DIVISORS ON HURWITZ SPACES: AN APPENDIX TO
‘THE CYCLE CLASSES OF DIVISORIAL MARONI LOCI’

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ABSTRACT. The Maroni stratification on the Hurwitz space of degree \(d\) covers of genus \(g\) has a stratum that is a divisor only if \(d - 1\) divides \(g\). Here we construct a stratification on the Hurwitz space that is analogous to the Maroni stratification, but has a divisor for all pairs \((d, g)\) with \(d \leq g\) with a few exceptions and we calculate the divisor class of an extension of these divisors to the compactified Hurwitz space.

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1. Introduction

The Hurwitz space \(H_{d,g}\) of simply-branched covers of genus \(g\) and degree \(d\) carries a stratification named after Maroni [7], which is defined as follows. If \(\gamma: C \to \mathbb{P}^1\) is a simply-branched cover one takes the dual of the cokernel of the natural map
\[
\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C,
\]
which is a vector bundle of rank \(d - 1\) on the projective line, hence is isomorphic to \(\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})\) for some \((d-1)\)-tuple \(\alpha = (a_1, \ldots, a_{d-1})\), where we assume that the \(a_i\) are non-decreasing. The loci of covers \(\gamma: C \to \mathbb{P}^1\) with fixed \(\alpha\) are the strata.

It is known (see [1]) that for general \(\gamma: C \to \mathbb{P}^1\) of genus
\[
g = k(d - 1) + s \quad \text{with} \quad 0 \leq s \leq d - 2
\]
the tuple \(\alpha\) takes the form \((k + 1, \ldots, k + 1, k + 2, \ldots, k + 2)\) with \(s\) entries equal to \(k + 2\). Only the case with \(s = 0\) yields a Maroni stratum that is a divisor (see [3] and [9, Thm. 1.15]).

In this paper we show how to define for the case that the genus \(g\) is not divisible by \(d - 1\) a stratification that has a stratum that is a divisor for \(g \geq d\) under exclusion of a few cases. If \(d - 1\) divides \(g\) then this reduces to the stratification of Maroni loci. It uses instead of the cokernel of \(\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C\) the cokernel of a natural map
\[
\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C(D),
\]

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where \( D \) is an appropriately chosen divisor of degree \( s \) with support in the ramification locus of \( \gamma \). The cycle classes of an extension of these divisors to the compactified Hurwitz space \( \tilde{H}_{d,g} \) can be calculated by using a global-to-local evaluation map \( p^*p_*V \rightarrow V \) of a vector bundle \( V \) on an extension of the \( \mathbb{P}^1 \)-fibration \( p: \mathbb{P} \rightarrow H_{d,g} \) to the compactified Hurwitz space that is trivial on the generic fibre of \( p \). The calculation and the answer are completely analogous to case of the cycle classes of the Maroni divisors calculated in [4]. The cycle classes are given in terms of an explicit sum of boundary divisors.

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2. The Setting

We recall the setting from [4]. We denote by \( \mathcal{P}_{d,g} \) the compactified Hurwitz space of admissible covers of degree \( d \) and genus \( g \). We have

\[
\mathcal{P}_{d,g} = \bigcup_{j, \mu} S_{j, \mu}.
\]

where the \( S_{j, \mu} = S_{b-j, \mu} \) are divisors indexed by \( 2 \leq j \leq b-2 \) and a partition \( \mu = (m_1, \ldots, m_n) \) of \( d \). These divisors can be reducible, but a generic point corresponds to an admissible cover \( \gamma: C \rightarrow P \), where \( P \) is a genus 0 curve consisting of two components \( P_1, P_2 \) of genus 0 intersecting in one point \( Q \) with \( j_1 = j \) or \( b-j \) branch points on \( P_1 \) (resp. \( j_2 = b-j \) or \( j \) branch points on \( P_2 \)) and the inverse image \( \gamma^{-1}(Q) \) consists of \( n \) points \( Q_1, \ldots, Q_n \) on \( C \) with ramification indices \( m_1, \ldots, m_n \).

Since \( \tilde{H}_{d,g} \) is not normal we normalize it and this results in a smooth stack \( \tilde{\mathcal{H}}_{d,g} \). We then have a diagram

\[
\tilde{C} \xrightarrow{\alpha} \tilde{\mathcal{M}}_{0,b+1} \xrightarrow{\pi_{b+1}} \tilde{\mathcal{H}}_{d,g} \xrightarrow{h} \mathcal{M}_{0,b},
\]

where \( \tilde{C} \) is the universal curve and \( \tilde{\mathcal{M}}_{0,b} \) is the moduli space of stable \( b \)-pointed curves of genus 0 and \( \pi_{b+1} \) is the map that forgets the \( (b+1) \)st point. With \( \mathbb{P} \) the fibre product of \( \tilde{\mathcal{M}}_{0,b+1} \) and \( \tilde{\mathcal{H}}_{d,g} \) over \( \mathcal{M}_{0,b} \) we have a diagram

\[
\tilde{C} \xrightarrow{\alpha} \mathbb{P} \xrightarrow{\pi'} \tilde{\mathcal{M}}_{0,b+1} \xrightarrow{\pi_{b+1}} \mathcal{M}_{0,b}.
\]

We now work over a base \( B \) (it can be \( \mathcal{H}_{d,g}, \tilde{\mathcal{H}}_{d,g} \) or often a 1-dimensional base). But we shall suppress the index \( B \). Note that normalization commutes with base change. In [4, Lemma 4.1] we showed that \( \tilde{C} \) and \( \mathbb{P} \) have only singularities of type \( A_k \). We resolve the singularities of \( \mathbb{P} \) obtaining a space \( \tilde{\mathbb{P}} \) and then let \( Y \) be the normalization of \( \tilde{C} \times_{\mathbb{P}} \tilde{\mathbb{P}} \) and let \( \tilde{Y} \) be the resolution of singularities of \( Y \). We then find the basic diagram as in [4].
We observe that the finite map $\pi: Y \to \bar{P}$ is flat as $Y$ is Cohen–Macaulay and $\bar{P}$ is smooth. Actually, $Y$ has rational singularities only.

3. Constructing Divisors

Let $D$ be an effective divisor on $Y$ of relative degree $s$ over $B$, supported on the sections. Then $D$ intersects the generic fibre $C$ of $\pi$ in the ramification locus of $\pi$ on that fibre $C$. This divisor $D$ is a Cartier divisor since the sections do not intersect the singular locus and so $\mathcal{O}(D)$ is a line bundle on $Y$. Therefore it follows that $\pi_*\mathcal{O}(D)$ is a locally free sheaf on $\bar{P}$. We denote by $\bar{\pi}$ the proper (and full) transform of $D$ under the resolution map $\nu$. Since $\nu_*\mathcal{O}(\bar{\pi}) = \nu_*\mathcal{O}_Y \otimes \mathcal{O}(D)$, we conclude that $\bar{\pi}_*\mathcal{O}(\bar{\pi}) = \pi_*\mathcal{O}(D)$. We can use the restriction of $\pi_*(\mathcal{O}(D))$ to the open part over $H_{d,g}$ to define a stratification by type of the bundle on $\mathbb{P}^1$ just as for the Maroni stratification. We are interested in the case we get a divisor.

For a divisor $D$ we have an inclusion

$$\iota_D: \mathcal{O}_{\bar{P}} \to \bar{\pi}_*\mathcal{O}(\bar{\pi}).$$

Note that the image $\iota_D(1)$ of the section 1 is a nowhere vanishing section of $\bar{\pi}_*\mathcal{O}(\bar{\pi})$.

We now introduce the vector bundle that we use to define a stratification.

**Definition 3.1.** We let $K_D$ be the cokernel of $\iota_D$. We define $V_D := K_D^\vee$ as the dual $\mathcal{O}_{\bar{P}}$-module. Since $\iota_D(1)$ is a nowhere vanishing section of $\bar{\pi}_*\mathcal{O}(\bar{\pi})$ the sheaf $K_D$ is locally free of rank $d-1$ on $\bar{P}$ and therefore so is $V_D$.

We start with a lemma which follows immediately from the Riemann-Roch theorem.

**Lemma 3.2.** Let $U = \bigoplus_{i=1}^r \mathcal{O}(a_i)$ be a vector bundle of rank $r$ and degree $n$ on $\mathbb{P}^1$. Suppose that $-1 \leq a_1 \leq \cdots \leq a_r$. Then $h^0(V) = r + n$. Moreover, this is the minimum dimension for the space of sections of a vector bundle of rank $r$ and degree $n$ on $\mathbb{P}^1$.

We recall that we write $g = k(d-1) + s$ with $0 \leq s \leq d - 2$. Given a cover $\gamma: C \to \mathbb{P}^1$ and an effective divisor $D$ of degree $s$ supported on the ramification divisor of $\gamma$, we write

$$\gamma_*\mathcal{O}_C(D) \cong \mathcal{O}_{\mathbb{P}^1}(-a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{d-1}),$$

with $a_0 \leq a_1 \leq \cdots \leq a_{d-1}$. Note that $\sum_{i=1}^{d-1} a_i = (k + 1)(d - 1)$.
We let \( \gamma : C \to \mathbb{P}^1 \) represent a general point of \( \mathcal{H}_{d,g} \). We want to determine the numbers \( a_i \). We know by results of Ballico [1] that
\[
\gamma_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(k + 1))^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-(2k + 2))^{\oplus s}.
\]
Thus we get
\[
\gamma_* \mathcal{O} \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-(1))^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-(2))^{\oplus s},
\]
so that \( h^0(kg_1) = k + 1 \) and \( h^1(kg_1^2) = s \).

**Proposition 3.3.** For \( \gamma : C \to \mathbb{P}^1 \) a general point of \( \mathcal{H}_{d,g} \) we can choose an effective divisor \( D \) of degree \( s \) supported on the ramification locus \( R \) of \( \gamma \) satisfying
\[
h^0(D) = 1 \quad \text{and} \quad h^0(kg_1^2 + D) = k + 1.
\]

**Proof.** Note that since for a general \( \gamma \) we have \( h^0(kg_1^1) = k + 1 \), the first condition is implied by the second. This is because for any effective divisors \( D_1, D_2 \) on \( C \) we have that \( h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1 \). Observe that \( b - (2g - 2) = 2d \).

We consider the linear system \([K_C - kg_1^1]\). For reasons of degree we can choose a ramification point \( p_1 \) which is not a base point of \([K_C - kg_1^1]\) and this gives \( h^0(K_C - kg_1^1 - p_1) = s - 1 \). Then by the same degree argument we can find a ramification point \( p_2 \) such that it is not a base point of \([K_C - kg_1^1 - p_1]\) such that \( h^0(K_C - kg_1^1 - p_1 - p_2) = s - 2 \). Repeating the argument we arrive at a divisor of degree \( s \) supported on the ramification locus such that \( h^0(K_C - kg_1^1 - D) = 0 \), hence by duality \( h^1(kg_1^2 + D) = 0 \). By Riemann–Roch we have \( h^0(kg_1^2 + D) = k + 1 \). \( \square \)

Now if we choose \( D \) as in Proposition 3.3 we have \( h^0(D) = 1 \) and therefore \( a_0 = 0 \) and \( a_i > 0 \) for \( i \geq 1 \).
Moreover \( h^0(kg_1^2 + D) = k + 1 \) and since
\[
\gamma_* \mathcal{O}_C(D) \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_{d-1})
\]
this implies that \( a_i \geq k + 1 \) for all \( i = 1, \cdots, d - 1 \). Since the \( a_i \) add up to \( (k + 1)(d - 1) \) we conclude that all \( a_i \) are equal to \( k + 1 \).

**Conclusion 3.4.** If we choose \( \gamma \) and \( D \) as in Proposition 3.3 then the dual of the cokernel of \( \mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C(D) \) has type \( \mathcal{O}(k + 1)^{\oplus d-1} \).

We now want to see that our degeneracy locus is non-empty in the open Hurwitz space \( \mathcal{H}_{d,g} \). According to Ohbuchi [8] only so-called acceptable \( (d - 1) \)-tuples \((a_1, \cdots, a_{d-1})\) can occur as the indices of the dual of the cokernel of \( \mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C \); here acceptable is defined as follows, see [9].

**Definition 3.5.** A non-decreasing \((d - 1)\)-tuple of natural numbers \((a_1, \cdots, a_{d-1})\) with \( \sum_{i=1}^{d-1} a_i = b/2 \) is said to be acceptable for \((d, g)\) if the \( a_i \) satisfy
\[
(1) \ a_1 \geq b/d(d - 1);
(2) \ a_{d-1} \leq b/d;
(3) \ a_{i+1} - a_i \leq a_1.
\]

Now we consider the unique acceptable \((d - 1)\)-tuple \( \alpha \) with \( a_1 = k \) and for which the sum
\[
\sum_{i=1}^{d-1} (d - i) a_i
\]
is maximal. This means that this $\alpha$ is the most balanced $(d-1)$-tuple with $a_1 = k$. In [2] Coppens proved the following existence result, see also [9].

**Theorem 3.6.** Let $a_1$ be an integer satisfying (2) of Definition 3.5. If $\alpha$ is the unique acceptable $(d-1)$-tuple $(a_1, \ldots, a_{d-1})$ for which the sum $\sum_{i=1}^{d-1} (d-i)a_i$ is maximal, then the locus in $H_{d,g}$ of covers $\gamma: C \to \mathbb{P}^1$ with invariant $\alpha$ is not empty.

We apply this in the case $a_1 = k$ and $\sum_{i=1}^{d-1} (d-i)a_i = (k+1)(d-1) + s$. We find the following maximizing $(d-1)$-tuples for the $a_i$ of $V = \bigoplus_{i=1}^{d-1} O(a_i)$ with $a_1 = k$.

**Lemma 3.7.** If $a_1 = k$ we have the following maximizing acceptable $(d-1)$-tuples:

1. if $s \leq d - 4$ the maximizing sequence is $(k, (k+1)^{d-s-3}, (k+2)^{s+1})$;
2. if $s = d - 3$ the maximizing sequence is $(k, (k+2)^{d-2})$, with $g \neq 2(d-2)$;
3. if $s = d - 2$ the maximizing sequence is $(k, (k+2)^{d-3}, k+3)$, with $g \neq 2d-3$ and $(d, g) \neq (3, 5)$.

We consider the three cases of Lemma 3.7.

**Lemma 3.8.** Let $\gamma: C \to \mathbb{P}^1$ be a general cover of type (1), (2) or (3) of Lemma 3.7. Then there exists a divisor of degree $s$ with support in the ramification locus of $\gamma$ such that $h^0(kg_d^1) = h^0(kg_d^2 + D)$ and $h^0(D) = 1$.

**Proof.** In the first case we consider a general cover $\gamma: C \to \mathbb{P}^1$ of this type. Then we have by [1]

$$\gamma_* O_C = O \oplus O(-k) \oplus O(-(k+1))^{\oplus d-s-3} \oplus O(-(k+2))^{\oplus s+1},$$

so that $h^0(kg_d^1) = k + 2$ and accordingly $h^1(kg_d^1) = s + 1$. We follow the argument of Proposition 3.3. The argument for the cases (2) and (3) is similar. □

For this pair $(\gamma, D)$ the conditions $h^0(D) = 1$, $h^0(kg_d^2 + D) = k + 2$ imply that for

$$\gamma_* O(D) = \bigoplus_{i=0}^{d-1} O(-a_i)$$

we must have $a_0 = 0$, $a_1 = k$ and $a_i \geq k + 1$ for $i = 2, \ldots, d-1$. Then the twisted (with $O(-k-1)$) dual of the cokernel of $O_{P_1} \to \gamma_* O_C(D)$ has the form

$$\bigoplus_{i=1}^{d-1} O(b_i) \quad \text{with} \quad b_1 = -1 \quad \text{and} \quad b_i \geq 0 \quad \text{for} \quad i = 2, \ldots, d-1$$

and thus is not balanced with a space of sections of minimum dimension (see Lemma 3.2). The other two cases are dealt with similarly.

**Conclusion 3.9.** Let $\gamma: C \to \mathbb{P}^1$ be a general cover such that the dual of the cokernel of $O_{P_1} \to \gamma_* O_C$ has type as in Lemma 3.7. Then there exists a divisor $D$ of degree $s$ with support in the ramification divisor of $\gamma$ such that the dual of the cokernel of $O_{P_1} \to \gamma_* O_C(D)$ has unbalanced type as above.

We define now our divisors that are analogous to the Maroni divisors. We consider the pullback of the diagram 2.1 to the open base $B = H_{d,g}$ and choose on $Y_B$ a reduced divisor $\mathcal{D}$ consisting of any $s$ sections of $\tilde{\pi}|_B$ and we let $V_{\mathcal{D}}$ be the dual
of the cokernel of the natural map $\iota_D|_B: \mathcal{O}_{\mathfrak{Y}} \to \pi_*\mathcal{O}(\mathcal{D})$ and tensor $V_p$ with a line bundle $M$ on $\mathfrak{Y}$ such that $V^\prime_D = V_D \otimes M$ has zero degree on the generic fibre of $\pi$. The sheaf $p_*V^\prime_D$ is reflexive (see [6]) and therefore it is a vector bundle on an open $U$ of $\mathcal{H}_{d,g}$ with complement of codimension $\geq 3$.

Consider an open subset $U'$ of $U$ such that $p_*V^\prime_D$ is a trivial of rank $r$, that is, isomorphic to $\mathcal{O}_{U'}$. Choose $r$ generating sections $s_1, \ldots, s_r$ of $p_*V^\prime_D$ on $U'$. If we consider their pullbacks under $p$ and restrict these for $x \in U'$ to $H^0(p^{-1}(x), V_D|p^{-1}(x))$ then they generate the stalk of $V^\prime_D$ at a point of $p^{-1}(x)$ if and only if $V^\prime_D|p^{-1}(x)$ is the trivial bundle of rank $r$ on $p^{-1}(x) = \mathbb{P}^1$; indeed, $V^\prime_D|p^{-1}(x) = \bigoplus_i \mathcal{O}(a_i)$ with $\sum a_i = 0$ and if some $a_i$ are negative, then these sections cannot generate it since these do not see the $\mathcal{O}(a_i)$ with $a_i$ negative; if $V^\prime_D|p^{-1}(x) = \mathcal{O}_{\mathbb{P}^1}^r$, by Grauert’s theorem they will generate it (see e.g. [5, III,Cor. 12.9]). Thus we arrive at the following result.

**Theorem 3.10.** Suppose that $g = k(d - 1) + s$ with $0 \leq s \leq d - 2$ and $3 \leq d \leq g$, satisfying the conditions that $g \neq 2d - 3$, $g \neq 2d - 4$ and $(d, g) \neq (3, 5)$. Then the vanishing locus of the determinant of the evaluation map $\text{ev}: p^*p_*V^\prime_D \to V^\prime_D$ defines a non-empty divisor $\mathfrak{d}_D$ on $\mathcal{H}_{d,g}$.

**Proof.** We take a cover as in Conclusion 3.4. Then there is a point in $\mathcal{H}_{d,g}$ representing this cover and such that the divisor $D$ of Conclusion 3.4 is given by the restriction of the above $D$ to the corresponding fiber. This is because the points of $\mathcal{H}_{d,g}$ parametrize simply branched coverings with ordered branch points, and hence the ramification points are ordered too. We conclude that $V^\prime_D$ is trivial on the generic fibre. But by Conclusion 3.9 above we see, for similar reasons, that the locus in $\mathcal{H}_{d,g}$ where $V^\prime_D$ restricted to a fibre is non-trivial is not empty. By Lemma 3.2 and Grauert’s theorem both the above loci belong to $U$. Thus the evaluation map on $p^{-1}U$ is a map of vector bundles of the same rank with degeneracy locus which is not the whole space nor empty. Therefore it defines a divisor on $p^{-1}U$ which extends uniquely to a divisor on $\mathcal{H}_{d,g}$. \qed

4. Cycle Classes

In this section we indicate how to calculate the classes of closed divisors in $\mathfrak{H}_{d,g}$ that extend the above defined divisors $\mathfrak{d}_D$. We proceed as in [4] for the Maroni case.

We take the standard line bundle $M$ constructed in [4, Definition 6.2] and we let $V^\prime_D = V_D \otimes M$. We then want to find a divisor $A_D$ supported on the singular fibres of $p: \mathfrak{Y} \to B$ such that $c_1(V^\prime_D) - A_D$ is a pullback under $p$ from the base $B$. In [4, Section 6] we constructed such a divisor $A_D$ in the case where the divisor $\mathcal{D}$ is zero. This is done locally on the base $B$. We restrict to 1-dimensional bases $B$ and consider the fibre of $p$ over a points $x$ where $B$ intersects the boundary. The fibre there consists of a chain of rational curves. In the present case the degrees of $V'$, the bundle used in [4] and corresponding to $\mathcal{D} = 0$, and $V^\prime_D$ differ at a chain $R_0 = P_1, R_1, \ldots, R_m = P_2$ only at $R_0$ and $R_m$. Adapting the result gives the following analogue of [4, Conclusion 6.5].
Proposition 4.1. For each irreducible component \( \Sigma \) of \( S_{j,\mu} \) we define
\[
c_{\Sigma, D} = d - n - 2(r - D \cdot P_2).
\]
If we let
\[
A^\Sigma = -\frac{1}{2} \sum_{i=0}^{m-1} ((m - i) c_{\Sigma, D} - \delta_i) R^\Sigma_i
\]
then the degree of \( A^\Sigma \) on \( R^\Sigma_i \) equals the degree of \( V'_D \) on \( R^\Sigma_i \).

The formalism described in [4] shows that if \( Q \) denotes the degeneracy locus of the evaluation map \( ev: p^* \pi_* V'_D \rightarrow V'_D \) we get that
\[
c_1(Q) + p^* R^1 \pi_* V'_D + ( - A_D )_{sh},
\]
where the index sh denotes the shift as in [4, Definition 3.7], is an effective class on \( \tilde{P} \) that is a pullback under \( p \), and pulling it back under a section of \( p \) gives us an effective class \( \mathfrak{d}_D \) which extends the divisor \( \mathfrak{d}_D \).

To calculate the class \( d_D \) of the degeneracy locus we wish to use the formula of [4, Theorem 3.10]. For this we need first a the following lemma (see [4, Proposition 10.1]).

Lemma 4.2. Let \( L \) be a line bundle on \( \tilde{Y} \) with first Chern class \( \ell \). If \( U \) denotes the ramification divisor of \( \tilde{\pi} \) we have
\[
c_1(\tilde{\pi}_* L) = \tilde{\pi}_* \ell - \frac{1}{2} \tilde{\pi}_* (U),
\]
\[
c_2(\tilde{\pi}_* L) - c_2(\pi_* \mathcal{O}_Y) = \frac{1}{2} ( (\tilde{\pi}_* \ell)^2 - \tilde{\pi}_* (\ell^2)) - \frac{1}{2} (\tilde{\pi}_* U \cdot \tilde{\pi}_* \ell - \tilde{\pi}_* (U \cdot \ell)).
\]

We denote the sections of \( p \) by \( \sigma_i \) (\( i = 1, \ldots, b \)) and their images by \( \Xi_i \) (\( i = 1, \ldots, b \)). These are divisors on \( \tilde{P} \). Over the section \( \Xi_i \) we have a ramification divisor \( U_i \). If \( D_1 \) is a reduced divisor with support on the sections \( \cup_i \Xi_i \) we let \( D \) be the corresponding divisor with support on the ramification divisor of \( \tilde{\pi} \). Then we have
\[
\tilde{\pi}_* D = D_1, \quad \tilde{\pi}^* D_1 = 2D + \Gamma_D,
\]
with \( \Gamma_D \) disjoint from \( D \). Another relation that we will use with \( W = \tilde{\pi}_* (U) \) is
\[
U \cdot D = \frac{1}{2} W \cdot D_1 = \frac{1}{2} D^2_D.
\]
Using Lemma 4.2 we thus find
\[
c_2(\tilde{\pi}_* \mathcal{O}_Y (D)) = c_2(\tilde{\pi}_* \mathcal{O}_Y), \quad c_2(\tilde{\pi}_* \mathcal{O}_Y (D)) = c_2(\tilde{\pi}_* \mathcal{O}_Y).
\]
Therefore, when we apply [4, Theorem 3.10] in order to calculate the class \( \mathfrak{d}_D \) the only thing that differs from the case treated there is the choice of the line bundle \( A_D \). This affects only the definition of \( c_{\Sigma, D} \) and therefore the class is calculated by the same formula given in [4, Theorem 8.3] with \( c_{j,\mu} \) determined by \( c_{\Sigma, D} \).

We can twist the bundle \( V'_D \) by a line bundle \( N \) corresponding to a divisor on \( \tilde{P} \) with support on the boundary of \( \tilde{P} \). This results in a divisor class \( \mathfrak{d}_{D, N} \) on \( \mathcal{H}_{d,g} \) extending the divisor \( \mathfrak{d}_D \) on \( \mathcal{H}_{d,g} \). We thus arrive as in [4, Section 9] at the following theorem.
Theorem 4.3. Suppose that \( 3 \leq d \leq g \neq 2d - 3, g \neq 2d - 4 \) and \((d, g) \neq (3, 5)\). Let \( \Sigma \) be an irreducible component of the boundary \( S_{j, \mu} \) of \( \overline{H}_{d,g} \). Then the coefficient \( \sigma_{j, \mu} \) of \( \Sigma \) of the locus \( \cap_N \overline{S}_{D,N} \) is equal to

\[
\begin{align*}
& m(\mu) \left( \frac{1}{12} \left( d - \sum_{\nu=1}^{m(\mu)} \frac{1}{m_\nu} \right) + \frac{j(b - j)(d - 2)}{8(b - 1)(d - 1)} \right) - \frac{1}{8(d - 1)} \sum_{i=1}^{m(\mu)} \left( \delta_{i-1} - \delta_i \right)^2 \\
& - \frac{d - 1}{2} \left( \frac{m(\mu)}{4} - \sum_{i=1}^{m(\mu)} \left( e_{i-1} - e_i \right)^2 \right).
\end{align*}
\]

For the numbers \( m(\mu), \delta_i \) and \( e_i \), we refer to [4]. Note that the rational numbers \( e_i \), that come from rounding off a rational solution to an integral one depend on the constants \( c_{j, \mu} \) that occur above.

In a similar way one can obtain analogues of the theorems of Sections 10 and 11 of [4] by adding to \( D \) an effective divisor \( Z \) supported on the boundary of \( \tilde{Y} \). One can then define an effective class \( \overline{S}_{D,N,Z} \) that extends the class of \( \overline{S}_D \). One may ask whether

\[
\bigcap_{N,Z} \overline{S}_{D,N,Z}
\]

is the class of the Zariski closure of \( \overline{S}_D \) on \( \overline{H}_{d,g} \).

The classes \( \overline{S}_D \) depend on \( D \), but using the monodromy one sees that their images on the Hurwitz space with unordered branch points do not.

References


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