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Optimal Reinsurance with Multiple Reinsurers: Distortion Risk Measures, Distortion Premium Principles, and Heterogeneous Beliefs *

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Abstract

This paper unifies the work on multiple reinsurers, distortion risk measures, premium budgets, and heterogeneous beliefs. An insurer minimizes a distortion risk measure, while seeking reinsurance with finitely many reinsurers. The reinsurers use distortion premium principles, and they are allowed to have heterogeneous beliefs regarding the underlying probability distribution. We provide a characterization of optimal reinsurance indemnities, and we show that they are of a layer-insurance type. This is done both with and without a budget constraint, i.e., an upper bound constraint on the aggregate premium. Moreover, the optimal reinsurance indemnities enable us to identify a representative reinsurer in both situations. Finally, two examples with the Conditional Value-at-Risk illustrate our results.

JEL-Classification: D86, G22.

Key Words: Optimal reinsurance design, distortion risk measures, distortion premium principle, heterogeneous beliefs, multiple reinsurers.

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1 Introduction

This paper presents an optimal reinsurance problem, where an insurer seeks a profile of reinsurance contracts with finitely many reinsurers. The insurer and all reinsurers are endowed with distortion risk measures, and they are all allowed to disagree about the distribution of the underlying insurance risk. Heterogeneity in such beliefs may arise from asymmetric information. For instance, the insurer may have different data about the insurance risk. Also, different insurance agents may deal with model risk in a different manner, as the "true" distribution is unknown by all agents.

As insurance premium principles, general distortion premium principles (also called generalized Wang's premium principles) are introduced by Wang (1996), and further characterized by Wu and Wang (2003). As an objective in economic decision theory, the same functional form is known as the dual theory of Yaari (1987), since it is the dual of the expected utility functional form in some sense. In general, we refer to dual utilities and Wang's premium principles as distortion risk measures. When insurance agents agree on the probability distribution of the insurable risk, distortion risk measures have gained recent popularity in the context of bilateral optimal reinsurance design since the work of Cui et al. (2013), Asimit, Badescu, and Cheung (2013), Assa (2015), and Zhuang et al. (2016). Boonen et al. (2016) extend these approaches to the case of multiple reinsurers[1], while Boonen (2016) extends these approaches to the case of heterogeneous beliefs about the underlying probability distribution. This paper generalizes and unifies both Boonen et al. (2016) and Boonen (2016) to the case of multiple reinsurers and heterogeneous beliefs, jointly.

To the best of our knowledge, the present work is the first to combine distortion risk measures and heterogeneous beliefs in an optimal reinsurance problem with multiple reinsurers. The presence of multiple reinsurers allows the insurer to allocate part of its risk to the reinsurers, and to find the lowest aggregate premium for the ceded risk. Our goal is to study optimality of coinsurance (that is, neither full insurance nor zero insurance), and we find that the optimal insurance indemnities are of a layer-type form. Here, all indemnities are implicitly provided by a mechanism where the marginal indemnities are allocated to specific insurance agents. This solution allows us to characterize a representative reinsurer. This means that all reinsurers can be treated collectively by means of a hypothetical premium principle in order to determine the optimal total risk that is ceded to all reinsurers. Note that this extends the findings of Boonen and Ghossoub (2019), who focus on expected-value premium principles for the reinsurers, since distortion risk measures are more general than expected-value premium principles.

In bilateral optimal reinsurance design, it is popular to study the effect of a premium budget. This means that the insurer has a maximum amount that it can spend on buying reinsurance[2]. The effect of a premium budget constraint is new in the context of optimal reinsurance with multiple reinsurers. In this paper, we show that using a Lagrangian dual method, the optimal reinsurance contracts have a similar structure to the unconstrained case, but all premium functions are penalized by the same factor. If no unconstrained solution is feasible for the constrained problem, then the aforementioned penalty is strictly positive and chosen such that the premium constraint is binding.

The rest of this paper is organized as follows. The model is defined in Section 2, where we introduce the heterogeneous beliefs. Section 3 presents the unconstrained optimal reinsurance problem that we study in this paper, and it characterizes the optimal reinsurance contracts and the repre-

[1] Earlier pioneering work on optimal reinsurance with multiple reinsurers was already done by Asimit, Badescu, and Verdonck (2013), Chi and Meng (2014), and Cong and Tan (2016).

[2] See, e.g., Amarante et al. (2015); Zhuang et al. (2016, 2017); Wang and Peng (2017); or Ghossoub (2019a,b,c,d).

sentative reinsurer. Section 4 extends these results to the constrained optimal reinsurance problem, where there is an upper bound on the aggregate premium that the insurer can spend. Section 5 provides examples of these two models with the Conditional Value-at-Risk (CVaR). Finally, Section 6 concludes. The proofs are presented in the Appendices.

2 Heterogeneous Beliefs

Let (S, Σ) be a measurable space, and assume that the insurer is subject to an exogenously given risk $X : S \to \mathbb{R}^+$ that we interpret as a loss at a given future reference time. Denote by Σ the σ-algebra $\sigma\{X\}$ generated by X on S. The insurer is endowed with beliefs represented by a probability measure $\mathbb{P}$ on (S, Σ). We index the finite set of reinsurers as $\{1, 2, \ldots, n\}$. The reinsurers are endowed with beliefs that are given by probability measures $\mathbb{Q}_i$ on (S, Σ), for $i \in \{1, 2, \ldots, n\}$. We do not impose any restriction on how the beliefs $\mathbb{P}, \mathbb{Q}_1, \mathbb{Q}_2, \ldots, \mathbb{Q}_n$ can differ, hence allowing for much flexibility. In particular, any couple of beliefs is allowed to exhibit disagreement about zero-probability events, and, as an extreme case, to be mutually singular.

Define the auxiliary probability measure μ on (S, Σ) by

$$\mu := \frac{1}{n+1}\left[\mathbb{P} + \sum_{i=1}^{n} \mathbb{Q}_i\right]. \tag{1}$$

By construction of μ, it follows that $\mathbb{P} \ll \mu$ and $\mathbb{Q}_i \ll \mu$, for each $i \in \{1, 2, \ldots, n\}$. All throughout, we assume that X is a non-negative integrable random variable on (S, Σ, μ).

For $i \in \{1, 2, \ldots, n\}$, the general distortion premium principle used by insurer i is given by

$$\pi^{\theta_i, T_i, \mathbb{Q}_i}(Y) := (1 + \theta_i) \int_0^\infty T_i\left(\mathbb{Q}_i(Y > z)\right) dz, \tag{2}$$

for all random variables Y on (S, Σ, μ), where $\theta_i \geq 0$ is interpreted as a risk-loading charged by insurer i, and $T_i : [0, 1] \to [0, 1]$ is the distortion function used by reinsurer i. A non-decreasing function $T_i : [0, 1] \to [0, 1]$ is called a distortion function when $T_i(0) = 0$ and $T_i(1) = 1$. This premium principle is also known as the generalized Wang's premium principle, and when $\theta_i = 0$ it is referred to as Wang's premium principle (Wang, 1996). Popular examples of this premium principle include the Expected-Value, the Value-at-Risk (VaR), and the Conditional Value-at-Risk (CVaR) premium principles. Section 5 provides an illustration of some of our main results in the case of CVaR premium principles.

The following proposition (the proof of which is given in Appendix A) shows that we can interpret this heterogeneity in beliefs as the existence of different pricing kernels in the market. Therefore, we can also interpret this premium principle similarly to the one studied by Chi et al. (2017) in the setting with one reinsurer, and preferences given by the risk-adjusted value of an insurer's liability.

Proposition 2.1 *Assume that, for each $i \in \{1, 2, \ldots, n\}$, the probability space $(S, \Sigma, \mathbb{Q}_i)$ is non-atomic and T_i is absolutely continuous. Then for each $i \in \{1, 2, \ldots, n\}$,*

$$\pi^{\theta_i, T_i, \mathbb{Q}_i}(Y) = \int F_{i,Y}^{-1}\left(U_i\right) \zeta_i \; d\mu, \tag{3}$$

where $\zeta_i := (1+\theta_i)\, T_i'\,(1-U_i)\,\frac{d\mathbb{Q}_i}{d\mu} = (1+\theta_i)\,\tilde{T}_i'\,(U_i)\,\frac{d\mathbb{Q}_i}{d\mu}$, $\tilde{T}_i(t) := 1 - T_i(1-t)$, *for each* $t \in [0,1]$, U_i *is a random variable on* $(S, \Sigma, \mathbb{Q}_i)$ *with a uniform distribution on* $(0,1)$, $F_{i,Y}$ *is the cumulative distribution function of* Y *with respect to the probability measure* $\mathbb{Q}_i$, *and*

$$F_{i,Y}^{-1}(t) = \inf\left\{z \in \mathbb{R} : F_{i,Y}(z) \geq t\right\},\ \forall t \in [0,1].$$

3 Unconstrained Optimal Reinsurance with Multiple Reinsurers

3.1 The Structure of Optimal Indemnity Profiles

The problem of optimal reinsurance is concerned with the optimal partitioning of X into $f_i(X), i \in \{1,2,\ldots,n\}$, and $X - \sum_{i=1}^n f_i(X)$, where $\sum_{i=1}^n f_i(X)$ represents the aggregate loss that is ceded to all n participating reinsurers, and $X - \sum_{i=1}^n f_i(X)$ is the loss that is retained by the insurer. The insurer's total exposure after reinsurance is given by

$$X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)).$$

In this paper, the feasible indemnity functions are such that $\{f_i\}_{i=1}^n \subset \mathcal{F}$ and $\sum_{i=1}^n f_i \in \mathcal{F}$, where:

$$\mathcal{F} := \left\{ f : [0,M) \to [0,M) \,\middle|\, 0 \leq f(X) \leq X,\ \mu\text{-a.s.};\ 0 \leq f'(z) \leq 1,\ \text{for a.e. } z \in [0,M) \right\}. \tag{4}$$

That is, the functions $z \mapsto f_i(z), i \in \{1,2,\ldots,n\}$, and $z \mapsto z - \sum_{i=1}^n f_i(z)$ are all non-decreasing, and any incremental compensation is never larger than the incremental loss. These constraints are referred to as the *no-sabotage* condition (Carlier and Dana, 2003). The assumption that aggregate indemnities $\sum_{i=1}^n f_i$ are in $\mathcal{F}$ is meant to prevent *ex post* moral hazard that could otherwise arise from possible misreporting of the loss by the insurer (Huberman et al., 1983; Denuit and Vermandele, 1998; Young, 1999; Chi and Tan, 2011). It has also gained recent popularity in dynamic optimal reinsurance design (Chen and Assa, 2019).

Under the beliefs represented by $\mathbb{P}$, we assume that the insurer uses a distortion risk measure $\rho^{\mathbb{P}}$, given by:

$$\rho^{\mathbb{P}}(Y) := \int_0^\infty T_0\left(\mathbb{P}(Y > z)\right) dz, \tag{5}$$

for all random variables Y on (S, Σ, μ), where $T_0 : [0,1] \to [0,1]$ is the distortion function used by the insurer[3]. We assume that $\rho^{\mathbb{P}}(X) < \infty$. The optimal strategy for the insurer of ceding part of its

[3] $T_0 : [0,1] \to [0,1]$ is a nondecreasing function that satisfies $T_0(0) = 0$ and $T_0(1) = 1$.

risk to n reinsures is determined by solving the following optimization problem:

$$\begin{array}{ll} \inf\limits_{\{f_i\}_{i=1}^n} & \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\Big) \\ \text{s.t.} & \{f_i\}_{i=1}^n \subset \mathcal{F},\ \sum_{i=1}^n f_i \in \mathcal{F}. \end{array} \tag{6}$$

The optimal indemnity profile $\{f_i\}_{i=1}^n$ therefore minimizes the risk exposure of the insurer. If $f_i(z) = 0$ for all $i \in \{1, \ldots, n\}$ and $z \in \mathbb{R}_+$, then the objective function in (6) is equal to $\rho^{\mathbb{P}}(X) < \infty$. Thus, the infimum in (6) always exists.

Define $M := \operatorname{esssup} X = \inf\{a \in \mathbb{R} : \mu(X > a) = 0\}$, which is possibly infinity. Note that for any profile $\{f_i\}_{i=1}^n \subset \mathcal{F}$ it holds that

$$\begin{aligned} \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) &= \sum_{i=1}^n (1+\theta_i) \int_0^{f_i(M)} T_i\left(\mathbb{Q}_i(f_i(X) > z)\right) dz \\ &= \sum_{i=1}^n (1+\theta_i) \int_0^{M} T_i\left(\mathbb{Q}_i(X > z)\right) df_i(z), \end{aligned} \tag{7}$$

where the second equality follows from Lemma 2.1 of Cheung and Lo (2017)[4]. Now, define the capacity[5] $\upsilon : \Sigma \to \mathbb{R}^+$ and the collection $\mathcal{I}(z)$ by

$$\upsilon\,(B) := \min_{1 \le i \le n} \Big\{ (1+\theta_i) T_i\left(\mathbb{Q}_i(B)\right) \Big\}, \quad \forall\ B \in \Sigma, \tag{8}$$

$$\mathcal{I}(z) := \operatorname*{argmin}_{1 \le j \le n}\, (1+\theta_j) T_j\left(\mathbb{Q}_j(X > z)\right), \quad \forall\ z \in [0, M). \tag{9}$$

The main result of this section is stated below, and its proof is given in Appendix B.

Theorem 3.1 *Profile $\{f_i\}_{i=1}^n$ is a solution to Problem (6) if and only if for each $x \in [0, M)$, $f_i(x) = \int_0^x h_i(z)dz$, where for $i = 1, \ldots, n$, and for a.e. $z \in [0, M)$,*

$$h_i(z) = \begin{cases} \gamma_i(z) & \text{if } \ i \in \mathcal{I}(z), \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

and for a.e. $z \in [0, M)$, $\gamma_i(z) \in [0, 1]$ is such that

$$\sum_{i=1}^n h_i(z) = \begin{cases} 1 & \text{if } \ \upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right), \\ \phi(z) & \text{if } \ \upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right), \\ 0 & \text{if } \ \upsilon(X > z) > T_0\left(\mathbb{P}(X > z)\right), \end{cases}$$

[4] Lemma 2.1 of Zhuang et al. (2016) provides a similar result, assuming left-continuity of the distortion function. However, an examination of the proof of this result shows that the left-continuity assumption is not needed. Lo (2017a,b) also provides a similar result.

[5] A (finite nonnegative) *capacity* on a measurable space (S, Σ) is a set function $\upsilon : \Sigma \to [0, \beta]$, for some $\beta \in [0, \infty)$ such that (i) $\upsilon\,(\varnothing) = 0$ and $\upsilon\,(S) = \beta$; and (ii) $A, B \in \Sigma$ *and* $A \subset B \Longrightarrow \upsilon\,(A) \le \upsilon\,(B)$. For more about capacities and Choquet integration, we refer to Marinacci and Montrucchio (2004).

and $\phi(z) \in [0,1]$.

Theorem 3.1 provides a full characterization of all reinsurance indemnities that solve Problem (6). For all $z \in [0, M)$, marginal indemnities $h_i(z)$ are strictly positive only for the reinsurers that have the smallest value among $T_0\left(\mathbb{P}(X > z)\right), (1+\theta_1)T_1\left(\mathbb{Q}_1(X > z)\right), \ldots,$ and $(1+\theta_n)T_n\left(\mathbb{Q}_n(X > z)\right)$. This generalizes the results of Boonen et al. (2016), where it was assumed that $\mathbb{Q}_1 = \cdots = \mathbb{Q}_n = \mathbb{P}$.

3.2 Existence of a Representative Reinsurer

In this section, we characterize the notion of a *representative reinsurer*. Suppose that an indemnity profile $\{f_i\}_{i=1}^n \subset \mathcal{F}$ is a solution to the Problem (6). Suppose further that there exists a reinsurer that uses a premium principle π such that:

- $\sum_{i=1}^n f_i$ solves the insurer's objective:

$$\inf_{f \in \mathcal{F}} \quad \rho^{\mathbb{P}}\Big(X - f(X) + \pi(f(X))\Big), \tag{11}$$

  with given premium principle π; and,

- $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) = \pi(\sum_{i=1}^n f_i(X))$.

Then, the reinsurer with premium functional π is referred to as a *representative reinsurer*. We can solve Problem (6) by first focusing on the single-reinsurer problem (11). Once we have established the representative reinsurer from Problem (11), this in turn gives us the total losses that will be ceded to the n reinsurers, jointly with the aggregate reinsurance premium for solving Problem (6). We first derive the following representative premium functional π of the n reinsurers.

Proposition 3.2 *If profile* $\{f_i\}_{i=1}^n$ *is a solution to Problem* (6)*, then*

$$\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) = \int_0^{f(M)} \upsilon(f(X) > z)dz := \pi\left(f\left(X\right)\right), \tag{12}$$

where $f := \sum_{i=1}^n f_i$. *Moreover, for any feasible profile* $\{g_i\}_{i=1}^n$ *for Problem* (6) *that satisfies* $\sum_{i=1}^n g_i\left(X\right) = \sum_{i=1}^n f_i\left(X\right) = f\left(X\right)$, μ*-a.s., we have*

$$\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X)) \geq \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)).$$

The proof of Proposition 3.2 is given in Appendix C. Note that from Proposition 3.2 it follows that

$$\pi\left(f\left(X\right)\right) = \int_0^{f(M)} \upsilon(f(X) > z)dz \leq \int_0^{f(M)} (1+\theta_i)T_i\left(\mathbb{Q}_i(f(X) > z)\right)dz = \pi^{\theta_i, T_i, \mathbb{Q}_i}(f(X)),$$

for all $i \in \{1, 2, \ldots, n\}$, whenever $\{f_i\}_{i=1}^n$ is a solution to Problem (6) and $f := \sum_{i=1}^n f_i$. Thus, multiple reinsurers and heterogeneous beliefs reduce the premium compared to the case with one insurer and homogeneous beliefs, studied for instance by Cui et al. (2013) and Assa (2015). Now, consider the following problem.

$$\begin{array}{ll} \inf_{\{f_i\}_{i=1}^n} & \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) \\ \text{s.t.} & \{f_i\}_{i=1}^n \subset \mathcal{F}, \text{ and } \sum_{i=1}^n f_i(X) = f(X), \ \mu\text{-a.s.} \end{array} \tag{13}$$

For each $f \in \mathcal{F}$, define the collection

$$\mathcal{F}(f) := \Big\{ \{f_i\}_{i=1}^n \subset \mathcal{F} : \{f_i\}_{i=1}^n \text{ solves Problem (13) for the given } f \Big\}.$$

The following theorem describes the set $\mathcal{F}(f)$. Its proof is given in Appendix D.

Theorem 3.3 *Fix $f \in \mathcal{F}$. Then, $\{f_i\}_{i=1}^n \in \mathcal{F}(f)$ if and only if the following two conditions hold simultaneously:*

(i) $\{f_i\}_{i=1}^n$ is such that for each $i \in \{1, 2, \ldots, n\}$ and for each $x \in [0, M)$, $f_i(x) = \int_0^x h_i(z)dz$, where for a.e. $z \in [0, M)$,

$$h_i(z) = 0 \quad \textit{whenever} \quad i \notin \mathcal{I}(z); \tag{14}$$

(ii) $\{f_i\}_{i=1}^n \subset \mathcal{F}$ and $\sum_{i=1}^n h_i(z) = f'(z)$, for a.e. $z \in [0, M)$.

Theorem 3.3 shows that reinsurance contracts in $\mathcal{F}(f)$ are determined by means of a characterization of $\{f_i'(z)\}_{i=1}^n$ for a.e. $z \in [0, M)$. The next theorem shows the existence of an optimal reinsurer. Its proof is given in Appendix E.

Theorem 3.4 *The following are equivalent:*

1. *$\{f_i\}_{i=1}^n$ is optimal for Problem (6);*

2. *$\sum_{i=1}^n f_i$ is optimal for Problem (11) with π as in eq. (12), and $\{f_i\}_{i=1}^n \in \mathcal{F}\left(\sum_{j=1}^n f_j\right)$.*

4 Budget-Constrained Optimal Reinsurance

4.1 The Structure of Optimal Indemnity Profiles

We now examine a situation in which the insurer is subject to a reinsurance premium budget. The insurer's problem then becomes:

$$\begin{array}{ll} \inf_{\{f_i\}_{i=1}^n} & \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\Big) \\ \text{s.t.} & \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) \le p, \\ & \{f_i\}_{i=1}^n \subset \mathcal{F}, \ \sum_{i=1}^n f_i \in \mathcal{F}, \end{array} \tag{15}$$

where $p \geq 0$ is an exogenously given premium budget. If there exists a profile $\{f_i^*\}_{i=1}^n$ solving Problem (6) such that $\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i^*(X)) \leq p$, then the profile $\{f_i^*\}_{i=1}^n$ also solves Problem (15). We define

$$\bar{\pi} := \inf\left\{ \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i^*(X)) : \{f_i^*\}_{i=1}^n \textit{ solves Problem } (6) \right\}. \tag{16}$$

Then, by Proposition 3.2,

$$\bar{\pi} = \inf\left\{ \pi\left(f^*\left(X\right)\right) : f^* = \sum_{i=1}^n f_i^*,\ \{f_i^*\}_{i=1}^n \textit{ solves Problem } (6) \right\}, \tag{17}$$

where the premium functional π is given in (12).

Remark 1 *If $p \geq \bar{\pi}$, then there exists a solution to the unconstrained Problem (6) that is also feasible in the constrained Problem (15). Thus, this will also be an optimal solution for Problem (15). Therefore, we proceed in this section with the more challenging problem of finding solutions to Problem (15) when $p \in [0, \bar{\pi})$.*

Note that the objective and the premium budget constraint in Problem (15) are linear in $\{f_i\}_{i=1}^n$. Using standard techniques (e.g., Zhuang et al. (2016, 2017)), we introduce a Lagrange multiplier $\lambda \geq 0$ associated with the constraint in Problem (15), and we consider the following auxiliary problem:

$$\begin{aligned} \inf_{\{f_i\}_{i=1}^n} \quad & \rho^{\mathbb{P}}\Big(X - \textstyle\sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big) + \lambda\left(\textstyle\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)) - p\right) \\ & \{f_i\}_{i=1}^n \subset \mathcal{F},\ \textstyle\sum_{i=1}^n f_i \in \mathcal{F}. \end{aligned} \tag{18}$$

The following result provides a link between solutions of Problem (18) and solutions of Problem (15). Its proof is given in Appendix F.

Lemma 4.1 *If there exists $\lambda^* \geq 0$ such that:*

(i) $\{f_{i,\lambda^*}^*\}_{i=1}^n$ is optimal for Problem (18) with $\lambda = \lambda^*$, and,

(ii) $\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_{i,\lambda^*}(X)) = p$,

then $\{f_{i,\lambda}^*\}_{i=1}^n$ is optimal for Problem (15).

Hence, by Lemma 4.1 solving Problem (18) will provide solutions for Problem (15). Now, for a given $\lambda \geq 0$, the objective function in Problem (18) can be written as

$$\begin{aligned} &\rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big) + \lambda\left(\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)) - p\right) \\ &\quad = \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X) + (1+\lambda)\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big) - \lambda p \end{aligned}$$

$$= \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i=1}^n \Big[(1+\lambda)(1+\theta_i)T_i(\mathbb{Q}_i(X > z)) - T_0\left(\mathbb{P}(X > z)\right)\Big] h_i(z)dz - \lambda p,$$

where $h_i := f_i'$, for $i = 1, \ldots, n$. The following result characterizes all solutions to Problem (18), and generalizes the unconstrained optimal reinsurance result of Theorem 3.1. Its proof is given in Appendix G, and it uses a lemma of Zhuang et al. (2017).

Corollary 4.2 *For a given $\lambda \geq 0$, the profile $\{f_{i,\lambda}^*\}_{i=1}^n$ solves Problem (18) with $p \in [0, \bar{\pi})$ if and only if for each $x \in [0, M)$, $f_{i,\lambda}^*(x) = \int_0^x h_{i,\lambda}^*(z)dz$, where for $i = 1, \ldots, n$, and for a.e. $z \in [0, M)$,*

$$h_{i,\lambda}^*(z) = \begin{cases} \gamma_{i,\lambda}(z) & \text{if } \ i \in \mathcal{I}(z), \\ 0 & \text{otherwise}, \end{cases} \tag{19}$$

and for a.e. $z \in [0, M)$, $\gamma_{i,\lambda}(z) \in [0, 1]$ is such that

$$\sum_{i=1}^n h_{i,\lambda}^*(z) = \begin{cases} 1 & \text{if } \ (1+\lambda)\upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right), \\ \phi_\lambda(z) & \text{if } \ (1+\lambda)\upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right), \\ 0 & \text{if } \ (1+\lambda)\upsilon(X > z) > T_0\left(\mathbb{P}(X > z)\right), \end{cases}$$

and $\phi_\lambda(z) \in [0, 1]$.

Moreover, there exists $\lambda^* > 0$ such that

$$\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_{i,\lambda^*}^*(X)) = p.$$

In Corollary 4.2, we can interpret the Lagrange parameter λ as a shadow price. This means that if we increase p by a small amount ε, the profit of the insurer is increased by $\lambda\varepsilon$. Moreover, Lemma 4.1 implies that for $\lambda^* > 0$ chosen such that $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_{i,\lambda^*}^*(X)) = p$, the profile $\{f_{i,\lambda^*}^*\}_{i=1}^n$ solves Problem (15).

4.2 Existence of a Representative Reinsurer

In this section, we show the existence of a representative reinsurer for the budget-constrained optimal reinsurance problem. In particular, we show that this problem reduces to a single-reinsurer problem, where the single reinsurer uses a premium functional π that is representative of the n reinsurers in the market. Fix $p \in [0, \bar{\pi})$, which we interpret as an exogenously given premium budget. First, by a proof similar to that of Proposition 3.2, we obtain the following result.

Proposition 4.3 *If profile $\{f_{i,\lambda^*}^*\}_{i=1}^n$ is a solution to Problem (18) with parameter $\lambda \geq 0$, then*

$$\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_{i,\lambda^*}^*(X)) = \int_0^{f_{\lambda^*}^*(M)} \upsilon(f_{\lambda^*}^*(X) > z)dz := \pi\left(f_{\lambda^*}^*(X)\right), \tag{20}$$

where $f^*_{\lambda^*} := \sum_{i=1}^n f^*_{i,\lambda^*}$. *Moreover, for any feasible profile* $\{g_i\}_{i=1}^n$ *for Problem* (18) *that satisfies* $\sum_{i=1}^n g_i(X) = \sum_{i=1}^n f^*_{i,\lambda^*}(X) = f^*_{\lambda^*}(X)$, μ*-a.s., we have*

$$\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(g_i(X)) \geq \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f^*_{i,\lambda^*}(X)).$$

Remark 2 *Lemma 4.1 and Proposition 4.3 imply that if* $\{f^*_{i,\lambda^*}\}_{i=1}^n$ *is a solution to Problem* (18) *with* $\lambda = \lambda^*$, *and* $\lambda^* > 0$ *is such that* $\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f^*_{i,\lambda}(X)) = p$, *then* $\{f^*_{i,\lambda^*}\}_{i=1}^n$ *is optimal for Problem* (15) *and* $\pi\left(\sum_{i=1}^n f^*_{i,\lambda^*}(X)\right) = p$, *where* π *is as in eq.* (20).

Now, consider the following problem.

$$\begin{aligned} \inf_{f\in\mathcal{F}} \quad & \rho^{\mathbb{P}}\Big(X - f(X) + \pi(f(X))\Big) \\ \text{s.t.} \quad & \pi(f(X)) \leq p, \end{aligned} \tag{21}$$

where π is as in eq. (20). Using a Lagrange multiplier $\eta \geq 0$ associated with the constraint in Problem (21), we consider the following auxiliary problem:

$$\inf_{f\in\mathcal{F}} \quad \rho^{\mathbb{P}}\Big(X - f(X) + \pi(f(X))\Big) + \eta\left(\pi(f(X)) - p\right). \tag{22}$$

By a proof similar to that of Lemma 4.1, we obtain the following result.

Lemma 4.4 *If there exists* $\eta^* \geq 0$ *such that:*

(i) f_{η^*} *is optimal for Problem* (22) *with* $\eta = \eta^*$, *and,*

(ii) $\pi\left(f_{\eta^*}(X)\right) = p$,

then f_{η^*} *is optimal for Problem* (21).

Now, define the collection $\mathcal{H}$ as follows:

$$\mathcal{H} := \left\{ g \in \mathcal{F} : g = \sum_{i=1}^n g_i \ \text{ for some } \{g_i\}_{i=1}^n \subset \mathcal{F}, \text{ and } \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(g_i(X)) = \pi\left(g(X)\right) \right\}. \tag{23}$$

For a given $\lambda \geq 0$, consider the following auxiliary problem:

$$\inf_{g\in\mathcal{H}} \quad \rho^{\mathbb{P}}\Big(X - g(X) + \pi(g(X))\Big) + \lambda\left(\pi(g(X)) - p\right). \tag{24}$$

The following result shows the existence of an optimal reinsurer. Its proof is given in Appendix H.

Theorem 4.5 *Fix $\lambda \geq 0$ and $p \in [0, \bar{\pi})$. Then the following hold.*

1. *If $\{f_i^*\}_{i=1}^n$ is optimal for Problem (18), then $\sum_{i=1}^n f_i^*$ is optimal for Problem (24);*

2. *If $\sum_{i=1}^n f_i^*$ is optimal for Problem (22) with $\eta = \lambda$ and $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i^*(X)) = \pi\left(\sum_{i=1}^n f_i^*(X)\right)$, then $\{f_i^*\}_{i=1}^n$ is optimal for Problem (18). If, moreover, $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i^*(X)) = p$, then $\{f_i^*\}_{i=1}^n$ is optimal for Problem (15).*

5 Examples with the Conditional Value-at-Risk

In this section, we illustrate our results with two examples: one unconstrained problem and one problem with an aggregate premium constraint. We assume that all reinsurers price risk with a Conditional Value-at-Risk (CVaR) premium principle, i.e., for all $i \in \{1, 2, \ldots, n\}$, we have $\theta_i = 0$, and $T_i(t) = \min\left\{\frac{t}{1-\alpha_i}, 1\right\}$ for all $t \in [0, 1]$, where $\alpha_i \in (0, 1)$. Then,

$$\pi^{\theta_i, T_i, \mathbb{Q}_i}(Y) = \mathrm{CVaR}_{\alpha_i}^{\mathbb{Q}_i}(Y) := F_{i,Y}^{-1}(\alpha_i) + \frac{1}{1-\alpha_i}\int_{F_{i,Y}^{-1}(\alpha_i)}^{M} \mathbb{Q}_i(Y > z)dz.$$

The CVaR risk measure has gained considerable interest among practitioners since the introduction of the Swiss Solvency Test and the Basel III regulations, for Swiss insurers and the global banking sector, respectively.

Moreover, we assume that the reinsurers believe that X has an exponential distribution with parameter $\beta_i > 0$, for all $i \in \{1, 2, \ldots, n\}$, i.e.,

$$\mathbb{Q}_i(X > z) = \exp(-\beta_i z), \text{ for } z \geq 0.$$

By virtue of Theorem 3.1, we are interested in $(1+\theta_i)T_i(\mathbb{Q}_i(X > z))$. It follows in this case that

$$(1+\theta_i)T_i(\mathbb{Q}_i(X > z)) = \min\left\{\frac{\exp(-\beta_i z)}{1-\alpha_i}, 1\right\}, \text{ for } z \geq 0.$$

Therefore,

$$\upsilon(X > z) = \min_{1 \leq i \leq n}\left\{\min\left\{\frac{\exp(-\beta_i z)}{1-\alpha_i}, 1\right\}\right\}, \text{ for } z \geq 0.$$

For instance, it holds that $\upsilon(X > z) = 1$ and $\mathcal{I}(z) = \{1, 2, \ldots, n\}$ for $z \in [0, a]$, where

$$a := \min_{1 \leq i \leq n}\left\{\frac{-\ln(1-\alpha)}{\beta_i}\right\} > 0.$$

Additionally, we assume that the insurer uses a CVaR risk measure, so that $\rho^{\mathbb{P}}(Y) = \mathrm{CVaR}_{\alpha_0}(Y)$, i.e., $T_0(t) = \min\left\{\frac{t}{1-\alpha_0}, 1\right\}$, for all $t \in [0, 1]$, for some $\alpha_0 \in (0, 1)$. The insurer believes that X has an exponential distribution with parameter $\beta_0 > 0$, i.e.,

$$\mathbb{P}(X > z) = \exp(-\beta_0 z), \text{ for } z \geq 0.$$

Consequently,

$$T_0(\mathbb{P}(X > z)) = \min\left\{\frac{\exp(-\beta_0 z)}{1-\alpha_0}, 1\right\}, \text{ for } z \geq 0.$$

By Theorem 3.1, optimal reinsurance contracts can be determined by analyzing

$$\underset{i\in\{0,1,\ldots,n\}}{\operatorname{argmin}} \left\{\min\left\{\frac{\exp(-\beta_i z)}{1-\alpha_i}, 1\right\}\right\}, \text{ for } z \geq 0. \tag{25}$$

Let $n = 2$, $\beta_1 = 2$, $\beta_2 = 1.7$, and $\beta_0 = 2.5$. Moreover, let $\alpha_1 = 95\%$, $\alpha_2 = 90\%$, and $\alpha_0 = 99\%$. Thus, the distribution of the risk X is the least risky (in the sense of first-order stochastic dominance) for the insurer, but the insurer is faced with the largest parameter in the CVaR risk measure. We display the relevant minimization problem (25) in Figure 1.

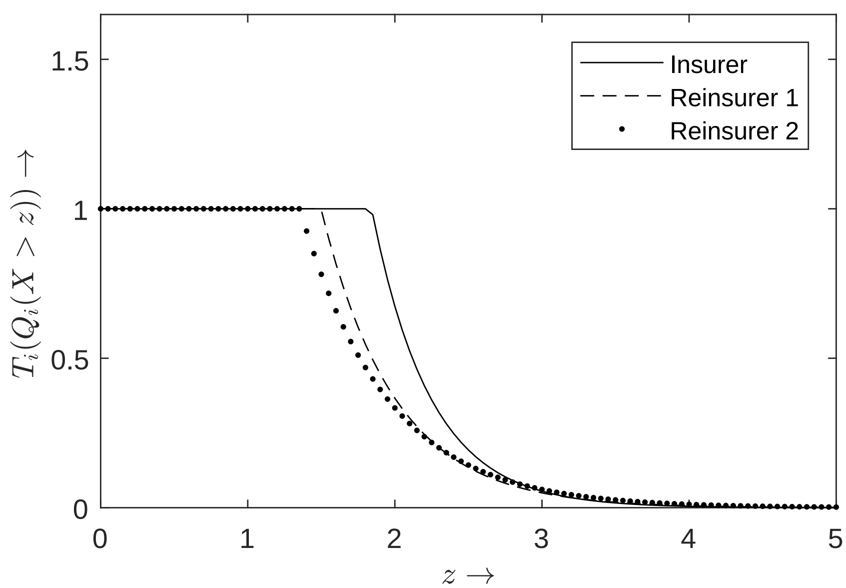

Figure 1: This figure displays the functions $\min\left\{\frac{\exp(-\beta_i z)}{1-\alpha_i}, 1\right\}$ for $i \in \{0, 1, 2\}$ and $z \in [0, 5]$, corresponding to Section 5.

For the collection $\mathcal{I}(z)$ defined in eq. (9), we find that, for $a = \ln(10)/1.7 \approx 1.35$ and $b = \ln(2)/0.3 \approx 2.31$,

$$\mathcal{I}(z) = \begin{cases} \{1,2\} & \text{if } z \in [0, a] \cup \{b\}, \\ \{2\} & \text{if } z \in (a, b), \\ \{1\} & \text{if } z \in (b, \infty). \end{cases}$$

Moreover, we define $c = 2\ln(5) \approx 3.22$. By Theorem 3.1, we find that for the optimal reinsurance contracts, the risk $\min\{X, 1.35\}$ can be allocated to any agent (say, it is kept by the insurer), the risk $f_2^*(X) := \min\{\max\{X - 1.35, 0\}, 0.96\}$ is allocated to Reinsurer 2, the risk $f_1^*(X) := \min\{\max\{X - 2.31, 0\}, 0.91\}$ is allocated to Reinsurer 1, and the remaining risk $\max\{X - 3.22, 0\}$

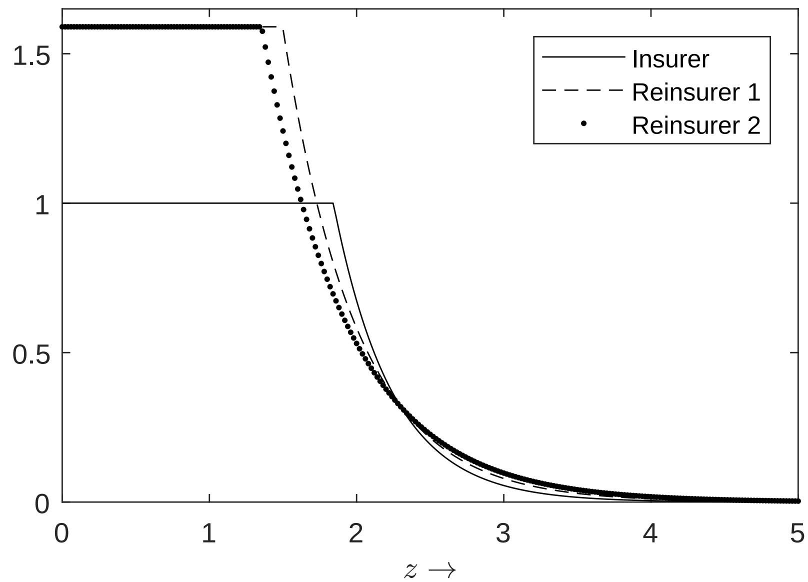

Figure 2: This figure displays the functions $\min\left\{\frac{\exp(-\beta_0 z)}{1-\alpha_0}, 1\right\}$ and $(1+\lambda^*)\min\left\{\frac{\exp(-\beta_i z)}{1-\alpha_i}, 1\right\}$ for $i \in \{1,2\}$ and $z \in [0,5]$, with $\lambda^* \approx 0.59$, corresponding to Section 5.

is kept by the insurer. The premium that Reinsurer 2 received for reinsuring this risk is given by $\pi^{0,T_2,\mathbb{Q}_2}(f_2^*(X)) = \int_a^b \frac{\exp(-1.7z)}{1-90\%}dz \approx 0.472$, and the premium that Reinsurer 1 received for reinsuring this risk is given by $\pi^{0,T_1,\mathbb{Q}_1}(f_1^*(X)) = \int_b^c \frac{\exp(-2z)}{1-95\%}dz \approx 0.082$.

Suppose now that the insurer is faced with an upper bound budget constraint of $p = 0.25$ on the aggregate reinsurance premium. Numerical computation yields $\bar{\pi} \approx 0.554$. Since $p \in [0, \bar{\pi})$, we penalize the functions $T_i(\mathbb{Q}_i(X > z))$ for $i \in \{1,2\}$ by $\lambda^* \geq 0$ (see Corollary 4.2). Numerically, we find that $\lambda^* \approx 0.59$. In Figure 2, we display the relevant minimization problem with the penalized functions.

By Corollary 4.2, we find that for the optimal reinsurance contracts, the risk $f_{2,\lambda^*}^*(X) := \min\{\max\{X - 1.63, 0\}, 0.67\}$ is allocated to Reinsurer 2, the risk $f_{1,\lambda^*}^*(X) := 0$ is allocated to Reinsurer 1, and the remaining risk $\min\{X, 1.63\} + \max\{X - 2.30, 0\}$ is kept by the insurer. The premium that Reinsurer 2 receives for this risk is given by $\pi^{0,T_2,\mathbb{Q}_2}(f_{2,\lambda^*}^*(X)) = 0.25$, and the premium that Reinsurer 1 received for this risk is given by $\pi^{0,T_1,\mathbb{Q}_1}(f_{1,\lambda^*}^*(X)) = 0$. Note that, because of the aggregate premium constraint, the first reinsurer is priced out of the market.

6 Conclusion

This paper presents a unifying theory of optimal reinsurance design, in the presence of finitely many reinsurers, distortion risk measures, and heterogeneous beliefs regarding the underlying risk distribution. Optimal indemnity contracts have a layer-type shape. Moreover, we characterize the optimal contracts in the presence of a budget constraint, i.e., an upper bound constraint on the aggregate premium. For both the unconstrained and the constrained reinsurance problems, we show the existence of a representative reinsurer.

Appendices

A Proof of Proposition 2.1

By non-atomicity of $(S, \Sigma, \mathbb{Q}_i)$, for each $i \in \{1, 2, \ldots, n\}$, there exists a random variable U_i on $(S, \Sigma, \mathbb{Q}_i)$ with a uniform distribution on $(0, 1)$. Following Dhaene et al. (2012), the absolute continuity of T_i implies that we can write the premium principle in eq. (2) as

$$\begin{aligned}
\pi^{\theta_i, T_i, \mathbb{Q}_i}(Y) &= (1+\theta_i) \int_0^{\infty} T_i \left(1 - F_{i,Y}(t)\right) dt = (1+\theta_i) \int_0^{\infty} t \; d\tilde{T}_i \left(F_{i,Y}(t)\right) \\
&= (1+\theta_i) \int_0^1 F_{i,Y}^{-1}(s) \, d\tilde{T}_i(s) = (1+\theta_i) \int_0^1 F_{i,Y}^{-1}(s) \, d\left[1 - T_i(1-s)\right] \\
&= (1+\theta_i) \int_0^1 F_{i,Y}^{-1}(s) \, d\tilde{T}_i(s) = (1+\theta_i) \int_0^1 F_{i,Y}^{-1}(s) \, \tilde{T}_i'(s) \, ds \\
&= (1+\theta_i) \int_0^1 F_{i,Y}^{-1}(s) \, T_i'(1-s) \, ds = (1+\theta_i) \int F_{i,Y}^{-1}(U_i) \, \tilde{T}_i'(U_i) \, d\,\mathbb{Q}_i \\
&= (1+\theta_i) \int F_{i,Y}^{-1}(U_i) \, T_i'(1-U_i) \, d\,\mathbb{Q}_i = (1+\theta_i) \int F_{i,Y}^{-1}(U_i) \, \tilde{T}_i'(U_i) \frac{d\,\mathbb{Q}_i}{d\mu} d\mu \\
&= (1+\theta_i) \int F_{i,Y}^{-1}(U_i) \, T_i'(1-U_i) \frac{d\,\mathbb{Q}_i}{d\mu} d\mu = \int F_{i,Y}^{-1}(U_i) \, \zeta_i \; d\mu.
\end{aligned}$$

□

B Proof of Theorem 3.1

First, suppose that the profile $\{f_i\}_{i=1}^n$ is such that $f_i(x) = \displaystyle\int_0^x h_i(z)dz$ for $x \in [0, M)$, where for all $i = 1, \ldots, n$, and for a.e. $z \in [0, M)$,

$$h_i(z) = \begin{cases} \gamma_i(z) & \text{if } \; i \in \mathcal{I}(z), \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

and for a.e. $z \in [0, M)$, $\gamma_i(z) \in [0, 1]$ is such that

$$\sum_{i=1}^n h_i(z) = \begin{cases} 1 & \text{if } \; \upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right), \\ \phi(z) & \text{if } \; \upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right), \\ 0 & \text{if } \; \upsilon(X > z) > T_0\left(\mathbb{P}(X > z)\right), \end{cases}$$

and $\phi(z) \in [0, 1]$. It then follows that for all $z \in [0, M)$, and for all $i = 1, \ldots, n$, $f_i'(z) = h_i(z) \geq 0$. Moreover, since $\sum_{i=1}^n h_i(z) \in [0, 1]$, it follows that $f_i'(z) = h_i(z) \leq 1$. Furthermore, by construction, $f_i(z) \geq 0$, and $f_i(z) = \int_0^z h_i(x)dx \leq \int_0^z dx = z$. Thus, $\{f_i\}_{i=1}^n \subset \mathcal{F}$. Moreover, for each $z \in [0, M)$,

$\left(\sum_{i=1}^n f_i\right)'(z) = \sum_{i=1}^n f_i'(z) = \sum_{i=1}^n h_i(z) \in [0,1]$. Consequently, it follows that for each $z \in [0,M)$

$$0 \le \sum_{i=1}^n f_i(z) = \sum_{i=1}^n \int_0^z h_i(x)dx = \int_0^z \sum_{i=1}^n h_i(x)dx \le \int_0^z dx = z.$$

Therefore, the profile $\{f_i\}_{i=1}^n$ is feasible for Problem (6). To show optimality of $\{f_i\}_{i=1}^n$ for Problem (6), let $\{g_i\}_{i=1}^n$ be any feasible profile for Problem (6). Since $\{f_i\}_{i=1}^n \subset \mathcal{F}$ and $\sum_{i=1}^n f_i \in \mathcal{F}$, it holds that

$$\begin{aligned}
\rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big) &= \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X)\Big) + \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)) \\
&= \rho^{\mathbb{P}}(X) - \rho^{\mathbb{P}}\Big(\sum_{i=1}^n f_i(X)\Big) + \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)) \\
&= \rho^{\mathbb{P}}(X) - \sum_{i=1}^n \rho^{\mathbb{P}}\left(f_i(X)\right) + \sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)) \\
&= \rho^{\mathbb{P}}(X) + \sum_{i=1}^n \left[\pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)) - \rho^{\mathbb{P}}\left(f_i(X)\right)\right] \\
&= \rho^{\mathbb{P}}(X) + \sum_{i=1}^n \left[(1+\theta_i)\int_0^M T_i\left(\mathbb{Q}_i(X>z)\right)h_i(z)dz - \int_0^\infty T_0\left(\mathbb{P}(X>z)\right)h_i(z)dz\right] \\
&= \rho^{\mathbb{P}}(X) + \sum_{i=1}^n \int_0^M \Big[(1+\theta_i)\,T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]h_i(z)dz \\
&= \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i=1}^n \Big[(1+\theta_i)\,T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]h_i(z)dz \\
&= \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i\in\mathcal{I}(z)} \Big[(1+\theta_i)\,T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]\gamma_i(z)dz \\
&= \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i\in\mathcal{I}(z)} \Big[\upsilon(X>z) - T_0\left(\mathbb{P}(X>z)\right)\Big]\gamma_i(z)dz \\
&= \rho^{\mathbb{P}}(X) + \int_0^M \Big[\upsilon(X>z) - T_0\left(\mathbb{P}(X>z)\right)\Big]\sum_{i\in\mathcal{I}(z)} \gamma_i(z)dz \\
&= \rho^{\mathbb{P}}(X) + \int_0^M \Big[\upsilon(X>z) - T_0\left(\mathbb{P}(X>z)\right)\Big]\sum_{i=1}^n h_i(z)dz \\
&= \rho^{\mathbb{P}}(X) + \int_{\mathcal{A}} \Big[\upsilon(X>z) - T_0\left(\mathbb{P}(X>z)\right)\Big]dz,
\end{aligned}$$

where $\mathcal{A} := \Big\{z \in [0,M) : \upsilon(X>z) < T_0\left(\mathbb{P}(X>z)\right)\Big\}$. Moreover, since $\{g_i\}_{i=1}^n$ is a feasible profile for Problem (6), it follows that $\{g_i\}_{i=1}^n \subset \mathcal{F}$ and $\sum_{i=1}^n g_i \in \mathcal{F}$. Consequently, for $i \in \{1,2,\ldots,n\}$ and for a.e. $z \in [0,M)$, $g_i(z) \in [0,M), g_i'(z) \in [0,1], \sum_{i=1}^n g_i(z) \in [0,M)$, and $\sum_{i=1}^n g_i'(z) \in [0,1]$.

Thus,

$$
\begin{aligned}
\rho^{\mathbb{P}}\Big(X-\sum_{i=1}^{n} f_i(X)+\sum_{i=1}^{n}\pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big) &= \rho^{\mathbb{P}}(X)+\int_{\mathcal{A}}\Big[\upsilon(X>z)-T_0\left(\mathbb{P}(X>z)\right)\Big]dz \\
&\leq \rho^{\mathbb{P}}(X)+\int_0^M \Big[\upsilon(X>z)-T_0\left(\mathbb{P}(X>z)\right)\Big]\sum_{i=1}^{n} g_i'(z)dz \\
&= \rho^{\mathbb{P}}(X)+\int_0^M \sum_{i=1}^{n}\Big[\upsilon(X>z)-T_0\left(\mathbb{P}(X>z)\right)\Big]g_i'(z)dz \\
&\leq \rho^{\mathbb{P}}(X)+\int_0^M \sum_{i=1}^{n}\Big[(1+\theta_i)\,T_i\left(\mathbb{Q}_i(X>z)\right)-T_0\left(\mathbb{P}(X>z)\right)\Big]g_i'(z)dz \\
&= \rho^{\mathbb{P}}\Big(X-\sum_{i=1}^{n} g_i(X)+\sum_{i=1}^{n}\pi^{\theta_i,T_i,\mathbb{Q}_i}(g_i(X))\Big).
\end{aligned}
$$

Hence, the profile $\{f_i\}_{i=1}^n$ is optimal for Problem (6).

We now show the converse statement. Let $\{f_i\}_{i=1}^n$ be an optimal profile for Problem (6). Then, in particular, $\{f_i\}_{i=1}^n \subset \mathcal{F}$ and $\sum_{i=1}^n f_i \in \mathcal{F}$. Therefore, for $i \in \{1,2,\ldots,n\}$ and for a.e. $z \in [0,M)$, $f_i(z) \in [0,M)$, $f_i'(z) \in [0,1]$, $\sum_{i=1}^n f_i(z) \in [0,M)$, and $\sum_{i=1}^n f_i'(z) \in [0,1]$. Let $h_i := f_i'$, for all $i \in \{1,2,\ldots,n\}$. For each $z \in [0,M)$, define the subcollection $\mathcal{J}(z)$ of $\{1,2,\ldots,n\}$ by

$$
\mathcal{J}(z) := \Big\{i \in \{1,2,\ldots,n\} : i \notin \mathcal{I}(z), h_i(z) > 0\Big\}.
$$

Suppose, by way of contradiction, that there exists $B^* \subset [0,M)$ such that $\mathcal{J}(z) \neq \varnothing$, for each $z \in B^*$, and $\int_{B^*} dz > 0$. Fix $z^* \in B^*$. Then $\mathcal{J}(z^*) \neq \varnothing$, and for each $i \in \mathcal{J}(z^*)$, we have $i \notin \mathcal{I}(z^*)$ and $h_i(z^*) > 0$. Define the collection $\{\kappa_i(z^*)\}_{i=1}^n \subset \mathbb{R}^+$ by

$$
\kappa_i(z^*) = \begin{cases} 0 & \text{for all } i \in \mathcal{J}(z^*), \\ h_i(z^*) = 0 & \text{for all } i \in \{1,2,\ldots,n\} \setminus (\mathcal{J}(z^*) \bigcup \mathcal{I}(z^*)), \\ h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} & \text{for all } i \in \mathcal{I}(z^*), \end{cases} \tag{27}
$$

where $|\mathcal{I}(z^*)|$ denotes the cardinality of the set $\mathcal{I}(z^*)$. Then, by construction, $\kappa_i(z^*) = 0$ for all $i \notin \mathcal{I}(z^*)$, and

$$
\sum_{i=1}^n \kappa_i(z^*) = \sum_{i \in \mathcal{I}(z^*)} \left(h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} \right) = \sum_{i=1}^n h_i(z^*).
$$

Moreover, by eq. (27), we have

$$
\sum_{i=1}^n \Big[(1+\theta_i)\,T_i\left(\mathbb{Q}_i(X>z^*)\right) - T_0\left(\mathbb{P}(X>z^*)\right)\Big]\kappa_i(z^*)
$$

$$
\begin{aligned}
&= \sum_{i\in\mathcal{I}(z^*)} \Big[(1+\theta_i)\, T_i\left(\mathbb{Q}_i(X>z^*)\right) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] \kappa_i(z^*) \\
&= \sum_{i\in\mathcal{I}(z^*)} \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] \kappa_i(z^*) \\
&= \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] \sum_{i\in\mathcal{I}(z^*)} \kappa_i(z^*) \\
&= \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] \sum_{i\in\mathcal{I}(z^*)} \left[h_i(z^*) + \frac{\sum_{j\in\mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} \right] \\
&= \sum_{i\in\mathcal{I}(z^*)} \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] h_i(z^*) \\
&\qquad + \sum_{i\in\mathcal{I}(z^*)} \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] \left(\frac{\sum_{j\in\mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} \right) \\
&= \sum_{i\in\mathcal{I}(z^*)} \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] h_i(z^*) + \sum_{j\in\mathcal{J}(z^*)} \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] h_j(z^*) \\
&< \sum_{i\in\mathcal{I}(z^*)} \Big[\upsilon(X>z^*) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] h_i(z^*) \\
&\qquad + \sum_{j\in\mathcal{J}(z^*)} \Big[(1+\theta_j) T_i\left(\mathbb{Q}_j(X>z^*)\right) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] h_j(z^*) \\
&= \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X>z^*)\right) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] h_i(z^*).
\end{aligned}
$$

For each $z \in B^*$, construct the collection $\{\kappa_i(z)\}_{i=1}^n \subset \mathbb{R}^+$ as in eq. (27), so that in particular

$$
\sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] \kappa_i(z) < \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z^*)\right) \Big] h_i(z), \tag{28}
$$

for all $z \in B^*$. Moreover, since $\sum_{i=1}^n \kappa_i(z) = \sum_{i=1}^n h_i(z)$ for each $z \in B^*$, and since each $\kappa_i(z) \geq 0$ for each $z \in B^*$, by construction, it follows that $\kappa_i(z) \in [0,1]$, for all $z \in B^*$. Now, let $\{\phi_i\}_{i=1}^n$ be defined by, for each $i \in \{1,2,\ldots,n\}$,

$$
\phi_i(z) = \begin{cases} h_i(z) & \text{if } z \notin B^*, \\ \kappa_i(z) & \text{if } z \in B^*. \end{cases} \tag{29}
$$

Then, by construction, we have $\phi_i(z) \in [0,1]$, for a.e. $z \in [0,M)$, and for each $i \in \{1,2,\ldots,n\}$.

Moreover,

$$\begin{cases} \sum_{i=1}^{n} \phi_i(z) = \sum_{i=1}^{n} \kappa_i(z) = \sum_{i=1}^{n} h_i(z) & \text{for each } z \in B^*, \\ \sum_{i=1}^{n} \phi_i(z) = \sum_{i=1}^{n} h_i(z) & \text{for each } z \notin B^*. \end{cases}$$

In particular, $\sum_{i=1}^{n} \phi_i(z) \in [0,1]$, for a.e. $z \in [0, M)$. Now, by eq. (28), we have

$$\begin{cases} \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z^*)\right)\Big]\phi_i(z) \\ \qquad < \ \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z^*)\right)\Big]h_i(z), \ \text{for all} \ \ z \in B^*, \\ \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z^*)\right)\Big]\phi_i(z) \\ \qquad = \ \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z^*)\right)\Big]h_i(z), \ \text{for all} \ \ z \notin B^*. \end{cases}$$

Now, letting the collection $\{\Psi_i\}_{i=1}^n$ be defined by $\Psi_i\left(X(s)\right) := \int_0^{X(s)} \phi_i(z)dz$, for each $s \in S$ and for each $i \in \{1, 2, \ldots, n\}$, it follows that for μ-a.e. $s \in S$,

$$\begin{aligned} \sum_{i=1}^{n} f_i\left(X(s)\right) &= \sum_{i=1}^{n} \int_0^{X(s)} h_i(z)dz = \int_0^{X(s)} \sum_{i=1}^{n} h_i(z)dz \\ &= \int_0^{X(s)} \sum_{i=1}^{n} \phi_i(z)dz = \sum_{i=1}^{n} \int_0^{X(s)} \phi_i(z)dz = \sum_{i=1}^{n} \Psi_i\left(X(s)\right). \end{aligned}$$

Therefore, $\Psi_i \in \mathcal{F}$, for each $i \in \{1, 2, \ldots, n\}$, and $\sum_{i=1}^{n} \Psi_i \in \mathcal{F}$. Hence $\{\Psi\}_{i=1}^{n}$ is feasible for Problem (6). Furthermore,

$$\begin{aligned} &\rho^{\mathbb{P}}\left(X - \sum_{i=1}^{n} \Psi_i(X) + \sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(\Psi_i(X))\right) \\ &\quad = \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i=1}^{n} \Big[\left(1+\theta_i\right) T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]\phi_i(z)dz \\ &\quad = \rho^{\mathbb{P}}(X) + \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]\phi_i(z)dz \\ &\qquad\qquad + \int_{[0,M)\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]\phi_i(z)dz \\ &\quad = \rho^{\mathbb{P}}(X) + \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]\kappa_i(z)dz \\ &\qquad\qquad + \int_{[0,M)\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big]h_i(z)dz \end{aligned}$$

$$
\begin{aligned}
&< \rho^{\mathbb{P}}(X) + \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big] h_i(z)dz \\
&\qquad + \int_{[0,M)\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big] h_i(z)dz \\
&= \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big] h_i(z)dz \\
&= \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^{n} f_i(X) + \sum_{i=1}^{n} \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big),
\end{aligned}
$$

contradicting with the assumption that $\{f_i\}_{i=1}^n$ is an optimal profile for Problem (6). Therefore, for a.e. $z \in [0, M)$,

$$
h_i(z) = \begin{cases} \gamma_i(z) & \text{if } \ i \in \mathcal{I}(z), \\ 0 & \text{otherwise}, \end{cases} \tag{30}
$$

for some $\gamma_i(z) \in [0, 1]$. It remains to show that for a.e. $z \in [0, M)$, $\gamma_i(z)$ is such that

$$
\sum_{i=1}^{n} h_i(z) = \begin{cases} 1 & \text{if } \ \upsilon(X>z) < T_0\left(\mathbb{P}(X>z)\right), \\ \phi(z) & \text{if } \ \upsilon(X>z) = T_0\left(\mathbb{P}(X>z)\right), \\ 0 & \text{if } \ \upsilon(X>z) > T_0\left(\mathbb{P}(X>z)\right), \end{cases}
$$

for some $\phi(z) \in [0, 1]$. Now, define the subsets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ of $[0, M)$ as follows:

$$
\begin{aligned}
&\mathcal{A} := \Big\{ z \in [0,M) : \upsilon(X>z) < T_0\left(\mathbb{P}(X>z)\right) \Big\}, \\
&\mathcal{B} := \Big\{ z \in [0,M) : \upsilon(X>z) = T_0\left(\mathbb{P}(X>z)\right) \Big\}, \\
&\mathcal{C} := \Big\{ z \in [0,M) : \upsilon(X>z) > T_0\left(\mathbb{P}(X>z)\right) \Big\}, \\
&\mathcal{D} := \Big\{ z \in \mathcal{A} : \sum_{i=1}^{n} h_i(z) < 1 \Big\}, \text{ and} \\
&\mathcal{E} := \Big\{ z \in \mathcal{C} : \sum_{i=1}^{n} h_i(z) > 0 \Big\}.
\end{aligned}
$$

Since h_i is as in eq. (30) and $\sum_{i=1}^{n} h_i(z) \in [0, 1]$ for a.e. $z \in [0, M)$, we have

$$
\begin{aligned}
&\rho^{\mathbb{P}}\Big(X - \sum_{i=1}^{n} f_i(X) + \sum_{i=1}^{n} \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big) \\
&\qquad = \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i=1}^{n} \Big[\,(1+\theta_i)\,T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big] h_i(z)dz \\
&\qquad = \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i\in\mathcal{I}(z)} \Big[\,(1+\theta_i)\,T_i\left(\mathbb{Q}_i(X>z)\right) - T_0\left(\mathbb{P}(X>z)\right)\Big] \gamma_i(z)dz
\end{aligned}
$$

$$= \rho^{\mathbb{P}}(X) + \int_0^M \sum_{i \in \mathcal{I}(z)} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \gamma_i(z) dz$$

$$= \rho^{\mathbb{P}}(X) + \int_0^M \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i \in \mathcal{I}(z)} \gamma_i(z) dz$$

$$= \rho^{\mathbb{P}}(X) + \int_0^M \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz$$

$$= \rho^{\mathbb{P}}(X) + \int_{\mathcal{A}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz + \int_{\mathcal{C}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz$$

$$= \rho^{\mathbb{P}}(X) + \int_{\mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz + \int_{\mathcal{A} \setminus \mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz + \int_{\mathcal{E}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz.$$

Suppose by way of contradiction that $\int_{\mathcal{D}} dz > 0$ or $\int_{\mathcal{E}} dz > 0$. Then since $\sum_{i=1}^n h_i(z) \in [0,1]$ for a.e. $z \in [0, M)$, it follows that

$$\rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) \Big)$$

$$= \rho^{\mathbb{P}}(X) + \int_{\mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz + \int_{\mathcal{A} \setminus \mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz + \int_{\mathcal{E}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz$$

$$> \rho^{\mathbb{P}}(X) + \int_{\mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] dz + \int_{\mathcal{A} \setminus \mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] dz + \int_{\mathcal{E}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] \sum_{i=1}^n h_i(z) dz$$

$$> \rho^{\mathbb{P}}(X) + \int_{\mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] dz + \int_{\mathcal{A} \setminus \mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right) \Big] dz$$

$$= \rho^{\mathbb{P}}(X) + \int_{\mathcal{A}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] dz.$$

Now, for each $i \in \{1, 2, \ldots, n\}$ and for a.e. $z \in [0, M)$, let

$$\overline{h}_i(z) := \begin{cases} \overline{\gamma}_i(z) & \text{if } \ i \in \mathcal{I}(z), \\ 0 & \text{otherwise,} \end{cases}$$

where for a.e. $z \in [0, M)$, $\overline{\gamma}_i(z)$ is such that

$$\sum_{i=1}^n \overline{h}_i(z) = \begin{cases} 1 & \text{if } \ z \in \mathcal{A}, \\ \overline{\phi}(z) & \text{if } \ z \in \mathcal{B}, \\ 0 & \text{if } \ z \in \mathcal{C}, \end{cases}$$

for some $\overline{\phi}(z) \in [0, 1]$. Then, we have $\overline{h}_i(z) \in [0, 1]$, for a.e. $z \in [0, M)$, and for each $i \in \{1, 2, \ldots, n\}$, and $\sum_{i=1}^n \overline{h}_i(z) \in [0, 1]$, for a.e. $z \in [0, M)$. Now, letting the collection $\{\overline{\Psi}_i\}_{i=1}^n$ be defined by $\overline{\Psi}_i\left(X(s)\right) := \int_0^{X(s)} \overline{h}_i(z) dz$, for each $s \in S$ and for each $i \in \{1, 2, \ldots, n\}$, it follows that $\{\overline{\Psi}_i\}_{i=1}^n$ is a feasible profile for Problem (6). Moreover, since $\sum_{i=1}^n h_i(z) \in [0, 1]$, for a.e. $z \in [0, M)$, it follows that

$$\begin{aligned}
&\rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n \overline{\Psi}_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(\overline{\Psi}_i(X))\Big) \\
&\quad = \rho^{\mathbb{P}}(X) + \int_{\mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] \sum_{i=1}^n \overline{h}_i(z) dz \\
&\qquad + \int_{\mathcal{A} \setminus \mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] \sum_{i=1}^n \overline{h}_i(z) dz \\
&\qquad + \int_{\mathcal{E}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] \sum_{i=1}^n \overline{h}_i(z) dz \\
&\qquad + \int_{\mathcal{C} \setminus \mathcal{E}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] \sum_{i=1}^n \overline{h}_i(z) dz \\
&\quad = \rho^{\mathbb{P}}(X) + \int_{\mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] dz + \int_{\mathcal{A} \setminus \mathcal{D}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] dz \\
&\quad = \rho^{\mathbb{P}}(X) + \int_{\mathcal{A}} \Big[\upsilon(X > z) - T_0\left(\mathbb{P}(X > z)\right)\Big] dz \\
&\quad < \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\Big),
\end{aligned}$$

contradicting with the assumption that $\{f_i\}_{i=1}^n$ is an optimal profile for Problem (6). This then completes the proof of Theorem 3.1. □

C Proof of Proposition 3.2

Let $\{f_i\}_{i=1}^n$ be a solution to Problem (6). Then, by Theorem 3.1, for each $x \in [0,M)$, $f_i(x) = \int_0^x h_i(z)dz$, where for $i = 1, \ldots, n$, and for a.e. $z \in [0, M)$,

$$h_i(z) = \begin{cases} \gamma_i(z) & \text{if } \ i \in \mathcal{I}(z), \\ 0 & \text{otherwise}, \end{cases} \tag{31}$$

and for a.e. $z \in [0, M)$, $\gamma_i(z) \in [0, 1]$ is such that

$$\sum_{i=1}^n h_i(z) = \begin{cases} 1 & \text{if } \ \upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right), \\ \phi(z) & \text{if } \ \upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right), \\ 0 & \text{if } \ \upsilon(X > z) > T_0\left(\mathbb{P}(X > z)\right), \end{cases}$$

and $\phi(z) \in [0, 1]$. Hence,

$$\begin{aligned}
&\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) \\
&= \sum_{i=1}^n (1+\theta_i) \int_0^{f_i(M)} T_i\left(\mathbb{Q}_i(f_i(X) > z)\right) dz = \sum_{i=1}^n (1+\theta_i) \int_0^M T_i\left(\mathbb{Q}_i(X > z)\right) df_i(z) \\
&= \sum_{i=1}^n \int_0^M (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z) dz = \int_0^M \sum_{i=1}^n \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z)\Big] dz \\
&= \int_0^M \sum_{i \in \mathcal{I}(z)} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \gamma_i(z)\Big] dz = \int_0^M \sum_{i \in \mathcal{I}(z)} \Big[\upsilon(X > z)\gamma_i(z)\Big] dz \\
&= \int_0^M \upsilon(X > z) \sum_{i \in \mathcal{I}(z)} \gamma_i(z) dz = \int_0^M \upsilon(X > z) \sum_{i=1}^n h_i(z) dz \\
&= \int_0^M \upsilon(X > z) f'(z) dz = \int_0^{f(M)} \upsilon(f(X) > z) dz.
\end{aligned}$$

Now, let $\{g_i\}_{i=1}^n$ be any feasible profile for Problem (6). Then $\{g_i\}_{i=1}^n \subset \mathcal{F}$ and $\sum_{i=1}^n g_i \in \mathcal{F}$. Consequently, for $i \in \{1, 2, \ldots, n\}$ and for a.e. $z \in [0, M)$, $g_i(z) \in [0, M), g_i'(z) \in [0, 1], \sum_{i=1}^n g_i(z) \in [0, M)$, and $\sum_{i=1}^n g_i'(z) \in [0, 1]$. Moreover, since $\sum_{i=1}^n f_i\left(X\right) = \sum_{i=1}^n g_i\left(X\right) = f\left(X\right)$, μ-a.s., it follows that for μ-a.e. $s \in S$,

$$\int_0^{X(s)} f'(z)dz = f\left(X(s)\right) = \sum_{i=1}^n g_i\left(X(s)\right) = \sum_{i=1}^n \int_0^{X(s)} g_i'(z)dz = \int_0^{X(s)} \sum_{i=1}^n g_i'(z)dz.$$

Consequently,

$$\sum_{i=1}^n g_i'(z) = f'(z), \text{ for a.e. } z \in [0, M). \tag{32}$$

Hence,

$$\begin{aligned}\sum_{i=1}^{n} \pi^{\theta_i,T_i,\mathbb{Q}_i}(g_i(X)) &= \int_0^M \sum_{i=1}^{n} \Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right) g_i'(z)\Big] dz \\ &\geq \int_0^M \sum_{i=1}^{n} \Big[\upsilon(X>z) g_i'(z)\Big] dz = \int_0^M \upsilon(X>z) \sum_{i=1}^{n} g_i'(z) dz \\ &= \int_0^M \upsilon(X>z) f'(z) dz = \int_0^{f(M)} \upsilon(f(X)>z) dz \\ &= \int_0^{f(M)} \upsilon(f(X)>z) dz = \sum_{i=1}^{n} \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)).\end{aligned}$$

□

D Proof of Theorem 3.3

We start with the "if" part. Fix $f \in \mathcal{F}$. Suppose that $\{f_i\}_{i=1}^n \subset \mathcal{F}$ is such that for each $i \in \{1, 2, \ldots, n\}$ and for each $x \in [0, M)$, $f_i(x) = \int_0^x h_i(z) dz$, where for a.e. $z \in [0, M)$,

$$h_i(z) = \begin{cases} 0 & \text{if } \ i \notin \mathcal{I}(z), \\ \gamma_i(z) & \text{if } \ i \in \mathcal{I}(z), \end{cases} \tag{33}$$

and where $\{\gamma_i\}_{i=1}^n$ is such that $\sum_{i=1}^n h_i(z) = f'(z)$, for a.e. $z \in [0, M)$. Then, in particular, for μ-a.e. $s \in S$,

$$\sum_{i=1}^{n} f_i\left(X(s)\right) = \sum_{i=1}^{n} \int_0^{X(s)} h_i(z) dz = \int_0^{X(s)} \sum_{i=1}^{n} h_i(z) dz \int_0^{X(s)} f'(z) dz = f\left(X(s)\right),$$

where the last equality follows from the fact that $f \in \mathcal{F}$ is absolutely continuous and is such that $f(0) = 0$, so that we can write $f\left(X(s)\right) = \int_0^{X(s)} f'(z) dz$, for each $s \in S$. Hence $\{f_i\}_{i=1}^n$ is feasible for Problem (13), by construction. To show optimality of $\{f_i\}_{i=1}^n$ for Problem (13), let $\{g_i\}_{i=1}^n \subset \mathcal{F}$ be also feasible for Problem (13). Then, in particular

$$\sum_{i=1}^{n} f_i\left(X\right) = \sum_{i=1}^{n} g_i\left(X\right) = f\left(X\right), \ \ \mu\text{-a.s.}$$

Moreover, since for each i, $g_i \in \mathcal{F}$ is absolutely continuous and is such that $g_i(0) = 0$, we can write $g_i(X(s)) = \int_0^{X(s)} g_i'(z) dz$, for each $s \in S$, where $g_i'(z) \in [0, 1]$, for a.e. $z \geq 0$. Hence, it follows that for μ-a.e. $s \in S$,

$$\int_0^{X(s)} f'(z) dz = f\left(X(s)\right) = \sum_{i=1}^{n} g_i\left(X(s)\right) = \sum_{i=1}^{n} \int_0^{X(s)} g_i'(z) dz = \int_0^{X(s)} \sum_{i=1}^{n} g_i'(z) dz.$$

Consequently,

$$\sum_{i=1}^{n} g_i'(z) = f'(z), \text{ for a.e. } z \in [0, M). \tag{34}$$

Now, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)) \\
&= \sum_{i=1}^{n}(1+\theta_i) \int_0^{f_i(M)} T_i\left(\mathbb{Q}_i(f_i(X) > z)\right) dz = \sum_{i=1}^{n}(1+\theta_i) \int_0^{M} T_i\left(\mathbb{Q}_i(X > z)\right) df_i(z) \\
&= \sum_{i=1}^{n} \int_0^{M} (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z) dz = \int_0^{M} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z)\Big] dz \\
&= \int_0^{M} \sum_{i \in \mathcal{I}(z)} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \gamma_i(z)\Big] dz = \int_0^{M} \sum_{i \in \mathcal{I}(z)} \Big[\upsilon(X > z) \gamma_i(z)\Big] dz \\
&= \int_0^{M} \upsilon(X > z) \sum_{i \in \mathcal{I}(z)} \gamma_i(z) dz = \int_0^{M} \upsilon(X > z) \sum_{i=1}^{n} h_i(z) dz \\
&= \int_0^{M} \upsilon(X > z) f'(z) dz = \int_0^{M} \upsilon(X > z) \sum_{i=1}^{n} g_i'(z) dz = \int_0^{M} \sum_{i=1}^{n} \Big[\upsilon(X > z) g_i'(z)\Big] dz \\
&\leq \int_0^{M} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) g_i'(z)\Big] dz = \sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X)).
\end{aligned}$$

Thus, $\{f_i\}_{i=1}^{n} \in \mathcal{F}(f)$.

We proceed with the "only if" part. Let $\{f_i\}_{i=1}^{n} \in \mathcal{F}(f)$ be such that $f_i \in \mathcal{F}$, for each i. Then $\sum_{j=1}^{n} f_j(X) = f(X)$, μ-a.s., by definition of $\mathcal{F}(f)$. Moreover, since for each i, $f_i \in \mathcal{F}$ is absolutely continuous and is such that $f_i(0) = 0$, we can write $f_i(X(s)) = \int_0^{X(s)} h_i(z) dz$, for each $s \in S$, where $h_i(z) = f_i'(z) \in [0, 1]$, for a.e. $z \geq 0$. Hence, for a given $f \in \mathcal{F}$, it follows that for μ-a.e. $s \in S$,

$$\int_0^{X(s)} f'(z) dz = f\left(X(s)\right) = \sum_{i=1}^{n} f_i\left(X(s)\right) = \sum_{i=1}^{n} \int_0^{X(s)} h_i(z) dz = \int_0^{X(s)} \sum_{i=1}^{n} h_i(z) dz.$$

Consequently, $\sum_{i=1}^{n} h_i(z) = f'(z)$ for a.e. $z \in [0, M)$. Now, for each $z \in [0, M)$, define

$$\mathcal{J}(z) := \Bigg\{ i \in \{1, 2, \ldots, n\} : i \notin \mathcal{I}(z), h_i(z) > 0 \Bigg\}.$$

Suppose, by way of contradiction, that there exists $B^* \subset [0, M)$ such that $\mathcal{J}(z) \neq \varnothing$, for each $z \in B^*$, and $\int_{B^*} dz > 0$. Fix $z^* \in B^*$. Then $\mathcal{J}(z^*) \neq \varnothing$, and for each $i \in \mathcal{J}(z^*)$, we have $i \notin \mathcal{I}(z^*)$ and

$h_i(z^*) > 0$. Define the collection $\{\kappa_i(z^*)\}_{i=1}^n \subset \mathbb{R}^+$ by

$$\kappa_i(z^*) = \begin{cases} 0 & \text{for all } i \in \mathcal{J}(z^*), \\ h_i(z^*) = 0 & \text{for all } i \in \{1, 2, \ldots, n\} \setminus (\mathcal{J}(z^*) \bigcup \mathcal{I}(z^*)), \\ h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} & \text{for all } i \in \mathcal{I}(z^*), \end{cases} \tag{35}$$

where $|\mathcal{I}(z^*)|$ denotes the cardinality of the set $\mathcal{I}(z^*)$. Then, by construction, $\kappa_i(z^*) = 0$ for all $i \notin \mathcal{I}(z^*)$, and

$$\sum_{i=1}^n \kappa_i(z^*) = \sum_{i \in \mathcal{I}(z^*)} \left(h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} \right) = \sum_{i=1}^n h_i(z^*).$$

Moreover, by eq. (35), we have

$$\begin{aligned} \sum_{i=1}^n (1+\theta_i) T_i \left(\mathbb{Q}_i(X > z^*)\right) \kappa_i(z^*) &= \sum_{i \in \mathcal{I}(z^*)} (1+\theta_i) T_i \left(\mathbb{Q}_i(X > z^*)\right) \kappa_i(z^*) = \sum_{i \in \mathcal{I}(z^*)} \upsilon(X > z^*) \kappa_i(z^*) \\ &= \upsilon(X > z^*) \sum_{i \in \mathcal{I}(z^*)} \left[h_i(z^*) + \frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} \right] \\ &= \sum_{i \in \mathcal{I}(z^*)} \upsilon(X > z^*) h_i(z^*) + \sum_{i \in \mathcal{I}(z^*)} \upsilon(X > z^*) \left(\frac{\sum_{j \in \mathcal{J}(z^*)} h_j(z^*)}{|\mathcal{I}(z^*)|} \right) \\ &= \sum_{i \in \mathcal{I}(z^*)} \upsilon(X > z^*) h_i(z^*) + \sum_{j \in \mathcal{J}(z^*)} \upsilon(X > z^*) h_j(z^*) \\ &< \sum_{i \in \mathcal{I}(z^*)} \upsilon(X > z^*) h_i(z^*) + \sum_{j \in \mathcal{J}(z^*)} (1+\theta_j) T_i \left(\mathbb{Q}_j(X > z^*)\right) h_j(z^*) \\ &= \sum_{i=1}^n (1+\theta_i) T_i \left(\mathbb{Q}_i(X > z^*)\right) h_i(z^*). \end{aligned}$$

For each $z \in B^*$, construct the collection $\{\kappa_i(z)\}_{i=1}^n \subset \mathbb{R}^+$ as in eq. (35), so that in particular

$$\sum_{i=1}^n (1+\theta_i) T_i \left(\mathbb{Q}_i(X > z)\right) \kappa_i(z) < \sum_{i=1}^n (1+\theta_i) T_i \left(\mathbb{Q}_i(X > z)\right) h_i(z), \quad \text{for all } \ z \in B^*. \tag{36}$$

Moreover, since $\sum_{i=1}^n h_i(z) = f'(z)$ for a.e. $z \in [0, M)$, since $\sum_{i=1}^n \kappa_i(z) = \sum_{i=1}^n h_i(z)$ for each $z \in B^*$, and since each $\kappa_i(z) \geq 0$ for each $z \in B^*$, by construction, it follows that $\kappa_i(z) \in [0, 1]$, for all $z \in B^*$. Now, let $\{\phi_i\}_{i=1}^n$ be defined by, for each $i \in \{1, 2, \ldots, n\}$,

$$\phi_i(z) = \begin{cases} h_i(z) & \text{if } z \notin B^*, \\ \kappa_i(z) & \text{if } z \in B^*. \end{cases} \tag{37}$$

Then

$$\begin{cases} \sum_{i=1}^{n} \phi_i(z) = \sum_{i=1}^{n} \kappa_i(z) = \sum_{i=1}^{n} h_i(z) & \text{for each } z \in B^*, \\ \sum_{i=1}^{n} \phi_i(z) = \sum_{i=1}^{n} h_i(z) & \text{for each } z \notin B^*. \end{cases}$$

Consequently, since $\sum_{j=1}^{n} h_j(z) = f'(z)$ for a.e. $z \in [0, M)$, it follows that $\sum_{j=1}^{n} \phi_j(z) = f'(z)$ for a.e. $z \in [0, M)$. Moreover, by construction, we have $\phi_i(z) \in [0, 1]$, for a.e. $z \geq 0$, for each $i \in \{1, 2, \ldots, n\}$. Now, by eq. (36), we have

$$\begin{cases} \sum_{i=1}^{n} (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \phi_i(z) < \sum_{i=1}^{n} (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z), & \text{for all } \; z \in B^*, \\ \sum_{i=1}^{n} (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \phi_i(z) = \sum_{i=1}^{n} (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z), & \text{for all } \; z \notin B^*. \end{cases}$$

Therefore, letting the collection $\{\Psi_i\}_{i=1}^{n}$ be defined by $\Psi_i\left(X(s)\right) := \int_0^{X(s)} \phi_i(z) dz$, for each $s \in S$ and for each $i \in \{1, 2, \ldots, n\}$, it follows that for μ-a.e. $s \in S$,

$$\begin{aligned} f\left(X(s)\right) &= \sum_{i=1}^{n} f_i\left(X(s)\right) = \sum_{i=1}^{n} \int_0^{X(s)} h_i(z) dz = \int_0^{X(s)} \sum_{i=1}^{n} h_i(z) dz \\ &= \int_0^{X(s)} \sum_{i=1}^{n} \phi_i(z) dz = \sum_{i=1}^{n} \int_0^{X(s)} \phi_i(z) dz = \sum_{i=1}^{n} \Psi_i\left(X(s)\right). \end{aligned}$$

Moreover, since for each $i \in \{1, 2, \ldots, n\}$, $\phi_i(z) \in [0, 1]$, for a.e. $z \geq 0$, it follows that $\Psi_i \in \mathcal{F}$, for each $i \in \{1, 2, \ldots, n\}$. Hence $\{\Psi\}_{i=1}^{n}$ is feasible for Problem (13). Finally,

$$\begin{aligned} &\sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(\Psi_i(X)) \\ &= \sum_{i=1}^{n} (1+\theta_i) \int_0^{\Psi_i(M)} T_i\left(\mathbb{Q}_i(\Psi_i(X) > z)\right) dz = \sum_{i=1}^{n} (1+\theta_i) \int_0^{M} T_i\left(\mathbb{Q}_i(X > z)\right) d\Psi_i(z) \\ &= \sum_{i=1}^{n} \int_0^{M} (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \phi_i(z) dz = \int_0^{M} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \phi_i(z)\Big] dz \\ &= \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \phi_i(z)\Big] dz + \int_{[0,M)\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \phi_i(z)\Big] dz \\ &= \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) \kappa_i(z)\Big] dz + \int_{[0,M)\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z)\Big] dz \\ &< \int_{B^*} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z)\Big] dz + \int_{[0,M)\setminus B^*} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z)\Big] dz \\ &= \int_0^{M} \sum_{i=1}^{n} \Big[(1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z)\Big] dz = \sum_{i=1}^{n} \int_0^{M} (1+\theta_i) T_i\left(\mathbb{Q}_i(X > z)\right) h_i(z) dz \end{aligned}$$

$$= \sum_{i=1}^{n} \int_0^M (1+\theta_i) T_i \left(\mathbb{Q}_i(X > z)\right) df_i(z) = \sum_{i=1}^{n} (1+\theta_i) \int_0^M T_i \left(\mathbb{Q}_i(X > z)\right) df_i(z)$$
$$= \sum_{i=1}^{n} (1+\theta_i) \int_0^M T_i \left(\mathbb{Q}_i(f_i(X) > z)\right) dz = \sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X)),$$

which contradicts with the assumption that $\{f_i\}_{i=1}^n \in \mathcal{F}(f)$. □

E Proof of Theorem 3.4

First, suppose that $\sum_{i=1}^n f_i$ is optimal for Problem (11) with π as in eq. (12), and that $\{f_i\}_{i=1}^n \in \mathcal{F}\left(\sum_{j=1}^n f_j\right)$. Then $\{f_i\}_{i=1}^n$ is clearly feasible for Problem (6). To show optimality of $\{f_i\}_{i=1}^n$ for Problem (6), suppose by way of contradiction that $\{f_i\}_{i=1}^n$ is not optimal for Problem (6). Then there exists some collection $\{\hat{f}_i\}_{i=1}^n \subset \mathcal{F}$ such that

$$\rho^{\mathbb{P}}\left(X - \sum_{i=1}^{n} \hat{f}_i(X) + \sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(\hat{f}_i(X))\right) < \rho^{\mathbb{P}}\left(X - \sum_{i=1}^{n} f_i(X) + \sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\right).$$

Now, letting $\hat{f} := \sum_{i=1}^n \hat{f}_i$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} \pi^{\theta_i, T_i, \mathbb{Q}_i}(\hat{f}_i(X)) &= \sum_{i=1}^{n} (1+\theta_i) \int_0^{\hat{f}_i(M)} T_i \left(\mathbb{Q}_i(\hat{f}_i(X) > z)\right) dz \\
&= \sum_{i=1}^{n} (1+\theta_i) \int_0^M T_i \left(\mathbb{Q}_i(X > z)\right) d\hat{f}_i(z) \\
&= \sum_{i=1}^{n} \int_0^M (1+\theta_i) T_i \left(\mathbb{Q}_i(X > z)\right) \hat{f}_i'(z) dz \\
&= \int_0^M \sum_{i=1}^{n} \Big[(1+\theta_i) T_i \left(\mathbb{Q}_i(X > z)\right) \hat{f}_i'(z)\Big] dz \\
&\geq \int_0^M \sum_{i=1}^{n} \min_{1 \leq j \leq n} \{(1+\theta_j) T_j \left(\mathbb{Q}_j(X > z)\right)\} \hat{f}_i'(z) dz \\
&= \int_0^M \sum_{i=1}^{n} \upsilon(X > z) \hat{f}_i'(z) dz = \int_0^M \upsilon(X > z) \sum_{i=1}^{n} \hat{f}_i'(z) dz \\
&= \int_0^M \upsilon(X > z) \hat{f}'(z) dz = \int_0^M \upsilon(X > z) d\hat{f}(z) \\
&= \int_0^{\hat{f}(M)} \upsilon(\hat{f}(X) > z) dz = \pi\left(\hat{f}(X)\right) = \pi\left(\sum_{i=1}^{n} \hat{f}_i(X)\right),
\end{aligned}$$

where π is as in eq. (12). Consequently,

$$\rho^{\mathbb{P}}\left(X-\sum_{i=1}^{n}\hat{f}_i(X)+\pi\left(\sum_{i=1}^{n}\hat{f}_i(X)\right)\right) \leq \rho^{\mathbb{P}}\Big(X-\sum_{i=1}^{n}\hat{f}_i(X)+\sum_{i=1}^{n}\pi^{\theta_i,T_i,\mathbb{Q}_i}(\hat{f}_i(X))\Big)$$
$$< \rho^{\mathbb{P}}\Big(X-\sum_{i=1}^{n}f_i(X)+\sum_{i=1}^{n}\pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\Big).$$

Now, since $\{f_i\}_{i=1}^n \in \mathcal{F}\left(\sum_{j=1}^n f_j\right)$, it follows from Theorem 3.3 that for each $i \in \{1,2,\dots,n\}$ and for each $x \in [0,M)$, $f_i(x) = \int_0^x h_i(z)dz$, where for a.e. $z \in [0,M)$, $h_i(z) = 0$ whenever $i \notin \mathcal{I}(z)$. Therefore, letting $f := \sum_{i=1}^n f_i$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n}\pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X)) &= \sum_{i=1}^{n}(1+\theta_i)\int_0^{f_i(M)} T_i\left(\mathbb{Q}_i(f_i(X)>z)\right)dz \\
&= \sum_{i=1}^{n}(1+\theta_i)\int_0^{M} T_i\left(\mathbb{Q}_i(X>z)\right)df_i(z) \\
&= \sum_{i=1}^{n}\int_0^{M}(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right)f_i'(z)dz \\
&= \int_0^{M}\sum_{i=1}^{n}\Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right)h_i(z)\Big]dz \\
&= \int_0^{M}\sum_{i\in\mathcal{I}(z)}\Big[(1+\theta_i)T_i\left(\mathbb{Q}_i(X>z)\right)h_i(z)\Big]dz \\
&= \int_0^{M}\sum_{i\in\mathcal{I}(z)}\upsilon\left(X>z\right)h_i(z)dz = \int_0^{M}\sum_{i=1}^{n}\upsilon\left(X>z\right)h_i(z)dz \\
&= \int_0^{M}\upsilon(X>z)f'(z)dz = \int_0^{M}\upsilon(X>z)df(z) \\
&= \int_0^{f(M)}\upsilon(f(X)>z)dz = \pi(f(X)) = \pi\left(\sum_{i=1}^{n}f_i(X)\right).
\end{aligned}$$

Hence,

$$\rho^{\mathbb{P}}\left(X-\sum_{i=1}^{n}\hat{f}_i(X)+\pi\left(\sum_{i=1}^{n}\hat{f}_i(X)\right)\right) < \rho^{\mathbb{P}}\left(X-\sum_{i=1}^{n}f_i(X)+\sum_{i=1}^{n}\pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i(X))\right)$$
$$= \rho^{\mathbb{P}}\left(X-\sum_{i=1}^{n}f_i(X)+\pi\left(\sum_{i=1}^{n}f_i(X)\right)\right),$$

contradicting the optimality of $\sum_{i=1}^n f_i$ for Problem (11) with π as in eq. (12). Hence, $\{f_i\}_{i=1}^n$ is optimal for Problem (6).

Conversely, let $\{f_i\}_{i=1}^n$ be optimal for Problem (6), and let $f^* := \sum_{j=1}^n f_j$. Suppose, by way of contradiction, that either f^* is not optimal for Problem (11) with π as in eq. (12), or that

$\{f_i\}_{i=1}^n \notin \mathcal{F}(f^*)$:

- First, we assume that f^* is not optimal for Problem (11) with π as in eq. (12). Then there exists some $\hat{f}^*$ that is optimal for Problem (11) with π as in eq. (12), and some $\{\hat{f}_i\}_{i=1}^n$ such that $\{\hat{f}_i\}_{i=1}^n \in \mathcal{F}(\hat{f}^*)$. It then follows from the first part of this proof that $\{\hat{f}_i\}_{i=1}^n$ is optimal for Problem (6). Hence, as in the first part of this proof,

$$\begin{aligned}
\rho^{\mathbb{P}}\left(X - \hat{f}^*(X) + \pi\left(\hat{f}^*(X)\right)\right) &= \rho^{\mathbb{P}}\left(X - \sum_{i=1}^n \hat{f}_i(X) + \pi\left(\sum_{i=1}^n \hat{f}_i(X)\right)\right) \\
&= \rho^{\mathbb{P}}\left(X - \sum_{i=1}^n \hat{f}_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(\hat{f}_i(X))\right) \\
&= \rho^{\mathbb{P}}\left(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\right) \\
&\geq \rho^{\mathbb{P}}\left(X - \sum_{i=1}^n f_i(X) + \pi\left(\sum_{i=1}^n f_i(X)\right)\right) \\
&= \rho^{\mathbb{P}}\left(X - f^*(X) + \pi\left(f^*(X)\right)\right),
\end{aligned}$$

  where the third equality follows from the assumption that $\{f_i\}_{i=1}^n$ is optimal for Problem (6). However, this contradicts the assumption that f^* is not optimal for Problem (11) with π as in eq. (12).

- Second, suppose that $\{f_i\}_{i=1}^n \notin \mathcal{F}(f^*)$, and, by Theorem 3.3, choose $\{\hat{f}_i\}_{i=1}^n \in \mathcal{F}(f^*)$. Then, by monotonicity of $\rho^{\mathbb{P}}$,

$$\rho^{\mathbb{P}}\left(X - \sum_{i=1}^n \hat{f}_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(\hat{f}_i(X))\right) < \rho^{\mathbb{P}}\left(X - \sum_{i=1}^n f_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i(X))\right),$$

  which contradicts the assumption that $\{f_i\}_{i=1}^n$ is optimal for Problem (6).

Consequently, f^* is optimal for Problem (11) with π as in eq. (12), and $\{f_i\}_{i=1}^n \in \mathcal{F}(f^*)$. □

F Proof of Lemma 4.1

Suppose that there exists $\lambda^* \geq 0$ such that:

(i) $\{f_{i,\lambda^*}^*\}_{i=1}^n$ is optimal for Problem (18) with $\lambda = \lambda^*$, and,

(ii) $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_{i,\lambda^*}(X)) = p$.

Then, in particular, $\{f_{i,\lambda^*}^*\}_{i=1}^n$ is feasible for Problem (15). Let $\{g_i\}_{i=1}^n$ be any feasible profile for Problem (15). Then $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X)) \leq p$, and therefore

$$
\begin{aligned}
\rho^{\mathbb{P}}\left(X-\sum_{i=1}^n g_i(X)+\pi\left(\sum_{i=1}^n g_i(X)\right)\right) &\geq \rho^{\mathbb{P}}\left(X-\sum_{i=1}^n g_i(X)+\pi\left(\sum_{i=1}^n g_i(X)\right)\right) \\
&\qquad + \lambda^*\left[\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(g_i(X)) - p\right] \\
&\geq \rho^{\mathbb{P}}\left(X-\sum_{i=1}^n f^*_{i,\lambda^*}(X)+\pi\left(\sum_{i=1}^n f^*_{i,\lambda^*}(X)\right)\right) \\
&\qquad + \lambda^*\left[\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f^*_{i,\lambda^*}(X)) - p\right] \\
&= \rho^{\mathbb{P}}\left(X-\sum_{i=1}^n f^*_{i,\lambda^*}(X)+\pi\left(\sum_{i=1}^n f^*_{i,\lambda^*}(X)\right)\right),
\end{aligned}
$$

where the second inequality follows from the optimality of $\{f^*_{i,\lambda^*}\}_{i=1}^n$ for Problem (18) with $\lambda = \lambda^*$. Hence, $\{f^*_{i,\lambda^*}\}_{i=1}^n$ is optimal for Problem (15). □

G Proof of Corollary 4.2

The assumption $p \in [0, \bar{\pi})$ implies that the premium constraint is binding. The proof follows from similar arguments as in the proof of Theorem 3.1. Following the proof of Theorem 3.1 by inserting the factor $(1+\lambda)$ to the component $\upsilon(X > z)$ yields a profile $\{f^*_{i,\lambda}\}_{i=1}^n$ that solves Problem (18) explicitly. Thus, it follows that $\{f^*_{i,\lambda}\}$ solves Problem (18) if and only if for each $x \in [0, M)$, $f^*_{i,\lambda}(x) = \int_0^x h^*_{i,\lambda}(z)dz$, where for $i = 1, \ldots, n$, and for a.e. $z \in [0, M)$,

$$
h^*_{i,\lambda}(z) = \begin{cases} \gamma_{i,\lambda}(z) & \text{if } \ i \in \mathcal{I}(z), \\ 0 & \text{otherwise,} \end{cases} \tag{38}
$$

and for a.e. $z \in [0, M)$, $\gamma_{i,\lambda}(z) \in [0, 1]$ is such that

$$
\sum_{i=1}^n h^*_{i,\lambda}(z) = \begin{cases} 1 & \text{if } \ (1+\lambda)\upsilon(X > z) < T_0\left(\mathbb{P}(X > z)\right), \\ \phi_\lambda(z) & \text{if } \ (1+\lambda)\upsilon(X > z) = T_0\left(\mathbb{P}(X > z)\right), \\ 0 & \text{if } \ (1+\lambda)\upsilon(X > z) > T_0\left(\mathbb{P}(X > z)\right), \end{cases} \tag{39}
$$

and $\phi_\lambda(z) \in [0, 1]$.

It remains to show that there exists a $\lambda^* > 0$ such that $\{f^*_{i,\lambda^*}\}_{i=1}^n$ is optimal for Problem (18) and

$$
\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f^*_{i,\lambda^*}(X)) = p.
$$

For any $p \in [0, \bar{\pi})$, let

$$A_p := \Big\{ \lambda > 0 : \text{ there exists } \{f_{i,\lambda}\}_{i=1}^n \text{ solving Problem (18) under } \lambda, \text{ with } \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_{i,\lambda}(X)) = p \Big\},$$

and

$$\lambda_p := \sup \big\{ \lambda > 0 : \lambda \in A_p \big\}. \tag{40}$$

The following result is similar to Zhuang et al. (2017, Lemma 4.1 therein), and its proof is therefore omitted.

Lemma G.1 *When $p \in [0, \bar{\pi})$, there exists $\lambda^* > 0$ such that $\{f^*_{i,\lambda^*}\}_{i=1}^n$ is optimal for Problem (18) with parameter λ^*, and*

$$\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f^*_{i,\lambda^*}(X)) = p.$$

Moreover, for all $p \in [0, \bar{\pi})$, the function $p \mapsto \lambda_p$ is non-increasing.

This concludes the proof of Corollary 4.2. □

H Proof of Theorem 4.5

Fix $\lambda \geq 0$ and $p \in [0, \bar{\pi})$. We first provide a proof of the second statement. Suppose that $f^* := \sum_{i=1}^n f_i^*$ is optimal for Problem (22) with $\eta = \lambda$ and $\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i^*(X)) = \pi\left(\sum_{i=1}^n f_i^*(X)\right)$. Then $\{f_i^*\}_{i=1}^n$ is feasible for Problem (18). To show optimality of $\{f_i^*\}_{i=1}^n$ for Problem (18), let $\{g_i\}_{i=1}^n$ be feasible for Problem (18) and $g := \sum_{i=1}^n g_i$. Then g is feasible for Problem (22) and $\pi(g(X)) \leq \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X))$, by definition of $\pi(\cdot)$, yielding

$$\begin{aligned}
&\rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i^*(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i^*(X))\Big) + \lambda\left(\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(f_i^*(X)) - p\right) \\
&\quad = \rho^{\mathbb{P}}\Big(X - f^*(X) + \pi(f^*(X))\Big) + \lambda\left(\pi(f^*(X)) - p\right) \\
&\quad = \rho^{\mathbb{P}}\Big(X - f^*(X) + (1+\lambda)\pi(f^*(X))\Big) - \lambda p \\
&\quad \leq \rho^{\mathbb{P}}\Big(X - g(X) + (1+\lambda)\pi(g(X))\Big) - \lambda p \\
&\quad \leq \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n g_i(X) + (1+\lambda)\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X))\Big) - \lambda p \\
&\quad = \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n g_i(X) + \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X))\Big) + \lambda\left(\sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X)) - p\right),
\end{aligned}$$

where the first inequality follows from the optimality of $f^* = \sum_{i=1}^n f_i^*$ is optimal for Problem (22) with $\eta = \lambda$, and the second inequality from the fact that $\lambda \geq 0$ and $\pi(g(X)) \leq \sum_{i=1}^n \pi^{\theta_i, T_i, \mathbb{Q}_i}(g_i(X))$.

Hence, $\{f_i^*\}_{i=1}^n$ is optimal for Problem (18). Now, if, moreover, $\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i^*(X)) = p$, then $\{f_i^*\}_{i=1}^n$ is optimal for Problem (15) by Lemma 4.1.

To prove the first statement, suppose that $\{f_i^*\}_{i=1}^n$ is optimal for Problem (18), and let $f^* := \sum_{i=1}^n f_i^*$. Then Proposition 4.3 implies that $\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i^*(X)) = \pi\left(f^*\left(X\right)\right)$, and so f^* is feasible for Problem (24). To show optimality of f^* for Problem (24), let g be feasible for Problem (24). Then there exists $\{g_i\}_{i=1}^n \subset \mathcal{F}$ such that $g = \sum_{i=1}^n g_i$ and $\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(g_i(X)) = \pi\left(g\left(X\right)\right)$. Hence, $\{g_i\}_{i=1}^n$ is feasible for Problem (18) and

$$\begin{aligned}
&\rho^{\mathbb{P}}\Big(X - f^*\left(X\right) + \pi\left(f^*\left(X\right)\right)\Big) + \lambda\left(\pi\left(f^*\left(X\right)\right) - p\right) \\
&\quad = \rho^{\mathbb{P}}\Big(X - f^*\left(X\right) + \left(1+\lambda\right)\pi\left(f^*\left(X\right)\right)\Big) - \lambda p \\
&\quad = \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n f_i^*\left(X\right) + \left(1+\lambda\right)\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(f_i^*(X))\Big) - \lambda p \\
&\quad \leq \rho^{\mathbb{P}}\Big(X - \sum_{i=1}^n g_i\left(X\right) + \left(1+\lambda\right)\sum_{i=1}^n \pi^{\theta_i,T_i,\mathbb{Q}_i}(g_i(X))\Big) - \lambda p \\
&\quad = \rho^{\mathbb{P}}\Big(X - g\left(X\right) + \left(1+\lambda\right)\pi\left(g\left(X\right)\right)\Big) - \lambda p \\
&\quad = \rho^{\mathbb{P}}\Big(X - g\left(X\right) + \pi\left(g\left(X\right)\right)\Big) + \lambda\left(\pi\left(g\left(X\right)\right) - p\right),
\end{aligned}$$

where the inequality follows from the optimality of $\{f_i^*\}_{i=1}^n$ for Problem (18). Hence, f^* is optimal for Problem (24), which concludes the proof of Theorem 4.5. □

References

Amarante, M., M. Ghossoub, and E. Phelps (2015). Ambiguity on the insurer's side: The demand for insurance. *Journal of Mathematical Economics 58*, 61–78.

Asimit, A. V., A. M. Badescu, and K. C. Cheung (2013). Optimal reinsurance in the presence of counterparty default risk. *Insurance: Mathematics and Economics 53*(3), 690–697.

Asimit, A. V., A. M. Badescu, and T. Verdonck (2013). Optimal risk transfer under quantile-based risk measurers. *Insurance: Mathematics and Economics 53*, 252–265.

Assa, H. (2015). On optimal reinsurance policy with distortion risk measures and premiums. *Insurance: Mathematics and Economics 61*, 70–75.

Boonen, T. J. (2016). Optimal reinsurance with heterogeneous reference probabilities. *Risks 4*(3), 26.

Boonen, T. J. and M. Ghossoub (2019). On the existence of a representative reinsurer under heterogeneous beliefs. *Insurance: Mathematics and Economics 88*, 209–225.

Boonen, T. J., K. S. Tan, and S. C. Zhuang (2016). The role of a representative reinsurer in optimal reinsurance. *Insurance: Mathematics and Economics 70*, 196–204.

Carlier, G. and R. Dana (2003). Pareto efficient insurance contracts when the insurer's cost function is discontinuous. *Economic Theory 21*(4), 871–893.

Chen, Y. and H. Assa (2019). Dynamic set-up for designing optimal reinsurance contracts. Available at SSRN: https://ssrn.com/abstract=3420090.

Cheung, K. and A. Lo (2017). Characterizations of Optimal Reinsurance Treaties: A Cost-Benefit Approach. *Scandinavian Actuarial Journal 2017*(1), 1–28.

Chi, Y., X. S. Lin, and K. S. Tan (2017). Optimal reinsurance under the risk-adjusted value of an insurer's liability and an economic reinsurance premium principle. *North American Actuarial Journal 21*(3), 417–432.

Chi, Y. and H. Meng (2014). Optimal reinsurance arrangements in the presence of two reinsurers. *Scandinavian Actuarial Journal 5*, 424–438.

Chi, Y. and K. S. Tan (2011). Optimal reinsurance under VaR and CVaR risk measurs: A simplified approach. *ASTIN Bulletin 41*(2), 487–509.

Cong, J. and K. S. Tan (2016). Optimal VaR-based risk management with reinsurance. *Annals of Operations Research 237*, 177–202.

Cui, W., J. Yang, and L. Wu (2013). Optimal reinsurance minimizing the distortion risk measure under general reinsurance premium principles. *Insurance: Mathematics and Economics 53*, 74–85.

Denuit, M. and C. Vermandele (1998). Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics 22*, 229–233.

Dhaene, J., A. Kukush, D. Linders, and Q. Tang (2012). Remarks on Quantiles and Distortion Risk Measures. *European Actuarial Journal 2*(2), 319–328.

Ghossoub, M. (2019a). Budget-constrained optimal insurance with belief heterogeneity. *Insurance Mathematics and Economics 89*, 79–91.

Ghossoub, M. (2019b). Budget-constrained optimal insurance without the nonnegativity constraint on indemnities. *Insurance Mathematics and Economics 84*, 22–39.

Ghossoub, M. (2019c). Budget-constrained optimal retention with an upper limit on the retained loss. *Scandinavian Actuarial Journal*, Forthcoming.

Ghossoub, M. (2019d). Optimal insurance under rank-dependent expected utility. *Insurance Mathematics and Economics 87*, 51–66.

Huberman, G., D. Mayers, and C. W. Smith Jr (1983). Optimal insurance policy indemnity schedules. *The Bell Journal of Economics 14*, 415–426.

Lo, A. (2017a). A Neyman-Pearson Perspective on Optimal Reinsurance with Constraints. *ASTIN Bulletin 47*(2), 467–499.

Lo, A. (2017b). A Unifying Approach to Risk-Measure-Based Optimal Reinsurance Problems with Practical Constraints. *Scandinavian Actuarial Journal 2017*(7), 584–605.

Marinacci, M. and L. Montrucchio (2004). Introduction to the Mathematics of Ambiguity. In *I. Gilboa (ed.), Uncertainty in Economic Theory: Essays in Honor of David Schmeidlerï£¡s 65th Birthday*, pp. 46–107. London: Routledge.

Wang, S. S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin 26*(1), 71–92.

Wang, W. and X. Peng (2017). Reinsurer's optimal reinsurance strategy with upper and lower premium constraints under distortion risk measures. *Journal of Computational and Applied Mathematics 315*, 142–160.

Wu, X. and J. Wang (2003). On characterization of distortion premium principle. *ASTIN Bulletin 33*(1), 1–10.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica 55*, 95–115.

Young, V. R. (1999). Optimal insurance under wang's premium principle. *Insurance: Mathematics and Economics 25*, 109–122.

Zhuang, S. C., T. J. Boonen, K. S. Tan, and Z. Q. Xu (2017). Optimal Insurance in the Presence of Reinsurance. *Scandinavian Actuarial Journal 2017*(6), 535–554.

Zhuang, S. C., C. Weng, K. S. Tan, and H. Assa (2016). Marginal indemnification function formulation for optimal reinsurance. *Insurance: Mathematics and Economics 67*, 65–76.