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AN EXAMPLE CONCERNING SADULLAEV’S BOUNDARY RELATIVE EXTREMAL FUNCTIONS

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In memory of Józef Siciak

Abstract. We exhibit a smoothly bounded domain $\Omega$ with the property that for suitable $K \subset \partial \Omega$ and $z \in \Omega$ the Sadullaev boundary relative extremal functions satisfy the inequality $\omega_1(z, K, \Omega) < \omega_2(z, K, \Omega) \leq \omega(z, K, \Omega)$.

1. Introduction

In [5] Sadullaev introduced several so-called boundary relative extremal functions for compact sets $K$ in the boundary of domains $D \subset \mathbb{C}^n$, and asked whether their regularizations are perhaps always equal. Recently Djire and the author [1, 2] gave a positive answer in certain cases where $D$ and $K$ are particularly nice.

In this note we show that in general equality does not hold. The example is formed by a suitable compact set in the boundary of the domain $\Omega$ that was constructed by Fornæss and the author [3] as an example of a domain $D$ where bounded plurisubharmonic functions that are continuous on $D$ cannot be approximated by plurisubharmonic functions that are continuous on $\overline{D}$. We start by briefly recalling the definitions of boundary relative extremal functions and the construction of the domain $\Omega$.

1.1. Boundary relative extremal functions. We follow Sadullaev [5, Section 27]. Let $D$ be a domain with smooth boundary in $\mathbb{C}^n$, $\xi \in \partial D$, and $A_{\alpha}(\xi) = \{z \in D; |z - \xi| < \alpha \delta_\xi(z)\}$, where $\alpha \geq 1$ and $\delta_\xi(z)$ is the distance from $z$ to the tangent plane at $\xi$ to $\partial D$. For a function $u$ defined on $D$, put

$$\tilde{u}(\xi) = \sup_{\alpha > 1} \limsup_{z \to \xi, z \in A_{\alpha}(\xi)} u(z), \quad \xi \in \partial D.$$ 

Definition 1.1. Let $\text{PSH}(D)$ denote the plurisubharmonic functions on $D$ and let $K \subset \partial D$ be compact. We define the following boundary relative extremal functions

\begin{enumerate}
  \item $\omega(z, K, D) = \sup\{u(z) : u \in \text{PSH}(D), u \leq 0, \tilde{u}|_K \leq -1\};$
  \item $\omega_1(z, K, D) = \sup\{u(z) : u \in \text{PSH}(D) \cap C(\overline{D}), u \leq 0, u|_K \leq -1\};$
  \item $\omega_2(z, K, D) = \sup\{u(z) : u \in \text{PSH}(D), u \leq 0, \limsup_{z \to \xi, z \in \overline{D}} u(z) \leq -1, \text{ for all } \xi \in K\}.$
\end{enumerate}

The upper semi-continuous regularization $u^*$ of a function $u$ on a domain $D$ is defined as

$$u^*(z) = \limsup_{w \to z} u(w).$$

The functions $\omega^*, \omega_1^*, \omega_2^*$ are plurisubharmonic. Observing that $\omega_1(z, K, D) \leq \omega_2(z, K, D) \leq \omega(z, K, D)$, Sadullaev’s question is for what $j$ is $\omega^*(z, K, D) = \omega_j^*(z, K, D)$?
1.2. The domain Ω. We briefly recall the construction and properties of the domain Ω from [3].

\[
\Omega = \{(z, w) \in \mathbb{C}^2; |w - e^{i\varphi(|z|)}|^2 < r(|z|)\}.
\]

Here \(r\) and \(\varphi\) are in \(C^\infty(\mathbb{R})\) with the following properties: \(-1 \leq r \leq 2; r(t) = 0\) for \(t \leq 1\) and for \(t \geq 17\); \(r(t) \equiv 1\) for \(3 \leq t \leq 8\) and for \(10 \leq t \leq 15\); \(r(t)\) takes its maximum value = 2 precisely at \(t = 2, 9,\) and \(16\). Moreover, \(r'(t) > 0\) on \(1 \leq t \leq 2, 8 < t < 9\) and \(15 < t < 16\), while \(f'(t) < 0\) on \(2 < t < 3, 9 < t < 10\), and \(16 < t < 17\). Next \(\varphi\) satisfies \(\varphi(t) < -\pi/2\) for \(t \leq 4\) and for \(t \geq 14\): \(\varphi(t) > \pi/2 + 100\) for \(5 \leq t \leq 6\) and for \(12 \leq t \leq 13\) and \(\varphi(t) < -\pi/2 + 100\) for \(7 < t < 10\), and we demand in addition that \(\varphi \leq 108\).

From [3] we recall that \(\Omega\) is a Hartogs domain with smooth boundary, and that the annulus

\[
A = \{(z, w); w = 0, 2 \leq |z| \leq 15\}
\]

is contained in \(\overline{\Omega}\).

2. Negative answer to Sadullaev’s question

**Theorem 2.1.** Let \(K = \{(z, w) \in \partial \Omega; |z| = 2\ or\ |z| = 16\}. Then

\[\omega_1((z, w), K, \Omega) < \omega_2((z, w), K, \Omega)\]

for \((z, w)\) in an open neighborhood of \(\{w = 0, |z| = 9\}\).

**Proof.** Let \(u \in \text{PSH}(\Omega) \cap C(\overline{\Omega}), u \leq 0, u|_K \leq -1\). Then by the maximum principle, \(|u| \leq -1\) on the discs \(|w - e^{i\varphi(|z|)}| \leq 2\), where \(z\) is fixed and satisfies \(|z| = 2\ or\ |z| = 16\), and in particular on the circles \(C_1(w) = \{(z, w): |z| = 2\}\) and \(C_2(w) = \{(z, w): |z| = 16\}\), where \(|w| < 1\). Because \(\Omega\) is a smoothly bounded domain, it follows from [3] Theorem 1 (see also [4] for recent extensions of this theorem), that \(u\) can be approximated uniformly on \(\overline{\Omega}\) by smooth plurisubharmonic functions \(v\) defined on shrinking neighborhoods of \(\overline{\Omega}\).

Let \(\Omega_\delta = \{\zeta \in \mathbb{C}^2; d(\zeta, \overline{\Omega}) < \delta\}. Then given \(\varepsilon > 0\), there exist \(\delta > 0\) and \(v \in \text{PSH}(\Omega_\delta)\), such that \(|u - v| < \varepsilon\) on \(\overline{\Omega}\). For \(|w| < \delta\) the annulus \(A_w = \{(z, w): 2 \leq |z| \leq 1\}\) is contained in \(\Omega_\delta\). On its boundary, which equals \(C_1(w) \cup C_2(w)\), we have that \(v < -1 + \varepsilon\), hence this also holds on \(A_w\). It follows that \(u < -1 + 2\varepsilon\) on \(A_w \cap \overline{\Omega}\), in particular \(u < -1 + 2\varepsilon\) on the open set \(V = \{(z, w): 8 < |z| < 10, |w| < \delta, |w| < r(|z|) - 1\} \subset \Omega\). It follows that \(\omega_1((z, w), K, \Omega) < -1 + 2\varepsilon\) on \(V\), and therefore also \(\omega_1((z, w), K, \Omega) < -1 + 2\varepsilon\) on \(V\).

Next we will construct a plurisubharmonic function in the family that determines \(\omega_2\). The construction is as in [3] Section 2). On \(\Omega \cap \{(3 < |z| < 8) \cup \{10 < |z| < 15\}\} there exists a continuous branch of \(\text{arg } w\), denoted by \(h(z, w)\), such that

\[\varphi(z) - \pi/2 \leq h(z, w) \leq \varphi(z) + \pi/2.\]

In [3] we constructed the following plurisubharmonic function.

\[
f(z, w) = \begin{cases} 
0 & \text{if } |z| < 4 \text{ or if } |z| > 14 \\
\max\{0, h(z, w)\} & \text{if } 3 < |z| < 6 \text{ or if } 12 < |z| < 14 \\
\max\{100, h(z, w)\} & \text{if } 5 < |z| < 8 \text{ or if } 10 < |z| < 13 \\
100 & \text{if } 7 < |z| < 11.
\end{cases}
\]

It satisfies \(f \leq 110\) on \(\Omega\), \(f \equiv 0\) on \(|z| \leq 3\) and on \(|z| \geq 14\), hence \(f\) extends continuously by \(0\) to \(\overline{\Omega} \cap \{(3 < |z| \leq 3) \cup \{|z| \geq 14\}\}\), and \(f = 100\) on \(V\). The plurisubharmonic function \(g\) on \(\Omega\) defined by

\[g(\zeta) = \frac{f(\zeta) - 110}{110}, \quad (\zeta = (z, w))\]

is negative, identically equal to \(-1\) on \(\overline{\Omega} \cap \{(3 < |z| \leq 3) \cup \{|z| \geq 14\}\}\), and equal to \(-10/11\) on \(V\). Hence also \(\omega_2((z, w), K, \Omega) \geq -10/11\) on \(V\). Choosing \(\varepsilon < 1/10\) completes the proof. \(\square\)
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