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*An Appendix to 'The Cycle Classes of Divisorial Maroni Loci'*

van der Geer, G.; Kouvidakis, A.

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# DIVISORS ON HURWITZ SPACES: AN APPENDIX TO 'THE CYCLE CLASSES OF DIVISORIAL MARONI LOCI'

GERARD VAN DER GEER AND ALEXIS KOUVIDAKIS

Abstract. The Maroni stratification on the Hurwitz space of degree $d$ covers of genus $g$ has a stratum that is a divisor only if $d-1$ divides $g$. Here we construct a stratification on the Hurwitz space that is analogous to the Maroni stratification, but has a divisor for all pairs $(d, g)$ with $d \leqslant g$ with a few exceptions and we calculate the divisor class of an extension of these divisors to the compactified Hurwitz space.

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## 1. Introduction

The Hurwitz space $\mathcal{H}_{d,g}$ of simply-branched covers of genus $g$ and degree $d$ carries a stratification named after Maroni [7], which is defined as follows. If $\gamma\colon C \to \mathbb{P}^1$ is a simply-branched cover one takes the dual of the cokernel of the natural map

$$\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C,$$

which is a vector bundle of rank $d-1$ on the projective line, hence is isomorphic to $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$ for some $(d-1)$-tuple $\alpha = (a_1, \ldots, a_{d-1})$, where we assume that the $a_i$ are non-decreasing. The loci of covers $\gamma\colon C \to \mathbb{P}^1$ with fixed $\alpha$ are the strata.

It is known (see [1]) that for general $\gamma\colon C \to \mathbb{P}^1$ of genus

$$g = k(d-1) + s \quad \text{with } 0 \leqslant s \leqslant d-2$$

the tuple $\alpha$ takes the form $(k+1, \ldots, k+1, k+2, \ldots, k+2)$ with $s$ entries equal to $k+2$. Only the case with $s = 0$ yields a Maroni stratum that is a divisor (see [3] and [9, Thm. 1.15]).

In this paper we show how to define for the case that the genus $g$ is not divisible by $d-1$ a stratification that has a stratum that is a divisor for $g \geqslant d$ under exclusion of a few cases. If $d-1$ divides $g$ then this reduces to the stratification of Maroni loci. It uses instead of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C$ the cokernel of a natural map

$$\mathcal{O}_{\mathbb{P}^1} \to \gamma_* \mathcal{O}_C(D)\,,$$

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where $D$ is an appropriately chosen divisor of degree $s$ with support in the ramification locus of $\gamma$. The cycle classes of an extension of these divisors to the compactified Hurwitz space $\overline{\mathcal{H}}_{d,g}$ can be calculated by using a global-to-local evaluation map $p^* p_* V \to V$ of a vector bundle $V$ on an extension of the $\mathbb{P}^1$-fibration $p\colon \mathbb{P} \to \mathcal{H}_{d,g}$ to the compactified Hurwitz space that is trivial on the generic fibre of $p$. The calculation and the answer are completely analogous to case of the cycle classes of the Maroni divisors calculated in [4]. The cycle classes are given in terms of an explicit sum of boundary divisors.

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## 2. The Setting

We recall the setting from [4]. We denote by $\overline{\mathcal{H}}_{d,g}$ the compactified Hurwitz space of admissible covers of degree $d$ and genus $g$. We have

$$\overline{\mathcal{H}}_{d,g} - \mathcal{H}_{d,g} = \bigcup_{j,\mu} S_{j,\mu},$$

where the $S_{j,\mu} = S_{b-j,\mu}$ are divisors indexed by $2 \leqslant j \leqslant b-2$ and a partition $\mu = (m_1, \ldots, m_n)$ of $d$. These divisors can be reducible, but a generic point corresponds to an admissible cover $\gamma\colon C \to P$, where $P$ is a genus 0 curve consisting of two components $P_1$, $P_2$ of genus 0 intersecting in one point $Q$ with $j_1 = j$ or $b-j$ branch points on $P_1$ (resp. $j_2 = b-j$ or $j$ branch points on $P_2$) and the inverse image $\gamma^{-1}(Q)$ consists of $n$ points $Q_1, \ldots, Q_n$ on $C$ with ramification indices $m_1, \ldots, m_n$. Since $\overline{\mathcal{H}}_{d,g}$ is not normal we normalize it and this results in a smooth stack $\widetilde{\mathcal{H}}_{d,g}$. We then have a diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "c"] \arrow[d, "\varpi"'] & \overline{\mathcal{M}}_{0,b+1} \arrow[d, "\pi_{b+1}"] \\ \widetilde{\mathcal{H}}_{d,g} \arrow[r, "h"] & \overline{\mathcal{M}}_{0,b}, \end{tikzcd}$$

where $\widetilde{\mathcal{C}}$ is the universal curve and $\overline{\mathcal{M}}_{0,b}$ is the moduli space of stable $b$-pointed curves of genus 0 and $\pi_{b+1}$ is the map that forgets the $(b+1)$st point. With $\mathbb{P}$ the fibre product of $\overline{\mathcal{M}}_{0,b+1}$ and $\widetilde{\mathcal{H}}_{d,g}$ over $\overline{\mathcal{M}}_{0,b}$ we have a diagram

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "\alpha"] \arrow[dr, "\varpi"'] & \mathbb{P} \arrow[r, "c'"] \arrow[d, "\varpi'"] & \overline{\mathcal{M}}_{0,b+1} \arrow[d, "\pi_{b+1}"] \\ & \widetilde{\mathcal{H}}_{d,g} \arrow[r, "h"] & \overline{\mathcal{M}}_{0,b}. \end{tikzcd}$$

We now work over a base $B$ (it can be $\mathcal{H}_{d,g}$, $\widetilde{\mathcal{H}}_{d,g}$ or often a 1-dimensional base). But we shall suppress the index $B$. Note that normalization commutes with base change. In [4, Lemma 4.1] we showed that $\widetilde{\mathcal{C}}$ and $\mathbb{P}$ have only singularities of type $A_k$. We resolve the singularities of $\mathbb{P}$ obtaining a space $\widetilde{\mathbb{P}}$ and then let $Y$ be the normalization of $\widetilde{\mathcal{C}} \times_{\mathbb{P}} \widetilde{\mathbb{P}}$ and let $\widetilde{Y}$ be the resolution of singularities of $Y$. We then find the basic diagram as in [4]:

**Diagram 2.1.**

$$\begin{tikzcd}
\widetilde{Y} \arrow[dr, "\nu"] \arrow[drr, bend left, "\rho"] \arrow[ddr, "\tilde{\pi}"'] & & & \\
& Y \arrow[r] \arrow[d, "\pi"] & \widetilde{\mathcal{C}} \arrow[d] \arrow[ddr, "\varpi"] & \\
& \widetilde{\mathbb{P}} \arrow[r] \arrow[drr, "p"'] & \mathbb{P} \arrow[dr] & \\
& & & B
\end{tikzcd}$$

We observe that the finite map $\pi\colon Y \to \widetilde{\mathbb{P}}$ is flat as $Y$ is Cohen–Macaulay and $\widetilde{\mathbb{P}}$ is smooth. Actually, $Y$ has rational singularities only.

## 3. Constructing Divisors

Let $\mathcal{D}$ be an effective divisor on $Y$ of relative degree $s$ over $B$, supported on the sections. Then $\mathcal{D}$ intersects the generic fibre $C$ of $\pi$ in the ramification locus of $\pi$ on that fibre $C$. This divisor $\mathcal{D}$ is a Cartier divisor since the sections do not intersect the singular locus and so $\mathcal{O}(\mathcal{D})$ is a line bundle on $Y$. Therefore it follows that $\pi_*\mathcal{O}(\mathcal{D})$ is a locally free sheaf on $\widetilde{\mathbb{P}}$. We denote by $\tilde{\mathcal{D}}$ the proper (and full) transform of $\mathcal{D}$ under the resolution map $\nu$. Since $\nu_*\mathcal{O}(\tilde{\mathcal{D}}) = \nu_*\mathcal{O}_{\tilde{Y}} \otimes \mathcal{O}(\mathcal{D})$, we conclude that $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}}) = \pi_*\mathcal{O}(\mathcal{D})$. We can use the restriction of $\pi_*(\mathcal{O}(\mathcal{D}))$ to the open part over $\mathcal{H}_{d,g}$ to define a stratification by type of the bundle on $\mathbb{P}^1$ just as for the Maroni stratification. We are interested in the case we get a divisor.

For a divisor $\mathcal{D}$ we have an inclusion

$$\iota_{\mathcal{D}} : \mathcal{O}_{\tilde{\mathbb{P}}} \to \tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})\,.$$

Note that the image $\iota_{\mathcal{D}}(1)$ of the section 1 is a nowhere vanishing section of $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$.

We now introduce the vector bundle that we use to define a stratification.

**Definition 3.1.** We let $\mathcal{K}_{\mathcal{D}}$ be the cokernel of $\iota_{\mathcal{D}}$. We define $V_{\mathcal{D}} := \mathcal{K}_{\mathcal{D}}^{\vee}$ as the dual $\mathcal{O}_{\tilde{\mathbb{P}}}$-module. Since $\iota_{\mathcal{D}}(1)$ is a nowhere vanishing section of $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$ the sheaf $\mathcal{K}_{\mathcal{D}}$ is locally free of rank $d-1$ on $\tilde{\mathbb{P}}$ and therefore so is $V_{\mathcal{D}}$.

We start with a lemma which follows immediately from the Riemann-Roch theorem.

**Lemma 3.2.** *Let $U = \bigoplus_{i=1}^{r} \mathcal{O}(a_i)$ be a vector bundle of rank $r$ and degree $n$ on $\mathbb{P}^1$. Suppose that $-1 \leqslant a_1 \leqslant \cdots \leqslant a_r$. Then $h^0(V) = r + n$. Moreover, this is the minimum dimension for the space of sections of a vector bundle of rank $r$ and degree $n$ on $\mathbb{P}^1$.*

We recall that we write $g = k(d-1) + s$ with $0 \leqslant s \leqslant d-2$. Given a cover $\gamma\colon C \to \mathbb{P}^1$ and an effective divisor $D$ of degree $s$ supported on the ramification divisor of $\gamma$, we write

$$\gamma_*\mathcal{O}_C(D) \cong \mathcal{O}_{\mathbb{P}^1}(-a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{d-1}),$$

with $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{d-1}$. Note that $\sum_{i=1}^{d-1} a_i = (k+1)(d-1)$.

We let $\gamma\colon C\to\mathbb{P}^1$ represent a general point of $\mathcal{H}_{d,g}$. We want to determine the numbers $a_i$. We know by results of Ballico [1] that

$$\gamma_*\mathcal{O}_C\cong\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-(k+1))^{\oplus d-1-s}\oplus\mathcal{O}_{\mathbb{P}^1}(-(k+2))^{\oplus s}.$$

Thus we get

$$\gamma_*\mathcal{O}\otimes\mathcal{O}(k)\cong\mathcal{O}_{\mathbb{P}^1}(k)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d-1-s}\oplus\mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus s},$$

so that $h^0(k\,g^1_d)=k+1$ and $h^1(k\,g^1_d)=s$.

**Proposition 3.3.** *For $\gamma\colon C\to\mathbb{P}^1$ a general point of $\mathcal{H}_{d,g}$ we can choose an effective divisor $D$ of degree $s$ supported on the ramification locus $R$ of $\gamma$ satisfying*

$$h^0(D)=1\quad\textit{and}\quad h^0(k\,g^1_d+D)=k+1.$$

*Proof.* Note that since for a general $\gamma$ we have $h^0(k\,g^1_d)=k+1$, the first condition is implied by the second. This is because for any effective divisors $D_1$, $D_2$ on $C$ we have that $h^0(D_1+D_2)\geqslant h^0(D_1)+h^0(D_2)-1$. Observe that $b-(2g-2)=2\,d$. We consider the linear system $|K_C-k\,g^1_d|$. For reasons of degree we can choose a ramification point $p_1$ which is not a base point of $|K_C-k\,g^1_d|$ and this gives $h^0(K_C-k\,g^1_d-p_1)=s-1$. Then by the same degree argument we can find a ramification point $p_2$ such that it is not a base point of $|K_C-k\,g^1_d-p_1|$ such that $h^0(K_C-k\,g^1_d-p_1-p_2)=s-2$. Repeating the argument we arrive at a divisor of degree $s$ supported on the ramification locus such that $h^0(K_C-k\,g^1_d-D)=0$, hence by duality $h^1(k\,g^1_d+D)=0$. By Riemann–Roch we have $h^0(k\,g^1_d+D)=k+1$. ☐

Now if we choose $D$ as in Proposition 3.3 we have $h^0(D)=1$ and therefore $a_0=0$ and $a_i>0$ for $i\geqslant 1$. Moreover $h^0(k\,g^1_d+D)=k+1$ and since

$$\gamma_*\mathcal{O}_C(D)\otimes\mathcal{O}(k)\cong\mathcal{O}_{\mathbb{P}^1}(k)\oplus\mathcal{O}_{\mathbb{P}^1}(k-a_1)\oplus\cdots\oplus\mathcal{O}_{\mathbb{P}^1}(k-a_{d-1})$$

this implies that $a_i\geqslant k+1$ for all $i=1,\cdots,d-1$. Since the $a_i$ add up to $(k+1)(d-1)$ we conclude that all $a_i$ are equal to $k+1$.

**Conclusion 3.4.** *If we choose $\gamma$ and $D$ as in Proposition 3.3 then the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1}\to\gamma_*\mathcal{O}_C(D)$ has type $\mathcal{O}(k+1)^{\oplus d-1}$.*

We now want to see that our degeneracy locus is non-empty in the open Hurwitz space $\mathcal{H}_{d,g}$. According to Ohbuchi [8] only so-called acceptable $(d-1)$-tuples $(a_1,\cdots,a_{d-1})$ can occur as the indices of the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1}\to\gamma_*\mathcal{O}_C$; here acceptable is defined as follows, see [9].

**Definition 3.5.** A non-decreasing $(d-1)$-tuple of natural numbers $(a_1,\cdots,a_{d-1})$ with $\sum_{i=1}^{d-1}a_i=b/2$ is said to be acceptable for $(d,g)$ if the $a_i$ satisfy

(1) $a_1\geqslant b/d(d-1)$;
(2) $a_{d-1}\leqslant b/d$;
(3) $a_{i+1}-a_1\leqslant a_1$.

Now we consider the unique acceptable $(d-1)$-tuple $\alpha$ with $a_1=k$ and for which the sum

$$\sum_{i=1}^{d-1}(d-i)a_i$$

is maximal. This means that this $\alpha$ is the most balanced $(d-1)$-tuple with $a_1 = k$. In [2] Coppens proved the following existence result, see also [9].

**Theorem 3.6.** *Let $a_1$ be an integer satisfying* (2) *of Definition* 3.5. *If $\alpha$ is the unique acceptable $(d-1)$-tuple $(a_1, \ldots, a_{d-1})$ for which the sum $\sum_{i=1}^{d-1}(d-i)a_i$ is maximal, then the locus in $\mathcal{H}_{d,g}$ of covers $\gamma\colon C \to \mathbb{P}^1$ with invariant $\alpha$ is not empty.*

We apply this in the case $a_1 = k$ and $\sum_{i=1}^{d-1}(d-i)a_i = (k+1)(d-1)+s$. We find the following maximizing $(d-1)$-tuples for the $a_i$ of $V = \bigoplus_{i=1}^{d-1}\mathcal{O}(a_i)$ with $a_1 = k$.

**Lemma 3.7.** *If $a_1 = k$ we have the following maximizing acceptable $(d-1)$-tuples*:

(1) *if $s \leqslant d-4$ the maximizing sequence is $(k, (k+1)^{d-s-3}, (k+2)^{s+1})$;*
(2) *if $s = d-3$ the maximizing sequence is $(k, (k+2)^{d-2})$, with $g \neq 2(d-2)$;*
(3) *if $s = d-2$ the maximizing sequence is $(k, (k+2)^{d-3}, k+3)$, with $g \neq 2d-3$ and $(d, g) \neq (3, 5)$.*

We consider the three cases of Lemma 3.7.

**Lemma 3.8.** *Let $\gamma\colon C \to \mathbb{P}^1$ be a general cover of type* (1), (2) *or* (3) *of Lemma* 3.7. *Then there exists a divisor of degree $s$ with support in the ramification locus of $\gamma$ such that $h^0(k\, g^1_d) = h^0(k\, g^1_d + D)$ and $h^0(D) = 1$.*

*Proof.* In the first case we consider a general cover $\gamma\colon C \to \mathbb{P}^1$ of this type. Then we have by [1]

$$\gamma_*\mathcal{O}_C = \mathcal{O} \oplus \mathcal{O}(-k) \oplus \mathcal{O}(-(k+1))^{\oplus d-s-3} \oplus \mathcal{O}(-(k+2))^{\oplus s+1},$$

so that $h^0(k\, g^1_d) = k+2$ and accordingly $h^1(k\, g^1_d) = s+1$. We follow the argument of Proposition 3.3. The argument for the cases (2) and (3) is similar. $\square$

For this pair $(\gamma, D)$ the conditions $h^0(D) = 1$, $h^0(kg^1_d + D) = k+2$ imply that for

$$\gamma_*\mathcal{O}(D) = \bigoplus_{i=0}^{d-1}\mathcal{O}(-a_i)$$

we must have $a_0 = 0$, $a_1 = k$ and $a_i \geqslant k+1$ for $i = 2, \ldots, d-1$. Then the twisted (with $\mathcal{O}(-k-1)$) dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_*\mathcal{O}_C(D)$ has the form

$$\bigoplus_{i=1}^{d-1}\mathcal{O}(b_i) \quad \text{with } b_1 = -1 \text{ and } b_i \geqslant 0 \text{ for } i = 2, \ldots, d-1$$

and thus is not balanced with a space of sections of minimum dimension (see Lemma 3.2). The other two cases are dealt with similarly.

**Conclusion 3.9.** *Let $\gamma\colon C \to \mathbb{P}^1$ be a general cover such that the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_*\mathcal{O}_C$ has type as in Lemma* 3.7. *Then there exists a divisor $D$ of degree $s$ with support in the ramification divisor of $\gamma$ such that the dual of the cokernel of $\mathcal{O}_{\mathbb{P}^1} \to \gamma_*\mathcal{O}_C(D)$ has unbalanced type as above.*

We define now our divisors that are analogous to the Maroni divisors. We consider the pullback of the diagram 2.1 to the open base $B = \mathcal{H}_{d,g}$ and choose on $Y_B$ a reduced divisor $\mathcal{D}$ consisting of any $s$ sections of $\tilde{\pi}|_B$ and we let $V_{\mathcal{D}}$ be the dual of the cokernel of the natural map $\iota_{\mathcal{D}}|_B \colon \mathcal{O}_{\tilde{Y}_B} \to \tilde{\pi}_* \mathcal{O}(\mathcal{D})$ and tensor $V_{\mathcal{D}}$ with a line bundle $M$ on $\tilde{\mathbb{P}}_B$ such that $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$ has zero degree on the generic fibre of $\pi$. The sheaf $p_* V'_{\mathcal{D}}$ is reflexive (see [6]) and therefore it is a vector bundle on an open $U$ of $\mathcal{H}_{d,g}$ with complement of codimension $\geqslant 3$.

Consider an open subset $U'$ of $U$ such that $p_* V'_{\mathcal{D}}$ is a trivial of rank $r$, that is, isomorphic to $\mathcal{O}_{U'}^r$. Choose $r$ generating sections $s_1, \ldots, s_r$ of $p_* V'_{\mathcal{D}}$ on $U'$. If we consider their pullbacks under $p$ and restrict these for $x \in U'$ to $H^0(p^{-1}(x), V_{\mathcal{D}}|p^{-1}(x))$ then they generate the stalk of $V'_{\mathcal{D}}$ at a point of $p^{-1}(x)$ if and only if $V'_{\mathcal{D}}|p^{-1}(x)$ is the trivial bundle of rank $r$ on $p^{-1}(x) = \mathbb{P}^1$; indeed, $V'_{\mathcal{D}}|p^{-1}(x)) = \bigoplus_i \mathcal{O}(a_i)$ with $\sum_i a_i = 0$ and if some $a_i$ are negative, then these sections cannot generate it since these do not see the $\mathcal{O}(a_i)$ with $a_i$ negative; if $V'_{\mathcal{D}}|p^{-1}(x) = \mathcal{O}_{\mathbb{P}^1}^r$ by Grauert's theorem they will generate it (see e.g. [5, III,Cor. 12.9]). Thus we arrive at the following result.

**Theorem 3.10.** *Suppose that $g = k(d-1) + s$ with $0 \leqslant s \leqslant d-2$ and $3 \leqslant d \leqslant g$, satisfying the conditions that $g \neq 2d-3$, $g \neq 2d-4$ and $(d, g) \neq (3, 5)$. Then the vanishing locus of the determinant of the evaluation map $\mathrm{ev} : p^* p_* V'_{\mathcal{D}} \to V'_{\mathcal{D}}$ defines a non-empty divisor $\mathfrak{d}_{\mathcal{D}}$ on $\mathcal{H}_{d,g}$.*

*Proof.* We take a cover as in Conclusion 3.4. Then there is a point in $\mathcal{H}_{d,g}$ representing this cover and such that the divisor $D$ of Conclusion 3.4 is given by the restriction of the above $\mathcal{D}$ to the corresponding fiber. This is because the points of $\mathcal{H}_{d,g}$ parametrize simply branched coverings with ordered branch points, and hence the ramification points are ordered too. We conclude that $V'_{\mathcal{D}}$ is trivial on the generic fibre. But by Conclusion 3.9 above we see, for similar reasons, that the locus in $\mathcal{H}_{d,g}$, where $V'_{\mathcal{D}}$ restricted to a fibre is non-trivial is not empty. By Lemma 3.2 and Grauert's theorem both the above loci belong to $U$. Thus the evaluation map on $p^{-1}U$ is a map of vector bundles of the same rank with degeneracy locus which is not the whole space nor empty. Therefore it defines a divisor on $p^{-1}U$ which extends uniquely to a divisor on $\mathcal{H}_{d,g}$. □

## 4. Cycle Classes

In this section we indicate how to calculate the classes of closed divisors in $\overline{\mathcal{H}}_{d,g}$ that extend the above defined divisors $\mathfrak{d}_{\mathcal{D}}$. We proceed as in [4] for the Maroni case.

We take the standard line bundle $M$ constructed in [4, Definition 6.2] and we let $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$. We then want to find a divisor $A_{\mathcal{D}}$ supported on the singular fibres of $p \colon \tilde{Y} \to B$ such that $c_1(V'_{\mathcal{D}}) - A_{\mathcal{D}}$ is a pullback under $p$ from the base $B$. In [4, Section 6] we constructed such a divisor $A_{\mathcal{D}}$ in the case where the divisor $\mathcal{D}$ is zero. This is done locally on the base $B$. We restrict to 1-dimensional bases $B$ and consider the fibre of $p$ over a points $x$ where $B$ intersects the boundary. The fibre there consists of a chain of rational curves. In the present case the degrees of $V'$, the bundle used in [4] and corresponding to $\mathcal{D} = 0$, and $V'_{\mathcal{D}}$ differ at a chain $R_0 = P_1, R_1, \ldots, R_m = P_2$ only at $R_0$ and $R_m$. Adapting the result gives the following analogue of [4, Conclusion 6.5].

**Proposition 4.1.** *For each irreducible component $\Sigma$ of $S_{j,\mu}$ we define*

$$c_{\Sigma,\mathcal{D}} = d - n - 2(r - \mathcal{D} \cdot P_2).$$

*If we let*

$$A_{\mathcal{D}}^{\Sigma} = -\frac{1}{2} \sum_{i=0}^{m-1} ((m-i)\, c_{\Sigma,\mathcal{D}} - \delta_i) R_i^{\Sigma}$$

*then the degree of $A_{\mathcal{D}}^{\Sigma}$ on $R_i^{\Sigma}$ equals the degree of $V_{\mathcal{D}}'$ on $R_i^{\Sigma}$.*

The formalism described in [4] shows that if $Q$ denotes the degeneracy locus of the evaluation map $\mathrm{ev}\colon p^* p_* V_{\mathcal{D}}' \to V_{\mathcal{D}}'$ we get that

$$c_1(Q) + p^* R^1 p_* V_{\mathcal{D}}' + (-A_{\mathcal{D}})_{\mathrm{sh}},$$

where the index sh denotes the shift as in [4, Definition 3.7], is an effective class on $\widetilde{\mathbb{P}}$ that is a pullback under $p$, and pulling it back under a section of $p$ gives us an effective class $\overline{\mathfrak{d}}_{\mathcal{D}}$ which extends the divisor $\mathfrak{d}_{\mathcal{D}}$.

To calculate the class $\overline{\mathfrak{d}}_{\mathcal{D}}$ of the degeneracy locus we wish to use the formula of [4, Theorem 3.10]. For this we need first a the following lemma (see [4, Proposition 10.1]).

**Lemma 4.2.** *Let $\mathcal{L}$ be a line bundle on $\tilde{Y}$ with first Chern class $\ell$. If $U$ denotes the ramification divisor of $\tilde{\pi}$ we have*

$$c_1(\tilde{\pi}_* \mathcal{L}) = \tilde{\pi}_* \ell - \frac{1}{2} \tilde{\pi}_*(U),$$

$$c_2(\tilde{\pi}_* \mathcal{L}) - c_2(\pi_* \mathcal{O}_{\tilde{Y}}) = \frac{1}{2}((\tilde{\pi}_* \ell)^2 - \tilde{\pi}_*(\ell^2)) - \frac{1}{2}(\tilde{\pi}_* U \cdot \tilde{\pi}_* \ell - \tilde{\pi}_*(U \cdot \ell)).$$

We denote the sections of $p$ by $\sigma_i$ $(i = 1, \ldots, b)$ and their images by $\Xi_i$ $(i = 1, \ldots, b)$. These are divisors on $\tilde{\mathbb{P}}$. Over the section $\Xi_i$ we have a ramification divisor $U_i$. If $\mathcal{D}_1$ is a reduced divisor with support on the sections $\cup_i \Xi_i$ we let $\mathcal{D}$ be the corresponding divisor with support on the ramification divisor of $\tilde{\pi}$. Then we have

$$\tilde{\pi}_* \mathcal{D} = \mathcal{D}_1, \quad \tilde{\pi}^* \mathcal{D}_1 = 2\,\mathcal{D} + \Gamma_{\mathcal{D}},$$

with $\Gamma_{\mathcal{D}}$ disjoint from $\mathcal{D}$. Another relation that we will use with $W = \tilde{\pi}_*(U)$ is

$$U \cdot \mathcal{D} = \frac{1}{2} W \cdot \mathcal{D}_1 = \frac{1}{2} \mathcal{D}_1^2.$$

Using Lemma 4.2 we thus find

$$c_2(\tilde{\pi}_* \mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_2(\tilde{\pi}_* \mathcal{O}_{\tilde{Y}}), \quad c_1^2(\tilde{\pi}_* \mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_1^2(\tilde{\pi}_* \mathcal{O}_{\tilde{Y}}).$$

Therefore, when we apply [4, Theorem 3.10] in order to calculate the class $\overline{\mathfrak{d}}_{\mathcal{D}}$ the only thing that differs from the case treated there is the choice of the line bundle $A_{\mathcal{D}}$. This affects only the definition of $c_{\Sigma,\mathcal{D}}$ and therefore the class is calculated by the same formula given in [4, Theorem 8.3] with $c_{j,\mu}$ determined by $c_{\Sigma,\mathcal{D}}$.

We can twist the bundle $V_{\mathcal{D}}'$ by a line bundle $N$ corresponding to a divisor on $\widetilde{\mathbb{P}}$ with support on the boundary of $\widetilde{\mathbb{P}}$. This results in a divisor class $\overline{\mathfrak{d}}_{\mathcal{D},N}$ on $\overline{\mathcal{H}}_{d,g}$ extending the divisor $\mathfrak{d}_{\mathcal{D}}$ on $\mathcal{H}_{d,g}$. We thus arrive as in [4, Section 9] at the following theorem.

**Theorem 4.3.** *Suppose that $3 \leqslant d \leqslant g$ and $g \neq 2d-3$, $g \neq 2d-4$ and $(d, g) \neq (3, 5)$. Let $\Sigma$ be an irreducible component of the boundary $S_{j,\mu}$ of $\overline{\mathcal{H}}_{d,g}$. Then the coefficient $\sigma_{j,\mu}$ of $\Sigma$ of the locus $\cap_N \overline{\mathfrak{d}}_{\mathcal{D},N}$ is equal to*

$$m(\mu)\left(\frac{1}{12}\left(d-\sum_{\nu=1}^{n(\mu)}\frac{1}{m_\nu}\right)+\frac{j(b-j)(d-2)}{8(b-1)(d-1)}\right)-\frac{1}{8(d-1)}\sum_{i=1}^{m(\mu)}(\delta_{i-1}-\delta_i)^2 - \frac{d-1}{2}\left(\frac{m(\mu)}{4}-\sum_{i=1}^{m(\mu)}(e_{i-1}-e_i)^2\right).$$

For the numbers $m(\mu)$, $\delta_i$ and $e_i$ we refer to [4]. Note that the rational numbers $e_i$ that come from rounding off a rational solution to an integral one depend on the constants $c_{j,\mu}$ that occur above.

In a similar way one can obtain analogues of the theorems of Sections 10 and 11 of [4] by adding to $\mathcal{D}$ an effective divisor $Z$ supported on the boundary of $\widetilde{Y}$. One can then define an effective class $\overline{\mathfrak{d}}_{\mathcal{D},N,Z}$ that extends the class of $\mathfrak{d}_{\mathcal{D}}$. One may ask whether

$$\bigcap_{N,Z}\overline{\mathfrak{d}}_{\mathcal{D},N,Z}$$

is the class of the Zariski closure of $\mathfrak{d}_{\mathcal{D}}$ on $\overline{\mathcal{H}}_{d,g}$.

The classes $\mathfrak{d}_{\mathcal{D}}$ depend on $\mathcal{D}$, but using the monodromy one sees that their images on the Hurwitz space with unordered branch points do not.

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Korteweg-de Vries Instituut, Universiteit van Amsterdam, Postbus 94248, 1090 GE Amsterdam, The Netherlands

*E-mail address*: G.B.M.vanderGeer@uva.nl

Department of Mathematics and Applied Mathematics, University of Crete, GR-70013 Heraklion, Greece

*E-mail address*: kouvid@uoc.gr