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AFFINE HECKE ALGEBRAS AND THE CONJECTURES OF HIRAGA, ICHINO AND IKEDA ON THE PLANCHEREL DENSITY

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Dedicated to Joseph Bernstein

ABSTRACT. Hiraga, Ichino and Ikeda have conjectured an explicit expression for the Plancherel density of the group of points of a reductive group defined over a local field F, in terms of local Langlands parameters. In these lectures we shall present a proof of these conjectures for Lusztig's class of representations of unipotent reduction if F is p-adic and G is of adjoint type and splits over an unramified extension of F. This is based on the author's paper [Spectral transfer morphisms for unipotent affine Hecke algebras, Selecta Math. (N.S.) **22** (2016), no. 4, 2143–2207].

More generally for G connected reductive (still assumed to be split over an unramified extension of F), we shall show that the requirement of compatibility with the conjectures of Hiraga, Ichino and Ikeda essentially determines the Langlands parameterisation for tempered representations of unipotent reduction. We shall show that there exist parameterisations for which the conjectures of Hiraga, Ichino and Ikeda hold up to rational constant factors. The main technical tool is that of spectral transfer maps between normalised affine Hecke algebras used in *op. cit.*

CONTENTS

1. Introduction

Let F be local field of characteristic 0, let $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F, and let G be a the group of points of a connected reductive algebraic group defined over F. Let $G^\vee$ denote the Langlands dual group of G (a complex Lie group with root system dual to that of G), and let $^L G := G^\vee \rtimes \Gamma_F$ be the Galois form of the L-group of G. The Langlands group L_F of F is defined to be W_F (the Weil group of F) if F is archimedean, and $W_F \times \mathrm{SL}_2(\mathbb{C})$ otherwise. Let ψ be a fixed additive character of F. To a finite dimensional complex representation V of $^L G$ one attaches epsilon factors $\epsilon(s, V, \psi)$ and L-functions $L(s, V)$, where $s \in \mathbb{C}$ is a complex variable (see [Tate]). A Langlands parameter for G is a homomorphism $\varphi : L_F \to {}^L G$ some natural conditions (cf. Section 2.3). With all this in place, the adjoint γ-factor of a Langlands parameter φ of G is defined as

$$\gamma(s, \mathrm{Ad} \circ \varphi, \psi) := \frac{\epsilon(s, \mathrm{Ad} \circ \varphi, \psi) L(1 - s, \mathrm{Ad} \circ \varphi)}{L(s, \mathrm{Ad} \circ \varphi)},$$

where Ad is the adjoint representation of $^L G$ on $\mathrm{Lie}(G^\vee)/\mathrm{Lie}(Z^\vee)$, with $Z^\vee$ the center of $G^\vee$. Let $\widehat{G}$ be the space of equivalence classes of irreducible unitary representations π of G, equipped with the Fell topology. We will denote by Θ_π the distribution character of π. Then Harish-Chandra's Plancherel formula states the existence of a unique positive measure ν_{Pl} on $\widehat{G}$, such that $f(e) = \int_{\widehat{G}} \Theta_\pi(f) d\nu_{Pl}(\pi)$, for all $f \in C_c^\infty(G)$. The measure $d\nu_{Pl}$ is called the Plancherel measure of G. If π is a discrete series representation of G, its formal degree is equal to $\mathrm{fdeg}(\pi) := \nu_{Pl}(\pi) > 0$. For more general tempered representations $d\nu_{Pl}$ is described in terms of a density function. Hiraga, Ichino, and Ikeda formulated two conjectures [HII, HIIcor] expressing the Plancherel density at a tempered representation π in terms of the conjectural enhanced Langlands parameter attached to π. For an essentially discrete series representation π_ρ in an L-packet $\Pi_\varphi(G)$ attached to a (discrete) Langlands parameter $\varphi : L_F \to {}^L G$, enhanced with a local system $\rho \in \mathrm{Irr}(\mathcal{S}_\varphi, \chi_G)$ on the $G^\vee$-orbit of φ (see Section 2.5 for unexplained notation

and more details), the conjecture reduces to the equality

$$\mathrm{fdeg}(\pi_\rho) = \frac{\dim(\rho)}{|\mathcal{S}_\varphi^\natural|}|\gamma(0, \mathrm{Ad} \circ \varphi, \psi)|.$$

Hiraga, Ichino, and Ikeda proved the conjecture for $F = \mathbb{R}$ [HII, HIIcor]. For F non-archimedean, it has been proved in several cases but not in general (see Section 2.7 for an overview of known results). From now on we assume that F is a non-archimedean field and that G splits over an unramified extension of F. When G is absolutely almost simple of adjoint type, the conjecture above and its extension to general tempered representations are known to hold for representations of G with unipotent reduction from previous works [R1], [R3], [HO1], [FO], [Opd5], [Fe2], [FOS]. The proof in [Opd5] uses the Lusztig classification (composed with the Iwahori-Matsumoto involution in order to map tempered representations to bounded parameters) as a Langlands parameterisation. The main goal of the present manuscript is to extend this result to a general connected reductive group G (still assumed to be split over an unramified extension of F). This is achieved in the main result Theorem 4.5.1.

The irreducible representations with unipotent reduction (a terminology introduced by Moeglin and Waldspurger), called unipotent by Lusztig, are the representations of G which admit non-zero invariant vectors by the pro-p unipotent radical of a parahoric subgroup of G. In particular, they are depth-zero representations. They are expected to correspond to unramified Langlands parameters. From their definition it follows that the category of representations with unipotent reduction of G is Morita equivalent to the module category of a direct sum of affine Hecke algebras $\mathcal{H}_t(G)$, where t runs over the set of equivalence classes of unipotent types of G. Let G^* be the quasi-split group in the inner class of G, and let $\mathbb{I}$ denote an Iwahori subgroup of G^*. Let $\mathcal{H}_\mathbb{I}(G^*)$ be the Iwahori Hecke algebra of G^*.

One of the main ingredients of the proof of Theorem 4.5.1 is the notion of spectral transfer morphism between normalised affine Hecke algebras [Opd4], [Opd5], which allows us to construct a bijection between the set $\widehat{G}_{uni}^{\mathrm{temp}}$ of equivalence classes of tempered irreducible representations of G with unipotent reduction and the set $\Phi_{nr}^{\mathrm{temp}}(G)$ of $G^\vee$-conjugacy classes of unramified bounded enhanced Langlands parameters for G. The construction of such a bijection has some interest in its own right. A key point is the fact that a spectral transfer morphism $\Psi : \mathcal{H}_t(G) \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$ from the Hecke algebra $\mathcal{H}_t(G)$ of a unipotent type t of G to the Iwahori Hecke algebra $\mathcal{H}_\mathbb{I}(G^*)$ defines Langlands parameters $\pi \to \varphi_\pi \in \Phi_{nr}^{\mathrm{temp}}(G)$ for the tempered representations covered by t such that the conjectures of Hiraga, Ichino and Ikeda hold (up to rational constant factors independent of the cardinality q of the residue field of F). This is explained in Corollary 4.3.5, which is itself based on the Iwahori-spherical case Theorem 3.8.1 and Theorem 3.8.2. Conversely, a Langlands parameterisation such that the conjectures of Hiraga, Ichino and Ikeda hold (up to rational constant factors independent of the cardinality q of the residue field of F) and satisfies a certain algebraic condition (see Theorem 4.5.1(a)) defines such STMs uniquely.

It is remarkable that for tempered representations of unipotent reduction the conditions imposed on a Langlands parameterisation by the conjectures of Hiraga, Ichino and Ikeda determine it up to twisting by certain diagram automorphism. This can be viewed

as a generalisation and strengthening of the principle expressed by Mark Reeder [R3] for discrete series L-packets, saying that "alleged L-packets can only be convicted upon circumstantial evidence, of which the formal degrees are one piece".

2. The conjecture of Hiraga, Ichino and Ikeda

Let F be a local field of characteristic 0, and let $\mathbf{G}$ be a connected reductive group defined over F. The group $G = \mathbf{G}(F)$ of F-points of $\mathbf{G}$ is a separable locally compact topological group which is unimodular. Let μ_G denote a Haar measure on G. Let $C^*(G)$ be the group C^*-algebra of G, i.e. the C^*-envelope of the Banach algebra $L^1(G, \mu_G)$ with respect to convolution. By famous results of Harish-Chandra [HC2] (if F is archimedean) and by Bernstein [Be] (if F is non-archimedean) we know that $C^*(G)$ is liminal, hence of Type I. Let $\hat{G}$ denote the space of equivalence classes of irreducible unitary representations of G, equipped with the Fell topology. For each $\pi \in \hat{G}$ we choose a representative denoted by (V_π, π). The abstract Plancherel formula for separable locally compact unimodular topological groups of Type I asserts that:

Theorem 2.0.1. *There exists a unique positive measure ν_{Pl} (called the Plancherel measure of G) on $\hat{G}$ such that:*

$$(2.1) \qquad L^2(G, \mu_G) \simeq \int_{\pi \in \hat{G}}^{\oplus} H(\pi) d\nu_{Pl}(\pi)$$

where $H(\pi) := V_\pi^* \hat{\otimes} V_\pi$ denotes the algebra of Hilbert-Schmidt operators on V_π.

Much of study of harmonic analysis on reductive groups is devoted to making the abstract Plancherel formula in this context explicit. This is a problem with many different facets, some of which are poorly understood or even unsolved even after more than 70 years into the subject. One part of this is conceptual. The tremendous success of the approach of Langlands towards harmonic analysis on reductive groups points out that number theory and algebraic geometry are inherent parts of this endeavour. An explicit Plancherel formula has to reflect the deep number theoretical problems which are conspiring in the background. There are formidable technical obstacles as well, stemming from the fact that one is forced to deal with representation theory on rather general topological vector spaces even if one's goal is the study unitary representations.

Harish-Chandra made deep contributions to our understanding of the structure of the explicit Plancherel formula ([HC3], [Wal]). He discovered that support of the Plancherel measure is not all of $\hat{G}$, except if G happens to be built from anisotropic and commutative almost factors. In general, the support of the Plancherel measure is the set of so-called irreducible *tempered* representations of G. A connected component (Harish-Chandra series) of this set consists of the irreducible summands of the representations obtained by unitary parabolic induction of a discrete series representation of a Levi subgroup of G. There are countably many Harish-Chandra series.

Assuming the local Langlands correspondence for tempered representations, Hiraga, Ichino and Ikeda conjectured an explicit formula for the Plancherel measure ν_{Pl} of G. The appeal of these conjectures is that they formulate the answers in terms of a natural number theoretical invariant which is associated with an irreducible representation π in the conjectural local Langlands correspondence, the so-called adjoint gamma factor.

2.1. **The decomposition of the trace.** The regular representation $L^2(G, \mu_G)$ corresponds to the semi-finite positive trace $\tau_G(f) := f(e)$ on $C^*(G)$, and in particular $C^*(G)$ is a Hilbert algebra. The dense subalgebra $C_c^\infty(G) \subset C^*(G)$ has the special property that for all $\pi \in \hat{G}$ and $f \in C_c^\infty(G)$ the operator $\pi(f) \in \mathcal{B}(V_\pi)$ is of trace class. This defines a distribution Θ_π defined by:

$$(2.2) \qquad \Theta_\pi(f) := \mathrm{Tr}_{V_\pi}(\pi(f))$$

This is called Harish-Chandra's distributional character of π.

The positive trace τ_G is defined on the dense subalgebra $C_c^\infty(G) \subset C^*(G)$, and the Plancherel measure ν_{Pl} is completely determined by the decomposition of τ_G as a superposition of the distributional characters Θ_π with $\pi \in \hat{G}$:

Corollary 2.1.1. *The Plancherel measure ν_{Pl} is the unique positive measure on $\hat{G}$ such that for all $f \in C_c^\infty(G)$:*

$$(2.3) \qquad f(e) = \int_{\pi \in \hat{G}} \Theta_\pi(f) d\nu_{Pl}(\pi)$$

Some remarks are in order:

(1) The measure ν_{Pl} depends on the normalization of the Haar measure. If we replace μ_G by $a\mu_G$ (for $a > 0$) then ν_{Pl} is replaced by $a^{-1}\nu_{Pl}$.

(2) The definition of the distributional character Θ_π can be extended naturally to the class of admissible representations of G. An irreducible admissible representation (V_π, π) is tempered iff the distribution Θ_π is tempered, by which we mean that Θ_π extends continuously to the Harish-Chandra Schwartz algebra $\mathcal{C}(G) \supset C_c^\infty(G)$ of G. In turn this is equivalent to the requirement that for every standard parabolic subgroup $P \subset G$, the exponents $\chi \in \mathrm{Exp}(\pi_P)$ of the Jacquet module $(V_{\pi,P}, \pi_P)$ satisfy the Casselman conditions $\mathrm{Re}(\chi) \in \overline{{}^+\mathfrak{a}_P^{G,*}}$ (see [Wal]). Here ${}^+\mathfrak{a}_P^{G,*}$ denotes the closed convex cone inside the real span $\mathfrak{a}_M^{G,*}$ of the set of G-roots $\Sigma(A_M, G)$ of the connected split center A_M of the standard Levi-factor M of P spanned by the set of roots associated to the unipotent radical of P.

(3) By a deep result of Harish-Chandra, the support of $d\nu_{Pl}$ is contained in the set $\hat{G}^{\mathrm{temp}}$ of equivalence classes of tempered representations of G. This was explained more conceptually by Joseph Bernstein [Be1].

(4) We call an admissible irreducible representation (V_π, π) of G a discrete series representation if the matrix coefficients of π are in $L^2(G, \mu_G)$. Let $\hat{G}^{\mathrm{disc}} \subset \hat{G}$ denote the subset of equivalence classes of discrete series representations. It is well known by Casselman's results (see [Wal]) that $\pi \in \hat{G}^{\mathrm{disc}}$ iff $\mathrm{Re}(\chi) \in {}^+\mathfrak{a}_P^{G,*}$ for all standard parabolic subgroups $P \subset G$ and all $\chi \in \mathrm{Exp}(\pi_P)$. The set $\hat{G}^{\mathrm{disc}}$ is not empty iff the center $Z(G)$ is anisotropic. By a well known characterisation of Dixmier we have $\pi \in \hat{G}^{\mathrm{disc}}$ iff $\nu_{Pl}(\{\pi\}) > 0$.

Definition 2.1.2. *If $\pi \in \hat{G}^{\mathrm{disc}}$ then we define the formal degree of π as* $\mathrm{fdeg}(\pi) := \nu_{Pl}(\{\pi\}) > 0$.

(5) Let $A \subset G$ be the maximal split component of $Z(G)$. Then G has discrete series only if A is trivial. More generally, we call an irreducible representation π a

discrete series modulo the center if π is tempered and its matrix coefficients are $L^2(G/A)$. If π is a discrete series representation modulo the center then one can show that there exists a constant $\mathrm{fdeg}(\pi) > 0$ such that for all $v, w \in V_\pi$:

$$(2.4) \qquad \int_{G/A} |(\pi(g)v, w)|^2 \, d\mu_G(g) = \mathrm{fdeg}(\pi)^{-1} \|v\|^2 \|w\|^2$$

This generalizes the notion of formal degree for a discrete series representation as in (3).

2.2. **Normalization of Haar measure.** As we have seen above, we need to fix the Haar measure μ_G in order to fix ν_{Pl} uniquely. The Haar measure depends on the choice of an additive character ψ of F. The construction is explained in [GG] (also see [G, Section 4], and the discussion in [HIIcor] of the differences between these two constructions). The corresponding Haar measure will be denoted by μ_G^ψ if we need to stress the dependence on the choice of ψ.

Lemma 2.2.1. (a) *Suppose that F is non-archimedean and that $\mathbf{G}$ is split over an unramified extension. Let ψ_0 be an additive character with conductor $\mathfrak{p} \subset \mathfrak{o}$. Let q be the cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$. Then for any parahoric subgroup $\mathbb{P} \subset G$ with reductive quotient $\overline{\mathbb{P}}$ we have*

$$(2.5) \qquad \mathrm{Vol}(\mathbb{P}, \mu_G^{\psi_0}) = q^{-\dim(\overline{\mathbb{P}})/2} |\overline{\mathbb{P}}|$$

(This is the normalization of Haar measure used in [DeRe].)

(b) *Suppose that $F = \mathbb{R}$, and that $\psi_0(x) = \exp(2\pi\sqrt{-1}x)$. Assume that G has discrete series representations. By Harish-Chandra's criterion this is equivalent to the existence of an anisotropic maximal torus $T \subset G$ of G. Let μ_G^0 be the Haar measure on G defined by the volume form on $\mathfrak{g} = \mathrm{Lie}(G)$ corresponding to invariant norm $\|x\|^2 = -B(x, \theta(x))$, where $x \in \mathfrak{g}$, θ denotes the Cartan involution, and B a nondegenerate bilinear form on $\mathfrak{g}$ as in [HC2]. Let μ_T^0 be the Haar measure on T defined similarly. We denote by $\Sigma^\vee$ the root system of $\mathfrak{g}^\vee := \mathrm{Lie}(G^\vee)$, with $|\Sigma^\vee| = 2N$ and $\dim(T) = l$. Then:*

$$(2.6) \qquad \mu_G^{\psi_0} = 2^N (2\pi)^l \mathrm{Vol}(T, \mu_T^0)^{-1} \prod_{\alpha^\vee \in \Sigma_+^\vee} (\alpha^\vee, \alpha^\vee) \mu_G^0$$

Proof. Part (a) follows from comparing [HII, (1.1) and further] with [DeRe]. Part (b) is an easy consequence of the computation in [HII, §2]. $\qquad \square$

In the sequel we will use $\mu_G := \mu_G^{\psi_0}$ as the standard normalization of the Haar measure on G.

2.3. **Local Langlands parameters.** Let $\Gamma := \mathrm{Gal}(\overline{F}/F)$ denote the absolute Galois group of F. Choose a Borel subgroup B and maximal torus $T \subset B$ of $G' := \mathbf{G}_{\overline{F}}$, and let $\beta(G) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$ be the corresponding based root datum. Choose a pinning $(G', B, T, \{x_\alpha\}_{\alpha \in \Delta})$, which induces a splitting of the exact sequence:

$$(2.7) \qquad 1 \to \mathrm{Int}(G') \to \mathrm{Aut}(G') \to \mathrm{Aut}(\beta(G')) \to 1$$

The action of Γ on $G(\overline{F})$ induces an action of Γ on $\beta(G')$, and via the above splitting this gives rise to an (algebraic) action a of Γ on G'. There is a unique split F-structure

on G' which fixes the chosen splitting, and clearly this commutes with a. Therefore the composition of these two actions defines a quasi-split F-structure $\mathbf{G}^*$ on G'. We denote by $G^* = \mathbf{G}^*(F)$ the corresponding group of points.

Let $G^\vee$ be the connected complex reductive group with $\beta(G^\vee) := \beta(G')^\vee$. Choose a pinning $(G^\vee, B^\vee, T^\vee, \{y_{\alpha^\vee}\}_{\alpha^\vee \in \Delta^\vee})$ of $G^\vee$. The action of Γ on $\beta(G^\vee)$ induced by the action on $\beta(G')$ gives rise to an (algebraic) action of Γ on $G^\vee$. We define $^L G := G^\vee \rtimes \Gamma$, the Langlands dual group of $\mathbf{G}$.

The Langlands group L_F of F is defined as follows:

(2.8)
$$L_F = \begin{cases} W_F \text{ if } F \text{ is archimedean,} \\ W_F \times \mathrm{SL}_2(\mathbb{C}) \text{ else.} \end{cases}$$

Here W_F denotes the Weil group of F (see [Tate]). A Langlands parameter is a homomorphism $\varphi : L_F \to {^L G}$ such that

(1) φ is continuous.
(2) In the non-archimedean case, $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.
(3) $\mathrm{pr}_2 \circ \varphi|_{W_F} \to \Gamma$ is the canonical homomorphism $W_F \to \Gamma$.
(4) $\varphi(W_F)$ is semisimple.
(5) If $\mathrm{Im}(\varphi)$ is contained in the Levi-subgroup of a parabolic subgroup P of $^L G$ then P is G-relevant (in the sense of [Bo, §8]).

Definition 2.3.1. *We call φ discrete if $C_{G^\vee}(\varphi)$ is finite. We call φ essentially discrete if $\mathrm{Im}(\varphi)$ is not contained in the Levi subgroup of a proper relevant parabolic subgroup of $^L G$.*

Lemma 2.3.2. *Discrete parameters exist iff the connected center of G is anisotropic and F is non-archimedean, or else if G admits an anisotropic maximal F-torus. In this situation φ is discrete iff φ is essentially discrete.*

Proof. It is easy to see that the connected center of G is F-anisotropic if and only if the center $^L Z := Z(G^\vee)^\Gamma$ is finite. The group $C := C_{G^\vee}(\varphi) \subset G^\vee$ is reductive since $\varphi(W_F)$ is semisimple and $\varphi(\mathrm{SL}_2(\mathbb{C}))$ is reductive. Hence C is finite iff C does not contain a nontrivial torus. By the above remark this can happen only if the connected center of G is anisotropic. In this case [Bo, Proposition 3.5, 3.6] implies that C does not contain a nontrivial torus iff $\mathrm{Im}(\varphi)$ is not contained in any proper Levi subgroup of $^L G$. By definition this is equivalent to saying that $\mathrm{Im}(\varphi)$ is not contained in a Levi subgroup of $^L G$ of any proper relevant parabolic subgroup of $^L G$. In the non-archimedean case we can define a discrete character φ_0 which is trivial on W_F and corresponds to the regular unipotent orbit on $\mathrm{SL}_2(\mathbb{C})$ (the principal parameter). In the case $F = \mathbb{C}$ there are no discrete parameters. If $F = \mathbb{R}$ and φ is discrete then $\varphi(\mathbb{C}^\times)$ (with $\mathbb{C}^\times = W_\mathbb{C} \subset W_\mathbb{R}$) must contain regular semsimple elements. Thus $\varphi(\mathbb{C}^\times)$ is contained in a unique maximal torus $T^\vee$ of $G^\vee$ which must be θ-stable (with θ the automorphism corresponding to the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$). Since $\theta\varphi(z)\theta^{-1} = \varphi(\bar{z})$, the discreteness of φ implies that $\varphi(\bar{z}) = \varphi(z)^{-1}$, and thus that θ restricted to $T^\vee$ is sending t to t^{-1}. This implies that G has an anisotropic F-torus (this is clear if $G = G^*$ is quasi-split, and it is well known that this condition is independent of the inner form [S, Corollary 2.9]). Conversely, when this condition holds it is easy to write down discrete parameters. $\square$

Definition 2.3.3. *We call a Langlands parameter φ tempered if $\varphi(W_F)$ is bounded.*

It is not difficult to see that φ is tempered if φ is discrete.

Definition 2.3.4. *Two Langlands parameters φ, φ' are called equivalent iff they are in the same orbit of $\mathrm{Int}(G^\vee)$ (acting on ${}^L G$). The set of equivalence classes of Langlands parameters of G is denoted by $\Phi(G)$ and the subset of equivalence classes of tempered Langlands parameters of G is denoted by $\Phi^{\mathrm{temp}}(G)$.*

2.4. L-functions and ϵ factors.

We associate an L-function and an ϵ-factor to a representation V of L_F in the usual way (see [Tate]). The L-function $L(s, V)$ is a meromorphic function of a parameter $s \in \mathbb{C}$ which only depends on the semisimplification of V, and satisfies by inductivity (i.e. if $F' \subset F$ then $L(s, V) = L(s, \mathrm{Ind}_{L_{F'}}^{L_F}(V))$) and additivity. It is known these properties determine $L(s, V)$ completely if $L(s, \chi)$ is known for all characters χ of L_F.

In the archimedean case the L-functions assigned to characters are as follows:

(a) If $F = \mathbb{R}$, $L_F^{ab} \simeq \mathbb{R}^\times$. A character has a unique representation of form $\chi(x) = x^{-n}|x|^{s_0}$ with $n \in \{0, 1\}$; then $L(s, \chi) = \pi^{-(s+s_0)/2}\Gamma((s + s_0)/2)$.

(b) If $F = \mathbb{C}$, $L_F^{ab} \simeq \mathbb{C}^\times$. A character has a unique representation of form $\chi(z) = \sigma(z)^{-n}\|z\|^{s_0}$ with $n \in \mathbb{Z}_{\geq 0}$ and $\sigma \in \Gamma(\mathbb{C}/\mathbb{R})$; then $L(s, \chi) = 2(2\pi)^{-(s+s_0)}\Gamma(s + s_0)$.

For the non-archimedean case, let Fr denote the Frobenius automorphism of the maximal unramified extension F_{ur} of F. We choose once and for all an extension of Fr to $\overline{F}$, defining an element of W_F which we will also denote by $\mathrm{Fr} \in W_F$. Define

$$(2.9) \qquad \tilde{\mathrm{Fr}} = (\mathrm{Fr}, \begin{pmatrix} v^{-1} & 0 \\ 0 & v \end{pmatrix}) \in L_F, \text{ with } v = q^{1/2}$$

Now we define for a representation (V, φ) of L_F:

$$L(s, V) = \det(1 - q^{-s}\varphi(\tilde{\mathrm{Fr}})|_{V_N^{I_F}})^{-1}$$
$$= \prod_{n \geq 0} \det(1 - q^{-s-n/2}\varphi(\mathrm{Fr})|_{V_n^{I_F}})^{-1}$$

where $I_F \subset W_F$ denotes the inertia subgroup $I_F := \Gamma(\overline{F}/F_{ur})$, and $V_N^{I_F}$ the space of highest weight vectors in the $(W_F/I_F) \times \mathrm{SL}_2(\mathbb{C})$-module V^{I_F}. In the second line we decomposed V as $V \simeq \oplus_{n \geq 0} V_n \otimes \mathrm{Sym}^n(\mathbb{C}^2)$ for certain representations V_n of W_F.

The ϵ-factors depend on the choice of the additive character ψ of F. It is known that ϵ is also additive, and inductive for virtual representations of degree 0. In the non-archimedean case we have $\epsilon(s, V, \psi) := \omega(V, \psi)q^{a(V)(1/2-s)}$ where $a(V)$ is the Artin conductor of V [GR, Section 2]. Here $\omega(V, \psi) \in \mathbb{C}^\times$ is independent of s. In the archimedean case we have $\epsilon(s, \chi, \psi) = c_\psi.(\sqrt{-1})^n$ where χ is a character of W_F expressed as above (see the discussion of the L-functions in the archimedean case).

Given a Langlands parameter $\varphi : L_F \to {}^L G$, we define the adjoint γ factor of φ as follows. Let Ad denote the adjoint representation of ${}^L G$ on $\mathrm{Lie}(G^\vee)/\mathrm{Lie}({}^L Z)$.

$$(2.10) \qquad \gamma(s, \mathrm{Ad} \circ \varphi, \psi) := \frac{\epsilon(s, \mathrm{Ad} \circ \varphi, \psi)L(1 - s, \mathrm{Ad} \circ \varphi)}{L(s, \mathrm{Ad} \circ \varphi)}$$

It is not difficult to show that [HII, Lemma 1.2]:

Proposition 2.4.1. *If φ is tempered then $\gamma(s, \mathrm{Ad} \circ \varphi, \psi)$ is regular at $s = 0$. Moreover φ is tempered and essentially discrete iff $\gamma(s, \mathrm{Ad} \circ \varphi, \psi)$ is nonzero at $s = 0$.*

2.5. **A conjectural tempered local Langlands correspondence.** The conjecture on Plancherel densities of Hiraga, Ichino and Ikeda presupposes the existence of a local Langlands correspondence for tempered representations. A satisfactory formulation of a refined local Langlands conjecture in full detail (including certain desired properties of transfer factors) at this level of generality seems not to be known. We refer the reader to [Ar1], [Vog], [ABV], [Ar2], [HII], [HS] and [Kal] for more background, discussion and overview of known results supporting various forms of the conjecture.

In this section we would like to formulate a more crude version of the local Langlands conjecture for tempered representations covering the aspects which are relevant to our goals. We mainly follow [Ar2, Section 3], [HII, Section 1], [HS, Section 9].

Put $\Pi(G)$ for the set of admissible irreducible representations of G. The local Langlands conjecture predicts that there exists a partition

(2.11) $$\Pi(G) = \sqcup_{[\varphi]} \Pi_{\varphi}(G)$$

where the disjoint union is over the set of equivalence classes $[\varphi]$ of local Langlands parameters $\varphi : L_F \to {}^L G$. The sets $\Pi_{\varphi}(G)$ are called L-packets. Some of the fundamental expected properties of this conjectural partitioning are:

(i) Π_{φ} is a non-empty finite set.

(ii) Π_{φ} contains tempered characters iff φ is tempered. In this case all members of Π_{φ} are tempered.

(iii) Π_{φ} contains characters which are discrete modulo center iff φ is essentially discrete. In this case all members of Π_{φ} are discrete modulo center.

(iv) Suppose that F is p-adic. Then Π_{φ} contains a character which is generic and supercuspidal iff $\varphi|_{W_F}$ is discrete. In this case all members of Π_{φ} are supercuspidal.

Let us now look in more detail into the conjectural parameterisation of the L-packets Π_{φ} for φ tempered, following [HII].

Let A denote the (group of points of) the maximal split torus of $Z(G)$. Let $(G_{ad})^{\vee} = G_{sc}^{\vee}$ be the simply connected cover of the derived group of $G^{\vee}$. Let $G^{\vee,\natural}$ denote the dual group of G/A, and let $G_{ad}^{\vee} = G_{sc}^{\vee}/Z(G_{sc}^{\vee})$ be the dual group of the simply connected cover of the derived group of G.

We have homomorphisms $G_{sc}^{\vee} \xrightarrow{\beta} G_{ad}^{\vee} \xleftarrow{\alpha} G^{\vee,\natural}$. Given a tempered Langlands parameter $\varphi : L_F \to {}^L G$ for G, we define $S_{\varphi}^{\natural} := \{s \in G^{\vee,\natural} \mid \mathrm{Ad}(s) \circ \varphi = \varphi\}$ and $S_{\varphi} := \beta^{-1}\alpha(S_{\varphi}^{\natural})$. Next we define $\mathcal{S}_{\varphi}^{\natural} = \pi_0(S_{\varphi}^{\natural})$ and $\mathcal{S}_{\varphi} = \pi_0(S_{\varphi})$.

Recall that G is (the group of F-points of) an inner form the quasi-split F-group G^*, which defines a class in $H^1(F, \mathbf{G}_{ad}^*)$. Kottwitz constructed a canonical map $H^1(F, \mathbf{G}_{ad}^*) \to ({}^L Z_{sc})^*$ (with ${}^L Z_{sc} = Z(G_{sc}^{\vee})^{\Gamma}$) which is bijective in the p-adic case. Let $\chi_G \in ({}^L Z_{sc})^*$ be the character that corresponds to G. We have $\chi_{G^*} = \mathrm{triv}$. By [Ar2, §3], [HS, Lemma 9.1] we have:

Lemma 2.5.1. *The kernel $\mathrm{Ker}(\chi_G)$ contains ${}^L Z_{sc} \cap S_{\varphi}^0$, hence χ_G descends to $\mathrm{Im}({}^L Z_{sc} \to \mathcal{S}_{\varphi}) \subset \mathcal{S}_{\varphi}$ (also called χ_G).*

Following [Ar2], we choose an extension of χ_G to $\mathrm{Im}(Z_{sc} \to \mathcal{S}_\varphi)$ (also denoted by χ_G) such that $\chi_{G^*} = \mathrm{triv}$ and define: $\Pi(\mathcal{S}_\varphi, \chi_G) = \{\rho \in \mathrm{Irr}(\mathcal{S}_\varphi) \mid \rho|_{\mathrm{Im}(Z_{sc} \to \mathcal{S}_\varphi)} = n\chi_G\}$.

From [Ar2, Section 3], [HII] and [HS, Section 9] we distill the following crude form of the local Langlands conjecture:

Conjecture 2.5.2. *There exists a bijection $\rho \to \pi_\rho$ between $\Pi(\mathcal{S}_\varphi, \chi_G)$ and $\Pi_\varphi(G)$ such that for all tempered local Langlands parameters φ for G,*

$$(2.12) \qquad \Theta_\varphi := \sum_{\rho \in \Pi(\mathcal{S}_\varphi, \chi_G)} \dim(\rho)\Theta_{\pi_\rho}$$

is a stable character of G. Here Θ_{π_ρ} denotes the distributional character of G corresponding to the tempered irreducible representation π_ρ. Any stable linear combination of characters from $\Pi(\mathcal{S}_\varphi, \chi_G)$ is a multiple of Θ_φ.

Definition 2.5.3. *We define $\tilde{\Phi}^{\mathrm{temp}}(G) = \{(\varphi, \rho) \mid \varphi \in \Phi^{\mathrm{temp}}(G) \text{ and } \rho \in \Pi(\mathcal{S}_\varphi, \chi_G)\}$. Suppose that for all $\varphi \in \Phi^{\mathrm{temp}}(G)$ a parameterisation $\rho \to \pi_\rho$ of $\Pi_\varphi(G)$ as in Conjecture 2.5.2 exists. The corresponding bijection $\hat{G}^{\mathrm{temp}} \to \tilde{\Phi}^{\mathrm{temp}}(G)$, $\pi \to (\varphi_\pi, \rho_\pi)$ such that for each $\varphi \in \Phi^{\mathrm{temp}}(G)$, $\Pi_\varphi(G) = \{\pi \in \tilde{\Phi}^{\mathrm{temp}}(G) \mid \varphi_\pi = \varphi\}$ and such that the bijection $\Pi_\varphi(G) \to \Pi(\mathcal{S}_\varphi, \chi_G)$, $\pi \to \rho_\pi$ is the inverse of the bijection $\rho \to \pi_\rho$ in Conjecture 2.5.2, is called an enhancement of the Langlands parameterisation $\pi \to \varphi_\pi$.*

2.6. **The conjectures of Hiraga, Ichino and Ikeda.** We now have everything in place in order to formulate the conjectures of Hiraga, Ichino and Ikeda [HII]. Suppose that we have given an enhanced Langlands parameterisation $\hat{G}^{\mathrm{temp}} \to \tilde{\Phi}^{\mathrm{temp}}(G)$.

Conjecture 2.6.1 (Conjecture 1.4 of [HII])**.** *Let $\varphi : L_F \to {}^L G$ be a discrete Langlands parameter for G, let $\rho \in \Pi(\mathcal{S}_\varphi, \chi_G)$ and let $\pi_\rho \in \Pi_\varphi$ be the tempered essentially discrete series representation corresponding to (φ, ρ). Then*

$$(2.13) \qquad \mathrm{fdeg}(\pi_\rho) = \frac{\dim(\rho)}{|\mathcal{S}_\varphi^\natural|} |\gamma(0, \mathrm{Ad} \circ \varphi, \psi)|$$

For general tempered representations [HII] formulate a conjecture expressing the Plancherel density. This amplification is based on Harish-Chandra's Plancherel Theorem ([HC3], [Wal]) and Langlands' conjecture on the Plancherel measure [L, Appendix II], [Sha].

Conjecture 2.6.2 (Conjecture 1.5 of [HII])**.** *Let $P = MN \subset G$ be a semi-standard F parabolic subgroup. Let $\mathcal{O}$ be an orbit of tempered essentially discrete series characters of M. Let $d\pi$ denote the Haar measure on $\mathcal{O}$, normalised as in [Wal], pages 239 and 302]. For $\pi \in \mathcal{O}$ we put*

$$(2.14) \qquad d\nu(\pi) = \frac{\dim(\rho)}{|\mathcal{S}_{\varphi_M}^\natural|} |\gamma(0, r_M \circ \varphi, \psi)| d\pi$$

where r_M denotes the adjoint representation of ${}^L M$ on $\mathrm{Lie}(G^\vee)/\mathrm{Lie}({}^L Z_M)$. Then the Plancherel density at $\mathrm{Ind}_P^G(\pi)$ is $c_M d\nu(\pi)$ for some explicit constants $c_M \in \mathbb{R}_+$ independent of F and $\mathcal{O}$.

2.7. **Known results and further comments.** Conjecture 2.6.1 is reduced to the case of generic tempered representations by Shahidi's paper [Sha], if one knows the stability of Θ_φ in Conjecture 2.5.2.

The Conjecture 2.6.1 is known for $F = \mathbb{R}$ [HII, Section 3]. For F non-archimedean Conjecture 2.6.1 is known in the following cases:

(a) G an inner form of GL_n ([SZ], [Z], [HII]).
(b) G an inner form of SL_n ([HS], [HII]).
(c) G arbitrary, π the Steinberg representation, φ the principal parameter (due essentially to Borel, [Bo1], [HII]).
(d) G split exceptional of adjoint type, π discrete series of unipotent reduction (due to Reeder, [R3]).
(e) G arbitrary, π depth 0 supercuspidal representation (tame regular semisimple case) (DeBacker and Reeder [DeRe], [HII]).
(f) Ichino, Lapid and Mao proved Conjecture 2.6.1 for odd orthogonal groups.
(g) Beuzart-Plessis proved Conjecture 2.6.1 for unitary groups.
(h) For supercuspidal representations of unipotent reduction of connected semisimple p-adic groups which split over an unramified extension [FOS, Theorem 1.3].

The main result we will discuss in these lectures is and extension to general connected reductive G of the following result: [1]

Theorem 2.7.1 ([R3], [Opd4], [Opd5], [Fe2]). *Let G be absolutely almost simple of adjoint type over a non-archimedean field F such that G splits over an unramified extension of F. Then Conjectures 2.6.1, 2.6.2 hold for representations of unipotent reduction, when we use Lusztig's classification [Lus4], [Lus5] as a Langlands parameterisation.*

The proof of Theorem 2.7.1 and its extension is based on two techniques for affine Hecke algebras:

(1) Spectral transfer maps between Hecke algebras [Opd4], [Opd5], [FO], [FOS]. We use these tools to deal with the q-rational factors of the formal degree.
(2) Dirac induction for affine Hecke algebras [COT], [CO]. This tool is useful to determine the precise rational constant factors of the formal degree.

In fact, by the theory of types and Theorem 4.5.1 I expect that these techniques may reduce the general case of Conjectures 2.6.1 and 2.6.2 to the case of generic supercuspidal representations.

The "converse results" Theorem 4.2.3, Theorem 4.5.1 are interesting in their own right, and are closely related to the theory of spectral transfer maps between normalised affine Hecke algebras [Opd5].

3. THE PLANCHEREL FORMULA FOR AFFINE HECKE ALGEBRAS

Let F be a nonarchimedean local field with residue field of cardinality q from here onwards.

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[1]The Lusztig parameterisation should be twisted by the Iwahori-Matsumoto involution in order to map tempered representations to bounded parameters (cf. [AMS, Text below Theorem 2]). Here and elsewhere we will tacitly assume this modification.

3.1. **The Bernstein center.** Let $\mathcal{C}(G)$ be the abelian category of smooth representations of G, and let $\Pi(G)$ denote the space of classes of irreducible objects of $\mathcal{C}(G)$. Let $\mathcal{B}(G)$ be the set of inertial equivalence classes of cuspidal pairs (M, τ) (with M a Levi subgroup, and τ a supercuspidal representation of M). For $s \in \mathcal{B}(G)$, let Ω_s be the corresponding set of cuspidal pairs in the class s, modulo G-conjugacy (an affine variety). We have a central character map

$$\text{(3.1)} \qquad \text{cc} : \Pi(G) \to \sqcup_{s \in \mathcal{B}(G)} \Omega_s$$

such that $\text{cc}(\pi) = (M, \tau)$ if π is a subquotient of $i_P^G(\tau)$, where $P = MU$ is a parabolic subgroup with Levi factor M. Let $\mathcal{O}(\Omega_s)$ be the ring of regular functions on Ω_s, and put $\mathfrak{z}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{O}(\Omega_s)$. We put $Z_{\mathcal{B}}(G) = \text{End}(\text{Id}_{\mathcal{C}(G)})$, the Bernstein center. As is well known, we can interpret $Z_{\mathcal{B}}(G)$ as the set of G-invariant distributions z on G such that $z\mathcal{H}(G) \subset \mathcal{H}(G)$ and $\mathcal{H}(G)z \subset \mathcal{H}(G)$, where $\mathcal{H}(G)$ denotes the Hecke algebra of G. The famous theorem of Bernstein and Deligne states:

Theorem 3.1.1 ([BeDe], Theorem 2.13). *There is a unique algebra isomorphism (the Fourier transform)*

$$\text{(3.2)} \qquad Z_{\mathcal{B}}(G) \to \mathfrak{z}(G)$$

$$\text{(3.3)} \qquad z \to \hat{z}$$

characterised by the property that for all $\pi \in \Pi(G)$, one has $z_\pi = \hat{z}(\text{cc}(\pi))\text{Id}_{V_\pi}$.

As an immediate consequence one obtains the Bernstein decomposition of $\mathcal{C}(G)$:

Corollary 3.1.2. *We have a family of orthogonal idempotents $e_s \in Z_{\mathcal{B}}(G)$ (with $s \in \mathcal{B}(G)$) such that $\widehat{e_s}$ is the characteristic function of Ω_s. We have corresponding decompositions*

$$\text{(3.4)} \qquad \mathcal{C}(G) = \prod_{s \in \mathcal{B}(G)} \mathcal{C}(G)_s, \quad \Pi(G) = \sqcup_{s \in \mathcal{B}(G)} \Pi(G)_s$$

where $\Pi(G)_s$ is the set of irreducible objects of $\mathcal{C}(G)_s$.

3.2. **Types, Hecke algebras and Plancherel measure.** A *type* is a pair $t = (J, \rho)$ such that:

(1) $J \subset G$ is a compact open subgroup of G.
(2) ρ is a finite dimensional irreducible representation of J.
(3) Let $e_t \in \mathcal{H}(G)$ be the idempotent given by

$$\text{(3.5)} \qquad e_t(x) = \begin{cases} \frac{\text{Tr}(\rho(x^{-1}))}{\text{Vol}(J)} & \text{if } x \in J \\ 0 & \text{else.} \end{cases}$$

Let $\mathcal{C}^t(G)$ be the full subcategory of $\mathcal{C}(G)$ consisting of representations (π, V_π) such that $\mathcal{H}(G)(e_t V_\pi) = V_\pi$, and let $\mathcal{H}_t = e_t \mathcal{H}(G) e_t$. Then the functor $m_t : \mathcal{C}^t(G) \to \mathcal{H}_t - \text{mod}$ given by $V_\pi \to e_t(V_\pi)$ is an equivalence of categories.

Theorem 3.2.1 (Bushnell and Kutzko [BK]). *Let t be a type. Then $\mathcal{C}^t(G) = \prod_{s \in \mathcal{B}^t(G)} \mathcal{C}(G)_s$ where $\mathcal{B}^t(G)$ is a finite set.*

As a consequence of Theorem 3.1.1 we see:

Corollary 3.2.2. *Let t be a type, and let $Z(\mathcal{H}_t)$ denote the center of $\mathcal{H}_t$. There exists a unique isomorphism:*

$$\beta^t : \Omega^t := \sqcup_{s \in \mathcal{B}^t(G)} \Omega_s \to \operatorname{Spec}(Z(\mathcal{H}_t)) \tag{3.6}$$

such that for all $\pi \in \Pi(G)^t := \sqcup_{s \in \mathcal{B}^t(G)} \Pi(G)_s$ we have: $\operatorname{cc}^t(m_t(\pi)) = \beta^t(cc(\pi))$. Here cc^t denotes the central character map of the algebra $\mathcal{H}_t$.

Theorem 3.2.3 (Yu, J-L Kim)**.** *If p is sufficiently large then for all $s \in \mathcal{B}(G)$ we can find a type t such that $\mathcal{C}^t(G) = \mathcal{C}(G)_s$.*

Example 3.2.4.   (1) The archetypical example is that of the Borel component: Let $\mathbb{B} \subset G$ be the Iwahori subgroup, then $(\mathbb{B}, \operatorname{triv})$ is a type. More generally, if $\mathbb{B}_m$ is the m-th filtration subgroup of $\mathbb{B}$ in the Moy-Prasad filtration, then $(\mathbb{B}_m, \operatorname{triv})$ is a type.
   (2) (Moy-Prasad, Morris, Lusztig) Let $t = (\mathbb{P}, \sigma)$ with $\mathbb{P} \subset G$ a parahoric subgroup, and σ a cuspidal unipotent representation of the reductive quotient $\overline{\mathbb{P}}(\mathbb{F}_q)$. We refer to such t as a "unipotent type", and to the objects in the associated categories $\mathcal{C}^t(G)$ as "representations of unipotent reduction".
   (3) Let $x \in B(G)$ be a point in the building of G, and let $r \geq 0$. Let $G_{x,r,+}$ denote the corresponding Moy-Prasad subgroups. Then $(G_{x,r,+}, \operatorname{triv})$ is a type (Bestvina-Savin).

The Hecke algebra $\mathcal{H}_t = e_t \mathcal{H}(G) e_t$ of a type t inherits a $*$ (an anti-linear anti-involution) and trace τ from $\mathcal{H}(G)$, defined by

   (1) $f^*(g) := \overline{f(g^{-1})}$ for $f \in \mathcal{H}_t$.
   (2) $\tau(f) := f(1)$ (observe that for the unit $e_t \in \mathcal{H}_t$ we have $\tau(e_t) = \frac{\dim(\rho)}{\operatorname{Vol}(J)}$).

The Hermitian form $(x,y) := \tau(x^*y)$ is positive definite, and defines a Hilbert space completion $L^2(\mathcal{H}_t)$ of $\mathcal{H}_t$. This turns $\mathcal{H}_t$ into a Hilbert algebra with trace τ. It is well known that the irreducible representations of $\mathcal{H}_t$ are finite dimensional, hence this Hilbert algebra has a type I C^*-algebra envelop. As a consequence of Dixmier's central decomposition theorem for Type I C^*-algebras we conclude:

Corollary 3.2.5. *Let $\hat{\mathcal{H}}_t = \{[\pi] \mid \pi \text{ is an irrducible } *\text{-unitary } \mathcal{H}_t\text{-mod}\}$. There exists a unique positive measure $\nu_{\mathcal{H}_t}$ on $\hat{\mathcal{H}}_t$ such that*

$$\tau = \int_{\pi \in \hat{\mathcal{H}}_t} \chi_\pi d\nu_{\mathcal{H}_t}(\pi) \tag{3.7}$$

The support $\hat{\mathcal{H}}_t^{\operatorname{temp}} := \operatorname{Supp}(\nu_{\mathcal{H}_t}) \subset \hat{\mathcal{H}}_t$ is called the tempered dual of $\mathcal{H}_t$.

Theorem 3.2.6 ([BKH])**.** *The functor m_t defines a homeomorphism $\hat{m}_t^{\operatorname{temp}} : \hat{G}^{t,\operatorname{temp}} := \Pi(G)^t \cap \hat{G}^{\operatorname{temp}} \to \hat{\mathcal{H}}_t^{\operatorname{temp}}$ such that $(\hat{m}_t^{\operatorname{temp}})_*(\nu_{Pl}|_{\hat{G}^{t,\operatorname{temp}}}) = \nu_{\mathcal{H}_t}$.*

Therefore we can compute ν_{Pl} by computing $\nu_{\mathcal{H}_t}$ for the Hecke algebras $\mathcal{H}_t$ of a collection of types t such that the open closed sets $\hat{G}^{t,\operatorname{temp}}$ cover $\hat{G}^{\operatorname{temp}}$. In this sense the measures $\nu_{\mathcal{H}_t}$ are the building blocks of the Plancherel measure of G.

3.3. **Affine Hecke algebras as Hilbert algebras.** The algebras $\mathcal{H}_t$ are slight gener-
alisations of affine Hecke algebras. In Lusztig's case of unipotent types ([Lus4], [Lus5];
also see [Mo], [MP1], [MP2]) the associated "unipotent Hecke algebras" are precisely
(extended) affine Hecke algebras.

Let us therefore review the theory of affine Hecke algebras and the spectral decompo-
sition of the corresponding Hilbert algebra.

Let $W = W^a \rtimes \Omega$ be an extended affine Weyl group. By this we mean that we
have an affine Coxeter group W^a with set of simple reflections S say, and a group of
special automorphisms $\Omega \subset \mathrm{Out}_S(W^a, S)$. (A diagram automorphism $\omega \in \mathrm{Out}_S(W^a, S)$
is called special if its restriction to the canonical normal subgroup $Q \subset W^a$ consisting
of the elements with a finite conjugacy class, equals the restriction to Q of an inner
automorphism of W^a.)

Let $\Lambda = \mathbb{Z}[v_s^{\pm 1} \mid s \in S; \ v_s = v_{s'} \text{ if } s \sim_W s']$ (the v_s are commuting indeterminates),
and put $\mathtt{Q} = \{\mathtt{v} \in \mathrm{Hom}_{alg}(\Lambda, \mathbb{C}^\times) \mid \mathtt{v}_s = \mathtt{v}(v_s) \in \mathbb{R}_+ \forall s \in S\} \simeq \mathbb{R}_+^N$ where N denotes the
number of conjugacy classes of affine reflections in W. We have a length function l on
W^a relative to the set S of simple reflections, which we extend to W by giving elements
of Ω length 0.

Definition 3.3.1 (Coxeter presentation of the Hecke algebra)**.** *Let $\mathcal{H}_\Lambda$ be the free Λ-
algebra with basis $\{N_w\}_{w \in W}$ such that*

(i) *$N_u N_v = N_{uv}$ for $u, v \in W$ such that $l(uv) = l(u) + l(v)$.*
(ii) *For all $s \in S$ we have $(N_s - v_s)(N_s + v_s^{-1}) = 0$.*

Given $\mathtt{v} \in \mathtt{Q}$ we define $\mathcal{H}_{\mathtt{v}} = \mathcal{H}_\Lambda \otimes \mathbb{C}_{\mathtt{v}}$. Let $d \in \Lambda$ be positive on $\mathtt{Q}$ (or on some subset
of $\mathtt{Q}$ which contains $\mathtt{v}$). Then we define a Hilbert algebra structure on $\mathcal{H}_{\mathtt{v}}$ by defining a
$*$-operator and a *positive trace* τ as we did before with $\mathcal{H}(G)$:

(i) $\tau(N_w) = \delta_{w,e} d(\mathtt{v})$.
(ii) $N_w^* = N_{w^{-1}}$.

We recall that the positivity of τ means that the Hermitian form $(x, y) = \tau(x^* y)$ on $\mathcal{H}_{\mathtt{v}}$
is positive definite. This elementary fact is crucial in all that follows.

Next we would like to express the Hilbert algebra stucture in terms of the Bernstein
presentation of $\mathcal{H}_{\mathtt{v}}$. Let $X \subset W$ be the canonical normal subgroup of W consisting of
the elements which have finitely many conjugates. This is the translation subgroup of
W, and we can choose a splitting of $W/X \simeq W_0$ by choosing a special point $0 \in C$,
where $C \subset V = \mathbb{R} \otimes X$ denotes the alcove.

The length $l(w)$ of $w \in W$ can be interpreted more geometrically as the number of
affine reflection hyperplanes of W^a separating the fundamental alcove $C \subset V = \mathbb{R} \otimes Q$
and $w(C)$. In particular we have $l(x) = 2\rho(x)$ for $x \in X^+$, the dominant cone in X.
By the defining relations of $\mathcal{H}_\Lambda$ this implies that the elements N_x with $x \in X^+$ form a
commutative monoid of invertible elements of $\mathcal{H}_\Lambda$. Bernstein and Zelevinski turned this
into a very important alternative presentation of $\mathcal{H}_\Lambda$, (see [Lus3] for further background).

Lemma 3.3.2. *There exists a unique homomorphism $X \ni x \to \theta_x \in \mathcal{H}_\Lambda$ such that
for $x \in X^+$ one has $\theta_x = N_x$. Let $A \subset \mathcal{H}_\Lambda$ be the commutative subalgebra $\Lambda[\theta_x \mid x \in
X] \subset \mathcal{H}_\Lambda$ generated by the θ_x. Let $\mathcal{H}_0 = \mathcal{H}_\Lambda(W_0, S_0) \subset \mathcal{H}_\Lambda$ be the finite type Hecke
subalgebra associated to the isotropy group (W_0, S_0) of the chosen special point $0 \in C$.
Then the multiplication map $A \otimes \mathcal{H}_0 \to \mathcal{H}_\Lambda$ is an isomorphism of $(A, \mathcal{H}_0)$-bimodules,*

and $\mathcal{H}_0 \otimes A \to \mathcal{H}_\Lambda$ is an isomorphism of $(\mathcal{H}_0, A)$-bimodules, and the algebra structure of $\mathcal{H}_\Lambda$ is determined by the Bernstein relation:

$$(3.8) \qquad \theta_x N_s - N_s \theta_{s(x)} = \left((v_s - v_s^{-1}) + (v_{s'} - v_{s'}^{-1})\theta_{-\alpha} \right) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-2\alpha}}$$

where $s = s_{\alpha^\vee} \in S_0$ for some simple root α_0, and $s' \in S$ is such that $s' \sim_W s_{\alpha^\vee + 1}$.

Remark 3.3.3.   (a) Let $\mathcal{R} = (\Delta_0, X, \Delta_0^\vee, Y)$ the based root datum associated with $W = W_0 \ltimes X$, where S_0 corresponds to Δ_0. For $\alpha \in \Delta_0$ one can put $q_\alpha^+ = v_s v_{s'}$ and $q_\alpha^- = v_s / v_{s'}$. The parameters $q_\alpha^\pm$ will be more convenient than the v_s in the spectral theory of $\mathcal{H}_v$.

(b) If $v_{s'} \neq v_s$ then $\alpha^\vee \in 2Y$. In irreducible cases this happens only if $\mathcal{R}$ is of type $C_n^{(1)}$. That means that Δ_0 is the basis $\{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}$ of the irreducible root system of type B_n, and $X = \mathbb{Z}^n$ is the root lattice of this root system. The affine Dynkin diagram of W is the untwisted affine extension of the Dynkin diagram of $\Delta_0^\vee$ of type C_n. Note that $q_\alpha^- = 1$ unless we are in the $C_n^{(1)}$ case, and $\alpha = e_n$.

Corollary 3.3.4. The center $Z := Z(\mathcal{H}_\Lambda)$ is equal to A^{W_0}, which is naturally isomorphic to the ring $\Lambda[\theta_x \mid x \in X]^{W_0}$. Let T denote the algebraic torus with character lattice $\mathbb{Z}^{S/\sim} \times X$, viewed as a split torus over Λ via $v_s^n \to (ne_s, 0)$. Then $Z(\mathcal{H}_\Lambda) = \mathbb{C}[W_0 \backslash\backslash T]$.

Definition 3.3.5. We have $T = T_0 \times \mathrm{Spec}(\Lambda)$ where T_0 is algebraic torus over $\mathbb{C}$ with character lattice X. If $\mathbf{v} \in \mathcal{Q}$ then we will write $T^{\mathbf{v}}$ for the fibre of T above $\mathbf{v}$, i.e. the spectrum of $\mathbb{C}_{\mathbf{v}}[X]$ where $\mathbb{C}_{\mathbf{v}}$ denotes the residue field of Λ at $\mathbf{v}$.

3.4. **A formula for the trace of an affine Hecke algebra.** We will now write the trace τ of $\mathcal{H}_{\mathbf{v}}$ (for some $\mathbf{v} \in \mathcal{Q}$) in terms of the Bernstein presentation of $\mathcal{H}_{\mathbf{v}}$. First we introduce "intertwining elements" $R_s \in \mathcal{H}_\Lambda$ for every $s = s_\alpha \in S_0$ by:

$$(3.9) \qquad R_s = v_s \left((1 - \theta_{-2\alpha})N_s - ((v_s - v_s^{-1}) + (v_{s'} - v_{s'}^{-1})\theta_{-\alpha}) \right)$$

These elements satisfy:

(i) For all $x \in X$ and $s = s_\alpha \in S_0$ we have: $R_s \theta_x = \theta_{s(x)} R_s$.

(ii) $R_s^2 = v_s^2 \Delta_{2\alpha} \Delta_{-2\alpha} c_\alpha c_{-\alpha}$, where $\Delta_{\pm 2\alpha} = (1 - \theta_{\pm 2\alpha})$ and where the rational functions c_α are the famous *Harish Chandra c-functions* in the present context: $c_\alpha = (\Delta_{-2\alpha})^{-1}(1 + \theta_{-\alpha}/q_\alpha^-)(1 - \theta_{-\alpha}/q_\alpha^+)$.

Corollary 3.4.1. We have $\mathcal{H}_\Lambda \otimes_{\mathcal{Z}} Z' \simeq A' \# W_0$, where Z' is the localization of Z on the open subset of $W_0 \backslash T$ which is the intersection of the open subsets $U_\alpha \subset T$ ($\alpha \in \Sigma_0$) where c_α is an invertible regular function (the complement of the union of the hyperplanes of the form $\alpha(t) = \pm q_\alpha^\pm$ and $\alpha(t) = \pm 1$).

Definition 3.4.2. Define

$$(3.10) \qquad \mu(t) = \frac{d}{q(w_0)} \frac{1}{\prod_{\alpha \in \Sigma_0^+} c_\alpha(t) c_{-\alpha}(t)} = \frac{1}{q(w_0)} \frac{d}{c(t) c(t^{-1})},$$

the μ-function of $\mathcal{H}_\Lambda$ with normalising factor $d \in \Lambda$.

Theorem 3.4.3 ([Opd1]). *Let $t \in T^{\mathbf{v}}$ where $\mathbf{v}$ is such that $q_\alpha^\pm > 0$ for all α. Via the nondegenerate symmetric bilinear form $\langle x, y \rangle := \tau(xy)$ we view $\mathcal{H}_{\mathbf{v}}$ as a subspace of $\mathcal{H}_{\mathbf{v}}^*$, and equip $\mathcal{H}_{\mathbf{v}}^*$ with the weak topology. Then $\mathcal{E}_t := \sum_{x \in X} t(-x)\theta_x \in \mathcal{H}_{\mathbf{v}}^*$ is convergent in $\mathcal{H}_{\mathbf{v}}^*$ if for all $\alpha \in \Delta_0$, we have $|\alpha(t)| < \min\{(q_\alpha^-)^{-1}, (q_\alpha^+)^{-1}\}$. Moreover,*

$$(3.11) \qquad\qquad \mathcal{E}_t = \frac{E_t}{q(w_0)\Delta(t)}\mu(t)$$

where $T^{\mathbf{v}} \ni t \to E_t \in \mathcal{H}_{\mathbf{v}}^*$ is a certain regular family of matrix coefficients of minimal principal series at t such that $E_t(1) = q(w_0)\Delta(t)$ and such that for all $a, b \in A$, $E_t(ahb) = a(t)b(t)E_t(h)$.

Corollary 3.4.4 ([Opd1]). *We have the following disintegration of τ on $\mathcal{H}_{\mathbf{v}}$:*

$$(3.12) \qquad\qquad \tau = \int_{t_0 T_u^{\mathbf{v}}} \frac{E_t}{q(w_0)\Delta(t)}\mu(t)dt$$

where $T_u^{\mathbf{v}}$ denotes the compact form of $T^{\mathbf{v}}$, and $t_0 \in T_v^{\mathbf{v}}$ is a real base point such that the inequality $|\alpha(t_0)| < \min\{(q_\alpha^-)^{-1}, (q_\alpha^+)^{-1}\}$ holds.

Proof. Immediate from Theorem 3.4.3 by the Fourier inversion formula on $T_u^{\mathbf{v}}$. $\square$

3.5. **Spectral decomposition of τ.** The disintegration of τ given in Corollary 3.4.4 is not yet a spectral decomposition because the matrix coefficients E_t are neither tempered on $t_0 T_u^{\mathbf{v}}$, nor tracial (i.e. they do not vanish on commutators). To arrive at the spectral decomposition of τ several steps of refinement are necessary. The first step uses "residue distributions" for integrals in the form (3.12), and symmetrizing the resulting distributions on T over W_0. This leads to a decomposition of the form [Opd2]:

$$(3.13) \qquad\qquad \tau = \int_{W_0 t \in W_0 \backslash T^{\mathbf{v}}} \chi_{W_0 t} d\nu(W_0 t)$$

where

   (i) ν denotes the spectral measure of the decomposition of $\tau|_{Z_{\mathbf{v}}}$, where $Z_{\mathbf{v}} \subset \mathcal{H}_{\mathbf{v}}$ denotes the center. We remark that $Z_{\mathbf{v}}$ is invariant for $*$, and the restriction of $*$ and τ to $Z_{\mathbf{v}}$ equips it with the structure of a commutative Hilbert algebra.

  (ii) The support of ν is denoted by $W_0 \backslash T^{\mathbf{v},\mathrm{temp}}$. For each $W_0 t$ in $W_0 \backslash T^{\mathbf{v},\mathrm{temp}}$, $\chi_{W_0 t}$ is a tempered positive trace of $\mathcal{H}_{\mathbf{v}}$, with central character $W_0 t$.

 (iii) We have $T^{\mathbf{v},\mathrm{temp}} = \cup_{L \text{ residual coset}} L^{\mathrm{temp}}$, where a coset $L \subset T^{\mathbf{v}}$ of a subtorus is called *residual* if

$$(3.14) \qquad \#\{\alpha \in \Sigma_0 \mid \alpha|_L = \pm q_\alpha^\pm\} - \#\{\alpha \in \Sigma_0 \mid \alpha|_L = \pm 1\} = \mathrm{codim}(L)$$

Furthermore if $L \subset T$ a residual coset, we define its tempered part $L^{\mathrm{temp}} \subset L$ as follows. Let $\Sigma_L \subset \Sigma_0$ be the parabolic subsystem of the roots which are constant on L. Let $T^L \subset T^{\mathbf{v}}$ be the identity component of the simultaneous kernel of the $\alpha \in \Sigma_L$. Let $T_L \subset T^{\mathbf{v}}$ be the subtorus associated with the subspace $\mathbb{C}\Sigma_L^\vee \subset \mathfrak{t} = \mathrm{Lie}(T^{\mathbf{v}})$. If $r_L \in T_L \cap L$ then $L = r_L T^L$. Now put $L^{\mathrm{temp}} = r_L T_u^L$ (this does not depend on the choice $r_L \in T_L \cap L$).

The computation of the spectral decomposition of τ as trace on $\mathcal{H}^{\mathbf{v}}$ now reduces to the problem of computing the measure ν explicitly, and for each $W_0 t \in W_0 \backslash T^{\mathbf{v},\mathrm{temp}}$, decomposing $\chi_{W_0 t}$ as a (positive) superposition of irreducible tempered characters (with central character $W_0 t$).

In the special case of the discrete series of $\mathcal{H}^{\mathbf{v}}$ we see that these correspond to the W_0-orbits of *residual points*. This case is the basic building block for the spectral decomposition:

Theorem 3.5.1 ([Opd2]).   (i) *An orbit $W_0 r \subset W_0 \backslash T^{\mathbf{v}}$ is the central character of a discrete series representation π of $\mathcal{H}^{\mathbf{v}}$ if and only if r is a residual point (a residual coset of dimension 0).*

(ii) *If $W_0 r \subset T^{\mathbf{v}}$ is an orbit of residual points then $\nu(\{W_0 r\}) = c\mu^{\{r\}}(r) := c\frac{d(\mathbf{v})}{q(w_0)} m_r(\mathbf{v})$ (the residue of μ at $W_0 r$), where $c \in \mathbb{Q}^{\times}$ and where the regularisation $\mu^{\{r\}}$ of μ at r is defined by:*

$$(3.15) \qquad \mu^{\{r\}} = \frac{d(\mathbf{v})}{q(w_0)} \prod_{\alpha \in \Sigma_0}{}' \frac{(1 - \alpha^{-2})}{(1 + \alpha^{-1}/q_\alpha^-)(1 - \alpha^{-1}/q_\alpha^+)} = \frac{d(\mathbf{v})}{q(w_0)} m_r(\mathbf{v})$$

where the symbol $\prod'$ means that all irreducible factors of the numerator and the denominator which become identically 0 upon evaluation at $r \in T^{\mathbf{v}}$ are omitted.

(iii) *We have $\chi_{W_0 r} = \sum_{\delta \ ds, \mathrm{cc}(\delta) = W_0 r} d_{\mathcal{H},\delta}(\mathbf{v}) \chi_\delta$ where $d_{\mathcal{H},\delta}(\mathbf{v}) > 0$.*

(iv) *(Scaling invariance.) Define $\mathbf{v}(\epsilon)$ by $\mathbf{v}_s(\epsilon) = \mathbf{v}_s^\epsilon$ for $\epsilon \in \mathbb{R}_+$. Every orbit of residual points $W_0 r \in T^{\mathbf{v}}$ has a unique extension to a real analytic ϵ-family of orbits of residual points $W_0 \tilde{r}$ such that $W_0 \tilde{r}(\mathbf{v}(1)) = W_0 r$. A discrete series character δ with $\mathrm{cc}(\delta) = W_0 r$ has a unique extension to a continuous ϵ-family of discrete series characters $\tilde{\delta}$ of $\mathcal{H}_{\mathbf{v}(\epsilon)}$, and we have $\mathrm{cc}(\tilde{\delta}(\epsilon)) = W_0 \tilde{r}(\mathbf{v}(\epsilon))$ for all $\epsilon > 0$. This yields for all $\epsilon > 0$ a canonical bijection between $\{\delta \ ds \ of \ \mathcal{H}_{\mathbf{v}} \mid \mathrm{cc}(\delta) = W_0 r\}$ and $\{\delta' \ ds \ of \ \mathcal{H}_{\mathbf{v}(\epsilon)} \mid \mathrm{cc}(\delta') = W_0 r(\epsilon)\}$. Then $d_{\mathcal{H},\tilde{\delta}(\epsilon)}(\mathbf{v}(\epsilon))$ is independent of $\epsilon > 0$.*

(v) *If δ is a discrete series representation of $\mathcal{H}_{\mathbf{v}}$ then $\mathrm{fdeg}(\delta) = d_{\mathcal{H},\delta} \frac{d(\mathbf{v})}{q(w_0)} |m_r(\mathbf{v})|$ where $d_{\mathcal{H},\delta} \in \mathbb{R}_+$ as defined in (iii) and m_r as defined in (ii) (we will see below that in fact $d_{\mathcal{H},\delta} \in \mathbb{Q}_+$).*

The non-discrete contributions to the spectral decomposition of τ can be obtained from the discrete summend of the spectral decomposition of the corresponding traces of "parabolic subalgebras" by a process of unitary parabolic induction, analogous to Harish-Chandra's theory of the Plancherel decomposition for reductive groups. More precisely we have [Opd2]:

Theorem 3.5.2. *Let $L = r_L T^L \subset T^{\mathbf{v}}$ be a residual coset, such that $\Sigma_L \subset \Sigma_0$ is a standard parabolic subsystem. Let $\mathcal{H}^{P(L)} \subset \mathcal{H}$ be the subalgebra corresponding to the based root datum $P(L) := (\Delta_L, X, \Delta_L^\vee, Y)$. Let X_L be the character lattice of T_L, and $Y_L \subset Y$ its dual. Let $\mathcal{H}_{P(L)}$ be the Hecke algebra with the semisimple based root datum $P_{ss}(L) = (\Delta_L, X_L, \Delta_L^\vee, Y_L)$ and Hecke parameters $q_\alpha^\pm$ obtained by restriction from Δ_0 to Δ_L. Given $t^L \in T^L$ there exists a homomorphism $\phi_{t^L} : \mathcal{H}^{P(L)} \to \mathcal{H}_{P(L)}$ defined by (in the Bernstein presentation) $N_w \to N_w$ for all $w \in W_L$, and $\theta_x \to x(t^L)\theta_{\mathrm{pr}(x)}$ where $\mathrm{pr} : X \to X_L$ is the canonical projection.*

(i) *Let $t = r_L t^L \in L^{\text{temp}}$ be a generic point, i.e. c_α defines a regular and invertible germ at t for all $\alpha \in \Sigma_0 \backslash \Sigma_L$. Then in (3.13) we have:*

$$(3.16) \qquad |W_0/W_L| \chi_{W_0 t} = \sum_{\delta \in \hat{\mathcal{H}}_{P(L),ds}, \text{cc}(\delta) = W_0 r_L} \text{Ind}_{\mathcal{H}^L}^{\mathcal{H}} (\chi_{L, W_L r_L} \circ \phi_{t^L})$$

(ii) *We have $\nu = \sum_{L \ residual \ coset} \nu_L$ where ν_L is the push forward of a measure on L^{temp} given by $d\nu_L(t) = \mu^L(t) dt = c_L \mu_{\mathcal{H}_L}^{\{r_L\}}(r_L) m^L(t) dt^L$ with $c_L \in \mathbb{Q}_+$ and*

$$(3.17) \qquad m^L(t) = \frac{1}{q(w^L)} \prod_{\alpha \in \Sigma_+^L := \Sigma_{0,+} \backslash \Sigma_{L,+}} \frac{1}{c_\alpha(t) c_\alpha(t^{-1})}$$

Corollary 3.5.3. *The explicit spectral decomposition of the trace τ on $\mathcal{H}_{\mathbf{v}}$ reduces, by Theorem 3.5.2, to the classification of the discrete series of the standard parabolic semisimple subquotient Hecke algebras $\mathcal{H}_P$ of $\mathcal{H}$, and the computation of their formal degree. This reduces further to the classification of the set of W_0-orbits of residual points $\{W_0 r\}$, and of the finite set of discrete series characters δ with $\text{cc}(\delta) = W_0 r$ (which has been carried out in [OS2]), and the computation of the constants $d_{\mathcal{H}_P, \delta} \in \mathbb{R}_+$ (carried out in [CO]).*

Remark 3.5.4. *In the context of Hecke algebras of a type of a reductive group over a non-archimedean local field F, changing the base field to an unramified extension of F of degree n corresponds to the scaling $\mathbf{v} \to \mathbf{v}(n)$. This explains the importance of the scaling invariance properties.*

3.6. **Residual cosets and their properties.** Given the importance of residual subspaces for the spectral decomposition of τ we discuss some of their properties [Opd2, Opd3, Opd4] and [OS2].

Theorem 3.6.1. *For every coset of a subtorus $L \subset T^{\mathbf{v}}$ we have*

$$(3.18) \qquad \#\{\alpha \in \Sigma_0 \mid \alpha|_L = \pm q_\alpha^\pm\} - \#\{\alpha \in \Sigma_0 \mid \alpha|_L = \pm 1\} \leq \text{codim}(L)$$

Proposition 3.6.2. *Let $L \subset T$ be a residual coset of the subtorus T^L, with $\mathfrak{t}^L := \text{Lie}(T^L) = \Sigma_L^\perp$. Let $T_L \subset T$ be the subtorus such that $\mathfrak{t}_L := \text{Lie}(T_L) = \mathbb{R}\Sigma_L^\vee$. Then $T = T_L T^L$ and $T^L \cap T_L = K_L$ is a finite abelian group. Moreover $L \cap T_L = K_L r_L$ for a residual point $r_L \in T_L$ of $\mathcal{H}_L$.*

Corollary 3.6.3. *There exists only finitely many residual cosets $L \subset T^{\mathbf{v}}$.*

Corollary 3.6.4. *If $L, M \subset T$ are residual cosets then $L^{\text{temp}} \subset M^{\text{temp}}$ if and only if $L = M$.*

Corollary 3.6.5. *The measure ν_L defined in 3.5.2 is smooth on L^{temp}.*

Theorem 3.6.6. *For $L \subset T$ residual, put $S_L := W_0 \backslash W_0 L^{\text{temp}} \subset \text{supp}(\nu) = S$ with $S = S(\mathcal{H}) := \text{cc}(\hat{\mathcal{H}}_{\mathbf{v}}^{\text{temp}}) \subset W_0 \backslash T^{\mathbf{v}}$. The sets $S_L \subset S$ are the connected components of S.*

Corollary 3.6.7. *For every connected component $C \subset \hat{\mathcal{H}}_{\mathbf{v}}^{\text{temp}}$ there exists a residual coset L such that $\text{cc}(C) = S_L$. Then $(\nu_{\mathcal{H}})|_C = c_C i_*(\text{cc}|_{C^{reg}})^*(\nu_L)$ for some constant $c_C > 0$. Here $C^{reg} \subset C$ is open and dense, and has a unique structure of a smooth manifold such that $\text{cc}|_{C^{reg}}$ is a smooth finite covering map to S_L^{reg}, where $S_L^{reg} \subset S_L$*

is the image of the largest stratum with respect to the action of W_0 on $W_0 L^{\text{temp}}$, and $i : C^{reg} \to C$ denotes the embedding.

Theorem 3.6.8 ([OS2]). *Let $L^{\mathrm{v}} \subset T^{\mathrm{v}}$ be a residual coset. Then there exists a residual coset $L_\Lambda \subset T = T_\Lambda$ defined over Λ such that $L^{\mathrm{v}} = \{\mathrm{v}\} \times_{\mathrm{Spec}(\Lambda)} L_\Lambda$.*

Theorem 3.6.9 ([OS2], [CO]). *Let $Q(\Lambda)$ denote the quotient field of Λ, and let $r_\Lambda \in T_\Lambda$ be a residual point defined over Λ. Then $m_{r_\Lambda} \in Q(\Lambda)$ is regular on $\mathbb{Q}$, and if $\mathrm{v} \in \mathbb{Q}$ then $r_\Lambda^{\mathrm{v}} \in T^{\mathrm{v}}$ is residual if and only if $m_{r_\Lambda}(\mathrm{v}) \neq 0$. The set $\{\mathrm{v} \in \mathbb{Q} \mid m_{r_\Lambda}(\mathrm{v}) = 0\}$ is a union of finitely many hyperplanes in the real vector group $\mathbb{Q}$.*

3.7. **Deformation of discrete series and the computation of $d_{\mathcal{H},\delta}$.** In this section we review a deformation principle in the parameters $\mathrm{v} \in \mathbb{Q}$ for the discrete series characters δ of an affine Hecke algebra $\mathcal{H}_{\mathrm{v}}$. As we will see, this leads to an important tool to compute the rational constants $d_{\mathcal{H}_v,\delta}$ for unequal parameter Hecke algebras $\mathcal{H}_{\mathbf{v}}$ (which are abundant among unipotent Hecke algebras).

Theorem 3.7.1 ([OS2]). *Assume that $\mathcal{H}$ is a semisimple affine Hecke algebra. Let $\mathcal{S} = \{\sum_{w \in W} c_w N_w \mid \forall N \in \mathbb{N} : W \ni w \to l(w)^N |c_w| \text{ is a bounded function}\}$ denote the Schwartz completion of $\mathcal{H}$. Note that this nuclear Frechet space is independent of the Hecke parameter v. Suppose that δ is a discrete series character of $\mathcal{H}_{\mathrm{v}}$ with $cc(\delta) = W_0 r$. There exists an (analytic) open neighbourhood $\mathbb{U} \subset \mathbb{Q}$ of v, a unique W_0-orbit $W_0 r_\Lambda$ of residual points defined over Λ, and a unique continuous family $\mathbb{U} \ni \mathrm{v}' \to \tilde{\delta}^{\mathrm{v}'} \in \mathcal{S}$ of discrete series characters $\tilde{\delta}^{\mathrm{v}'}$ of $\mathcal{H}_{\mathrm{v}'}$ such that $\tilde{\delta}^{\mathrm{v}} = \delta$.*

We now review a remarkable rationality property of the formal degree of $\tilde{\delta}$ as in Theorem 3.7.1. According to [CO] there exists an orthonormal set $\mathcal{B}_{gm}$ of elliptic virtual characters of the affine Weyl group $W = X \rtimes W_0$ (cf. [CO, 2A1]) (with respect to the Euler-Poincaré pairing) which naturally parameterises the generic families of discrete series characters. More precisely, to each $b \in \mathcal{B}_{gm}$ we assign (based in part on the previous subsection):

(i) An orbit of generic residual points $W_0 r_b \in W_0 \backslash T(\Lambda)$.
(ii) The open set $\mathbb{Q}_b^{reg} = \{\mathrm{v} \in \mathbb{Q} \mid m_b(\mathrm{v}) := m_{W_0 r_b}(\mathrm{v}) \neq 0\}$ (the complement of finitely many hyperplanes of $\mathbb{Q}$).
(iii) A continuous family $\mathbb{Q}_b^{reg} \ni \mathrm{v} \to \mathrm{Ind}_D(b, \mathrm{v})$ of virtual characters of $\mathcal{H}_{\mathrm{v}}$ (the "Dirac induction" of b, cf. [CO, 2B4]) with the properties that:
   (a) For each $b \in \mathcal{B}_{gm}$ there exists a locally constant function $\epsilon(b, \mathrm{v}) \in \{\pm 1\}$ on $\mathbb{Q}_b^{reg}$ such that for all $\mathrm{v} \in \mathbb{Q}_b^{reg}$, $\epsilon(b, \mathrm{v}) \mathrm{Ind}_D(b, \mathrm{v})$ is an irreducible discrete series character of $\mathcal{H}_{\mathrm{v}}$.
   (b) We have $cc(\mathrm{Ind}_D(b, \mathrm{v})) = W_0 r_b$, for all $\mathrm{v} \in \mathbb{Q}_b^{reg}$.
   (c) For all $\mathrm{v} \in \mathbb{Q}$, the set of irreducible discrete series character of $\mathcal{H}_{\mathrm{v}}$ is equal to $\{\epsilon(b, \mathrm{v}) \mathrm{Ind}_D(b, \mathrm{v}) \mid b \in \mathcal{B}_{gm}^{\mathrm{v}}\}$, where $\mathcal{B}_{gm}^{\mathrm{v}} := \{b \in Bc_{gm} \mid \mathrm{v} \in \mathbb{Q}_b^{reg}\}$.
   (d) Let $[\pi]$ denote the elliptic class of a virtual character π of W. Then we have $[\lim_{\epsilon \to 0} \mathrm{Ind}_D(b, \mathrm{v}^\epsilon)] = b$.

Remark 3.7.2. *By the Langlands classification, virtual elliptic characters of $\mathcal{H}_{\mathrm{v}}$ can be written as linear combinations of* tempered *characters of $\mathcal{H}$. Hence it makes sense to view* fdeg *as a linear function on the space of virtual elliptic characters of $\mathcal{H}_{\mathrm{v}}$.*

Theorem 3.7.3 ([CO]). *Assume that τ is normalised by $\tau(1) = 1$. For all $b \in \mathcal{B}_{gm}$ there exist a constant $d_b \in \mathbb{Q}_+$ such that for all $\mathbf{v} \in \mathbb{Q}_b^{reg}$: $\mathrm{fdeg}(\mathrm{Ind}_D(b, \mathbf{v})) = d_b m_b(\mathbf{v})$.*

Hence for each $b \in \mathcal{B}_{gm}$ the function $\mathrm{fdeg}(\mathrm{Ind}_D(b, \mathbf{v}))$ is a rational function of $\mathbf{v}$ which is regular on $\mathbb{Q}$. This rationality is remarkable, because the family of characters $\mathbb{Q}_b^{reg} \ni \mathbf{v} \to \mathrm{Ind}_D(b, \mathbf{v})$ does not extend continuously to $\mathbb{Q}$. The rationality is very powerful to compute the rational constants $d_{\mathcal{H}_{\mathbf{v}}, \delta}$ for the discrete series characters δ of $\mathcal{H}_{\mathbf{v}}$ in the unequal parameters cases, because we see that it is enough to compute the single constant d_b in each generic family. These "generic constants" d_b are known for all irreducible root data and are quite simple (for example, for the classical Hecke algebras of the type $C_n[m_-, m_+](q^\beta)$ we have $d_b = 1$ for all b, (cf. [CO], [CK], [Opd5])).

3.8. **Central characters and Langlands parameters.** The orbits $S_L = W_0 \backslash W_0 L^{\mathrm{temp}}$ of tempered residual cosets for affine Hecke algebras can be viewed as "parameter deformations of unramified Langlands parameters". This is a crucial point in order to be able to cast the results on spectral decompositions of traces of affine Hecke algebras as discussed above in terms of adjoint gamma factors.

The basic result is the following ([KL], [HO1], [Opd2], [Opd5]):

Theorem 3.8.1. *Consider the special case of the Iwahori Hecke algebra $\mathcal{H}_{\mathbb{I}, \mathbf{v}}$ of (the group of points of) an unramified connected reductive group G over F. By [Bo, Lemma 6.5, Proposition 6.7] we have $\mathrm{Spec}(Z(\mathcal{H}_{\mathbb{I}, \mathbf{v}})) \simeq (G^\vee \theta)_{ss} = W_{\mathbb{I}, 0} \backslash T_{\mathbb{I}, \mathbf{v}}$ where $W_{\mathbb{I}, 0} := W^\theta$ is the F-Weyl group of G, and where the torus $T_{\mathbb{I}, \mathbf{v}}$ can be identified with the quotient $T_{\mathbb{I}, \mathbf{v}} = T^\vee / (1 - \theta) T^\vee$ of the maximal torus $T^\vee$ of $G^\vee$. Let*

$$S_{\mathbb{I}, \mathbf{v}} := S(\mathcal{H}_{\mathbb{I}, \mathbf{v}}) = \bigsqcup_{L \subset T_{\mathbb{I}, \mathbf{v}} \; residual} S_L \subset W_{\mathbb{I}, 0} \backslash T_{\mathbb{I}, \mathbf{v}} = (G^\vee \theta)_{ss}$$

denote the central support of the tempered spectrum of $\mathcal{H}_{\mathbb{I}, \mathbf{v}}$, and let $\Phi_{nr}^{\mathrm{temp}}(G)$ denote the set of equivalence classes of unramified tempered Langlands parameters for G. The map

$$(3.19) \qquad \gamma^{\mathbb{I}} : \Phi_{nr}^{\mathrm{temp}}(G) \to S_{\mathbb{I}, \mathbf{v}} \subset W_{\mathbb{I}, 0} \backslash T_{\mathbb{I}, \mathbf{v}}$$

$$(3.20) \qquad [\varphi] \to W_{\mathbb{I}, 0} \overline{\varphi(\tilde{\mathrm{Fr}})}$$

is a bijection.

Using this fact it is not difficult to translate the results on the spectral decomposition of τ in this special case using adjoint γ-factors, a remark that essentially goes back to [HII]. In fact, using the work of Reeder [R5] and results from [CO] one can deduce:

Theorem 3.8.2 ([HII], [HO1], [Opd2], [KL], [R3], [R5], [CO]). *Suppose $\mathbf{G}$ is unramified over F. There exists an enhanced Langlands parameterisation of the tempered Iwahori-spherical representations of the packets $\Pi_\varphi(G)$ such that the conjectures 2.6.1 and 2.6.2 hold true for Iwahori-spherical representations.*

Proof. Based on the results of [R3] this was shown in [HII, §3.4] for Iwahori spherical discrete series of a group G which is split of adjoint type. The results of [R3] have been extended to Iwahori-spherical representations of general semisimple unramified groups G in [CO, Proposition 4.9] using [R5], [Opd2], [HO1], and Theorem 3.2.6. Applying the same proof as in [HII, §3.4] shows the result for the discrete series in the general

unramified semisimple case. By Theorems 3.5.1, 3.5.2 this proves the required results for Iwahori spherical tempered representations of an arbitrary unramified group G. $\square$

4. Lusztig's representations of unipotent reduction and spectral transfer maps. Main result.

Let $\mathbf{G}$ be a connected reductive group over F which is split over an unramified extension of F. Our main theorem is a slight sharpening and extension of the main result of [Opd5].

In the formulation of the main result, the action of the group $X_{wur}(G)$ of weakly unramified characters of G plays an important role. Let F_{ur}/F be an unramified extension of F over which G splits, and let $\mathrm{Fr} \in \mathrm{Gal}(F_{ur}/F)$ denote the geometric Frobenius map. We also denote by Fr the corresponding automorphism on $G(F_{ur})$, and by Fr^* an inner twist of Fr which defines an F-quasi-split structure (denoted by $\mathbf{G}^*$, with group of points G^*) on $\mathbf{G}$. We denote by θ the action of Fr on $G^\vee$, to that $^LG = G^\vee \rtimes \langle \theta \rangle$. Let $\Omega = \mathrm{Hom}(Z(G^\vee), \mathbb{C}^\times)$. A complex character χ of G is called *weakly unramified* if χ is trivial on the kernel G_1 of the Kottwitz homomorphism $w_G : G \to \mathrm{Hom}(Z(G^\vee), \mathbb{C}^\times)^\theta = \Omega^\theta$ (cf. [Kot1, PR]). We denote by $X_{wur}(G) = (\Omega^\theta)^* = Z(G^\vee)/(1-\theta)Z(G^\vee)$ the diagonalizable group of weakly unramified characters. This is the group of characters π_α of G attached to the set of unramified Langlands parameters $H^1_{nr}(W_F, Z(G^\vee))$ as constructed in [Bo, 10.2]. We note that if $\mathbb{I}_1 \subset \mathbf{G}(F_{ur})_1$ is a Fr-stable Iwahori subgroup (such exist by [Tits, 1.10.3]), then there is a canonical isomorphism $\Omega = N_{\mathbf{G}(F_{ur})}(\mathbb{I}_1)/\mathbb{I}_1$. Put $\mathbb{I} = \mathbb{I}_1^{\mathrm{Fr}}$, then $\Omega^\theta = N_G(\mathbb{I})/\mathbb{I}$.

Denote by $X_{wur}^{\mathrm{temp}}(G) \subset X_{wur}(G)$ the subgroup of tempered weakly unramified characters. Tensoring by (tempered) weakly unramified characters defines a natural action of $X_{wur}^{\mathrm{temp}}(G)$ on the set of (tempered) irreducible characters of G of unipotent reduction which is Plancherel density preserving. There is also a natural action of $X_{wur}(G)$ on $\Phi_{nr}(G)$ as follows. If $\omega \in \Omega^* = Z(G^\vee)$ represents a weakly unramified character $[\omega] \in X_{wur}(G) = (\Omega^\theta)^* = \Omega^*/(1-\theta)\Omega^*$ and $\varphi : \mathbb{Z} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ represents a class $[\varphi] \in \Phi_{nr}(G)$, then we define $[\omega].[\varphi] = [\varphi']$ where $\varphi'(\mathrm{Fr}) = \omega\varphi(\mathrm{Fr})$ while φ' coincides with φ on $\mathrm{SL}_2(\mathbb{C})$. One easily verifies that φ' defines an unramified Langlands parameter of G, that $[\varphi']$ is independent of the lift ω of $[\omega]$, and that this defines by restriction an action of $X_{wur}^{\mathrm{temp}}(G)$ on $\Phi_{nr}^{\mathrm{temp}}(G)$ which preserves the adjoint γ-factors. The following lemma is obvious:

Lemma 4.0.1. *The bijection $\gamma^{\mathbb{I}}$ of Theorem 3.8.1 is equivariant with respect to the action of $X_{wur}^{\mathrm{temp}}(G)$ on $\Phi_{nr}^{\mathrm{temp}}(G)$ and on the set $S_{\mathbb{I},\mathbf{v}}$ of central characters of the tempered irreducible $\mathcal{H}_{\mathbb{I},\mathbf{v}}(G^*)$-modules corresponding to the Iwahori-spherical tempered irreducible representations of unipotent reduction of G^*.*

The *unramified* characters of G are the complex (quasi-)characters of G which are trivial on the intersection G^1 of the kernels of the compositions $\mathrm{Val}_F \circ \chi$ with $\chi \in X^*(G)$. It is clear that G/G^1 can be identified with a sublattice of the dual of $X^*(G)$, thus $X_{nr}(G)$ is the group of complex points of an algebraic torus. By the functoriality of Kottwitz's map w_G it follows easily that $G_{der} \subset G_1 \subset G^1$, so we have a natural embedding $X_{nr}(G) \subset X_{wur}(G)$. If $A \subset Z(G)$ is the group of F-points of the maximal

F-split subtorus of the center $Z(G)$ of G then the restriction map gives a canonical embedding $X^*(G) \subset X^*(A)$ with finite cokernel. It follows that there is a canonical surjection with finite kernel $X_{nr}(G) \to X_{nr}(A)$ which we will denote by m. Again by the functoriality of Kottwitz's map we also see easily that there exits a canonical surjection with finite kernel $X_{wur}(G) \to X_{wur}(A) = X_{nr}(A)$. Therefore $X_{nr}(G) \subset X_{wur}(G)$ is the identity component. An irreducible representation of unipotent reduction π canonically defines an unramified character $z_\pi \in X_{nr}(A)$ since the scalar action of A on V_π is clearly unramified. For $\omega \in X_{wur}(G)$ we have $z_{\omega.\pi} = m(\omega) z_\pi$.

We mention in passing that $X_{wur}(G)$ and $X_{nr}(G)$ are not sensitive to inner twists; in particular we have canonical isomorphisms $X_{wur}(G) = X_{wur}(G^*)$ and $X_{nr}(G) = X_{nr}(G^*)$.

Our main result Theorem 4.5.1 deals with existence and uniqueness of *a parameterisation* (the precise meaning of the set $\hat{G}_{uni}^{\text{temp}}$ is explained in Section 4.1):

$$\varphi : \hat{G}_{uni}^{\text{temp}} \to \Phi_{nr}^{\text{temp}}, \ \pi \to \varphi_\pi$$

such that the conjectures 2.6.1 and 2.6.2 hold. By a parameterisation we mean:

Definition 4.0.2. *A map $\hat{G}_{uni}^{\text{temp}} \to \Phi_{nr}^{\text{temp}}$, $\pi \to \varphi_\pi$ is called "a parameterisation" only if it satisfies following properties:*

  (i) *The map $\pi \to \varphi_\pi$ is equivariant for the actions of the group of tempered weakly unramified characters $X_{wur}^{\text{temp}}(G)$ as defined above (cf. [Bo, §10.2; 10.3(2)]).*

 (ii) *We can express the character by which the center $Z(G)$ acts on V_π in terms of φ_π, as in [Bo, 10.1; 10.3(1)].*

(iii) *Compatibility with unitary parabolic induction as in [Bo, §10.3(3), 11.3, 11.7]. In particular, an irreducible tempered representation is a direct summand of an induced tempered representation $i_P^G(\delta)$ for some relevant parabolic $P = MN$ if and only if φ_π is equivalent to a parameter $\varphi_\delta^M \in \Phi_{nr}^{\text{temp}}(M)$ for M (considering $^L M$ as a subgroup of $^L G$), where φ^M is a parameterisation for M.*

(iv) *The parameterisation is compatible with restriction of scalars [Bo, §10.1], and taking products of reductive groups.*

 (v) *Let $\eta : H \to G$ is an F-morphism of connected reductive groups with commutative kernel and cokernel, and let $\varphi \in \Phi_{nr}^{\text{temp}}(G)$. Given $\pi \in \Pi_\varphi(G)^{\text{temp}}$, the pullback of π to H is a finite direct sum of tempered irreducible representations in $\Pi_{^L\eta \circ \varphi}^{\text{temp}}(H)$, where $^L\eta : {}^L G \to {}^L H$ denotes the natural map (cf. [Bo, §10.3(5)]). It follows in particular that a representation $\pi \in \hat{G}_{uni}^{\text{temp}}$ factorizes through a representation of G/A (with A as above) if and only if $\text{Im}(\varphi_\pi) \subset {}^L G^\natural$ (the L-group of G/A). (We use here that G maps surjectively to $(\mathbf{G}/\mathbf{A})(F)$.)*

(vi) *For Iwahori-spherical representations of unramified connected reductive groups, the correspondence equals that of Theorem 3.8.2 (compare with [Bo, 10.4]).*

4.1. Unipotent types and unipotent affine Hecke algebras.
Let $\Lambda_0 = \mathbb{C}[v^\pm]$, with v a formal variable. Let $\mathtt{v} \in \mathbb{R}_+$ be such that $\mathtt{v}^2 = \mathtt{q} = |\mathfrak{O}/\mathfrak{p}|$. We remark that there are no "bad primes" for representations of unipotent reduction [Lus1], and we may and will often replace $\mathtt{v}$ by the indeterminate v in the theory. For example, we can view the Hecke algebra over $\mathbb{C}$ with parameter $\mathtt{v}$ as specialization of the corresponding generic

Hecke algebra over Λ_0.

We assume from now on that $\mathbf{G}$ is connected reductive over F and splits over an unramified extension of F. Let $F_{ur} \supset F$ be a maximal unramified extension of F. Let $\mathbf{T} \subset \mathbf{G}$ be a maximally split F-torus which is F_{ur}-split. In the apartment $\mathcal{A}_1$ of $\mathbf{T}(F_{ur})$ we can choose an Fr-stable alcove, by [Tits, 1.10.3]. Let $\mathbb{I}_1 \subset G(F_{ur})$ be the corresponding Fr-stable Iwahori subgroup, and let $\mathbb{I} = \mathbb{I}_1^{\mathrm{Fr}}$ be the corresponding Iwahori subgroup of G.

Steinberg's vanishing theorem $H^1(F_{ur}, \mathbf{G}) = 1$ implies that:
$$H^1(F, \mathbf{G}) = H^1(\mathrm{Gal}(F_{ur}/F), \mathbf{G}(F_{ur}))$$
. Kottwitz's Theorem expresses this in terms of the center of the Langlands dual group:
$$H^1(F, \mathbf{G}) = H^1(\mathrm{Fr}, \mathbf{G}(F_{ur})) = [\Omega/(1-\theta)\Omega]_{tor} = \mathrm{Irr}(\pi_0(Z(G^\vee)^\theta))$$
with $\Omega = X_*(T)/Q$, where Q denotes the root lattice of the dual group $G^\vee$. [2] The inner forms of G are parametrised by $H^1(F, \mathbf{G}_{ad}) = (Z(G_{sc}^\vee)^\theta)^* = \Omega_{sc}/(1-\theta)\Omega_{sc}$ with $\Omega_{sc} = P/Q$, where P is the weight lattice of Q. Given $\omega \in H^1(F, \mathbf{G}_{ad})$ we may choose a representative $u \in N_{G_{ad}(F_{ad})}(\mathbb{I}_1)$ whose image in $H^1(F, \mathbf{G}_{ad}) = \Omega_{sc}/(1-\theta)\Omega_{sc}$ is ω. Then the inner twist $\mathrm{Fr}_u := \mathrm{Ad}(u) \circ \mathrm{Fr}^*$ of Fr^* defines an inner form of G^* corresponding to ω, which we will often denoted by G^u.

Let $\mathbb{P} \subset G$ be a parahoric subgroup. There exists an Fr-stable parahoric $\mathbf{P} \subset \mathbf{G}(F_{ur})$ such that $\mathbb{P} = \mathbf{P}^{\mathrm{Fr}}$. We put as before
$$\mathrm{Vol}(\mathbb{P}) := \mathbf{v}^{-\dim(\overline{\mathbb{P}})}|\overline{\mathbb{P}}|$$
This is the specialization at $v = \mathbf{v}$ of a Laurent polynomial $\mathrm{Vol}(\mathbb{P}) \in \Lambda_0$ in v. Let σ be a cuspidal unipotent representation of $\overline{\mathbb{P}}$, lifted to $\mathbb{P}$. Let $\tilde{\mathbb{P}} := N_G(\mathbb{P})$, and choose an extension of σ to a representation $\tilde{\sigma}$ of $\tilde{\mathbb{P}} \subset G$ (such extensions exist [Lus4]). Then $t = (\mathbb{P}, \sigma)$ is a type (see Theorem 4.1.1) for a finite set of Bernstein components of representations of unipotent reduction of G. Notice that t and the extension $\tilde{t} = (\tilde{\mathbb{P}}, \tilde{\sigma})$ are determined by data (the local Tits index of G (with trivial action of the inertia group), a facet of the apartment of T, a cuspidal unipotent representation σ of the corresponding parahoric subgroup $\mathbb{P}$, and an extension to its normalizer in G) which are independent of the base field F of G. (We use here that the classification of cuspidal unipotent characters of finite groups of Lie type is independent of the field of definition [Lus1]). We write $t = (\mathbb{P}, \sigma)$ to refer to this "abstract" unipotent type (in which the base field F is undetermined, and the cardinality of its residue field considered as indeterminate v), while we often write $t_{\mathbf{v}} = (\mathbb{P}_{\mathbf{v}}, \sigma_{\mathbf{v}})$ if we want to refer to the "concrete" type of G "specialised at $v = \mathbf{v}$". Similarly for $\tilde{t}$. Such "families of unipotent types" t (with varying base field F) have explicit meaning on the Langlands dual side, as we will see shortly.

Theorem 4.1.1 ([MP1], [MP2], [Mo], [Lus4], [Lus5])**.** *Let $t = (\mathbb{P}, \sigma)$ and $\tilde{t} = (\tilde{\mathbb{P}}, \tilde{\sigma})$ be as above, and let $\Omega^{\mathbb{P}, \theta} = \tilde{\mathbb{P}}/\mathbb{P} \subset \Omega^\theta$ be the stabilizer of $\mathbb{P}$ (see [Lus4], 1.16]).*

---

[2]From now on we will call roots of $(G^\vee, T^\vee)$ "roots", and roots of $(\mathbf{G}, \mathbf{T})$ "coroots". We apologize for the incovenience this may cause.

Then $t_{\mathbf{v}}$ is a type for G with Hecke algebra $\mathcal{H}_{t,\mathbf{v}}$ which is of the form $\mathcal{H}_{t,\mathbf{v}} = \mathcal{H}_{t,\mathbf{v}}^a \rtimes \Omega^{\mathbb{P},\theta}$, where $\mathcal{H}_{t,\mathbf{v}}^a$ is the specialization at $v = \mathbf{v}$ of a generic (unextended) affine Hecke algebra $\mathcal{H}_t^a$ defined over Λ_0 (depending on t only), on which $\Omega^{\mathbb{P},\theta}$ acts via diagram automorphisms. If $\Omega_1^{\mathbb{P},\theta} \subset \Omega^{\mathbb{P},\theta}$ denotes the subgroup which acts trivially on $\mathcal{H}_t^a$, and $\Omega_2^{\mathbb{P},\theta} = \Omega^{\mathbb{P},\theta}/\Omega_1^{\mathbb{P},\theta}$ then

$$(4.1) \qquad \mathcal{H}_t = \mathcal{H}_{\tilde{t}}^e \otimes \mathbb{C}[\Omega_1^{\mathbb{P},\theta}]; \quad \mathcal{H}_{\tilde{t}}^e = \mathcal{H}_t^a \rtimes \Omega_2^{\mathbb{P},\theta}$$

where $\mathcal{H}_{\tilde{t}}^e$ is an extended affine Hecke algebra. The set of extensions $\tilde{t}$ of t is a torsor for the quotient $(\Omega^{\mathbb{P},\theta})^*$ of the group $X_{wur}(G) = (\Omega^\theta)^*$, and $\mathcal{H}_{\tilde{t}}^e$ corresponds to the orbit of $\tilde{t}$ under the subgroup $(\Omega^{\mathbb{P},\theta}/\Omega_1^{\mathbb{P},\theta})^*$. Thus the summands $\mathcal{H}_{\tilde{t}}^e$ of $\mathcal{H}_t$ form a torsor for the quotient $(\Omega_1^{\mathbb{P},\theta})^*$, and $X_{wur}(G) = (\Omega^\theta)^*$ acts on the center of $\mathcal{H}_t$.

Let $\mathrm{Vol}(\mathbb{P}) \in \Lambda_0$ be the Laurent polynomial defined above, let $\deg(\sigma) \in \Lambda_0$ be the polynomials such that $\deg(\sigma)(\mathbf{v}) = \deg(\sigma_{\mathbf{v}})$, then the trace τ of $\mathcal{H}_{\tilde{t}}^e$ is normalised by $\tau(1) = d^t = |\Omega_1^{\mathbb{P},\theta}|^{-1} \frac{\deg(\sigma)}{\mathrm{Vol}(\mathbb{P}^{\mathrm{Fr}u})}$. This turns the summands $\mathcal{H}_{\tilde{t}}^e$ into normalised affine Hecke algebras in the sense of [Opd5].

Proof. We refer the reader to [Opd5, Section 2.4, Theorem 2.8], where this is worked out in detail, and to the discussion in [Lus4]. $\square$

Definition 4.1.2 (Lusztig [Lus4]). *The category $\mathcal{C}_{uni}(G)$ of smooth representations of G of unipotent reduction is the direct product over all conjugacy classes of unipotent types $t = (\mathbb{P}, \sigma)$ of the abelian subcategories $\mathcal{C}^t(G)$ of $\mathcal{C}(G)$, where $\mathcal{C}^t(G)$ is Morita equivalent to the Hecke algebra $\mathcal{H}_{t,\mathbf{v}}$ of t. Since $\mathcal{H}_t$ is a direct sum (4.1) of (mutually isomorphic) extended affine Hecke algebras $\mathcal{H}_{\tilde{t}}^e$ parameterised by the set of characters of $\Omega_1^{\mathbb{P},\theta}$, each subcategory $\mathcal{C}^t(G)$ decomposes as a product over a finite set of Bernstein components $\mathcal{C}^{\tilde{t},e}(G)$ parameterised by the set of extensions $\tilde{t}$ of t to the inverse image of $\Omega_1^{\mathbb{P},\theta} \subset \Omega^{\mathbb{P},\theta}$ in $\tilde{\mathbb{P}}$. This is a torsor for $(\Omega_1^{\mathbb{P},\theta})^*$, such that for each extended type $\tilde{t}$, the map $(\pi, V_\pi) \to \mathrm{Hom}_{N_G(\mathbb{P}^{\mathrm{Fr}u})}(\tilde{\sigma}_{\mathbf{v}}, V_\pi|_{N_G(\mathbb{P}^{\mathrm{Fr}u})})$ is a Morita equivalence from $\mathcal{C}^{\tilde{t},e}(G)$ to $\mathcal{H}_{\tilde{t},\mathbf{v}}^e$.*

Using his arithmetic-geometric diagram correspondences Lusztig constructed [Lus4], [Lus5] a parameterisation of the irreducible objects of $\mathcal{C}_{uni}(G)$ if G is simple of adjoint type. In particular:

Theorem 4.1.3 (Lusztig). *Let $G = G^u$ be (the group of points of) a simple group of adjoint type defined over F. There exists a partitioning*

$$(4.2) \qquad \mathrm{Irr}_{uni}^{\mathrm{temp}}(G) = \bigsqcup_{\varphi \in \Phi_{nr}^{\mathrm{temp}}} \Pi_\varphi$$

such that for all $\varphi \in \Phi_{nr}^{\mathrm{temp}}$, there is a bijection between Π_φ and $\Pi(\mathcal{S}_\varphi, \chi_G)$ (where $\chi_G \in \Omega = (Z_{sc}^\theta)^*$ and χ_G and $\Pi(\mathcal{S}_\varphi, \chi_G)$ as in Conjecture 2.5.2). This map $\pi \to \varphi_\pi$ can be taken $X_{wur}(G)$-equivariantly.

For $G = G^*$ and $t_{\mathbb{I}} = (\mathbb{I}, \mathrm{triv})$, $\mathcal{C}^{t_{\mathbb{I}}}(G)$ is the Bernstein component of the minimal unramified principal series. Indeed, by Borel's classical result this abelian subcategory of $\mathcal{C}(G)$ is equal to the category of smooth representations of G which are generated by their Iwahori-fixed vectors. Then the equivalence of Definition 4.1.2 is the classical

equivalence between $\mathcal{C}^{t_\mathbb{I}}(G)$ and the module category of the (extended) Iwahori-spherical Hecke algebra $\mathcal{H}^e_{t_\mathbb{I},\mathbf{v}} = \mathcal{H}_{\mathbb{I},\mathbf{v}}$. Via this equivalence, the restriction of the correspondence of Theorem 4.1.3 to $\mathcal{C}^{t_\mathbb{I}}(G)$ becomes the classical Kazhdan-Lusztig parameterisation [KL]. Theorem 3.8.2 (and its proof) shows that in this special case, this parameterisation restricted to tempered representations satisfies the conjectures 2.6.1 and 2.6.2, and that this can in fact be extended to general reductive groups.

The basic problem one is facing when trying to extend this result to all tempered representations of $\mathcal{C}_{uni}(G^u)$ (with G^u an arbitrary inner twist of G), is how one should parameterise the tempered irreducible representations of the affine Hecke algebras of the form $\mathcal{H}^e_{\tilde{t}}$ (with $t = (\mathbb{P}, \sigma)$ a unipotent type of G^u, and $\tilde{t}$ an extension) in terms of tempered unramified Langlands parameters for G^u. Lusztig [Lus4], [Lus5] does this via his theory of local systems on $G^\vee$-orbits of Langlands parameters, and the remarkable isomorphisms between arithmetic diagrams (related to the affine diagram of $\mathcal{H}^e_{\tilde{t}}$) and geometric diagrams related to graded affine Hecke algebras associated with a local system. We follow a different approach in which the conjectures 2.6.1 and 2.6.2 play a central role. In view of Theorem 3.8.1, it is clear that one would like to map the spectrum of the center $Z(\mathcal{H}^e_{\tilde{t},\mathbf{v}})$ of $\mathcal{H}^e_{\tilde{t},\mathbf{v}}$ to the spectrum of the center of $Z(\mathcal{H}_{\mathbb{I},\mathbf{v}})$, so that $S(\mathcal{H}^e_{\tilde{t},\mathbf{v}})$ maps to $S(\mathcal{H}_{\mathbb{I},\mathbf{v}})$, and such that the Plancherel measures (see Theorem 3.5.1 and Theorem 3.5.2) up to constant factors correspond. We call such a map a *spectral transfer map* (STM) $\mathcal{H}^e_{\tilde{t},\mathbf{v}} \rightsquigarrow \mathcal{H}_{\mathbb{I},\mathbf{v}}$. Such maps turns out to exist and turn out to be *essentially unique*. Moreover, these STMs are essentially "the same" as the maps implicit in Lusztig's arithmetic-geometric correspondences. Seeing this through (cf. [Opd5], [FO], [FOS]) recovers Lusztig parameterisation $\pi \to \varphi_\pi$, proving at the same time the conjectures 2.6.1 and 2.6.2 for this parameterisation. In our Main Theorem 4.5.1 we have further extended this to general connected reductive group.

4.2. Langlands parameters and residual cosets.
Let us first consider the case of cuspidal representations of unipotent reduction.

Let $t = (\mathbb{P}, \sigma)$ be a unipotent type of G, with $\mathbb{P} \subset G$ a maximal parahoric subgroup. Assume that G has anisotropic center. Then $\mathcal{H}_t = \Lambda_0[\Omega^{\mathbb{P},\theta}]$ has rank 0, and its trace is normalised by $\tau(1) = d^t = \frac{\deg(\sigma)}{\mathrm{vol}(\mathbb{P})}$. Let π be a supercuspidal unipotent character of G^u belonging to the finite set $\mathcal{C}^t(G)$. Then, in view of Theorems 3.5.1, 3.8.1 and 3.8.2 we need to find a Λ_0-valued residual point $r \in T_\mathbb{I}(\Lambda_0)$ such that:

$$(4.3) \qquad \mathrm{fdeg}(\pi) = c\mu^{\{r\}}(r) = \frac{c}{q(w_0)\mathrm{Vol}(\mathbb{I})}m_r(v)$$

for some constant factor $c \in \mathbb{R}_+$, where

$$(4.4) \qquad \mathrm{fdeg}(\pi) = \frac{\deg(\sigma)}{|\Omega^{\mathbb{P},\theta}|\mathrm{Vol}(\mathbb{P})}$$

Recall that by Theorem 3.8.1, an orbit of residual points $W_0 r \in W_0 \backslash T_\mathbb{I}$ corresponds to a unique orbit of discrete unramified Langlands parameters $\varphi \in \Phi^{\mathrm{temp}}_{nr}(G^*)$ such that $W_0 r = [\varphi(\tilde{\mathrm{Fr}})] \in W_0 \backslash T_\mathbb{I}$. By Theorem 3.8.2 we then have, in terms of φ, for some

constant factor c':

$$\text{(4.5)} \qquad \text{fdeg}(\pi) = c\mu^{\{r\}}(r) = c'\gamma(0, \text{Ad} \circ \varphi, \psi)$$

viewed as identity of rational functions in v (where $v^2 = q$).

Example 4.2.1. We begin with a very basic example. Let $G = \text{PGL}_{m+1}(F)$. The only inner form of G which has a cuspidal type in this case is the anisotropic inner form $G^u = \mathbb{D}^\times/F^\times$ with $\mathbb{D}$ the tame division algebra of degree $m + 1$ over F. The unique parahoric subgroup of G^u is $\mathbb{P}^u = G_1^u = \cap_{\chi \in X_{wur}(G^u)}\text{Ker}(\chi)$, which has a unique unipotent cuspidal $\sigma = \text{triv}$. This gives a maximal unipotent type $t = (\mathbb{P}^u, \sigma)$ whose extensions $\tilde{t}$ to G^u are given by $X_{wur}(G^u) = \Omega^*$. Let us call the corresponding cuspidal unipotent characters of G^u: π_χ (with $\chi \in X_{wur}(G^u)$). Now G has essentially only one unramified discrete Langlands parameter, the regular parameter φ_0, up to the action of $X_{wur}(G) = \Omega = C_{m+1}$. We have $\varphi_0|_{W_F} = 1$, and $\varphi_0(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ is a regular unipotent element. Hence $W_0 r = [\varphi_0(\tilde{\text{Fr}})] \in W_0\backslash T_{\mathbb{I}}$, and the Ω^*-orbit $\Omega^* r$ is defined by the equations $\alpha_i(r) = \mathsf{q}$ for $i = 1 \ldots m$. We check simply from (4.3) and (4.4) that

$$\text{(4.6)} \qquad \text{fdeg}(\pi_\chi) = \mu^r(r) = \frac{1}{(m+1)[m+1]_q}, \quad \text{with } [n]_q := \frac{v^{m+1} - v^{-(m+1)}}{v - v^{-1}}.$$

Example 4.2.2. Another example which is known for a long time [R1]. Let $G = \mathbf{G}_2(F)$. Let π be the cuspidal unipotent character of G which is compactly induced from the cuspidal unipotent representation $\mathbf{G}_2(\mathbb{F}_{\mathsf{q}})$ which is denoted by $\mathbf{G}_2[1]$, inflated to $\mathbb{P} = \mathbf{G}_2(\mathcal{O}_F)$. Then

$$\text{(4.7)} \qquad \text{fdeg}(\pi) = \frac{1}{6}\mu^r(r) = \frac{1}{6(v + v^{-1})^2(v^2 + 1 + v^{-2})}.$$

where $W_0 r = [\varphi_{sub}(\tilde{\text{Fr}})] \in W_0\backslash T_{\mathbb{I}}$ is the Λ_0-point associated with the real discrete unramified Langlands parameter φ_{sub}, associated with the subregular unipotent orbit of $\mathbf{G_2}$. This uniquely determines $W_0 r$ and thus φ_{sub}. Indeed, there is a cuspidal local system supported by $G^\vee \dot{\varphi}_{sub}$, and Lusztig maps π to this cuspidal local system.

It turns out that this always works, at least in the case of G being absolutely simple and of adjoint type:

Theorem 4.2.3 (Reeder [R1], [R3] (split exceptional groups), [HO1], [FO], [Fe2], [FOS]). *Let G be a simple group of adjoint type defined over F and split over an unramified extension of F. Let π be a supercuspidal representation of G of unipotent reduction. Let $t = (\mathbb{P}, \sigma)$ be a unipotent type for G such that π belongs to $\mathcal{C}^t(G)$. Then $\mathbb{P} \subset G$ is a maximal parahoric subgroup. Conversely, if $\mathbb{P} \subset G$ is maximal then $\mathcal{C}^t(G)$ consists of supercuspidal unipotent representations. In this situation there exists a* unique *$X_{wur}(G)$-orbit of discrete unramified Langlands parameters $[\varphi] \in \Phi_{nr}^{\text{temp}}(G)$ such that:*

$$\text{(4.8)} \qquad \text{fdeg}(\pi) = c\gamma(0, \text{Ad} \circ \varphi, \psi)$$

The collection of classes of discrete unramified Langlands parameters $[\varphi] \in \Phi_{nr}^{\text{temp}}(G)$ thus obtained is exactly equal to the set of classes of discrete unramified Langlands parameters which support a cuspidal local system.

Recall that $\mathcal{H}_{\tilde{t}}^e = \Lambda_0$, and that the trace of this Hecke algebra is normalised by $\tau^e(1) = \frac{\deg(\sigma)}{|\Omega^{\mathbb{P},\theta}|\mathrm{Vol}(\mathbb{P})}$. In the terminology of STMs we view a Λ_0-valued point $r : \mathrm{Spec}(\Lambda_0) \to T_{\mathbb{I}}(\Lambda_0)$ such that (4.8) holds as realizing a cuspidal STM $\mathcal{H}_{\tilde{t}}^e \rightsquigarrow \mathcal{H}_{\mathbb{I}}$. In the next section we will discuss the notion of STM in the general case.

The proof of Theorem 4.2.3 is difficult. In the exceptional cases it reduces to explicit case by case computations due to [HO1], [R3], [Fe2]. In the classical case, see [FO]. This is based on the existence an explicit set of generators between the unipotent normalised affine Hecke algebras of type C_n, and the observation that for classical cases other than $\mathrm{PGL}_{m+1}(F)$, the expressions $\frac{\deg(\sigma)}{\mathrm{Vol}(\mathbb{P})}$ do not contain odd cyclotomic factors in the numerator or the denominator. This property turns out to eliminate most of the discrete unramified Langlands parameters φ for which an identity of the form (4.8) could hold. Analysing the remaining cases carefully using the STMs whose *existence* was established directly in [Opd5], [Fe2] completes the proof.

4.3. **Spectral transfer maps.** Let us now introduce the notion of STM formally. Assume we are given two normalised affine Hecke algebras $\mathcal{H}_1, \mathcal{H}_2$ defined over Λ_0. For $i = 1, 2$ we have the torus T_i defined over Λ_0 associated with the character lattice X_i of the root datum of $\mathcal{H}_1$. We have the Weyl groups $W_{0,i}$ acting on T_i, and the μ-function μ_i, a rational function on T_i.

Given a residual coset $L \subset T_2$ defined over Λ_0, we know that $L = r_L T^L$ for some residual point $r_L \in T_L$, by Proposition 3.6.2. Let $L_n = L/K_L^n$ be the smallest quotient of L such that for each $K_L^n t \in L_n$, the orbit $W_{0,2} K_L^n t = W_{0,2} t$ is a single W_0-orbit in T_2. Then $K_L^n \subset K_L = T_L \cap T^L$ is the finite abelian subgroup $K_L^n = N_{W(\Sigma_L)}(L)/Z_{W(\Sigma_L)}(L) \subset K_L$. Then μ_2^L (see Theorem 3.5.2) is K_L^n invariant, and descends to a μ-function $\mu_2^{L_n}$.

Definition 4.3.1. *A spectral transfer map $\mathcal{H}_1 \rightsquigarrow \mathcal{H}_2$ between the normalised affine Hecke algebras over Λ_0 is an equivalence class of morphisms $\Psi : T_1 \to L_n$ of torsors of algebraic tori defined over Λ_0, where $L \subset T_2$ is a residual coset for $\mathcal{H}_2$, such that*

(1) *Ψ is surjective with finite fibres.*
(2) *For all $w_1 \in W_{0,1}$ there exists a $w_2 \in N_{W_{0,2}}(L)$ such that $\Psi \circ w_1 = w_2 \circ \Psi$.*
(3) *$\Psi^*(\mu_2^{L_n}) = D\mu_1$ for some constant $D \in \mathbb{Q}^\times$.*

We call Ψ and Ψ' equivalent if $\Psi' = w \circ \Psi$ for some $w \in W_{0,2}$.

Remark 4.3.2.   (i) *In the special case $\dim(T_1) = 0$ this recovers the notion cuspidal STMs as discussed above for unipotent affine Hecke algebras $\mathcal{H}_{\tilde{t}}^e$.*
(ii) *The property (2) in Defnition 4.3.1 is almost always superfluous, except in degenerate cases [Opd4, cf. Proposition 5.4]. For unipotent affine Hecke algebras this property is always automatic.*
(iii) *Note that Ψ defines a morphism $\Psi_Z : \mathrm{Spec}(Z_1) \to \mathrm{Spec}(Z_2)$ between the spectra of the centers of the $\mathcal{H}_i$.*
(iv) *Suppose that $L_1 \subset T_1$ is a residual coset for $\mathcal{H}_1$, and let $\mathrm{Im}(\Psi) = L_n$. Then $\Psi_Z(W_{1,0}L_1) = N_{W_{0,2}}(L)(L_2)$ for some residual coset $L_2 \subset L$, and all residual cosets $L_2 \subset L$ are the image under Ψ of a residual coset of $\mathcal{H}_1$ in this sense.*

(v) *We can compose STMs. This is useful as we may generate in this way all STMs between the unipotent affine Hecke algebras of the various groups G^u from a small number of generators.*

The most important property of STMs with respect to the spectral decomposition of affine Hecke algebras is the following result [Opd4, Theorem 6.1]:

Theorem 4.3.3. *Let $\Psi : \mathcal{H}_1 \rightsquigarrow \mathcal{H}_2$ be an STM. Suppose that $C_{1,\mathbf{v}} \subset \hat{\mathcal{H}}_{1,\mathbf{v}}^{\mathrm{tem}}$ is a component of the space of irreducible tempered characters of $\mathcal{H}_{1,\mathbf{v}}$. Let $cc_{i,\mathbf{v}} : \hat{\mathcal{H}}_i^{\mathrm{temp},\mathbf{v}} \to \mathrm{Spec}(Z_{i,\mathbf{v}})$ be the central character map of $\mathcal{H}_{i,\mathbf{v}}$. Then $cc_1(C_{1,\mathbf{v}}) = S_{L_{1,\mathbf{v}}}$ where $L_1 \subset T_1$ is a residual coset. There exists a unique orbit $W_{0,2}L_2 \subset T_2$ of residual cosets of $\mathcal{H}_2$ such that $\Psi_Z(S_{L_{1,\mathbf{v}}}) = S_{L_{2,\mathbf{v}}}$. If $C_{2,\mathbf{v}} \subset \hat{\mathcal{H}}_{2,\mathbf{v}}^{\mathrm{tem}}$ is such that $cc_2(C_{2,\mathbf{v}}) = S_{L_{2,\mathbf{v}}}$ then consider the fibred product C_{12} defined by the diagram*

(4.9)
$$
\begin{array}{ccc}
C_{12} & \xrightarrow{\ P_2\ } & C_2 \\
{\scriptstyle P_1}\big\downarrow & & \big\downarrow{\scriptstyle cc_2} \\
C_1 & \xrightarrow[\ \Psi_Z \circ cc_1\ ]{} & S_{L_2}
\end{array}
$$

Then there exist constants $r_i \in \mathbb{Q}_+$ and a family of positive measures $\nu_{\mathbf{v}}$ on $C_{12,\mathbf{v}}$ (for all $\mathbf{v} > 0$) such that for $i = 1, 2$ we have $r_i.P_{i,*}(\nu_{\mathbf{v}}) = \nu_{i\mathbf{v}}|_{C_{1,\mathbf{v}}}$.

This result states that an STM $\Psi : \mathcal{H}_1 \rightsquigarrow \mathcal{H}_2$ defines a correspondence between the tempered spectrum of $\mathcal{H}_1$ and of $\mathcal{H}_2$ which respects the connected components in these tempered irreducible spectra and, up to constant rational factors only depending on the components, the Plancherel measures $\nu_{i,\mathbf{v}}$ of $\mathcal{H}_i$.

Let $\mathbf{G}$ be a connected reductive group defined over F which splits over an unramified extension of F. Let $t = (\mathbb{P}, \sigma)$ be a unipotent type of G, i.e. $\mathbb{P} \subset G$ is a parahoric subgroup, and σ is a cuspidal unipotent representation of $\mathbb{P}$. Suppose in Theorem 4.3.3 that $\mathcal{H}_1 = \mathcal{H}_t(G)$ (a finite direct sum of extended affine Hecke algebras of the form $\mathcal{H}_{\tilde{t}}^e$), and that $\mathcal{H}_2 = \mathcal{H}_{\mathbb{I}}(G^*)$, the Iwahori Hecke algebra of a quasi-split inner form G^* of G (with the obvious extension of the notion of an STM on a finite direct sum of extended affine Hecke algebras such as $\mathcal{H}_t(G)$, by allowing T_1 to be disjoint union of algebraic tori over Λ_0).

Let $M \subset G$ be a Levi subgroup such that the set Ω^t of G-conjugacy classes of cuspidal pairs which belong in the inertial classes covered by t are of the form $[(M, \delta)]$. By Corollary 3.2.2 the diagonalizable group $T_{1,\mathbf{v}}$ over $\mathbb{C}$ can be identified with the space $\Omega^t(M)$ of M-conjugacy classes of such cuspidal pairs. The natural action (by taking tensor products) of $X_{wur}(M)$ on $\Omega^t(M)$ turns $T_{1,\mathbf{v}}$ into a $X_{wur}(M)$-space, and in fact each component of $T_{1,\mathbf{v}}$ is a quotient of $X_{wur}(M)$ with finite kernel (because $X_{nr}(M) \subset X_{wur}(M)$ is the identity component, and the components of $\Omega^t(M)$ are by definition already homogeneous for $X_{nr}(M)$ with finite kernel).

Remark 4.3.4. *Recall that (cf. Theorem 3.8.1) the semisimple conjugacy classes of the set $G^\vee \theta \subset {}^L G$ are in natural bijection with the set of W^θ-orbits in the complex torus*

$T_\mathbb{I} = \mathrm{Hom}(X_*(T)^\theta, \mathbb{C}^\times)$. *Observe that*

$$X_{wur}(M) = \mathrm{Hom}((X_*(T)/Q_M)^\theta, \mathbb{C}^\times) \to \mathrm{Hom}(X_*(T)^\theta, \mathbb{C}^\times) = T_2 = T_\mathbb{I}.$$

Hence $X_{wur}(M)$ also acts naturally on $T_\mathbb{I}$ (and faithfully, in fact).

Corollary 4.3.5. *Let G, $\mathcal{H}_1$ and $\mathcal{H}_2$ be as above. Suppose that $\Psi : \mathcal{H}_t(G) \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$ is a $X_{wur}(G)$-equivariant STM. Let $\psi : \hat{G}^{t,temp} \to \Phi_{nr}^{temp}(G)$ be the map $(\gamma^\mathbb{I})^{-1} \circ \Psi_Z \circ \beta^t \circ cc$ (cf. (3.1), Corollary 3.2.2, Theorem 3.8.1 and Definition 4.3.1).*

(1) *Via the map $\pi \to \psi_\pi$ on $\hat{G}^{t,temp}$, the restriction of the Plancherel measure of G to $\hat{G}^{t,temp}$ is expressed as in conjectures 2.6.1 and 2.6.2, up to rational constant factors.*

(2) *ψ satisfies the conditions (i), (ii), and (vi) of a parameterisation if and only if the following additional compatibility conditions hold:*

   (a) *$\Psi(\eta.t) = \eta.\Psi(t)$ for all $\eta \in X_{wur}(M)$, $t \in T_{1,\mathbf{v}}$.*

   (b) *If $t = (\mathbb{I}, \mathrm{triv})$, with $\mathbb{I} \subset G^*$ the Iwahori subgroup of a quasi-split group G^* which is an inner twist of G. Then we require that $\Psi : \mathcal{H}_\mathbb{I} \rightsquigarrow \mathcal{H}_\mathbb{I}$ is the identity.*

Proof. By Theorem 3.2.6 $\hat{m}_t$ defines a Plancherel measure preserving homeomorphism from $\hat{G}^{t,temp}$ to $\hat{\mathcal{H}}_t(G)$. By Theorem 3.5.1, Theorem 3.5.2 a component C_1 in the tempered irreducible spectrum of $\mathcal{H}_t(G)$ is defined by unitary parabolic induction of discrete series characters modulo center of Levi-subalgebras. The image $cc^t(C_1)$ under the central character map cc^t is the image $S_{L_1} \subset \mathrm{mSpec}(Z(\mathcal{H}_t(G)))$ of the tempered form of a residual coset L_1^{temp}. Moreover the Plancherel measure of $\mathcal{H}_t(G)$ on C_1 is given, up to a constant factor, by the pull back $cc^{t,*}(\mu_t^{L_1}|_{S_{L_1}})$ on L_1 to C_1 (using that cc^t is a smooth finite covering map on a dense open subset of C_1). Hence modulo constant factors, the Plancherel density on C_1 is a function of the central character only.

By Theorem 4.3.3 $\Psi_Z(S_{L_1}) = S_{L_2} \subset W_{\mathbb{I},0}\backslash T_\mathbb{I}$ for a unique residual coset $L_2 \subset L = \mathrm{Im}(\Psi)$, and up to rational constant factors $\Psi_Z^*(\mu_{\mathbb{I},L_2}) = c\mu_t^{L_1}$. Hence Theorem 3.8.1 and Theorem 3.8.2 imply that in this way we can express the Plancherel density at $\pi \in \hat{G}^{t,\mathrm{remp}}$ up to locally constant rational factors by the appropriate adjoint γ-factors in ψ_π, proving (1).

For (2): It is easy to see that ψ satisfies (i), (ii) iff Ψ satisfies the stated compatibility condition (a). Clearly (vi) makes sense only in the case of the Borel component of G^*. In this case, (2)(b) forces (vi) by insisting that Ψ is identical. □

Remark 4.3.6. *Without this condition (2)(b) it would be allowed in the case $t = (\mathbb{I}, \mathrm{triv})$ that our map $\pi \to \psi_\pi$ is that of Theorem 3.8.2 (as required in (vi)) but twisted by an STM $\Psi : \mathcal{H}_\mathbb{I} \rightsquigarrow \mathcal{H}_\mathbb{I}$ satisfying (2)(a). Such STMs are given by the action of $X_{wur}(G) = (\Omega^\theta)^*$. For general components of $\hat{G}^t$ in general I do not know of a preferred choice for Ψ within its $X_{wur}(G)$-orbit.*

4.4. Lusztig's geometric-arithmetic correspondences and STMs.

In this subsection we will assume that **G** is absolutely simple and adjoint (we will reduce the general case to this case), and that **G** is split over an unramified extension of F.

The following result was essentially proven in [Opd4, Theorem 3.4]:

Theorem 4.4.1. *Let* $t = (\mathbb{P}, \sigma)$ *be a unipotent type for* $G = \mathbf{G}(F)$, *and let* $\mathcal{H}_\mathbb{I}(G^*)$ *denote the Iwahori Hecke algebra of the quasi-split inner twist* G^* *of* G. *Let* $\tilde{t} = (\tilde{\mathbb{P}}, \tilde{\sigma})$ *be an extension of* t *to* $\tilde{\mathbb{P}} = N_G(\mathbb{P})$.

   (a) *There exist STMs* $\Phi_{\tilde{t}} : \mathcal{H}_{\tilde{t}}^e \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$.

   (b) *If* $\Phi_{\tilde{t}}' : \mathcal{H}_{\tilde{t}}^e \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$ *is also an STM, then there exists an* $\omega \in X_{wur}(G) = X_{wur}(G^*)$ *and a spectral automorphism* α *of* $\mathcal{H}_t$ *such that* $\Phi_{\tilde{t}}' = m_\omega \circ \Phi_{\tilde{t}} \circ \alpha$. *Here* $m_\omega : \mathcal{H}_\mathbb{I}(G^*) \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$ *denotes the STM given by multiplication with* ω.

   (c) *There exists a* $\Phi_t : \mathcal{H}_t^e \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$ *which is* $X_{wur}(G)$-*equivariant and which satisfies the conditions (2)(a) and (b) of Corollary 4.3.5. Such* Φ_t *is unique up to the action of* $X_{wur}(G)$.

   (d) *The parameterisation* $\varphi_t : \hat{G}^t \to \Phi_{nr}^{\text{temp}}(G)$ *associated to* Φ_t *as in Corollary 4.3.5 is, up to a twist by* $X_{wur}(G)$, *the same as Lusztig's parameterisation* $\varphi_{Lu} : \hat{G}^t \to \Phi_{nr}^{\text{temp}}(G)$ *of* [Lus4], [Lus5].

In this subsection I will sketch the proof of the Theorem, which is quite involved. For details the reader is referred to [Opd5], [Opd4], [FO], [Fe2] and [FOS].

First the existence (a) is proved in [Opd5] and [Fe2].

In the exceptional cases the existence is shown by constructing STMs associated to Lusztig's arithmetic-geometric correspondences, in a way we will explain below. Given a arithmetic-geometric correspondence of diagrams as in Lusztig, it is not difficult to find the candidate map $\Phi_{\tilde{t},T} : T_{\tilde{t}} \to L_n$ underlying the alleged STM $\Phi_{\tilde{t}} : \mathcal{H}_{\tilde{t}}^e \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$. To verify the main property (3) of Definition 4.3.1 one needs to do a rather cumbersome computation.

In the classical cases this is not practical. Fortunately the required STMs can be obtained from a small set of generators of STMs between certain unipotent affine Hecke algebras of the form $C_d[m_-, m_+](q^\beta)$ (see [Opd5], 3.2.6; 3.2.7]). There are three kind of generating STMs: The translation STMs which decrease one of $m_\pm$ by 1 (if this parameter is in $\mathbb{Z}_+ + 1/2$) or by 2 (if this parameter is in $\mathbb{Z}_+$), increasing the rank accordingly, and do not change β; the spectral isomorphisms, which interchange m_- and m_+, or give one of these a minus sign, and the extraspecial STMs. In this case $m_\pm \in \mathbb{Z} \pm 1/4$ and $\beta = 2$, while the target Hecke algebra is of the form $C_n[\delta_-, \delta_+](q)$ with $\delta_\pm \in \{0, 1\}$. The latter STMs correspond to Hecke algebras of types of inner forms G^u of G^* of even orthogonal or symplectic groups where $u \in \Omega/(1 - \theta)\Omega$ has maximal order.

Let us now explain the construction of STMs associated with a correspondence of diagrams as in Lusztig in the exceptional cases. For more details, see [Opd5].

First one needs to establish the cuspidal case of Theorem 4.4.1 in the exceptional case. This is an explicit case by case verification based on the classification of the cusipdal unipotent characters of finite groups of Lie type [Lus1], the classification of the residual points for $\mathcal{H}_\mathbb{I}$, and the computation of the formal degree (up to constant factors) for discrete series representations of $\mathcal{H}_\mathbb{I}$ supported by the corresponding central characters [HO1], [OS2]. This was carried out in [Opd5], 3.2.2].

Next we discuss the diagram of an STM $\Phi : \mathcal{H}_{\tilde{t}}^e \rightsquigarrow \mathcal{H}_\mathbb{I}(G^*)$. Consider the underlying map $\Phi : T_{\tilde{t}} \to L_n$ with $L \subset T = T_\mathbb{I} = T^\vee/(1 - \theta)T^\vee$ a residual coset. Such a map is determined by assigning weights to the nodes of the Kac diagram $D(\mathfrak{g}^\vee, \theta)$ [R4] of

$^L G = G^\vee \rtimes \langle \theta \rangle$ according to the following steps (i)-(vii).

(i) Lift $\tilde{\Phi}$ to an affine linear map $\mathfrak{t}_{\tilde{t}} \to \mathfrak{t} = \mathfrak{t}_{\mathbb{I}}$ and use the action of the dual affine Weyl group $W_{\mathbb{I}}^{a,\vee} = Q^\vee \rtimes W_{\mathbb{I},0}$ to replace $\tilde{\Phi}$ by the lift $\tilde{\Phi}'$ of a map equivalent to Φ, such that $P_L := \mathrm{Im}(\tilde{\Phi}'_{\mathbf{v}=1})$ meets the closure $\overline{C^\vee}$ of the dual alcove $C^\vee$ in a facet of dimension equal to $\dim(T_{\tilde{t}})$. We assume from now on that $\Phi = \Phi'$ has this property.

(ii) We have $\mathrm{Im}(\Phi) = L_n$ for a unique residual coset $L_\Phi = r_L T^L \subset T_{\mathbb{I}}$, where $r_L = s_L c_L \in T_L$ is a residual point. Then P_L is a lift of $L_{\Phi,\mathbf{v}=1}$.

(iii) Let us denote the spectral diagram [OS2, Section 8], [Opd4, 2.3] of $\mathcal{H}_{\tilde{t}}^e$ by Σ_t^{spec}. Let $(W_t^{a,\vee}, S_t^{a,\vee})$ be the associated affine Coxeter group. It turns out that property (3) of Definition 4.3.1 implies that there exists an isomorphism of affine Coxeter groups $(W_t^{a,\vee}, S_t^{a,\vee}) \to W_L^* := N_{W_{\mathbb{I}}^{a,\vee}}(P_L)/C_{W_{\mathbb{I}}^{a,\vee}}(P_L)$. Here $W_{\mathbb{I}}^{a,\vee}$ is the affine Weyl group associated to the Kac diagram $D(\mathfrak{g}^\vee, \theta)$ acting on $\mathfrak{t}_{\mathbb{I}} = \mathfrak{t}^\theta$ (see [R4]).

Indeed, using the theory of intertwiners Harish-Chandra proved that the reflection in a hyperplane on which the Plancherel density of a generalised principal series vanishes $i_{P=LN}^{G^*}(\delta)$ is an element of $N_{W_{\mathbb{I}}}(L, \delta)$. Apply this to the generalised principal series of G^* supported by L_Φ, then we see that such hyperplane reflection is in $N_{W_{\mathbb{I}}}(L_\Phi)$. On the other hand, by (3) of Definition 4.3.1 all reflection hyperplanes of $W_t^{a,\vee}$ correspond to such zeroes of the Plancherel density μ^{L_Φ} via Φ_T. This implies that the simple affine reflections in P_L associated with the faces of P_L are restrctions to P_L of elements of the normalizer of P_L in $W_{\mathbb{I}}^{a,\vee}$.

(iv) By (iii), the material in [Lus4, Section 2] applies. We see that if P_L corresponds to the set $J \subset I$ (with I an index set for the set of nodes of the Kac diagram $D(\mathfrak{g}^\vee, \theta)$), then $J \subset I$ is an *excellent* subset, and the set of affine simple reflections of $W^* \simeq (W_t^{a,\vee}, S_t^{a,\vee})$ is in natural bijection with $K = I \backslash J$. Hence the set I decomposes in a subset J correspond to the affine simple roots constant which are constant on L, and $K = I \backslash J$, the which is in canonical bijection with the set $\tilde{K}$ of simple reflections of Σ_t^{spec} via the isomorphism $(W_t^{a,\vee}, S_t^{a,\vee}) \to W_L^*$ induced by Φ.

(v) Assign weights to the nodes of I: If $i \in I$ let $a_i^\vee$ be the corresponding affine simple Kac root, and let $Da_i^\vee$ be its gradient, viewed as character of $T_{\mathbb{I}}$ (defined over Λ_0).

We define $w_i(s) := D(a_i^\vee)(\Phi(s))$, viewed as a function of $s \in T_{\tilde{t},v}$ (the real vector group $T_{\tilde{t},v} \subset T_{\tilde{t}}$). We note that w_i is independent of $s \in T_{\tilde{t},v}$ if and only if $i \in J$, and that $\prod_{i \in I} w_i^{n_i} = 1$, where n_i are such that $\sum_{i \in I} n_i a_i^\vee = 1$.

Observe that for all $j \in J$, $w_j = v^{\mathbf{d}_j}$ for certain $\mathbf{d}_j \in \mathbb{Z}_{\geq 0}$, and these coordinates determine the residue point $r_L \in T_L = T_J$ modulo the action of $(\Omega_{J,ad})^*$.

We identify $\tilde{K}$ and K via the isomorphism of (iv). Suppose that $\sum_{k \in K} \tilde{n}_k b_k^\vee = 1$ is the affine relation for the affine simple roots of Σ_t^{Spec}. It turns out that $\mathbf{z}_k := n_k/\tilde{n}_k \in \mathbb{N}$ for all $k \in \tilde{K} = K$. We then have for all $k \in K$ that $w_k(s) = \zeta_k v^{c_k}(Db_k^\vee(s))^{1/\mathbf{z}_k}$, where $b_k^\vee$ is the affine root of $(W_t^{a,\vee}, S_t^{a,\vee})$ corresponding to k via Φ. Moreover, if k_0 denotes

the unique affine extension root of Σ_t^{Spec} then ζ_{k_0} is a primitive n_{k_0}-th root of unity, $\zeta_k = 1$ if $k \neq k_0$, and $c_k \in \mathbb{Z}$ can be computed by $Da_k^\vee(c_L) = v^{c_k}$.

(vi) We call the diagram $D(\mathfrak{g}^\vee, \theta)$ so obtained, with the weights w_i attached, the diagram $D(\Phi)$ of the STM Φ. Note that Φ is determined by $D(\Phi)$, since $\tilde{\Phi}$ is.

(vii) Given such a diagram, to check that the map it defines is an STM comes down to checking first of all that the constant weights w_j on J define a residue point of T_J, and secondly that (3) of Definition 4.3.1 holds. The latter verification is straightforward but cumbersome (a lot of cancellations are taking place).

We claim that the diagram $D(\Phi)$ without the weights w_i, but with the subsets J and K remembered, is the "geometric diagram" which Lustig [Lus4], [Lus5] attaches to the arithmetic diagram of $\mathcal{H}_{\tilde{t}}^e$.

Conversely, given the "geometric diagram" Lusztig attaches to the arithmetic diagram of $\mathcal{H}_{\tilde{t}}^e$. Now one needs to assign the appropriate weights w_i to the nodes of the diagram so that it becomes the diagram of an STM. This is done as follows. From $(\mathbb{P}, \sigma)$ we obtain a corresponding cuspidal pair (M, δ) with a Levi subgroup $M \subset G$ and a supercuspidal representations of unipotent reduction δ of M/A_M (with A_M the maximal split torus in $Z(M)$). Note that M/A_M is again of adjoint type. Thus by the cuspidal case of Theorem 4.4.1 (which we have established already for exceptional groups) we obtain a *unique* $X_{wur}(M)$-orbit of central characters $W_M r_M$ with $r_M \in T_M$ a residue points for M/A_M associated to δ such that (4.3) holds (for (M, δ) instead of (G, π)). Let $L = r_M T^M \subset T_{\mathbb{I}}$ be the corresponding residual coset. Using $W_{\mathbb{I}}$ we may assume that L is in the position as in (a),(b). We observe that $X_{wur}(M)/(T_M \cap T^M)$ is a quotient of $X_{wur}(G)$. Thus (M, δ) determines up to the action of $W_{\mathbb{I}}$ a unique $X_{wur}(G)$-orbit of residual subspaces, and we have a representative $L = r_M T^M \subset T_{\mathbb{I}}$ of this orbit in the position described in (b) above. We now assign weights w_i to the nodes of $D(\mathfrak{g}^\vee, \theta)$, exactly as has been described above in (a)-(g), see [Opd5, 3.2.4; 3.2.5]. This defines a map $\Phi : T_{\tilde{t}} \to L_n$. Finally one needs to verify the property (3) of Definition 4.3.1 for this map, which amounts to case by case computations [Opd5].

At this stage we have established the existence of enough STMs, Theorem 4.4.1(a).

Next we now look at the essential uniqueness statement Theorem 4.4.1(b) for $\Phi : \mathcal{H}_{\tilde{t}}^e \rightsquigarrow \mathcal{H}_{\mathbb{I}}(G^*)$ This reduces to the cuspidal case using STM diagrams as follows: By Theorem [Opd4, Proposition 7.13] if suffices to show that one can find a $\omega \in X_{wur}(G)$ such that $\Phi'_{\tilde{t}}$ and $m_\omega \circ \Phi_{\tilde{t}}$ have the same image (in the sense of [Opd5, Definition 5.10]). Assume that the existence and uniqueness property (b) for the cuspidal case has been solved already, and let $\mathcal{H}_{\tilde{t}}^e$ be given. Let (M, δ) be the cuspidal pair associated to t by [Mo], then $\mathbb{P}_M = M \cap \mathbb{P}$ is a maximal parahoric subgroup of M, and $\sigma_M = \mathbb{P}|_{\mathbb{P}_M}$ is a cuspidal unipotent character for $\mathbb{P}_M$. By the already established cuspidal case of Theorem 4.4.1(a),(b), equation (4.3) determines a unique $X_{wur}(M)$-orbit of W_M-orbits $W_M r_M$ of residual points $r_M = s_M c_M$ for $\mathcal{H}_{\mathbb{I}_M}(M^*)$. Let $\tilde{J}$ be the maximal proper subset of the spectral diagram $D(\mathfrak{m}, \theta_M)$ of $\mathcal{H}_{\mathbb{I}_M}(M^*)$ which corresponds to r_M, i.e. the set of affine simple roots of $D(\mathfrak{m}, \theta_M)$ which are trivial on s_M. Then it turns out that in

all cases, $\tilde{J}$ fits as excellent subdiagram of $D(\mathfrak{g}^\vee, \theta)$, with image $J \subset D(\mathfrak{g}^\vee, \theta)$ which is *unique* up to the action of $X_{wur}(G)$. This determines T_L and the image $W_0 r_L \in T_L \subset T_{\mathbb{I}}$ of $W_M r_M \in T_{\mathbb{I}_M}$, up to the action of $X_{wur}(G)$. Hence the image $L = r_L L$ of Φ is uniquely determined by $t = (\mathbb{P}, \sigma)$ up to the action of $X_{wur}(G)$. By [Opd4, Proposition 7.13] this proves the desired uniqueness.

The above reasoning reduces the general uniqueness statement Theorem 4.4.1(b) to the existence and uniqueness 4.4.1(a), (b) for the cuspidal case. This existence and uniqueness for the cuspidal case was shown in general in [FO] (using the general existence statement Theorem 4.4.1(a)), thereby finishing the proof of Theorem 4.4.1(b).

The existence and essential uniqueness of a $X_{wur}(G)$-equivariant STM, and its compatibility to Lusztig's parameterisation (Theorem 4.4.1(c), (d)) also reduces to the cuspidal case, by using the same construction of "inducing higher rank STMs from cuspidal ones" as discussed above. In the cuspidal case, these statements follow from a case by case verification [FO], [FOS]. This finishes the discussion of the proof of Theorem 4.4.1.

4.5. **Main Theorem.** The following Theorem is the main result. It is a generalisation of results of [R3], [Opd5], [FO], [FOS].

Theorem 4.5.1. *Let $G = \mathbf{G}(F)$ be the group of points of a connected reductive group $\mathbf{G}$ defined over a non-archimedean local field F which splits over an unramified extension of F. Let $\hat{G}_{uni}^{\text{temp}} = \sqcup_{s \in \mathcal{B}_{uni}(G)} \hat{G}_s^{\text{temp}}$ denote the set of equivalence classes of irreducible tempered representations of G of unipotent reduction. Here $\mathcal{B}_{uni}(G) \subset \mathcal{B}(G)$ is the finite subset of unipotent inertial equivalence classes, and for each $s \in \mathcal{B}_{uni}(G)$, $\hat{G}_s^{\text{temp}} \subset \Pi(G)_s$ is the subsets of equivalence classes of tempered irreducible representations in $\mathcal{C}(G)_s$. For a unipotent type $t = (\mathbb{P}, \sigma)$, $\mathcal{B}^t(G)$ is an $X_{wur}(G)$-orbit in $\mathcal{B}_{uni}(G)$. For $s \in \mathcal{B}^t(G)$ choose a corresponding extension $\tilde{t}(s)$ of t to $N_G(\mathbb{P})$, and extend this choice $X_{wur}(G)$-equivariantly. Let $\mathcal{H}_t$ be the Hecke algebra of the type t, then $\mathcal{H}_t = \oplus_{s \in \mathcal{B}^t(G)} \mathcal{H}_{\tilde{t}(s)}^e$. Put $\mathcal{H}_{uni}(G) := \oplus_{s \in \mathcal{B}_{uni}(G)} \mathcal{H}_{\tilde{t}(s)}^e$. Let $\mathcal{H}_{\mathbb{I}} = \mathcal{H}_{\mathbb{I}}(G^*)$ be the Iwahori Hecke algebra of the quasi-split group G^* in the inner class of G.*

(a) *There exists a parameterisation $\varphi : \hat{G}_{uni}^{\text{temp}} \to \Phi_{nr}^{\text{temp}}(G)$, $\pi \to \varphi_\pi$ such that:*
   (1) *Conjectures 2.6.1 and 2.6.2 hold, up to constant factors independent of q.*
   (2) *For every unipotent type t of G there exists a morphism $\Phi_Z^t : \text{Spec}(Z(\mathcal{H}_t)) \to \text{Spec}(Z(\mathcal{H}_{\mathbb{I}}))$ such that the map $\gamma^{\mathbb{I}} \circ \varphi : \hat{G}^{t,\text{temp}} \to S_{\mathbb{I}} \subset \text{Spec}(Z(\mathcal{H}_{\mathbb{I}}))$ factorises as $\gamma^{\mathbb{I}} \circ \varphi = \Phi_Z^t \circ \beta^t \circ \text{cc}$.*
   The morphism Φ_Z^t is associated to an STM $\Phi^t : \mathcal{H}_t \rightsquigarrow \mathcal{H}_{\mathbb{I}}$ satisfying the conditions (2)(a) and (b) of Corollary 4.3.5.
(b) *Such a parameterisation is unique up to automorphisms $\alpha \in \text{Aut}(\mathcal{H}_{uni})$ such that $\varphi' := \varphi \circ \alpha^*$ is again a parameterisation.*
(c) *If G is of adjoint type then Lusztig's enhanced parameterisation [Lus4, Lus5] $\tilde{\varphi}_{Lu} : \hat{G}_{uni}^{\text{temp}} \to \tilde{\Phi}_{nr}^{\text{temp}}(G)$ satisfies the conjectures 2.6.1 and 2.6.2 of Hiraga, Ichino and Ikeda [HII], and moreover satisfies property (a)(2) above. (For general unramified G: see Theorem 3.8.2.)*

Proof. (c) For G simple and of adjoint type, Conjecture 2.6.1 for discrete series representations is [Opd5, Theorem 4.11]. The proof is based on Theorem 4.4.1 (to show

the validity modulo rational constant factors), and the precise computation of the rational constant $d_{\mathcal{H},\delta}$ for the discrete series of affine Hecke algebras based on Theorem 3.7.3. Obviously this implies the result for all semisimple groups G of adjoint type. If G is of adjoint type and $M \subset G$ is a Levi subgroup with maximal split central torus $A_M \subset Z_M$, then M/A_M is also of adjoint type. Hence Conjecture 2.6.1 also holds for the discrete series of M modulo the center and of unipotent reduction. Thus Conjecture 2.6.2 follows from the Plancherel formula of Harish-Chandra [Wal, Théorème VIII.1.1]. By Theorem 4.4.1 (see also [Opd5, Theorem 3.4]), Lusztig's parameterisation $\varphi_{Lu} : \hat{G}_{uni}^{\text{temp}}(G) \to \Phi_{nr}^{\text{temp}}(G)$ of the irreducible tempered representations of unipotent reduction gives rise to a spectral transfer map $\Phi_{uni} : \mathcal{H}_{uni}(G) \rightsquigarrow \mathcal{H}_{\mathbb{I}}(G^*)$. By [Opd4, Theorem 6.1] and [BKH] this implies that, up to constants independent of q, the map $\gamma^{\mathbb{I}} \circ \varphi_{Lu} : \hat{G}_{uni}^{\text{temp}} \to S_{\mathbb{I}} \subset \text{mSpec}(Z(\mathcal{H}_{\mathbb{I}}(G^*)_\mathbf{v}))$ also satisfies (a)(2).

(a) and (b): For an absolutely semisimple group G of adjoint type, (c) implies the existence of a parameterisation $\pi \to \varphi_\pi$ of tempered representations of unipotent reduction satisfying (a)(1) and (2). Let us also prove the uniqueness property (b) for such G. Given a parameterisation satisfying (a)(1) and (2) it follows from Definition 4.0.2(iv) of the parameterisation and [Bo, Proposition 8.4], that we may assume that G is absolutely simple. For all unipotent types t of G, the morphism Φ_Z^t defines a spectral transfer map $\Phi^t : \mathcal{H}_t \rightsquigarrow \mathcal{H}_{\mathbb{I}}$ by definition. Indeed, by Definition 4.0.2 (i), (iii) and (v) of a parameterisation, the morphism Φ_Z^t comes from an affine morphism of algebraic tori $\Phi^t : T^t = X_{nr}(M) \to L'$ where L' is an intermediate quotient $L \to L' \to L_n$ of a residual coset $L \subset T_{\mathbb{I},\mathbf{v}}$ (indeed, the points of L' are in bijection with the orbit of $X_{nr}(M)$ twists of δ, which is a finite quotient of $X_{nr}(M)$, and L_n corresponds by definition to the central character map for representations of M of this type, hence is a quotient of L'). By (a)(1), [BKH], Theorems 3.5.1 and 3.5.2, and [Opd4, Definition 5.1] it follows that Φ^t defines a spectral transfer map $\mathcal{H}_t \rightsquigarrow \mathcal{H}_{\mathbb{I}}$. By Definition 4.0.2(i) and by Theorem 4.4.1 or [Opd5, Theorem 3.4] the required uniqueness property follows. Hence we have established the existence and uniqueness of parameterisations satisfying (a)(1) and (2) for G absolutely simple of adjoint type.

Let us now show (a) and (b) when G is connected reductive with anisotropic center. We write $\mathbf{G} = \mathbf{Z}^0\mathcal{D}(\mathbf{G})$ with $\mathcal{D}(\mathbf{G})$ the derived subgroup (which is a connected semisimple group) and $\mathbf{Z}^0$ an anisotropic torus. Then the quotient $\mathbf{Z} = \mathbf{G}/\mathcal{D}(\mathbf{G})$ is an isogenous quotient of $\mathbf{Z}^0$, in particular an anisotropic torus. Let $\mathbf{G}_{ad}$ be the adjoint quotient of $\mathbf{G}$, a connected semisimple group of adjoint type. Now consider the isogeny $\psi : \mathbf{G} \to \mathbf{Z} \times \mathbf{G}_{ad}$, and the corresponding dual isogeny $\psi^\vee : Z^\vee \times G_{sc}^\vee \to G^\vee$. Let $T^\vee$ be a maximal torus of $G^\vee$, and $T_{sc}^\vee$ its inverse image in $G_{sc}^\vee$. The kernel ψ can be expressed as $\text{Hom}(K_G^*, F_{ur}^\times)^{\text{Fr}}$, where $K_G = X^*(T_{sc}^\vee)/X^*(T^\vee)$.

We have a surjection with finite kernel $^L(Z^\vee \times G_{sc}^\vee) \to {}^L G$. Now $\Phi_{nr}^{\text{temp}}(Z \times G_{ad}) = \Phi_{nr}^{\text{temp}}(Z) \times \Phi_{nr}^{\text{temp}}(G_{ad}) \simeq \Phi_{nr}^{\text{temp}}(G_{ad})$, since for the anisotropic torus Z, $\Phi_{nr}^{\text{temp}}(Z) = \{[\varphi_{triv}]\}$. The corresponding map $\Phi_{nr}^{\text{temp}}(Z \times G_{ad}) \to \Phi_{nr}^{\text{temp}}(G)$ can be identified, via these isomorphisms, with the natural map $\Phi_{nr}^{\text{temp}}(G_{ad}) \to \Phi_{nr}^{\text{temp}}(G)$. Via the isomorphism of Theorem 3.8.1 this map comes from the covering map $T_{sc}^\vee/(1-\theta)T_{sc}^\vee \to T^\vee/(1-\theta)T^\vee$

applied to $S_{sc,\mathbb{I},\mathbf{v}}$. Therefore this map is surjective, with fibres that are orbits for the canonical image $I_G \subset (\Omega_{sc}^\theta)^*$ of the finite abelian group $(K_G^\theta)^*$ in $X_{wur}(G_{ad}) = (\Omega_{sc}^\theta)^*$. Then I_G is the kernel of the canonical surjection $(\Omega_{sc}^\theta)^* \to (\Omega_G^\theta)^*$, and we have an identification $\Phi_{nr}^{\mathrm{temp}}(G) = I_G \backslash \Phi_{nr}^{\mathrm{temp}}(G_{ad})$.

We propose the following parameterisation $\hat{G}_{uni}^{\mathrm{temp}} \to \Phi_{nr}^{\mathrm{temp}}(G)$. Given an irreducible tempered representation (π, V_π) of unipotent reduction of G. The kernel $\mathrm{Hom}(K_G^*, F_{nr}^\times)^{\mathrm{Fr}}$ of the isogeny $G \to Z \times G_{ad}$ acts trivially in V_π, hence (π, V_π) descends to a representation of the image of G in $Z \times G_{ad}$. Since the image in Z acts trivially on V_π, we can extend to a representation $\overline{\pi}$ of $Z \times \mathrm{Im}(G)$ (with trivial action of the Z-factor) where $\mathrm{Im}(G) \subset G_{ad}$. We *claim* that $\overline{\pi}$ is a summand of the restriction of a representation $\tilde{\pi}$ of unipotent reduction of G_{ad}, and that this uniquely determines the Lusztig-Langlands parameter $\varphi_{\tilde{\pi}}$ of $\tilde{\pi}$ up to the action of I_G. We define, in view of the above identification, $\varphi_\pi := I_G \varphi_{\pi_{ad}}$. In order that this definition makes sense we need to verify the above claim.

Let $t = (\mathbb{P}, \delta)$ be a "unipotent type" for the unipotent Bernstein component of G, where $\mathbb{P}$ is a parahoric subgroup, δ a cuspidal unipotent of $\mathbb{P}$, and $\tilde{\delta}$ an extension of δ to $N_G \mathbb{P}$. The Hecke algebra $\mathcal{H}_t$ of t is a direct sum of extended affine Hecke algebras $\mathcal{H}_{\tilde{t}}^e$. We have (see [PR]) $\mathbb{P} = G_1 \cap N_G \mathbb{P}$, and this group is self-normalizing within G_1, the kernel of the Kottwitz homomorphism w_G. The set of Bernstein components described by t corresponds bijectively to the set of extensions $\tilde{t} = (N_G \mathbb{P}, \tilde{\delta}')$ of δ to $N_G \mathbb{P}$. This is a torsor for the character group $(\Omega_G^{\mathbb{P},\theta})^*$ of the abelian group $\Omega_G^{\mathbb{P},\theta}$, the subgroup of $\Omega_G^\theta = G/G_1$ fixing $\mathbb{P}$ under conjugation. The subalgebra $\mathcal{H}_t^a$ of the extended affine Hecke algebra $\mathcal{H}_{\tilde{t}}^e$ of functions with support in G_1 is the (unextended) underlying affine Hecke algebra, by the arguments of Lusztig [Lus4, 1.10–1.20]. The Hecke algebra $\mathcal{H}_t$ of the type t is isomorphic to $\mathcal{H}_t^a \# (\Omega_G^{\mathbb{P},\theta})^*$. This algebra can be written as a direct sum of the various extended affine Hecke algebras $\mathcal{H}_{\tilde{t}}^e$ associated with the extensions $\tilde{t}$. This is a direct sum of mutually isomorphic extended affine Hecke algebras $\mathcal{H}_{\tilde{t}}^e$ whose set of summands form a torsor for the group of characters of the subgroup $(\Omega_G^{\mathbb{P},\theta})_1$ of $\Omega_G^{\mathbb{P},\theta}$ of elements acting trivially on $\mathcal{H}_t^a$. In any case, one obtains a canonical action of $(\Omega_G^{\mathbb{P},\theta})^*$, and thus of $(\Omega_G^\theta)^*$, on the disjoint union of the centers of the summands $\mathcal{H}_{\tilde{t}}^e$ of $\mathcal{H}_t$.

There is a bijection between the set of parahoric subgroups of G and of $Z \times G_{ad}$ defined as follows. First choose a maximally F-split maximal F-torus T of G which splits over F_{ur}. Let $\mathcal{A}$ be the apartment of G associated with T, which can be embedded in the apartment of $\mathbf{G}(F_{ur})$ as the set of Fr-invariant elements. By [PR, Definition 1; Remark 16], [Opd5, Corollary 2.2], the stabilizer of a facet f in $\mathcal{A}$ in $G_1 = \mathbf{G}(F_{ur})^{\mathrm{Fr}}$ is also the pointwise fixator of f in G_1, since $\mathbf{G}(F_{ur})$ is generated by $\mathbf{G}_{der}(F_{ur})$ and elements acting trivially on $\mathcal{A}$. Hence by [PR] it follows that the parahoric subgroup $\mathbb{P}_f$ of G is also equal to the normalizer $\mathbb{P}_f = N_{G_1}(\mathbb{P}_f \cap G_{der})$. Indeed, $\subset$ follows since $\mathbb{P}_f$ is self-normalizing in G_1, while $\supset$ follows since the right hand side is stabilizing, hence pointwise fixing f. By the functoriality of Kottwitz's map w_G we have a homomorphism $G_1 \to (Z \times G_{ad})_1$. The bijection between the parahoric subgroups is defined as follows: Given $\mathbb{P}$ of G, we have $\mathbb{P}' := N_{(Z \times G_{ad})_1}(\psi(\mathbb{P}))$ is a parahoric group of $Z \times G_{ad}$, as follows from the discussion

above and the remark that $(Z \times G_{ad})_{der} \subset \psi(G)$; conversely if $\mathbb{P}' \subset Z \times G_{ad}$ is parahoric then $\mathbb{P} := \psi^{-1}(\mathbb{P}') \cap G_1$ is parahoric in G. These maps are inverse to each other. Moreover, the cuspidal unipotent representations of $\mathbb{P}$ and $\mathbb{P}'$ correspond bijectively to each other, since it is known that the set of cuspidal unipotent characters of a connected reductive group $\mathcal{G}$ over a finite field is independent of the type of $\mathcal{G}$ within its isogeny class (cf. [Lus1]). It follows that the type $t_1 := (\mathbb{P}, \delta)$ of G corresponds to a unique type $\tilde{t}_1 = (\mathbb{P}', \delta')$ of $Z \times G_{ad}$, and the affine Hecke algebras $\mathcal{H}'_{t_1}$ and $\mathcal{H}'_{\tilde{t}_1}$ are isomorphic unextended affine Hecke algebras via ψ, with the same normalization of their respective traces.

Now let us return to the verification of the above claim. Consider an irreducible tempered representation (π, V_π) of G of unipotent reduction, and let $\tilde{t} = (N_G\mathbb{P}, \tilde{\delta})$ be an extension of a unipotent type t associated to the Bernstein component to which π belongs. The induction of $\overline{\pi}$ to $Z \times G_{ad}$ is a unitary tempered representation of $Z \times G_{ad}$ with finitely many irreducible summands $\tilde{\pi}$, and this gives an obvious construction of tempered irreducible representations $\tilde{\pi}$ which contain $\overline{\pi}$ when restricted to the image of G in $Z \times G_{ad}$. By restriction to $\psi : G_1 \to (Z \times G_{ad})_1$ it follows that an extension $\tilde{t}$ describing the Bernstein component of $\tilde{\pi}$ must restrict to $\tilde{t}_1 = (\mathbb{P}', \delta')$ as described above.

Hence $\tilde{t}$ is of the form $\tilde{t} = (N_{Z \times G_{ad}}\mathbb{P}', \tilde{\delta}')$. As before, the algebra $\mathcal{H}_{\tilde{t}_1}$ is a direct sum of isomorphic extended affine Hecke algebras. Via ψ we can view $\mathcal{H}_{t_1}$ as a subalgebra of $\mathcal{H}_{\tilde{t}_1}$, and by the Bernstein description of the center of an affine Hecke algebra it follows that $Z(\mathcal{H}_{t_1}) \subset Z(\mathcal{H}_{\tilde{t}_1})$ is a subalgebra of finite index. The kernel of the corresponding surjective homomorphism $\mathrm{Spec}(Z(\mathcal{H}_{\tilde{t}_1})) \to \mathrm{Spec}(Z(\mathcal{H}_{t_1}))$ is the image of $I_G \subset (\Omega_{sc}^\theta)^*$ under the surjection $(\Omega_{sc}^\theta)^* \to (\Omega_{sc}^{\mathbb{P}, \theta})^*$, and the natural action by the group $(\Omega_{sc}^\theta)^*$ on the spectrum of $\mathrm{Spec}Z(\mathcal{H}_{\tilde{t}_1})$ corresponds to the natural action of $(\Omega_G^\theta)^*$ on $\mathrm{Spec}(Z(\mathcal{H}_{\tilde{t}_1}))$. Clearly $cc(\tilde{\pi})$ belongs to the I_G-orbit of central characters in the fibre above $cc(\pi)$ under this map.

Now recall that for the representation $\tilde{\pi}$ of G_{ad} (or $Z \times G_{ad}$) the Langlands parameter $\varphi_{\tilde{\pi}}$ is defined by $\varphi_{\tilde{\pi}} := (\gamma^{\mathbb{I}})^{-1}(\Phi_Z(cc^t(m_t(\tilde{\pi}))))$, where $\Phi : \mathcal{H}_{uni}(G_{ad}) \rightsquigarrow \mathcal{H}_{\mathbb{I}}(G_{ad}^*)$ is the spectral transfer map of [Opd5, Theorem 3.4]. We just explained above that $cc^t(m_t(\tilde{\pi}))$ belongs to a single I_G-orbit, and then the $X_{wur}(G_{ad}) := (\Omega_{sc}^\theta)^*$-equivariance of Φ and of $\gamma^{\mathbb{I}}$ (by Lemma 4.0.1) finally establishes the claim.

Hence for G connected reductive with anisotropic kernel, we have now shown existence of a Langlands correspondence $\varphi : \hat{G}_{uni}^{temp} \to \Phi_{nr}^{temp}(G)$ which obviously satisfies (a)(1) (we reduced this to the case G_{ad} where we know (a)(1) already, via spectral transfer maps which preserve Plancherel densities up to constants by [Opd4]) and (a)(2) (since spectral transfer maps yield such morphisms by definition). The uniqueness property (b) follows from the case of G_{ad} if we can show that there always exists a lift of a $X_{wur}(G) = I_G \backslash X_{wur}(G_{ad})$-equivariant parameterisation $\varphi : \hat{G}_{uni}^{temp} \to \Phi_{nr}^{temp}(G)$ to a $X_{wur}(G_{ad})$-equivariant parameterisation $\varphi_{ad} : (\hat{G}_{ad})_{uni}^{temp} \to \Phi_{nr}^{temp}(G_{ad})$. For this we need to invoke a stronger form of the uniqueness property of STMs $\Phi^t : \mathcal{H}_t \to \mathcal{H}_{\mathbb{I}, \mathbf{v}}(G_{ad})$ (where t is an (extended) unipotent type which represents a Bernstein component of representations of unipotent reduction for an inner form of G_{ad}) as explained in [Opd5,

3.2.8] and [Opd4, Proposition 7.13]: If $\Phi_1^t : \mathcal{H}_t \to \mathcal{H}_{\mathbb{I},\mathbf{v}}(G_{ad})$ is another such STM, then there exists an isomorphism $\alpha : \mathcal{H}_t \to \mathcal{H}_t$, $w \in W_{\mathbb{I},0}$, and $\omega \in X_{wur}(G_{ad}) = (\Omega_{sc}^\theta)^*$ such that $\Phi_1^t = \omega \circ w \circ \Phi^t \circ \alpha$. It follows that any two matching I_G-orbits of connected components under φ must also correspond under any equivariant correspondence $\tilde{\varphi}_{ad}$ (which we know exists) up to the action of $X_{wur}(G_{ad})$. In this way we can compose $\tilde{\varphi}_{ad}$ with an element of $X_{wur}(G_{ad})$ to obtain an I_G-equivariant lifting of φ defined on this I_G-orbit of components. Clearly we can do this for any I_G-orbit, thus defining the desired equivariant lifting φ_{ad} of φ.

Finally let us consider the general case. Let G be connected reductive and let A be the maximal F-split torus in the center of G. Then $H = G/A$ is the group of F-points of the quotient $\mathbf{G}/\mathbf{A}$, whose center is anisotropic. In particular the functoriality of the Kottwitz homomorphism implies that G_1 maps surjectively to H_1.

Given a tempered irreducible representation π of unipotent reduction of G, choose a $x \in X_{wur}(G)$ such that $x^{-1} \otimes \pi$ is trivial on A. Hence $x^{-1} \otimes \pi$ descends to a tempered irreducible representation of unipotent reduction $\overline{x^{-1} \otimes \pi}$ of H. Let $\varphi_{\overline{x^{-1} \otimes \pi}} \in \Phi_{nr}^{\text{temp}}(H)$. Let $p_G(\varphi_{\overline{x^{-1} \otimes \pi}})$ in $\Phi_{nr}^{\text{temp}}(G)$ be the image of $\varphi_{\overline{x^{-1} \otimes \pi}}$ under the canonical map $^L H \to {}^L G$. Now define $\varphi_\pi := x.p_G(\varphi_{\overline{x^{-1} \otimes \pi}}) \in \Phi_{nr}^{\text{temp}}(G)$. In order that this makes sense, we need to show the independence of the choice of $x \in X_{wur}(G)$ such that $x^{-1} \otimes \pi$ is trivial on A. Suppose that $z \in X_{wur}(G)$ such that $z|_A = 1$. Then z restricts to $\overline{z}$, a character of H. Since G_1 surjects onto H_1, we see that $\overline{z} \in X_{wur}(H)$. Hence if we would replace x by zx then we would get $\varphi_\pi' := (zx).p_G((\varphi_{\overline{(z^{-1}x^{-1}) \otimes \pi}}))$. The equivariance for $X_{wur}(H)$ of the parameterisation of H and of p_G (which is obvious) implies that $\varphi_\pi' = \varphi_\pi$. Hence our definition is sound, $X_{wur}(G)$-equivariant by construction, and is the only possible extension of a given equivariant parameterisation for H. The properties (a)(1), (a)(2) and (b) easily follow.

This finishes the proof of the existence and essential uniqueness of STM's $\Phi_Z^t :$ $\text{Spec}(Z(\mathcal{H}_t)) \to \text{Spec}(Z(\mathcal{H}_{\mathbb{I}}))$ in the general case for all unipotent types t, satisfying the conditions (2)(a) and (b) of Corollary 4.3.5. We finally need to show that the corresponding map $\hat{G}_{uni}^{\text{temp}} \to \Phi_{nr}^{\text{temp}}$, $\pi \to \varphi_\pi$ defines a parameterisation in the sense of Definition 4.0.2. Of course, in conditions (iii) and (v) where two such maps play a role we need to allow for the fact that the maps are only uniquely determined up to certain spectral automorphisms of $\mathcal{H}_{uni}$, which means that we may need to choose the relevant morphisms Φ_Z^t judiciously.

The conditions (i), (ii) and (vi) hold because Φ_Z^t satisfies conditions (2)(a) and (b) of Corollary 4.3.5. Furthermore condition (iv) follows easily from [Bo, Proposition 8.4].

Let us show that condition (iii) (the compatibility with unitary parabolic induction) holds. Let $\pi \in \hat{G}_{uni}^{\text{temp}}$ be a tempered representation of unipotent reduction of G, and suppose that π is a summand of $i_P^G(\pi')$ (unitary induction), for an F-parabolic subgroup $P = HU$ with H a standard F-Levi subgroup of G, and $\pi' \in \hat{H}_{uni}^{\text{temp}}$. Let $\varphi_{\pi'}^H \in \Phi_{nr}^{\text{temp}}(H)$ be the parameter for π', defined by a morphism $\Phi_Z^{t_H}$ as in (a) (thus originating from an STM $\Phi^{t_H} : \mathcal{H}_{t_H} \rightsquigarrow \mathcal{H}_{\mathbb{I}}(H^*)$ which satisfies the conditions (2)(a) and (b) of Corollary 4.3.5) for H). Then we want to show that we can find such a morphism $\Phi_Z^{t_G}$ as in (a)

for G, such that it attaches the parameter $\varphi_{\pi'}^H \in \Phi_{nr}^{temp}(H) \subset \Phi_{nr}^{temp}(G)$ to π. In order to do so, suppose that $\pi \in \hat{G}_{t_G}^{\text{temp}}$ for some unipotent type t_G for G (such type exists by [Mo],[MP1]). Then t_G is a G-cover of a cuspidal unipotent type t_M for a cuspidal pair (M, τ) in the cuspidal support of π ([BK], [Mo], [MP2]), where we choose $M \subset G$ to be a standard parabolic subgroup. Since the cuspidal support of π' equals that of π, we see that we may assume $M \subset H$, and that we have an intermediate H-cover t_H of t_M such that $\pi' \in \Pi(H)^{t_H}$. In this situation we have ([BK], [Mo])) an injective homomorphism $j : \mathcal{H}_{t_H} \to \mathcal{H}_{t_G}$ (a "parabolic subalgebra" in the sense of [Opd2]) inducing a canonical embedding of commutative algebras $Z(\mathcal{H}_{t_G}) \subset Z(\mathcal{H}_{t_H})$. Similarly we have $\mathcal{H}_{\mathbb{I}}(H^*) \subset \mathcal{H}_{\mathbb{I}}(G^*)$ (likewise a parabolic subalgebra), inducing a canonical embedding $Z(\mathcal{H}_{\mathbb{I}}(G^*)) \subset Z(\mathcal{H}_{\mathbb{I}}(H^*))$. The morphisms defined by induction on the spaces of central characters of these Hecke algebras are given by these inclusion maps.

It follows from [Opd5, 3.1.1] that $\Phi^{t_H} : \mathcal{H}_{t_H} \to \mathcal{H}_{\mathbb{I}}(H^*)$ is induced form a cuspidal STM $\Phi'_M : \mathcal{H}_{t_M} \to \mathcal{H}_{\mathbb{I}}(M^*)_M$. Similarly Φ_{t_G} (whose existence we have established in the first part of the proof) is induced from a cuspidal STM $\Phi_M : \mathcal{H}_{t_M} \to \mathcal{H}_{\mathbb{I}}(M^*)_M$. But it is clear that Φ^{t_G} also defines an STM $\mathcal{H}_{t_H} \rightsquigarrow \mathcal{H}_{\mathbb{I}}(H^*)$ satisfying the conditions (2)(a) and (b) of Corollary 4.3.5 for H (defined by the same map as Φ^{t_G} on $T_{t_H} = T_{t_G}$; indeed this clearly defines a spectral preserving map for the parabolic subalgebra $\mathcal{H}_{t_H}$ if it does so for $\mathcal{H}_{t_G}$). In particular, we may just as well choose $\Phi'_M = \Phi_M$. The resulting commuting diagram

$$\begin{array}{ccc}
\mathrm{Spec}(Z(\mathcal{H}_{t_H})) & \longrightarrow & \mathrm{Spec}(Z(\mathcal{H}_{t_G})) \\
{\scriptstyle \Phi_Z^{t_H}} \downarrow & & \downarrow {\scriptstyle \Phi_Z^{t_G}} \\
\mathrm{Spec}(Z(\mathcal{H}_{\mathbb{I}}(H^*))) & \longrightarrow & \mathrm{Spec}(Z(\mathcal{H}_{\mathbb{I}}(G^*)))
\end{array}$$

now easily shows the desired compatibility with unitary induction for the associated maps φ^H and φ^G.

The proof of (v) is comparable to that of (iv). Suppose that $\eta : H \to G$ is an F-morphism with abelian kernel and co-kernel. We want to show that if $\varphi \in \Phi_{nr}^{\text{temp}}(G)$ is given and $\pi \in \Pi_\varphi(G)^{\text{temp}}$, the pull-back of π to H is a finite direct sum of tempered irreducible representations in $\Pi_{^L\eta \circ \varphi}^{\text{temp}}(H)$, where $^L\eta : {}^LG \to {}^LH$ denotes the natural map.

We choose maximally F-split tori T_H and T_G such that $\eta(T_H) \subset T_G$. The induced map by η on the (absolute) root data of (H, T_H) and (G, T_G) is an isomorphism on the root systems, but the lattices may differ. This defines a bijective correspondence between the sets of standard F-Levi subgroups of H and G. Let M_G and M_H be matching standard Levi subgroups of G and H respectively. Then $\eta^M := \eta|_{M_H} : M_H \to M_G$ also has abelian kernel and co-kernel. The pull-back of a cuspidal character of M_G of unipotent reduction defines a cuspidal character of unipotent reduction of M_H, and conversely, every cuspidal character of unipotent reduction of M_H is obtained like this. Suppose that we have such matching unipotent inertial equivalence classes $\mathfrak{s}_H$ and $\mathfrak{s}_G$, and let t_H and t_G be unipotent types for H and G respectively covering the corresponding Bernstein components. Let $\Phi^{t_H} : \mathcal{H}_{t_H} \rightsquigarrow \mathcal{H}_{\mathbb{I}}(H^*)$ and $\Phi^{t_G} : \mathcal{H}_{t_G} \rightsquigarrow \mathcal{H}_{\mathbb{I}}(G^*)$ be corresponding STMs which satisfy the conditions (2)(a) and (b) of Corollary 4.3.5 (the existence and essential

uniqueness of such STMs was proved in the first part of the proof). By [Opd5, 3.1.1] these STMs are induced from cuspidal unipotent STMs $\Phi_{M_H} : \mathcal{H}_{t_{M_H}} \to \mathcal{H}_{\mathbb{I}}(M_H^*)_{M_H}$ and $\Phi_{M_G} : \mathcal{H}_{t_{M_G}} \to \mathcal{H}_{\mathbb{I}}(M_G^*)_{M_H}$ of the reductive groups $(M_H)_{ssa} := M_H/A_H^M$ and $(M_G)_{ssa} = M_G/A_G^M$ respectively, which are almost direct products of a semisimple and an anisotropic groupn (here A_G^M denotes the connected split center of M_G, and likewise A_H^M is the connected split center of M_H). Hence the latter Hecke algebras are all semisimple. By the equivariance property (2)(a) of Corollary 4.3.5, the STMs Φ^{t_H} and Φ^{t_G} are completely determined by Φ_{M_H} and Φ_{M_G} respectively. Note that η induces a spectral isomorphism $\Phi_\eta^t : \mathcal{H}_{t_{M_H}} \rightsquigarrow \mathcal{H}_{t_{M_G}}$ (since these Hecke algebras have rank zero, if an STM between them exists it is an isomorphism) and an embedding (in particular an STM–see [Opd4, 7.1.3]) $\Phi_\eta^{\mathbb{I}} : \mathcal{H}(M_H^*)_{M_H} \rightsquigarrow \mathcal{H}_{t_{M_G}}(M_G^*)_{M_G}$.

Because the root systems of G and H are isomorphic via η, it is clear that $\Phi'_{M_G} = \Phi_\eta^{\mathbb{I}} \circ \Phi_{M_H} \circ (\Phi_\eta^t)^{-1}$ also defines an STM $\mathcal{H}_{t_{M_G}} \to \mathcal{H}_{\mathbb{I}}(M_G^*)_{M_H}$ that can be induced to an STM $(\Phi^{t_G})' : \mathcal{H}_{t_G} \rightsquigarrow \mathcal{H}_{\mathbb{I}}(G^*)$. (Indeed, the μ-functions of the $\mathcal{H}_{\mathbb{I}}(G^*)$ and $\mathcal{H}_{\mathbb{I}}(H^*)$ are given by "the same" formula.) Thus we may and will from now on assume that $\Psi^{t_G} = (\Phi^{t_G})'$ is constructed like this. It follows easily that we have a commutative diagram of morphisms

$$
\begin{array}{ccc}
\mathrm{Spec}(Z(\mathcal{H}_{t_G})) & \xrightarrow{(\eta_Z^t)^*} & \mathrm{Spec}(Z(\mathcal{H}_{t_H})) \\
{\scriptstyle \Phi_Z^{t_G}}\downarrow & & \downarrow{\scriptstyle \Phi_Z^{t_H}} \\
\mathrm{Spec}(Z(\mathcal{H}_{\mathbb{I}}(G^*))) & \xrightarrow{(\eta_Z^{\mathbb{I}})^*} & \mathrm{Spec}(Z(\mathcal{H}_{\mathbb{I}}(H^*)))
\end{array}
$$

where η_Z^t and $\eta_Z^{\mathbb{I}}$ denote the natural morphisms which η induces on the centers of the unipotent affine Hecke algebras involved. By comparing the central characters of a tempered representation of $\mathcal{H}_{t_G}$ and a summand of its pull back via η the desired property (v) now follows in a straightforward fashion. $\square$

Remark 4.5.2. *In* [FOS, *Theorem 1.3*] *the conjecture* 2.6.1 *(including the rational constants) was proved for supercuspidal representations of unipotent reduction of semisimple groups. We hope that this can be extended to prove the conjectures* 2.6.1 *and* 2.6.2 *of* [HII] *for all tempered representations of unipotent reduction of a general connected reductive group over F (split over an unramified extension), using* [CO] *(in particular Theorem* 3.7.3*) and drawing on ideas from* [R5].*

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