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AN EXAMPLE CONCERNING SADULLAEV’S BOUNDARY RELATIVE EXTREMAL FUNCTIONS

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In memory of Józef Siciak

Abstract. We exhibit a smoothly bounded domain Ω with the property that for suitable $K \subset \partial \Omega$ and $z \in \Omega$ the Sadullaev boundary relative extremal functions satisfy the inequality $\omega_1(z, K, \Omega) < \omega_2(z, K, \Omega) \leq \omega(z, K, \Omega)$.

1. Introduction

In [5] Sadullaev introduced several so-called boundary relative extremal functions for compact sets $K$ in the boundary of domains $D \subset \mathbb{C}^n$, and asked whether their regularizations are perhaps always equal. Recently Djire and the author \cite{1, 2} gave a positive answer in certain cases where $D$ and $K$ are particularly nice.

In this note we show that in general equality does not hold. The example is formed by a suitable compact set in the boundary of the domain $\Omega$ that was constructed by Fornæss and the author \cite{3} as an example of a domain $D$ where bounded plurisubharmonic functions that are continuous on $D$ cannot be approximated by plurisubharmonic functions that are continuous on $\overline{D}$. We start by briefly recalling the definitions of boundary relative extremal functions and the construction of the domain $\Omega$.

1.1. Boundary relative extremal functions. We follow Sadullaev \cite{5, Section 27}. Let $D$ be a domain with smooth boundary in $\mathbb{C}^n$, $\xi \in \partial D$, and $A_{\alpha}(\xi) = \{ z \in D : |z - \xi| < \alpha \delta_{\xi}(z) \}$, where $\alpha \geq 1$ and $\delta_{\xi}(z)$ is the distance from $z$ to the tangent plane at $\xi$ to $\partial D$. For a function $u$ defined on $D$, put

$$\tilde{u}(\xi) = \sup_{\alpha > 1} \limsup_{z \to \xi} \{ u(z) : z \in A_{\alpha}(\xi) \}, \quad \xi \in \partial D.$$ 

**Definition 1.1.** Let $\text{PSH}(D)$ denote the plurisubharmonic functions on $D$ and let $K \subset \partial D$ be compact. We define the following boundary relative extremal functions

1. $\omega(z, K, D) = \sup\{ u(z) : u \in \text{PSH}(D), u \leq 0, \tilde{u}|_{\partial D} \leq -1 \};$

2. $\omega_1(z, K, D) = \sup\{ u(z) : u \in \text{PSH}(D) \cap C(\overline{D}), u \leq 0, u|_{\partial D} \leq -1 \};$

3. $\omega_2(z, K, D) = \sup\{ u(z) : u \in \text{PSH}(D), u \leq 0, \limsup_{z \to \xi} u(z) \leq -1, \text{ for all } \xi \in K \}.$

The upper semi-continuous regularization $u^*$ of a function $u$ on a domain $D$ is defined as

$$u^*(z) = \limsup_{w \to z} u(w).$$

The functions $\omega^*$, $\omega^*_1$, $\omega^*_2$ are plurisubharmonic. Observing that $\omega_1(z, K, D) \leq \omega_2(z, K, D) \leq \omega(z, K, D)$, Sadullaev’s question is for what $j$ is $\omega^*(z, K, D) \equiv \omega^*_j(z, K, D)$?

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1.2. **The domain** \( \Omega \). We briefly recall the construction and properties of the domain \( \Omega \) from \cite{3}.

\[
\Omega = \{(z, w) \in \mathbb{C}^2; |w - e^{i\varphi(z)}|^2 < r(|z|)\}.
\]

Here \( r \) and \( \varphi \) are in \( \mathbb{C}^\infty(\mathbb{R}) \) with the following properties: \(-1 \leq r \leq 2; r(t) \equiv 0 \) for \( t \leq 1 \) and for \( t \geq 17; r(t) \equiv 1 \) for \( 3 \leq t \leq 8 \) and for \( 10 \leq t \leq 15 \); \( r(t) \) takes its maximum value = 2 precisely at \( t = 2, 9, \) and 16. Moreover, \( r'(t) > 0 \) on \( 1 \leq t < 2, 8 < t < 9 \) and \( 15 < t < 16 \), while \( f'(t) < 0 \) on \( 2 < t < 3, 9 < t < 10, \) and \( 16 < t < 17 \). Next \( \varphi \) satisfies \( \varphi(t) < -\pi/2 \) for \( t \leq 4 \) and for \( t \geq 14; \varphi(t) > \pi/2 + 100 \) for \( 5 \leq t \leq 6 \) and for \( 12 \leq t \leq 13 \) and \( \varphi(t) < -\pi/2 + 100 \) for \( 7 < t < 10 \), and we demand in addition that \( \varphi \leq 108 \).

From \cite{3} we recall that \( \Omega \) is a Hartogs domain with smooth boundary, and that the annulus

\[
A = \{(z, w); w = 0, 2 \leq |z| \leq 15\}
\]

is contained in \( \overline{\Omega} \).

2. **Negative answer to Sadullaev’s question**

**Theorem 2.1.** Let \( K = \{(z, w) \in \partial \Omega; |z| = 2 \text{ or } |z| = 16\} \). Then

\[
\omega_1((z, w), K, \Omega) < \omega_2((z, w), K, \Omega)
\]

for \((z, w)\) in an open neighborhood of \( \{w = 0, |z| = 9\} \).

**Proof.** Let \( u \in \text{PSH}(\Omega) \cap C(\overline{\Omega}), u \leq 0, u|_K \leq -1 \). Then by the maximum principle, \( |u| \leq -1 \) on the discs \( |w - e^{i\varphi(z)}| \leq 2, \) where \( z \) is fixed and satisfies \( |z| = 2 \) or \( |z| = 16 \), and in particular on the circles \( C_1(w) = \{(z, w); |z| = 2\} \) and \( C_2(w) = \{(z, w); |z| = 16\} \), where \( |w| < 1 \). Because \( \Omega \) is a smoothly bounded domain, it follows from \cite{3} Theorem 1 (see also \cite{4} for recent extensions of this theorem), that \( u \) can be approximated uniformly on \( \overline{\Omega} \) by smooth plurisubharmonic functions \( v \) defined on shrinking neighborhoods of \( \overline{\Omega} \).

Let \( \Omega_\delta = \{\zeta \in \mathbb{C}^2; d(\zeta, \overline{\Omega}) < \delta\} \). Then given \( \varepsilon > 0 \), there exist \( \delta > 0 \) and \( v \in \text{PSH}(\Omega_\delta) \), such that \( |u - v| < \varepsilon \) on \( \overline{\Omega} \). For \( |w| < \delta \) the annulus \( A_w = \{(z, w) : 2 \leq |z| \leq 16\} \) is contained in \( \Omega_\delta \). On its boundary, which equals \( C_1(w) \cup C_2(w) \), we have that \( v < -1 + \varepsilon \), hence this also holds on \( A_w \). It follows that \( u < -1 + 2\varepsilon \) on \( A_w \cap \overline{\Omega} \), in particular \( u < -1 + 2\varepsilon \) on the open set \( V = \{(z, w); 8 < |z| < 10, |w| < \delta, |w| < r(|z|) - 1\} \subset \Omega \). It follows that \( \omega_1((z, w), K, \Omega) \leq -1 + 2\varepsilon \) on \( V \), and therefore also \( \omega_1((z, w), K, \Omega) \leq -1 + 2\varepsilon \) on \( V \).

Next we will construct a plurisubharmonic function in the family that determines \( \omega_2 \). The construction is as in \cite{3} Section 2). On \( \Omega \cap \{(3 < |z| < 8) \cup \{10 < |z| < 15\} \) there exists a continuous branch of \( \arg w \), denoted by \( h(z, w) \), such that

\[
\varphi(z) - \pi/2 \leq h(z, w) \leq \varphi(z) + \pi/2.
\]

In \cite{3} we constructed the following plurisubharmonic function.

\[
(2.1) f(z, w) = \begin{cases} 
0 & \text{if } |z| < 4 \text{ or if } |z| > 14 \\
\max\{0, h(z, w)\} & \text{if } 3 < |z| < 6 \text{ or if } 12 < |z| < 14 \\
\max\{100, h(z, w)\} & \text{if } 5 < |z| < 8 \text{ or if } 10 < |z| < 13 \\
100 & \text{if } 7 < |z| < 11.
\end{cases}
\]

It satisfies \( f \leq 110 \) on \( \Omega \), \( f \equiv 0 \) on \( \{|z| \leq 3\} \) and on \( \{|z| \geq 14\} \), hence \( f \) extends continuously by 0 to \( \overline{\Omega} \cap \{(3 \leq |z| \leq 16) \cup \{|z| \geq 14\}\} \), and \( f = 100 \) on \( V \). The plurisubharmonic function \( g \) on \( \Omega \) defined by

\[
g(\zeta) = \frac{f(\zeta) - 110}{110}, \quad (\zeta = (z, w))
\]

is negative, identically equal to \( -1 \) on \( \overline{\Omega} \cap \{(3 \leq |z| \leq 16) \cup \{|z| \geq 14\}\} \), and equal to \( -10/11 \) on \( V \). Hence also \( \omega_2((z, w), K, \Omega) \geq \omega_2((z, w), K, \Omega) \geq -10/11 \) on \( V \). Choosing \( \varepsilon < 1/10 \) completes the proof. \( \Box \)
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