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AN EXAMPLE CONCERNING SADULLAEV’S BOUNDARY RELATIVE EXTRMEAL FUNCTIONS

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In memory of Józef Siciak

Abstract. We exhibit a smoothly bounded domain \( \Omega \) with the property that for suitable \( K \subset \partial \Omega \) and \( z \in \Omega \) the Sadullaev boundary relative extremal functions satisfy the inequality
\[
\omega_1(z, K, \Omega) < \omega_2(z, K, \Omega) \leq \omega(z, K, \Omega).
\]

1. Introduction

In [5] Sadullaev introduced several so-called boundary relative extremal functions for compact sets \( K \) in the boundary of domains \( D \subset \mathbb{C}^n \), and asked whether their regularizations are perhaps always equal. Recently Djire and the author [1, 2] gave a positive answer in certain cases where \( D \) and \( K \) are particularly nice.

In this note we show that in general equality does not hold. The example is formed by a suitable compact set in the boundary of the domain \( \Omega \) that was constructed by Fornæss and the author [3] as an example of a domain \( D \) where bounded plurisubharmonic functions that are continuous on \( D \) cannot be approximated by plurisubharmonic functions that are continuous on \( \overline{D} \). We start by briefly recalling the definitions of boundary relative extremal functions and the construction of the domain \( \Omega \).

1.1. Boundary relative extremal functions. We follow Sadullaev [5, Section 27]. Let \( D \) be a domain with smooth boundary in \( \mathbb{C}^n \), \( \xi \in \partial D \), and \( A_\alpha(\xi) = \{ z \in D : |z - \xi| < \alpha \delta_\xi(z) \} \), where \( \alpha \geq 1 \) and \( \delta_\xi(z) \) is the distance from \( z \) to the tangent plane at \( \xi \) to \( \partial D \). For a function \( u \) defined on \( D \), put
\[
\hat{u}(\xi) = \sup_{\alpha > 1} \limsup_{z \to \xi} u(z), \quad \xi \in \partial D.
\]

Definition 1.1. Let \( \text{PSH}(D) \) denote the plurisubharmonic functions on \( D \) and let \( K \subset \partial D \) be compact. We define the following boundary relative extremal functions
\[
\begin{align*}
\omega(z, K, D) &= \sup\{ u(z) : u \in \text{PSH}(D), u \leq 0, \hat{u}|_K \leq -1 \}; \\
\omega_1(z, K, D) &= \sup\{ u(z) : u \in \text{PSH}(D) \cap C(\overline{D}), u \leq 0, u|_K \leq -1 \}; \\
\omega_2(z, K, D) &= \sup\{ u(z) : u \in \text{PSH}(D), u \leq 0, \limsup_{z \to \xi} u(z) \leq -1, \text{ for all } \xi \in K \}.
\end{align*}
\]

The upper semi-continuous regularization \( u^* \) of a function \( u \) on a domain \( D \) is defined as
\[
u^*(z) = \limsup_{w \to z} u(w).
\]
The functions \( \omega^*, \omega_1^*, \omega_2^* \) are plurisubharmonic. Observing that \( \omega_1(z, K, D) \leq \omega_2(z, K, D) \leq \omega(z, K, D) \), Sadullaev’s question is for what \( j \) is \( \omega^*(z, K, D) \equiv \omega_j^*(z, K, D)? \)

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1.2. The domain $\Omega$. We briefly recall the construction and properties of the domain $\Omega$ from [3].

\begin{equation}
\Omega = \{(z, w) \in \mathbb{C}^2; |w - e^{i\varphi(|z|)}|^2 < r(|z|)\}.
\end{equation}

Here $r$ and $\varphi$ are in $C^\infty(\mathbb{R})$ with the following properties: $-1 \leq r \leq 2$; $r(t) \leq 0$ for $t \leq 1$ and for $t \geq 17$; $r(t) \equiv 1$ for $3 \leq t \leq 8$ and for $10 \leq t \leq 15$; $r(t)$ takes its maximum value = 2 precisely at $t = 2, 9,$ and 16. Moreover, $r'(t) > 0$ on $1 \leq t < 2, 8 < t < 9$ and $15 < t < 16$, while $f'(t) < 0$ on $2 < t < 3, 9 < t < 10,$ and $16 < t < 17$. Next $\varphi$ satisfies $\varphi(t) < -\pi/2$ for $t \leq 4$ and for $t \geq 14$; $\varphi(t) > \pi/2 + 100$ for $5 \leq t \leq 6$ and for $12 \leq t \leq 13$ and $\varphi(t) < -\pi/2 + 100$ for $7 < t < 10,$ and we demand in addition that $\varphi \leq 108$.

From [3] we recall that $\Omega$ is a Hartogs domain with smooth boundary, and that the annulus
\begin{equation}
A = \{(z, w); w = 0, 2 \leq |z| \leq 15\}
\end{equation}
is contained in $\Omega$.

2. Negative answer to Sadullaev’s question

**Theorem 2.1.** Let $K = \{(z, w) \in \partial \Omega; |z| = 2$ or $|z| = 16\}$. Then

$\omega_1((z, w), K, \Omega) < \omega_2((z, w), K, \Omega)$

for $(z, w)$ in an open neighborhood of $\{w = 0, |z| = 9\}$.

**Proof.** Let $u \in \text{PSH}(\Omega) \cap C(\overline{\Omega})$, $u \leq 0$, $u|_{K} \leq -1$. Then by the maximum principle, $|u| \leq -1$ on the discs $|w - e^{i\varphi(|z|)}| \leq 2$, where $z$ is fixed and satisfies $|z| = 2$ or $|z| = 16$, and in particular on the circles $C_1(w) = \{(z, w): |z| = 2\}$ and $C_2(w) = \{(z, w): |z| = 16\}$, where $|w| < 1$. Because $\Omega$ is a smoothly bounded domain, it follows from [3] Theorem 1 (see also [4] for recent extensions of this theorem), that $u$ can be approximated uniformly on $\Omega$ by smooth plurisubharmonic functions $v$ defined on shrinking neighborhoods of $\Omega$.

Let $\Omega_\delta = \{\zeta \in \mathbb{C}^2; d(\zeta, \overline{\Omega}) < \delta\}$. Then given $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \text{PSH}(\Omega_\delta)$, such that $|u - v| < \varepsilon$ on $\Omega_\delta$. For $|w| < \delta$ the annulus $A_w = \{(z, w); 2 \leq |z| \leq 16\}$ is contained in $\Omega_\delta$. On its boundary, which equals $C_1(w) \cup C_2(w)$, we have that $v < -1 + \varepsilon$, hence this also holds on $A_w$. It follows that $u < -1 + 2\varepsilon$ on $A_w \cap \overline{\Omega}$, in particular $u < -1 + 2\varepsilon$ on the open set $V = \{(z, w); 8 < |z| < 10, |w| < \delta, |w| < r(|z|) - 1\} \subset \Omega$. It follows that $\omega_1((z, w), K, \Omega) \leq -1 + 2\varepsilon$ on $V$, and therefore also $\omega_1((z, w), K, \Omega) \leq -1 + 2\varepsilon$ on $V$.

Next we will construct a plurisubharmonic function in the family that determines $\omega_2$. The construction is as in [3] Section 2]. On $\Omega \cap \{(3 < |z| < 8) \cup \{10 < |z| < 15\}$ there exists a continuous branch of $\arg w$, denoted by $h(z, w)$, such that

$\varphi(z) - \pi/2 \leq h(z, w) \leq \varphi(z) + \pi/2$.

In [3] we constructed the following plurisubharmonic function.

\begin{equation}
f(z, w) = \begin{cases} 
0 & \text{if } |z| < 4 \text{ or if } |z| > 14 \\
\max \{0, h(z, w)\} & \text{if } 3 < |z| < 6 \text{ or if } 12 < |z| < 16 \\
\max \{100, h(z, w)\} & \text{if } 5 < |z| < 8 \text{ or if } 10 < |z| < 13 \\
100 & \text{if } 7 < |z| < 11.
\end{cases}
\end{equation}

It satisfies $f \leq 110$ on $\Omega$, $f \equiv 0$ on $\{|z| \leq 3\}$ and on $\{|z| \geq 14\}$, hence $f$ extends continuously by 0 to $\overline{\Omega} \cap (\{|z| \leq 3\} \cup \{|z| \geq 14\})$, and $f = 100$ on $V$. The plurisubharmonic function $g$ on $\Omega$ defined by

$g(\zeta) = \frac{f(\zeta) - 110}{110}, \quad (\zeta = (z, w))$

is negative, identically equal to $-1$ on $\overline{\Omega} \cap (\{|z| \leq 3\} \cup \{|z| \geq 14\})$, and equal to $-10/11$ on $V$. Hence also $\omega_2((z, w), K, \Omega) \geq -10/11$ on $V$. Choosing $\varepsilon < 1/10$ completes the proof. □
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