An example concerning Sadullaev's boundary relative extremal functions

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AN EXAMPLE CONCERNING SADULLAEV’S BOUNDARY RELATIVE EXTREMAL FUNCTIONS

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In memory of Józef Siciak

Abstract. We exhibit a smoothly bounded domain Ω with the property that for suitable $K \subset \partial \Omega$ and $z \in \Omega$ the Sadullaev boundary relative extremal functions satisfy the inequality $\omega_1(z, K, \Omega) < \omega_2(z, K, \Omega) \leq \omega(z, K, \Omega)$.

1. Introduction

In [5] Sadullaev introduced several so-called boundary relative extremal functions for compact sets $K$ in the boundary of domains $D \subset \mathbb{C}^n$, and asked whether their regularizations are perhaps always equal. Recently Djire and the author [1, 2] gave a positive answer in certain cases where $D$ and $K$ are particularly nice.

In this note we show that in general equality does not hold. The example is formed by a suitable compact set in the boundary of the domain $\Omega$ that was constructed by Fornæss and the author [3] as an example of a domain $D$ where bounded plurisubharmonic functions that are continuous on $D$ cannot be approximated by plurisubharmonic functions that are continuous on $\overline{D}$. We start by briefly recalling the definitions of boundary relative extremal functions and the construction of the domain $\Omega$.

1.1. Boundary relative extremal functions. We follow Sadullaev [5, Section 27]. Let $D$ be a domain with smooth boundary in $\mathbb{C}^n$, $\xi \in \partial D$, and $A_\alpha(\xi) = \{ z \in D ; |z - \xi| < \alpha \delta(\xi) \}$, where $\alpha \geq 1$ and $\delta(\xi)$ is the distance from $z$ to the tangent plane at $\xi$ to $\partial D$. For a function $u$ defined on $D$, put

$$
\tilde{u}(\xi) = \sup_{\alpha > 1} \limsup_{z \to \xi} \frac{u(z)}{\alpha}, \quad \xi \in \partial D.
$$

Definition 1.1. Let $\text{PSH}(D)$ denote the plurisubharmonic functions on $D$ and let $K \subset \partial D$ be compact. We define the following boundary relative extremal functions

(1) $\omega(z, K, D) = \sup \{ u(z) : u \in \text{PSH}(D), u \leq 0, \tilde{u}|_K \leq -1 \};$

(2) $\omega_1(z, K, D) = \sup \{ u(z) : u \in \text{PSH}(D) \cap C(\overline{D}), u \leq 0, u|_K \leq -1 \};$

(3) $\omega_2(z, K, D) = \sup \{ u(z) : u \in \text{PSH}(D), u \leq 0, \limsup_{z \to \xi} u(z) \leq -1, \text{ for all } \xi \in K \}.$

The upper semi-continuous regularization $u^*$ of a function $u$ on a domain $D$ is defined as

$$
u^*(z) = \limsup_{w \to z} \{ u(w) \}.
$$

The functions $\omega^*$, $\omega_1^*$, $\omega_2^*$ are plurisubharmonic. Observing that $\omega_1(z, K, D) \leq \omega_2(z, K, D) \leq \omega(z, K, D)$, Sadullaev’s question is for what $j$ is $\omega^*(z, K, D) \equiv \omega^*_j(z, K, D)$?
1.2. The domain $\Omega$. We briefly recall the construction and properties of the domain $\Omega$ from [3].

\[(1.1) \quad \Omega = \{(z, w) \in \mathbb{C}^2; |w - e^{i\varphi(|z|)}|^2 < r(|z|)\}.\]

Here $r$ and $\varphi$ are in $C^\infty(\mathbb{R})$ with the following properties: $-1 \leq r \leq 2$; $r(t) \leq 0$ for $t \leq 1$ and for $t \geq 17$; $r(t) \equiv 1$ for $3 \leq t \leq 8$ and for $10 \leq t \leq 15$; $r(t)$ takes its maximum value $= 2$ precisely at $t = 2, 9, \text{and } 16$. Moreover, $r'(t) > 0$ on $1 \leq t < 2, 8 < t < 9$ and $15 < t < 16$, while $f'(t) < 0$ on $2 < t < 3, 9 < t < 10,$ and $16 < t \leq 17$. Next $\varphi$ satisfies $\varphi(t) < -\pi/2$ for $t \leq 4$ and for $t \geq 14$; $\varphi(t) > \pi/2 + 100$ for $5 \leq t \leq 6$ and for $12 \leq t \leq 13$ and $\varphi(t) < -\pi/2 + 100$ for $7 < t < 10$, and we demand in addition that $\varphi \leq 108$.

From [3] we recall that $\Omega$ is a Hartogs domain with smooth boundary, and that the annulus
\[(1.2) \quad A = \{(z, w); w = 0, 2 \leq |z| \leq 15\}\]
is contained in $\overline{\Omega}$.

2. Negative answer to Sadullaev's question

**Theorem 2.1.** Let $K = \{(z, w) \in \partial\Omega; |z| = 2 \text{ or } |z| = 16\}$. Then
\[
\omega_1((z, w), K, \Omega) < \omega_2((z, w), K, \Omega)
\]
for $(z, w)$ in an open neighborhood of $\{w = 0, |z| = 9\}$.

**Proof.** Let $u \in \text{PSH}(\Omega) \cap C(\overline{\Omega})$, $u \leq 0$, $u|_K \leq -1$. Then by the maximum principle, $|u| \leq -1$ on the discs $|w - e^{i\varphi(|z|)}| \leq 2$, where $z$ is fixed and satisfies $|z| = 2$ or $|z| = 16$, and in particular on the circles $C_1(w) = \{(z, w); |z| = 2\}$ and $C_2(w) = \{(z, w); |z| = 16\}$, where $|w| < 1$. Because $\Omega$ is a smoothly bounded domain, it follows from [3] Theorem 1 (see also [4] for recent extensions of this theorem), that $u$ can be approximated uniformly on $\overline{\Omega}$ by smooth plurisubharmonic functions $v$ defined on shrinking neighborhoods of $\overline{\Omega}$.

Let $\Omega_\delta = \{\zeta \in \mathbb{C}^2; d(\zeta, \overline{\Omega}) < \delta\}$. Then given $\varepsilon > 0$, there exist $\delta > 0$ and $v \in \text{PSH}(\Omega_\delta)$, such that $|u - v| < \varepsilon$ on $\overline{\Omega}$. For $|w| < \delta$ the annulus $A_w = \{(z, w); 2 \leq |z| \leq 16\}$ is contained in $\Omega_\delta$. On its boundary, which equals $C_1(w) \cup C_2(w)$, we have that $v < -1 + \varepsilon$, hence this also holds on $A_w$. It follows that $u < -1 + 2\varepsilon$ on $A_w \cap \overline{\Omega}$, in particular $u < -1 + 2\varepsilon$ on the open set $V = \{(z, w); 8 < |z| < 10, |w| < \delta, |w| < r(|z|) - 1\} \subset \Omega$. It follows that $\omega_1((z, w), K, \Omega) < -1 + 2\varepsilon$ on $V$, and therefore also $\omega_1((z, w), K, \Omega) < -1 + 2\varepsilon$ on $V$.

Next we will construct a plurisubharmonic function in the family that determines $\omega_2$. The construction is as in [3] Section 2]. On $\Omega \cap \{(3 < |z| < 8) \cup \{10 < |z| < 15\}$ there exists a continuous branch of $\arg w$, denoted by $h(z, w)$, such that
\[
\varphi(z) - \pi/2 < h(z, w) < \varphi(z) + \pi/2.
\]
In [3] we constructed the following plurisubharmonic function.

\[
(2.1) \quad f(z, w) = \begin{cases} 0 & \text{if } |z| < 4 \text{ or if } |z| > 14 \\ \max\{0, h(z, w)\} & \text{if } 3 < |z| < 6 \text{ or if } 12 < |z| < 14 \\ \max\{100, h(z, w)\} & \text{if } 5 < |z| < 8 \text{ or if } 10 < |z| < 13 \\ 100 & \text{if } 7 < |z| < 11. \end{cases}
\]

It satisfies $f \leq 110$ on $\Omega$, $f \equiv 0$ on $\{|z| \leq 3\}$ and on $\{|z| \geq 14\}$, hence $f$ extends continuously by $0$ to $\overline{\Omega} \cap \{|z| \leq 3\} \cup \{|z| \geq 14\}$, and $f = 100$ on $V$. The plurisubharmonic function $g$ on $\Omega$ defined by
\[
g(\zeta) = \frac{f(\zeta) - 110}{110}, \quad (\zeta = (z, w))
\]
is negative, identically equal to $-1$ on $\overline{\Omega} \cap \{|z| \leq 3\} \cup \{|z| \geq 14\}$, and equal to $-10/11$ on $V$. Hence also $\omega_2((z, w), K, \Omega) \geq \omega_2((z, w), K, \Omega) \geq -10/11$ on $V$. Choosing $\varepsilon < 1/10$ completes the proof.
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