullaev’s question

**Theorem 2.1.** Let $K = \{(z,w) \in \partial \Omega; |z| = 2 \text{ or } |z| = 16\}$. Then

$\omega_1((z,w), K, \Omega) < \omega_2((z,w), K, \Omega)$

for $(z,w)$ in an open neighborhood of $\{w = 0, |z| = 9\}$.

**Proof.** Let $u \in \text{PSH}(\Omega) \cap C(\overline{\Omega})$, $u \leq 0$, $u|_K \leq -1$. Then by the maximum principle, $|u| \leq 1$ on the discs $|w - e^{i\varphi(|z|)}| \leq 2$, where $z$ is fixed and satisfies $|z| = 2$ or $|z| = 16$, and in particular on the circles $C_1(w) = \{(z,w); |z| = 2\}$ and $C_2(w) = \{(z,w); |z| = 16\}$, where $|w| < 1$. Because $\Omega$ is a smoothly bounded domain, it follows from $[3]$ Theorem 1 (see also $[4]$ for recent extensions of this theorem), that $u$ can be approximated uniformly on $\overline{\Omega}$ by smooth plurisubharmonic functions $v$ defined on shrinking neighborhoods of $\overline{\Omega}$.

Let $\Omega_\delta = \{z \in \mathbb{C}^2; d(\zeta, \overline{\Omega}) < \delta\}$. Then given $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \text{PSH}(\Omega_\delta)$, such that $|u - v| < \varepsilon$ on $\overline{\Omega}$. For $|w| \leq \delta$ the annulus $A_w = \{(z,w); 2 \leq |z| \leq 16\}$ is contained in $\Omega_\delta$. On its boundary, which equals $C_1(w) \cup C_2(w)$, we have that $v < -1 + \varepsilon$, hence this also holds on $A_w$. It follows that $u < -1 + 2\varepsilon$ on $A_w \cap \overline{\Omega}$, in particular $u < -1 + 2\varepsilon$ on the open set $V = \{(z,w); 8 < |z| < 10, |w| < \delta, |w| < r(|z|) - 1\} \subset \Omega$. It follows that $\omega_1((z,w), K, \Omega) < -1 + 2\varepsilon$ on $V$, and therefore also $\omega_1((z,w), K, \Omega) < -1 + 2\varepsilon$ on $V$.

Next we will construct a plurisubharmonic function in the family that determines $\omega_2$. The construction is as in $[3]$ Section 2]. On $\Omega \cap \{(3 < |z| < 8) \cup \{10 < |z| < 15\}$ there exists a continuous branch of $\arg w$, denoted by $h(z,w)$, such that $\varphi(z) - \pi/2 \leq h(z,w) \leq \varphi(z) + \pi/2$.

In $[3]$ we constructed the following plurisubharmonic function.

\begin{equation}
\begin{aligned}
f(z,w) &= \begin{cases} 
0 & \text{if } |z| < 4 \text{ or if } |z| > 14 \\
\max\{0,h(z,w)\} & \text{if } 3 < |z| < 6 \text{ or if } 12 < |z| < 14 \\
\max\{100,h(z,w)\} & \text{if } 5 < |z| < 8 \text{ or if } 10 < |z| < 13 \\
100 & \text{if } 7 < |z| < 11.
\end{cases}
\end{aligned}
\end{equation}

It satisfies $f \leq 110$ on $\Omega$, $f \equiv 0$ on $\{|z| \leq 3\}$ and on $\{|z| \geq 14\}$, hence $f$ extends continuously by 0 to $\overline{\Omega} \cap (\{|z| \leq 3\} \cup \{|z| \geq 14\})$, and $f = 100$ on $V$. The plurisubharmonic function $g$ on $\Omega$ defined by

$g(\zeta) = \frac{f(\zeta) - 110}{110}, \quad (\zeta = (z,w))$

is negative, identically equal to $-1$ on $\overline{\Omega} \cap (\{|z| \leq 3\} \cup \{|z| \geq 14\})$, and equal to $-10/11$ on $V$. Hence also $\omega_2((z,w), K, \Omega) \geq \omega_2((z,w), K, \Omega) \geq -10/11$ on $V$. Choosing $\varepsilon < 1/10$ completes the proof. \qed
References


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