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**DIVISORS ON HURWITZ SPACES: AN APPENDIX TO  
‘THE CYCLE CLASSES OF DIVISORIAL MARONI LOCI’**

GERARD VAN DER GEER AND ALEXIS KOUVIDAKIS

**ABSTRACT.** The Maroni stratification on the Hurwitz space of degree  $d$  covers of genus  $g$  has a stratum that is a divisor only if  $d - 1$  divides  $g$ . Here we construct a stratification on the Hurwitz space that is analogous to the Maroni stratification, but has a divisor for all pairs  $(d, g)$  with  $d \leq g$  with a few exceptions and we calculate the divisor class of an extension of these divisors to the compactified Hurwitz space.

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1. INTRODUCTION

The Hurwitz space  $\mathcal{H}_{d,g}$  of simply-branched covers of genus  $g$  and degree  $d$  carries a stratification named after Maroni [7], which is defined as follows. If  $\gamma: C \rightarrow \mathbb{P}^1$  is a simply-branched cover one takes the dual of the cokernel of the natural map

$$\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C,$$

which is a vector bundle of rank  $d - 1$  on the projective line, hence is isomorphic to  $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{d-1})$  for some  $(d - 1)$ -tuple  $\alpha = (a_1, \dots, a_{d-1})$ , where we assume that the  $a_i$  are non-decreasing. The loci of covers  $\gamma: C \rightarrow \mathbb{P}^1$  with fixed  $\alpha$  are the strata.

It is known (see [1]) that for general  $\gamma: C \rightarrow \mathbb{P}^1$  of genus

$$g = k(d - 1) + s \quad \text{with } 0 \leq s \leq d - 2$$

the tuple  $\alpha$  takes the form  $(k + 1, \dots, k + 1, k + 2, \dots, k + 2)$  with  $s$  entries equal to  $k + 2$ . Only the case with  $s = 0$  yields a Maroni stratum that is a divisor (see [3] and [9, Thm. 1.15]).

In this paper we show how to define for the case that the genus  $g$  is not divisible by  $d - 1$  a stratification that has a stratum that is a divisor for  $g \geq d$  under exclusion of a few cases. If  $d - 1$  divides  $g$  then this reduces to the stratification of Maroni loci. It uses instead of the cokernel of  $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C$  the cokernel of a natural map

$$\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C(D),$$

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where  $D$  is an appropriately chosen divisor of degree  $s$  with support in the ramification locus of  $\gamma$ . The cycle classes of an extension of these divisors to the compactified Hurwitz space  $\overline{\mathcal{H}}_{d,g}$  can be calculated by using a global-to-local evaluation map  $p^*p_*V \rightarrow V$  of a vector bundle  $V$  on an extension of the  $\mathbb{P}^1$ -fibration  $p: \mathbb{P} \rightarrow \mathcal{H}_{d,g}$  to the compactified Hurwitz space that is trivial on the generic fibre of  $p$ . The calculation and the answer are completely analogous to case of the cycle classes of the Maroni divisors calculated in [4]. The cycle classes are given in terms of an explicit sum of boundary divisors.

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2. THE SETTING

We recall the setting from [4]. We denote by  $\overline{\mathcal{H}}_{d,g}$  the compactified Hurwitz space of admissible covers of degree  $d$  and genus  $g$ . We have

$$\overline{\mathcal{H}}_{d,g} - \mathcal{H}_{d,g} = \bigcup_{j,\mu} S_{j,\mu},$$

where the  $S_{j,\mu} = S_{b-j,\mu}$  are divisors indexed by  $2 \leq j \leq b - 2$  and a partition  $\mu = (m_1, \dots, m_n)$  of  $d$ . These divisors can be reducible, but a generic point corresponds to an admissible cover  $\gamma: C \rightarrow P$ , where  $P$  is a genus 0 curve consisting of two components  $P_1, P_2$  of genus 0 intersecting in one point  $Q$  with  $j_1 = j$  or  $b - j$  branch points on  $P_1$  (resp.  $j_2 = b - j$  or  $j$  branch points on  $P_2$ ) and the inverse image  $\gamma^{-1}(Q)$  consists of  $n$  points  $Q_1, \dots, Q_n$  on  $C$  with ramification indices  $m_1, \dots, m_n$ . Since  $\overline{\mathcal{H}}_{d,g}$  is not normal we normalize it and this results in a smooth stack  $\tilde{\mathcal{H}}_{d,g}$ . We then have a diagram

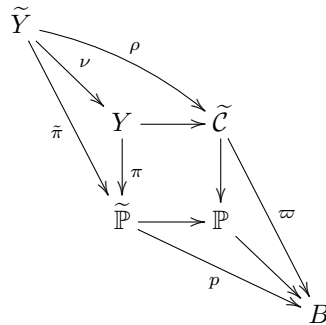
$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{c} & \overline{\mathcal{M}}_{0,b+1} \\ \varpi \downarrow & & \downarrow \pi_{b+1} \\ \tilde{\mathcal{H}}_{d,g} & \xrightarrow{h} & \overline{\mathcal{M}}_{0,b}, \end{array}$$

where  $\tilde{\mathcal{C}}$  is the universal curve and  $\overline{\mathcal{M}}_{0,b}$  is the moduli space of stable  $b$ -pointed curves of genus 0 and  $\pi_{b+1}$  is the map that forgets the  $(b + 1)$ st point. With  $\mathbb{P}$  the fibre product of  $\overline{\mathcal{M}}_{0,b+1}$  and  $\tilde{\mathcal{H}}_{d,g}$  over  $\overline{\mathcal{M}}_{0,b}$  we have a diagram

$$\begin{array}{ccccc} \tilde{\mathcal{C}} & \xrightarrow{\alpha} & \mathbb{P} & \xrightarrow{c'} & \overline{\mathcal{M}}_{0,b+1} \\ & \searrow \varpi & \downarrow \varpi' & & \downarrow \pi_{b+1} \\ & & \tilde{\mathcal{H}}_{d,g} & \xrightarrow{h} & \overline{\mathcal{M}}_{0,b}. \end{array}$$

We now work over a base  $B$  (it can be  $\mathcal{H}_{d,g}, \tilde{\mathcal{H}}_{d,g}$  or often a 1-dimensional base). But we shall suppress the index  $B$ . Note that normalization commutes with base change. In [4, Lemma 4.1] we showed that  $\tilde{\mathcal{C}}$  and  $\mathbb{P}$  have only singularities of type  $A_k$ . We resolve the singularities of  $\mathbb{P}$  obtaining a space  $\tilde{\mathbb{P}}$  and then let  $Y$  be the normalization of  $\tilde{\mathcal{C}} \times_{\mathbb{P}} \tilde{\mathbb{P}}$  and let  $\tilde{Y}$  be the resolution of singularities of  $Y$ . We then find the basic diagram as in [4]:

Diagram 2.1.



We observe that the finite map  $\pi: Y \rightarrow \tilde{\mathbb{P}}$  is flat as  $Y$  is Cohen–Macaulay and  $\tilde{\mathbb{P}}$  is smooth. Actually,  $Y$  has rational singularities only.

3. CONSTRUCTING DIVISORS

Let  $\mathcal{D}$  be an effective divisor on  $Y$  of relative degree  $s$  over  $B$ , supported on the sections. Then  $\mathcal{D}$  intersects the generic fibre  $C$  of  $\pi$  in the ramification locus of  $\pi$  on that fibre  $C$ . This divisor  $\mathcal{D}$  is a Cartier divisor since the sections do not intersect the singular locus and so  $\mathcal{O}(\mathcal{D})$  is a line bundle on  $Y$ . Therefore it follows that  $\pi_*\mathcal{O}(\mathcal{D})$  is a locally free sheaf on  $\tilde{\mathbb{P}}$ . We denote by  $\tilde{\mathcal{D}}$  the proper (and full) transform of  $\mathcal{D}$  under the resolution map  $\nu$ . Since  $\nu_*\mathcal{O}(\tilde{\mathcal{D}}) = \nu_*\mathcal{O}_{\tilde{Y}} \otimes \mathcal{O}(\mathcal{D})$ , we conclude that  $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}}) = \pi_*\mathcal{O}(\mathcal{D})$ . We can use the restriction of  $\pi_*(\mathcal{O}(\mathcal{D}))$  to the open part over  $\mathcal{H}_{d,g}$  to define a stratification by type of the bundle on  $\mathbb{P}^1$  just as for the Maroni stratification. We are interested in the case we get a divisor.

For a divisor  $\mathcal{D}$  we have an inclusion

$$\iota_{\mathcal{D}} : \mathcal{O}_{\tilde{\mathbb{P}}} \rightarrow \tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}}).$$

Note that the image  $\iota_{\mathcal{D}}(1)$  of the section 1 is a nowhere vanishing section of  $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$ .

We now introduce the vector bundle that we use to define a stratification.

**Definition 3.1.** We let  $\mathcal{K}_{\mathcal{D}}$  be the cokernel of  $\iota_{\mathcal{D}}$ . We define  $V_{\mathcal{D}} := \mathcal{K}_{\mathcal{D}}^{\vee}$  as the dual  $\mathcal{O}_{\tilde{\mathbb{P}}}$ -module. Since  $\iota_{\mathcal{D}}(1)$  is a nowhere vanishing section of  $\tilde{\pi}_*\mathcal{O}(\tilde{\mathcal{D}})$  the sheaf  $\mathcal{K}_{\mathcal{D}}$  is locally free of rank  $d - 1$  on  $\tilde{\mathbb{P}}$  and therefore so is  $V_{\mathcal{D}}$ .

We start with a lemma which follows immediately from the Riemann-Roch theorem.

**Lemma 3.2.** Let  $U = \bigoplus_{i=1}^r \mathcal{O}(a_i)$  be a vector bundle of rank  $r$  and degree  $n$  on  $\mathbb{P}^1$ . Suppose that  $-1 \leq a_1 \leq \dots \leq a_r$ . Then  $h^0(U) = r + n$ . Moreover, this is the minimum dimension for the space of sections of a vector bundle of rank  $r$  and degree  $n$  on  $\mathbb{P}^1$ .

We recall that we write  $g = k(d - 1) + s$  with  $0 \leq s \leq d - 2$ . Given a cover  $\gamma: C \rightarrow \mathbb{P}^1$  and an effective divisor  $D$  of degree  $s$  supported on the ramification divisor of  $\gamma$ , we write

$$\gamma_*\mathcal{O}_C(D) \cong \mathcal{O}_{\mathbb{P}^1}(-a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(-a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(-a_{d-1}),$$

with  $a_0 \leq a_1 \leq \dots \leq a_{d-1}$ . Note that  $\sum_{i=1}^{d-1} a_i = (k + 1)(d - 1)$ .

We let  $\gamma: C \rightarrow \mathbb{P}^1$  represent a general point of  $\mathcal{H}_{d,g}$ . We want to determine the numbers  $a_i$ . We know by results of Ballico [1] that

$$\gamma_* \mathcal{O}_C \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-(k+1))^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-(k+2))^{\oplus s}.$$

Thus we get

$$\gamma_* \mathcal{O} \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d-1-s} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus s},$$

so that  $h^0(kg_d^1) = k + 1$  and  $h^1(kg_d^1) = s$ .

**Proposition 3.3.** *For  $\gamma: C \rightarrow \mathbb{P}^1$  a general point of  $\mathcal{H}_{d,g}$  we can choose an effective divisor  $D$  of degree  $s$  supported on the ramification locus  $R$  of  $\gamma$  satisfying*

$$h^0(D) = 1 \quad \text{and} \quad h^0(kg_d^1 + D) = k + 1.$$

*Proof.* Note that since for a general  $\gamma$  we have  $h^0(kg_d^1) = k + 1$ , the first condition is implied by the second. This is because for any effective divisors  $D_1, D_2$  on  $C$  we have that  $h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1$ . Observe that  $b - (2g - 2) = 2d$ . We consider the linear system  $|K_C - kg_d^1|$ . For reasons of degree we can choose a ramification point  $p_1$  which is not a base point of  $|K_C - kg_d^1|$  and this gives  $h^0(K_C - kg_d^1 - p_1) = s - 1$ . Then by the same degree argument we can find a ramification point  $p_2$  such that it is not a base point of  $|K_C - kg_d^1 - p_1|$  such that  $h^0(K_C - kg_d^1 - p_1 - p_2) = s - 2$ . Repeating the argument we arrive at a divisor of degree  $s$  supported on the ramification locus such that  $h^0(K_C - kg_d^1 - D) = 0$ , hence by duality  $h^1(kg_d^1 + D) = 0$ . By Riemann–Roch we have  $h^0(kg_d^1 + D) = k + 1$ .  $\square$

Now if we choose  $D$  as in Proposition 3.3 we have  $h^0(D) = 1$  and therefore  $a_0 = 0$  and  $a_i > 0$  for  $i \geq 1$ . Moreover  $h^0(kg_d^1 + D) = k + 1$  and since

$$\gamma_* \mathcal{O}_C(D) \otimes \mathcal{O}(k) \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(k - a_{d-1})$$

this implies that  $a_i \geq k + 1$  for all  $i = 1, \dots, d - 1$ . Since the  $a_i$  add up to  $(k + 1)(d - 1)$  we conclude that all  $a_i$  are equal to  $k + 1$ .

**Conclusion 3.4.** *If we choose  $\gamma$  and  $D$  as in Proposition 3.3 then the dual of the cokernel of  $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C(D)$  has type  $\mathcal{O}(k + 1)^{\oplus d-1}$ .*

We now want to see that our degeneracy locus is non-empty in the open Hurwitz space  $\mathcal{H}_{d,g}$ . According to Ohbuchi [8] only so-called acceptable  $(d - 1)$ -tuples  $(a_1, \dots, a_{d-1})$  can occur as the indices of the dual of the cokernel of  $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_* \mathcal{O}_C$ ; here acceptable is defined as follows, see [9].

**Definition 3.5.** A non-decreasing  $(d - 1)$ -tuple of natural numbers  $(a_1, \dots, a_{d-1})$  with  $\sum_{i=1}^{d-1} a_i = b/2$  is said to be acceptable for  $(d, g)$  if the  $a_i$  satisfy

- (1)  $a_1 \geq b/d(d - 1)$ ;
- (2)  $a_{d-1} \leq b/d$ ;
- (3)  $a_{i+1} - a_1 \leq a_1$ .

Now we consider the unique acceptable  $(d - 1)$ -tuple  $\alpha$  with  $a_1 = k$  and for which the sum

$$\sum_{i=1}^{d-1} (d - i)a_i$$

is maximal. This means that this  $\alpha$  is the most balanced  $(d - 1)$ -tuple with  $a_1 = k$ . In [2] Coppens proved the following existence result, see also [9].

**Theorem 3.6.** *Let  $a_1$  be an integer satisfying (2) of Definition 3.5. If  $\alpha$  is the unique acceptable  $(d - 1)$ -tuple  $(a_1, \dots, a_{d-1})$  for which the sum  $\sum_{i=1}^{d-1} (d - i)a_i$  is maximal, then the locus in  $\mathcal{H}_{d,g}$  of covers  $\gamma: C \rightarrow \mathbb{P}^1$  with invariant  $\alpha$  is not empty.*

We apply this in the case  $a_1 = k$  and  $\sum_{i=1}^{d-1} (d - i)a_i = (k + 1)(d - 1) + s$ . We find the following maximizing  $(d - 1)$ -tuples for the  $a_i$  of  $V = \bigoplus_{i=1}^{d-1} \mathcal{O}(a_i)$  with  $a_1 = k$ .

**Lemma 3.7.** *If  $a_1 = k$  we have the following maximizing acceptable  $(d - 1)$ -tuples:*

- (1) *if  $s \leq d - 4$  the maximizing sequence is  $(k, (k + 1)^{d-s-3}, (k + 2)^{s+1})$ ;*
- (2) *if  $s = d - 3$  the maximizing sequence is  $(k, (k + 2)^{d-2})$ , with  $g \neq 2(d - 2)$ ;*
- (3) *if  $s = d - 2$  the maximizing sequence is  $(k, (k + 2)^{d-3}, k + 3)$ , with  $g \neq 2d - 3$  and  $(d, g) \neq (3, 5)$ .*

We consider the three cases of Lemma 3.7.

**Lemma 3.8.** *Let  $\gamma: C \rightarrow \mathbb{P}^1$  be a general cover of type (1), (2) or (3) of Lemma 3.7. Then there exists a divisor of degree  $s$  with support in the ramification locus of  $\gamma$  such that  $h^0(kg_d^1) = h^0(kg_d^1 + D)$  and  $h^0(D) = 1$ .*

*Proof.* In the first case we consider a general cover  $\gamma: C \rightarrow \mathbb{P}^1$  of this type. Then we have by [1]

$$\gamma_*\mathcal{O}_C = \mathcal{O} \oplus \mathcal{O}(-k) \oplus \mathcal{O}(-(k + 1))^{\oplus d-s-3} \oplus \mathcal{O}(-(k + 2))^{\oplus s+1},$$

so that  $h^0(kg_d^1) = k + 2$  and accordingly  $h^1(kg_d^1) = s + 1$ . We follow the argument of Proposition 3.3. The argument for the cases (2) and (3) is similar.  $\square$

For this pair  $(\gamma, D)$  the conditions  $h^0(D) = 1, h^0(kg_d^1 + D) = k + 2$  imply that for

$$\gamma_*\mathcal{O}(D) = \bigoplus_{i=0}^{d-1} \mathcal{O}(-a_i)$$

we must have  $a_0 = 0, a_1 = k$  and  $a_i \geq k + 1$  for  $i = 2, \dots, d - 1$ . Then the twisted (with  $\mathcal{O}(-k - 1)$ ) dual of the cokernel of  $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_*\mathcal{O}_C(D)$  has the form

$$\bigoplus_{i=1}^{d-1} \mathcal{O}(b_i) \quad \text{with } b_1 = -1 \text{ and } b_i \geq 0 \text{ for } i = 2, \dots, d - 1$$

and thus is not balanced with a space of sections of minimum dimension (see Lemma 3.2). The other two cases are dealt with similarly.

**Conclusion 3.9.** *Let  $\gamma: C \rightarrow \mathbb{P}^1$  be a general cover such that the dual of the cokernel of  $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_*\mathcal{O}_C$  has type as in Lemma 3.7. Then there exists a divisor  $D$  of degree  $s$  with support in the ramification divisor of  $\gamma$  such that the dual of the cokernel of  $\mathcal{O}_{\mathbb{P}^1} \rightarrow \gamma_*\mathcal{O}_C(D)$  has unbalanced type as above.*

We define now our divisors that are analogous to the Maroni divisors. We consider the pullback of the diagram 2.1 to the open base  $B = \mathcal{H}_{d,g}$  and choose on  $Y_B$  a reduced divisor  $\mathcal{D}$  consisting of any  $s$  sections of  $\tilde{\pi}|_B$  and we let  $V_{\mathcal{D}}$  be the dual

of the cokernel of the natural map  $\iota_{\mathcal{D}}|_B: \mathcal{O}_{\tilde{Y}_B} \rightarrow \tilde{\pi}_*\mathcal{O}(\mathcal{D})$  and tensor  $V_{\mathcal{D}}$  with a line bundle  $M$  on  $\tilde{\mathbb{P}}_B$  such that  $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$  has zero degree on the generic fibre of  $\pi$ . The sheaf  $p_*V'_{\mathcal{D}}$  is reflexive (see [6]) and therefore it is a vector bundle on an open  $U$  of  $\mathcal{H}_{d,g}$  with complement of codimension  $\geq 3$ .

Consider an open subset  $U'$  of  $U$  such that  $p_*V'_{\mathcal{D}}$  is a trivial of rank  $r$ , that is, isomorphic to  $\mathcal{O}_{U'}^r$ . Choose  $r$  generating sections  $s_1, \dots, s_r$  of  $p_*V'_{\mathcal{D}}$  on  $U'$ . If we consider their pullbacks under  $p$  and restrict these for  $x \in U'$  to  $H^0(p^{-1}(x), V_{\mathcal{D}}|_{p^{-1}(x)})$  then they generate the stalk of  $V'_{\mathcal{D}}$  at a point of  $p^{-1}(x)$  if and only if  $V'_{\mathcal{D}}|_{p^{-1}(x)}$  is the trivial bundle of rank  $r$  on  $p^{-1}(x) = \mathbb{P}^1$ ; indeed,  $V'_{\mathcal{D}}|_{p^{-1}(x)} = \bigoplus_i \mathcal{O}(a_i)$  with  $\sum_i a_i = 0$  and if some  $a_i$  are negative, then these sections cannot generate it since these do not see the  $\mathcal{O}(a_i)$  with  $a_i$  negative; if  $V'_{\mathcal{D}}|_{p^{-1}(x)} = \mathcal{O}_{\mathbb{P}^1}^r$  by Grauert's theorem they will generate it (see e.g. [5, III, Cor. 12.9]). Thus we arrive at the following result.

**Theorem 3.10.** *Suppose that  $g = k(d - 1) + s$  with  $0 \leq s \leq d - 2$  and  $3 \leq d \leq g$ , satisfying the conditions that  $g \neq 2d - 3$ ,  $g \neq 2d - 4$  and  $(d, g) \neq (3, 5)$ . Then the vanishing locus of the determinant of the evaluation map  $ev : p^*p_*V'_{\mathcal{D}} \rightarrow V'_{\mathcal{D}}$  defines a non-empty divisor  $\mathfrak{d}_{\mathcal{D}}$  on  $\mathcal{H}_{d,g}$ .*

*Proof.* We take a cover as in Conclusion 3.4. Then there is a point in  $\mathcal{H}_{d,g}$  representing this cover and such that the divisor  $D$  of Conclusion 3.4 is given by the restriction of the above  $\mathcal{D}$  to the corresponding fiber. This is because the points of  $\mathcal{H}_{d,g}$  parametrize simply branched coverings with ordered branch points, and hence the ramification points are ordered too. We conclude that  $V'_{\mathcal{D}}$  is trivial on the generic fibre. But by Conclusion 3.9 above we see, for similar reasons, that the locus in  $\mathcal{H}_{d,g}$ , where  $V'_{\mathcal{D}}$  restricted to a fibre is non-trivial is not empty. By Lemma 3.2 and Grauert's theorem both the above loci belong to  $U$ . Thus the evaluation map on  $p^{-1}U$  is a map of vector bundles of the same rank with degeneracy locus which is not the whole space nor empty. Therefore it defines a divisor on  $p^{-1}U$  which extends uniquely to a divisor on  $\mathcal{H}_{d,g}$ .  $\square$

#### 4. CYCLE CLASSES

In this section we indicate how to calculate the classes of closed divisors in  $\overline{\mathcal{H}}_{d,g}$  that extend the above defined divisors  $\mathfrak{d}_{\mathcal{D}}$ . We proceed as in [4] for the Maroni case.

We take the standard line bundle  $M$  constructed in [4, Definition 6.2] and we let  $V'_{\mathcal{D}} = V_{\mathcal{D}} \otimes M$ . We then want to find a divisor  $A_{\mathcal{D}}$  supported on the singular fibres of  $p: \tilde{Y} \rightarrow B$  such that  $c_1(V'_{\mathcal{D}}) - A_{\mathcal{D}}$  is a pullback under  $p$  from the base  $B$ . In [4, Section 6] we constructed such a divisor  $A_{\mathcal{D}}$  in the case where the divisor  $\mathcal{D}$  is zero. This is done locally on the base  $B$ . We restrict to 1-dimensional bases  $B$  and consider the fibre of  $p$  over a points  $x$  where  $B$  intersects the boundary. The fibre there consists of a chain of rational curves. In the present case the degrees of  $V'$ , the bundle used in [4] and corresponding to  $\mathcal{D} = 0$ , and  $V'_{\mathcal{D}}$  differ at a chain  $R_0 = P_1, R_1, \dots, R_m = P_2$  only at  $R_0$  and  $R_m$ . Adapting the result gives the following analogue of [4, Conclusion 6.5].

**Proposition 4.1.** *For each irreducible component  $\Sigma$  of  $S_{j,\mu}$  we define*

$$c_{\Sigma, \mathcal{D}} = d - n - 2(r - \mathcal{D} \cdot P_2).$$

If we let

$$A_{\mathcal{D}}^{\Sigma} = -\frac{1}{2} \sum_{i=0}^{m-1} ((m-i)c_{\Sigma, \mathcal{D}} - \delta_i) R_i^{\Sigma}$$

then the degree of  $A_{\mathcal{D}}^{\Sigma}$  on  $R_i^{\Sigma}$  equals the degree of  $V'_{\mathcal{D}}$  on  $R_i^{\Sigma}$ .

The formalism described in [4] shows that if  $Q$  denotes the degeneracy locus of the evaluation map  $\text{ev}: p^*p_*V'_{\mathcal{D}} \rightarrow V'_{\mathcal{D}}$  we get that

$$c_1(Q) + p^*R^1p_*V'_{\mathcal{D}} + (-A_{\mathcal{D}})_{\text{sh}},$$

where the index sh denotes the shift as in [4, Definition 3.7], is an effective class on  $\tilde{\mathbb{P}}$  that is a pullback under  $p$ , and pulling it back under a section of  $p$  gives us an effective class  $\bar{\mathfrak{d}}_{\mathcal{D}}$  which extends the divisor  $\mathfrak{d}_{\mathcal{D}}$ .

To calculate the class  $\bar{\mathfrak{d}}_{\mathcal{D}}$  of the degeneracy locus we wish to use the formula of [4, Theorem 3.10]. For this we need first a the following lemma (see [4, Proposition 10.1]).

**Lemma 4.2.** *Let  $\mathcal{L}$  be a line bundle on  $\tilde{Y}$  with first Chern class  $\ell$ . If  $U$  denotes the ramification divisor of  $\tilde{\pi}$  we have*

$$c_1(\tilde{\pi}_*\mathcal{L}) = \tilde{\pi}_*\ell - \frac{1}{2}\tilde{\pi}_*(U),$$

$$c_2(\tilde{\pi}_*\mathcal{L}) - c_2(\pi_*\mathcal{O}_{\tilde{Y}}) = \frac{1}{2}((\tilde{\pi}_*\ell)^2 - \tilde{\pi}_*(\ell^2)) - \frac{1}{2}(\tilde{\pi}_*U \cdot \tilde{\pi}_*\ell - \tilde{\pi}_*(U \cdot \ell)).$$

We denote the sections of  $p$  by  $\sigma_i$  ( $i = 1, \dots, b$ ) and their images by  $\Xi_i$  ( $i = 1, \dots, b$ ). These are divisors on  $\tilde{\mathbb{P}}$ . Over the section  $\Xi_i$  we have a ramification divisor  $U_i$ . If  $\mathcal{D}_1$  is a reduced divisor with support on the sections  $\cup_i \Xi_i$  we let  $\mathcal{D}$  be the corresponding divisor with support on the ramification divisor of  $\tilde{\pi}$ . Then we have

$$\tilde{\pi}_*\mathcal{D} = \mathcal{D}_1, \quad \tilde{\pi}^*\mathcal{D}_1 = 2\mathcal{D} + \Gamma_{\mathcal{D}},$$

with  $\Gamma_{\mathcal{D}}$  disjoint from  $\mathcal{D}$ . Another relation that we will use with  $W = \tilde{\pi}_*(U)$  is

$$U \cdot \mathcal{D} = \frac{1}{2}W \cdot \mathcal{D}_1 = \frac{1}{2}\mathcal{D}_1^2.$$

Using Lemma 4.2 we thus find

$$c_2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}), \quad c_1^2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}(\mathcal{D})) = c_1^2(\tilde{\pi}_*\mathcal{O}_{\tilde{Y}}).$$

Therefore, when we apply [4, Theorem 3.10] in order to calculate the class  $\bar{\mathfrak{d}}_{\mathcal{D}}$  the only thing that differs from the case treated there is the choice of the line bundle  $A_{\mathcal{D}}$ . This affects only the definition of  $c_{\Sigma, \mathcal{D}}$  and therefore the class is calculated by the same formula given in [4, Theorem 8.3] with  $c_{j,\mu}$  determined by  $c_{\Sigma, \mathcal{D}}$ .

We can twist the bundle  $V'_{\mathcal{D}}$  by a line bundle  $N$  corresponding to a divisor on  $\tilde{\mathbb{P}}$  with support on the boundary of  $\tilde{\mathbb{P}}$ . This results in a divisor class  $\bar{\mathfrak{d}}_{\mathcal{D}, N}$  on  $\tilde{\mathcal{H}}_{d,g}$  extending the divisor  $\mathfrak{d}_{\mathcal{D}}$  on  $\mathcal{H}_{d,g}$ . We thus arrive as in [4, Section 9] at the following theorem.



**Theorem 4.3.** *Suppose that  $3 \leq d \leq g$  and  $g \neq 2d - 3$ ,  $g \neq 2d - 4$  and  $(d, g) \neq (3, 5)$ . Let  $\Sigma$  be an irreducible component of the boundary  $S_{j,\mu}$  of  $\overline{\mathcal{H}}_{d,g}$ . Then the coefficient  $\sigma_{j,\mu}$  of  $\Sigma$  of the locus  $\cap_N \overline{\mathfrak{d}}_{\mathcal{D},N}$  is equal to*

$$m(\mu) \left( \frac{1}{12} \left( d - \sum_{\nu=1}^{n(\mu)} \frac{1}{m_\nu} \right) + \frac{j(b-j)(d-2)}{8(b-1)(d-1)} \right) - \frac{1}{8(d-1)} \sum_{i=1}^{m(\mu)} (\delta_{i-1} - \delta_i)^2 - \frac{d-1}{2} \left( \frac{m(\mu)}{4} - \sum_{i=1}^{m(\mu)} (e_{i-1} - e_i)^2 \right).$$

For the numbers  $m(\mu)$ ,  $\delta_i$  and  $e_i$  we refer to [4]. Note that the rational numbers  $e_i$  that come from rounding off a rational solution to an integral one depend on the constants  $c_{j,\mu}$  that occur above.

In a similar way one can obtain analogues of the theorems of Sections 10 and 11 of [4] by adding to  $\mathcal{D}$  an effective divisor  $Z$  supported on the boundary of  $\tilde{Y}$ . One can then define an effective class  $\overline{\mathfrak{d}}_{\mathcal{D},N,Z}$  that extends the class of  $\mathfrak{d}_{\mathcal{D}}$ . One may ask whether

$$\bigcap_{N,Z} \overline{\mathfrak{d}}_{\mathcal{D},N,Z}$$

is the class of the Zariski closure of  $\mathfrak{d}_{\mathcal{D}}$  on  $\overline{\mathcal{H}}_{d,g}$ .

The classes  $\mathfrak{d}_{\mathcal{D}}$  depend on  $\mathcal{D}$ , but using the monodromy one sees that their images on the Hurwitz space with unordered branch points do not.

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