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Most General Algebraic Specifications for an Abstract Datatype of Rational Numbers

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Abstract

The notion of a most general algebraic specification of an arithmetical datatype of characteristic zero is introduced. Three examples of such specifications are given. A preference is formulated for a specification by means of infinitely many equations which can be presented via a finite number of so-called schematic equations phrased in terms of an infinite signature. On the basis of the latter specification three topics are discussed: (i) fracterm decomposition operators and the numerator paradox, (ii) foundational specifications of arithmetical datatypes, and (iii) poly-infix operations.
Keywords: Algebraic specification, fracterm decomposition operations, arithmetic datatypes, poly-infex operations.

1 Introduction

In [16] a finite initial algebra specification is given for an abstract datatype of rational numbers. The axioms are given with inversive notation (inverse rather than division) and inverse is made total by requiring that $0^{-1} = 0$. I will write $\mathbb{Q}_0$ for the abstract datatype of rational numbers with 0-totalized inverse.

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The specification given in [16] consists of the equations for a commutative ring plus the equations $x \cdot (x^{-1} \cdot x) = x$ and $(x^{-1})^{-1} = x$, together constituting the axioms Md of meadows in the terminology of [9], plus the equation $(x^2 + y^2 + z^2 + u^2 + 1) \cdot (x^2 + y^2 + z^2 + u^2 + 1)^{-1} = 1$. Below I will make use of divisive notation (division instead of inverse, see also [10]). Using divisive notation the axioms for meadows are the axioms (1,...,11) of Table 2. This set of 11 equations is referred to as Md^d (equations for meadows in divisive notation).

In [10] it was shown that improving upon the result of [16] the simpler equation $(x^2+y^2+1)/(x^2+y^2+1) = 1$ can be used in combination with Md^d for giving an initial algebra specification of the abstract datatype $\mathbb{Q}_0^d$. In [4] it is shown that replacing the latter equation by a finite set of equations of the form $(x^2 + \underline{n} + 1)/(x^2 + \underline{n} + 1) = 1$ cannot provide a specification of $\mathbb{Q}_0^d = 1$. In [5] it is shown that the following equation, with a single variable x only suffices:

$$(2 \cdot (x^2 - 3) \cdot (x^2 - 5) \cdot (x^2 - 15))/((2 \cdot (x^2 - 3) \cdot (x^2 - 5) \cdot (x^2 - 15)) = 1.$$

It is shown in [5] that for each finite sequence of equations $e_1, \ldots, e_n$ there is an equation e which, relative to Md^d is logically equivalent to $e_1 \wedge \cdots \wedge e_n$, which implies that when looking for logically weak extensions of Md^d as specifications of $\mathbb{Q}_0^d$ considering single additional equations suffices. Following [7] a (divisive) cancellation meadow is an algebra which comes about from a field K by enriching it with an inverse function (division function) which is made total by choosing $0^{-1} = 0$ $(1/0 = 0)$. Trivially a (divisive) cancellation meadow satisfies the axioms for (divisive) meadows.

Proposition 1 *Suppose* $\mathsf{Md} + e$ *constitutes an initial algebra specification of* $\mathbb{Q}_0$*. Then* e *is not valid in some cancellation meadow.*

Proof: The completeness theorem of [9] states that equations valid in all cancellation meadows are derivable from Md. Let e be such that $\mathsf{Md}^d + e$ constitutes an initial algebra specification of $\mathbb{Q}_0^d$ and such that e is true in all cancellation meadows. Then with the completeness theorem $\mathsf{Md} \vdash e$ so that already Md constitutes an initial algebra specification of $\mathbb{Q}_0$, a conclusion which contradicts the results of [17]. A contradiction is found and the theorem follows. □

Next I will consider the special case of characteristic 0, which allows a strengthening of the above proposition. Following the notation of [5] $\mathsf{Inv}_P = \{\underline{n}/\underline{n} = 1 | n \text{ a prime number}\}$. Now according to [5] $\mathsf{Md}^d + \mathsf{Inv}_P$ is an initial algebra specification of $\mathbb{Q}_0^d$ and moreover it is a weakest possible specification because each such specification must imply Inv_P given that $\mathbb{Q}_0^d \models \mathsf{Inv}_P$. The following observation is shown in [5].

Proposition 2 *Suppose* $\mathsf{Md}^d + e$ *constitutes an initial algebra specification of* $\mathbb{Q}_0$*. Then* e *is not valid in some cancellation meadow with characteristic* 0*.*

I will use module algebra notation (following the module algebra as defined in [8]) for specification which may involve auxiliary functions: $\Sigma \Box X$ exports from the specification (module) only the sorts, constants and functions of Σ. Further $\_ + \_$ is used for the combination (union) of signatures, of collections of equations and of modules.

1.1 Most General (Conditional) Equational Specifications

The objective of this paper is to investigate initial algebra specifications for $\mathbb{Q}_0$ which are general to the extent that the axioms of the specification are compatible with each cancellation meadow of characteristic zero. This idea motivates the following definition.

Definition 1 *A specification* $\Sigma_{\mathsf{Md}} \Box (\Sigma, \mathsf{Md}^d + E)$*, with* $\Sigma = \Sigma_{\mathsf{Md}} \cup \Sigma(E)$*, of* $\mathbb{Q}_0^d$ *is most general, if the following two conditions are met:*

(*i*) *its initial algebra is isomorphic to* $\mathbb{Q}_0^d$ *or (in case one or more auxiliary constants or functions are used) to an expansion of* $\mathbb{Q}_0^d$*, and*

(*ii*) *each cancellation meadow of characteristic* 0 *is a model of* E*, or (in case one or more auxiliary constants or functions are used) each cancellation meadow of characteristic* 0 *can be expanded to a model of* E*.*

The following proposition provides a sufficient and necessary criterion for the first condition that a most general specification of $\mathbb{Q}_0^d$ must meet.

Proposition 3 *Given an equational (or conditional equational) specification* $\Sigma_{\mathsf{Md}} \Box (\Sigma, \mathsf{Md}^d + E)$ *which satisfies these properties:*

(*i*) $\mathsf{Md}^d + E$ *implies all assertions in* Inv_P*,*

(*ii*) *There is a* Σ *expansion* $\mathbb{R}_0^d$ *of* $\mathbb{Q}_0^d$ *which satisfies* E*, and*

(*iii*) *Each closed* Σ *expression is provably equal in* $\mathsf{Md}^d + E$ *to a closed* Σ_{Md} *expression.*

Then $\Sigma_{\mathsf{Md}}\Box(\Sigma, \mathsf{Md}^d + E)$ *is an initial algebra specification of* $\mathbb{Q}_0^d$*.*

Proof: From (ii) it follows that the initial algebra $\mathbb{A}$ of $\Sigma_{\mathsf{Md}}\Box(\Sigma, \mathsf{Md}^d + E)$ does not identify $\underline{n}$ and $\underline{m}$ for different positive natural numbers n and m. From (i) in the presence of Md it follows that the Σ_{Md} reduct $\mathbb{A}|_{\Sigma_{\mathsf{Md}}}$ of $\mathbb{A}$ has the rational numbers as a substructure. From (iii) it follows that all elements of the domain of $\mathbb{A}$ a re in $\mathbb{A}|_{\Sigma_{\mathsf{Md}}}$ which is isomorphic to $\mathbb{Q}_0^d$, so that $\mathbb{A}$ is an expansion of $\mathbb{Q}_0^d$ as desired. □

2 Three Most General Initial Algebra Specifications of $\mathbb{Q}_0^d$

That the specification of $\mathbb{Q}_0^d$ mentioned in the introduction is not most general follows from the propositions mentioned above as well as from the counter example one finds when working in the complex numbers with $x = \mathbf{i}, y, z, u = 0$. In this Section some most general specifications of $\mathbb{Q}_0^d$ are discussed. The following fact is immediate.

Proposition 4 $(\Sigma_{\mathsf{Md}}, \mathsf{Md} + \mathsf{Inv}_P)$ *constitutes a most general initial algebra specification of* $\mathbb{Q}_0^d$*.*

Clearly in the light of Proposition 2 a most general specification of $\mathbb{Q}_0^d$ must be either infinite (such as $\mathsf{Md} + \mathsf{Inv}_P$) or it must make use of one or more auxiliary functions, or it must enjoy both of these features. Theorem 1 is concerned with the existence of a most general initial algebra specification of $\mathbb{Q}_0^d$ s pecification with an auxiliary function. The resulting specification, however, seems to be a mere curiosity without much explanatory value and which hardly serves the practical purpose of specification. For that reason I will mainly consider χ_0-general specifications with infinitely many equations in more detail, focussing on a particular framework for representing such specifications as a finite collection of schemes.

I will consider divisive meadows extended with decimal constants, for digits as well as for sequences of digits, which are understood as natural numbers. The successor function on the decimal digits $\{0, \ldots, 8\}$ is specified in Table 1 as a transformation on syntax.

Definition 2 C_{D} *denotes the collection of non-empty sequences of decimal digits starting with a non-zero digit. Elements of* C_{D} *are referred to as (positive) decimal numbers.*

Table 1: DGS: enumeration and successor notation of decimal digits

$0' \equiv 1$	$3' \equiv 4$	$6' \equiv 7$
$1' \equiv 2$	$4' \equiv 5$	$7' \equiv 8$
$2' \equiv 3$	$5' \equiv 6$	$8' \equiv 9$

I will make no distinction between say 87 as a number and 87 as a constant. When adopting the latter convention the notions of number and constant (more specifically natural number and decimal number constant) are being associated according to terminology of [25]. I will make use of schematic equations of the form, say $e(u,v,w) \equiv t(u,v,w) = r(u,v,w)$, which are supposed to be asserted for each instantiation with constants σ_1, σ_2, σ_3 for u, v, w. Thus the equation $u/u = 1$ stands for $\{1/1 = 1, 2/2 = 1, \ldots, 9/9 = 1, 10/10 = 1, 11/11 = 1, \ldots\}$.

Table 2 provides an infinite algebraic (initial algebra) specification of the meadow rational numbers with decimal number constants $\mathbb{Q}^d_{0,\mathsf{C_D}}$. The specification may be restricted to the signature of meadows by hiding constants in $\mathsf{C_D}$ except 1. Module algebra as defined in [8] can be extended with the infinite signature $\mathsf{C_D}$, and with mechanisms fo schematic equations and schematic conditional equations taking metavariables in $\mathsf{C_D}$ without any problem. Using the module algebra notation for hiding a part of a signature (and extended to infinite signatures) the specification can be denoted as a module expression: $\Sigma_{\mathsf{Md}} \square \mathsf{Md}^d_{\mathsf{C_D}} = (\mathsf{C_D} \backslash \{1\}) \Delta \mathsf{Md}^d_{\mathsf{C_D}}$.

Proposition 5 $\Sigma_{\mathsf{Md}} \square \mathsf{Md}^d_{\mathsf{C_D}}$ *constitutes a most general initial algebra specification of* $\mathbb{Q}^d_0$.

Proof: With induction on n it is easily shown that for a natural number $n > 0$ $\mathsf{Md}^d_{\mathsf{C_D}} \vdash \underline{n}/\underline{n} = 1$. In combination with $\mathbb{Q}^d_{0,\mathsf{C_D}} \models \mathsf{Md}^d_{\mathsf{C_D}}$, which is immediate by inspection, this observation proves that the initial algebra of $\mathsf{Md}^d_{\mathsf{C_D}}$ is $\mathbb{Q}^d_{0,\mathsf{C_D}}$. In order to see that, upon viewing the constants in $\mathsf{C_D} \backslash \{1\}$ as auxiliary constants (hidden constants), the specification is most general for $\mathbb{Q}^d_0$, one may notice that the axioms of $\mathbb{Q}^d_{0,\mathsf{C_D}}$ after expansion of the metavariables constitute a chain of explicit definitions. Thus each cancellation meadow of characteristic 0 can be expanded to a model of $\mathsf{Md}^d_{\mathsf{C_D}}$. $\square$

Table 3 provides a specification module which may be combined with $\mathsf{Md}^d_{\mathsf{C_D}}$ into a specification of an expansion with $f{:}V \to V$ of $\mathbb{Q}^d_0$.

Table 2: $\mathsf{Md}^d_{\mathsf{C_D}}$: initial algebra specification of the abstract datatype $\mathbb{Q}^d_{0,\mathsf{C_D}}$

$$(x+y)+z = x+(y+z) \tag{1}$$
$$x+y = y+x \tag{2}$$
$$x+0 = x \tag{3}$$
$$x+(-x) = 0 \tag{4}$$
$$(x\cdot y)\cdot z = x\cdot(y\cdot z) \tag{5}$$
$$x\cdot y = y\cdot x \tag{6}$$
$$1\cdot x = x \tag{7}$$
$$x\cdot(y+z) = (x\cdot y)+(x\cdot z) \tag{8}$$
$$1/(1/x) = x \tag{9}$$
$$(x\cdot x)/x = x \tag{10}$$
$$x/y = x\cdot y^{-1} \tag{11}$$
$$d' = d+1 \quad (\text{for } d\in\{0,\ldots,8\}) \tag{12}$$
$$u0 = ((((((((u+u)+u)+u)+u)+u)+u)+u)+u)+u \quad (\text{for } u\in \mathsf{C_D}) \tag{13}$$
$$ud = u0+d \quad (\text{for } d\in\{1,\ldots,9\}, u\in\mathsf{C_D}) \tag{14}$$
$$u/u = 1 \quad (\text{for } u\in\mathsf{C_D}) \tag{15}$$

Theorem 1 *$\Sigma_{\mathsf{Md}^d}\square(\Sigma_{\mathsf{Md}^d}+f{:}V\to V, \mathsf{Md}^d+E_f)$ is a most general initial algebra specification of $\mathbb{Q}^d_0$.*

Proof: Proposition 3 is used to show that the initial algebra is $\mathbb{Q}^d_0$. The conditional equations E^c_f in Table 4 are consequences of the equations E_f in Table 3. It follows with induction on n that for each n, $E^c_f \vdash f(\underline{n}) = 1$ and thus $E^c_f \vdash \underline{n}/\underline{n} = 1$ and also $E_f \vdash \underline{n}/\underline{n} = 1$, whence $E_f \vdash \mathsf{Inv}_P$. With Proposition 4 it follows that the reduct of the initial algebra of the specification $(\Sigma_{\mathsf{Md}^d}+f{:}V\to V, \mathsf{Md}+E_f)$ to the signature Σ_{Md^d} is a homomorphic image of $\mathbb{Q}^d_0$, or is an extension of a homomorphic image of $\mathbb{Q}^d_0$.

Further $\mathbb{Q}^d_0$ can be expanded with a function f to $\mathbb{Q}^d_{0,f}$ of E_f by defining f as follows: $f(p) = 1$ for positive natural numbers p and $f(p) = 0$

Table 3: E_f

$$f(0) = 0 \tag{16}$$
$$f(1) = 1 \tag{17}$$
$$f(x) \cdot (1 - x/x) = 0 \tag{18}$$
$$f(x) \cdot (1 - f(x+1)) = 0 \tag{19}$$
$$(1 - f(x)) \cdot f(x-1) = 0 \tag{20}$$
$$(f(x) \cdot f(y) \cdot f(x/(x+y)) = 0 \tag{21}$$
$$((f(x) \cdot f(y)) \cdot f(z)) \cdot f(z + (x/(x+y))) = 0 \tag{22}$$
$$(f(x) \cdot f(y)) \cdot f(-(x/(x+y))) = 0 \tag{23}$$

Table 4: E_f^c

$$f(0) = 0 \tag{24}$$
$$f(1) = 1 \tag{25}$$
$$f(x) = 1 \to x/x = 1 \tag{26}$$
$$f(x) = 1 \to f(x+1) = 1 \tag{27}$$
$$f(x) = 0 \to f(x-1) = 0 \tag{28}$$
$$f(x) = 1 \wedge f(y) = 1 \to f(x/(x+y)) = 0 \tag{29}$$
$$f(x) = 1 \wedge f(y) = 1 \wedge f(z) = 1 \to f(z + x/(x+y)) = 0 \tag{30}$$
$$f(x) = 1 \wedge f(y) = 1 \to f(-(x/(x+y))) = 0 \tag{31}$$

on all other arguments. Inspection of the equations suffices to infer that $\mathbb{Q}_{0,f}^d \models E_f$. Every cancellation meadow of characteristic 0 can be expanded to a model of E_f in a corresponding manner. It follows that the reduct of the initial algebra of the specification $(\Sigma_{\mathsf{Md}^d} + f{:}V \to V, \mathsf{Md} + E_f)$ to the signature Σ_{Md^d} is an extension of $\mathbb{Q}_0^d$. It remains to be shown that the extension is in fact an expansion.

To demonstrate the latter it must be shown that closed terms involving f can be proven equal to closed terms not involving f. It suffices to consider terms of the form $t \equiv f(s)$ with a single occurrence of f at top level only.

A case distinction on $[\![s]\!]$ the rational number denoted by s is required: if $s = 0$, $f(s) = 0$, if $s = 1$, then $f(s) = 1$. If s is provably equal to $\underline{n}$ for some positive natural number n then $f(s) = 1$, if $s = -\underline{n}$ for some positive natural number n then induction starting from $n = 0$ yields that $f(s) = 0$. If $[\![s]\!]$ is not an integer and is between 0 and 1 then for some positive n and m: $\mathbb{Q}_0^d \models s = \underline{n}/(\underline{n} + \underline{m})$ from which it can be inferred that $\mathsf{Md}{+}E_f^c \vdash s = \underline{n}/(\underline{n}{+}\underline{m})$ and therefore that $\mathsf{Md}{+}E_f^c \vdash f(s) = 0$. If s denotes a non-integral positive rational number above 1 then besides n and m, a positive integer k can be found so that $s = \underline{k}{+}\underline{n}/(\underline{n}{+}\underline{m})$ which allows to infer that $f(s) = 0$. In case s denotes a negative rational number between -1 and 0 positive naturals n and m can be found so that $s = -(\underline{n}/(\underline{n} + \underline{m}))$ which allows to infer $f(s) = 0$, and for smaller non-integer negative rational values of s the conditional equation $f(x) = 0 \to f(x - 1) = 0$ allows an inductive proof that $f(s) = 0$ using the cases that $[\![s]\!] \in (-1, 0)$ as a basis. □

Viewed as a term rewrite system, $\mathsf{Md}^d_{\mathsf{C_D}}$ fails to have the decimal numbers as normal forms. However, I will not discuss the technically rather involved topic of term rewriting for arithmetical datatypes in the presence of schematic rules in this paper.

Most general specifications of the rational numbers do not have finite models, whereas the equations for meadows allow for a wide diversity of finite models, see [18].

The expressive power of equational schemes shows limitations in the context of elementary arithmetic. In particular the use of addition in the specification of Table 2 can hardly be avoided. This observation is detailed in the following Paragraph.

2.1 Specifying Natural Numbers with Successor Function

The specification in Table 5 specifies a datatype of atural numbers with successor function and with all constants in $\mathsf{C_D}$. It turns out that the use of conditional equation(s) can not be avoided for this specification.

Table 5: Initial algebra specification of $\mathbb{N}_{0,C_D}(S(\_))$: natural numbers with successor function and decimal number constants

$d' = S(d)$	(for $d \in \{0, \dots, 8\}$)	(32)
$10 = S(9)$		(33)
$d'0 = S(d9)$	(for $d \in \{1, \dots, 8\}$)	(34)
$ud' = S(ud)$	(for $u \in C_D, d \in \{0, \dots, 8\}$)	(35)
$v = S(u) \to v0 = S(u9)$	(for $u, v \in C_D$)	(36)

Theorem 2 $\mathbb{N}_{0,\mathsf{C_D}}(S(\_))$ *does not have an equational specification that consists of finitely many schemes.*

Proof: Let, for $d \in D$ and $k > 0$, $d^{[1]} = d, d^{[k+1]} = d^{[k]}d$. With this notation: $\mathbb{N}_{0,\mathsf{C_D}}(S(\_)) \models 10^{[k]} = S^{10^k}(0)$. Let E_s constitute an initial algebra specification of $\mathbb{N}_{0,C_D}(S(\_))$ made up from a finite number of equation schemes. It will be shown that for sufficiently large k the equation $10^{[k]} = S^{10^k}(0)$ cannot be derived from E_s. Now choose k larger than the length of any constant of C_D occurring in any equation in E_s plus the number of occurrences of digits occurring in all equations plus the number of occurrences of the successor function in all equations. Because an expression σ for $\sigma \in \mathsf{C_D}$ is a constant it has no proper subterms. In particular σ is not considered a subterm of its extension by postfixing the decimal digit d: σd.

First notice that for syntactic reasons an expression t may contain at most a single meta-variable u ranging over $\mathsf{C_D}$. Secondly an equation $t(u) = r(v)$ (with by assumption u and v different variables) cannot be valid in $\mathbb{N}_{0,\mathsf{C_D}}(S(\_))$, a counter example is found by taking $u = 0$ and v larger than $[\![t(0)]\!]$. Therefore an equation involving u has the form $t(u) = r(u)$, and in more detail: $S^a(ud_1 \dots d_m) = S^b(ue_1 \dots e_n)$, for natural numbers a, b, m, n. Each equation of E_s involves at most one metavariable for $\mathsf{C_D}$ and therefore it may be assumed that all schematic equations involve the same metavariable u. Equational logic is understood as term rewriting where all equations may be used in both directions. It is to be shown that for sufficiently large k the $10^{[k]}$ cannot be rewritten into the expression $S^{10^k}(0)$. The proof uses induction on the number of equational schemes in E_s so it is assumed that this number is minimal. It cannot be zero because say $1 = s(0)$ must be derivable from E_s.

For a rewrite rule $t(u) = r(u)$ to match with $S(10^{[k]})$ it is needed that t results by substituting some $\sigma \in \mathsf{C_D}$ for u in t because otherwise the constant $10^{[k]}$ is too long to match with any subterm of any lefthand side of an equation in E_s. So there is an equation $t(u) = r(u)$ in E_s (i.e. $t(u) = r(u)$ is in E_s or $r(u) = t(u)$ is in E_s) such that for some $\sigma \in C_D$ $t(\sigma) = r(\sigma)$ is a rewrite rule, say of the form $S^a(\sigma d_1 \dots d_m) = S^b(\sigma e_1 \dots e_n)$, which applies to $10^{[k]}$. It follows that $a = 0$ and that $\sigma d_1 \dots d_m = 10^{[k]}$, whence $\sigma = 10^{[k-m]}$. Thus $10^{[k]} = S^b(10^{[k-m]}e_1 \dots e_n)$. For $b > 0$ his equation cannot be valid in $\mathbb{N}_{0,C_D}(S(\_))$, however, because k has been chosen so large that it exceeds $b+m+n$ with the effect that the $S^b(10^{[k-m]}e_1 \dots e_n)$ cannot be a power of 2. Therefore $b = 0$ so that $10^{[k]} = S^b(10^{[k-m]}e_1 \dots e_n)$ with as a consequence that $n = m$ and $e_1 = \ldots = e_n = 0$. It follows that the equation $t(u) = r(u)$ is (syntactically) equal to $\sigma 0 \ldots 0 = \sigma 0 \ldots 0$ where both sequences of zeros at the tail have equal length. It follows that $t(u) \equiv r(u)$ for each substitution σ for u so that the equation scheme is redundant and can be deleted, which contradicts the minimality of E_s. □

3 Fracterm Decomposition Functions

For meadows the so-called fracterms plays a central role. Fracterms are discussed in much detail in [3] and appear in [14]. It is tempting to refer to a fracterm as fraction, as was done in [11] but I will refrain from the identification of fracterms with fractions. Fractions have been defined in the literature in disparate ways and there is no indication that defining fractions as fracterms amounts to a common understanding of fractions. For instance in [28] a fraction is understood as a value, rather than as an expression. In [21] is it suggested that unlike rational number fraction is not a mathematical notion. In [23] a variety of different perspectives on fractions are discussed, [24] suggests that fractions are expressions, while [29] indicates that fractions are pairs of integers.

3.1 Fracterms, Fracsigns, and Fracmarks

I prefer to remain uncommitted to any specific notion or definition of fractions and this neutral position can be maintained by introducing different names for similar but potentially different notions, starting with fracterm. Besides fracterm, fracsign and fracmark are useful related notions. A fracsign is a physical sign which refers to a fracterm, which itself may but need not refer to a number. A fracmark is a mark on a number line which denotes a rational number relative to the line. Below I will focus on fracterms only. Technically a fracterm can be defined as follows:

Definition 3 *A fracterm is a term t for the signature Σ_{Md}, or for any larger signature, which has division as its leading function symbol, i.e. a fracterm has the form r/s.*

A fracterm may be open or closed. The class of open fracterms includes the closed ones but not conversely. When needed, the presence of a variable must be positively asserted e.g. with t is not closed, or x occurs in t. The components r and s of a fracterm are referred to as the numerator and the denominator of the fracterm respectively. In this Section I will consider options for extending the specification $\mathsf{Md}^d_{\mathsf{C_D}}$ with another module that introduces selector functions for numerator and denominator on fracterms.

A first attempt for the design of such a module is given Table 6. It is immediate, however, that this extension is inconsistent: $1 = \mathsf{Num}(1/2) = \mathsf{Num}(2/4) = 2$. I propose to name this argument (and similar arguments) the numerator paradox. This paradox results if a numerator selector operation is introduced without due care. The paradox demonstrates that extending $\mathsf{Md}^d_{\mathsf{C_D}}$ to $\mathsf{Md}^d_{\mathsf{C_D}}(\mathsf{Num}, \mathsf{Denom})$ is not a valid approach and is, for that reason, in need of modification. Working in an arithmetical datatype such as $\mathbb{Q}^d_{0,\mathsf{C_D}}$, and restricting attention to closed terms only, a suitable improvement of the datatype expansion with fracterm decomposition functions can be obtained.

Table 6: $\mathsf{Md}^d_{\mathsf{C_D}}(\mathsf{Num}, \mathsf{Denom})$: naive equational specification of fracterm decomposition operations

$$\texttt{include } \mathsf{Md}^d_{\mathsf{C_D}}$$

$$\mathsf{Num}(x/y) = x \tag{37}$$

$$\mathsf{Denom}(x/y) = y \tag{38}$$

3.2 The Numerator Paradox and Its Informal Solution

The contradiction that comes with adopting Table 6 is obvious, but at the same time it is equally "obvious" that human informal reasoning allows to ignore the reasoning pattern leading to $1 = 2$ and to work with numerators and denominators as if these satisfy the equations in Table 7 while avoiding to draw wrong conclusions. For instance: it may be asserted that "the denominator of the fraction $31/(-7)$ is negative". This assertion is potentially problematic because $31/(-7) = (-31)/7$ so that said denominator may also be considered positive. However, the human mind seems able to avoid making consequential mistakes about such matters by (implicitly) viewing each argument of Num and Denom as a fracterm rather than as a quotient, and by dismissing texts which discourage such reading.

Understanding the mechanism of this remarkable ability to make moderate and safe use of terminology requires further attention. It may be the case that reasoning about numerators and denominators is mainly used to describe and justify which rewrite rules may be applied, and in addition that $\mathsf{Num}(-)$ and $\mathsf{Denom}(-)$ are understood as operating on syntax rightaway. Formalising that idea is possible by introducing the convention that underlined operations work on syntax (terms) rather than on semantics, which is formalized in Table 7 from which a numerator paradox cannot be inferred. If it is preferred that $\underline{\mathsf{Num}}$ and $\underline{\mathsf{Denom}}$ are total operations, 0 may be chosen as the result for all non-fracterm arguments.

Table 7: $\mathsf{Md}^d_{\mathsf{C_D}}(\underline{\mathsf{Num}}, \underline{\mathsf{Denom}})$: equational specification of an arithmetical abstract datatype $\mathbb{Q}^d_{0,\mathsf{C_D}}(\underline{\mathsf{Num}}, \underline{\mathsf{Denom}})$ expanded with syntactic operations $\underline{\mathsf{Num}}$ and $\underline{\mathsf{Denom}}$

$$\texttt{include } \mathsf{Md}^d_{\mathsf{C_D}}$$

$$\underline{\mathsf{Num}}(x/y) \equiv x \tag{39}$$

$$\underline{\mathsf{Denom}}(x/y) \equiv y \tag{40}$$

$$\underline{\mathsf{Num}}(u) \equiv 0 \tag{41}$$

$$\underline{\mathsf{Num}}(-x) \equiv 0 \tag{42}$$

$$\underline{\mathsf{Num}}(x+y) \equiv 0 \tag{43}$$

$$\underline{\mathsf{Num}}(x \cdot y) \equiv 0 \tag{44}$$

$$\underline{\mathsf{Denom}}(u) \equiv 0 \tag{45}$$

$$\underline{\mathsf{Denom}}(-x) \equiv 0 \tag{46}$$

$$\underline{\mathsf{Denom}}(x+y) \equiv 0 \tag{47}$$

$$\underline{\mathsf{Denom}}(x \cdot y) \equiv 0 \tag{48}$$

3.3 Equational/Equivalence Logic

The combination of mathematical functions and syntactic functions suggests a different logic, which I call equational/equivalence logic, and which applies in the context of arithmetical datatypes.

Consider an arithmetical abstract datatype $\mathbb{A}$ which is a canonical term algebra for its signature. For instance $\mathbb{A} = \mathbb{Q}^d_{0,\mathsf{C_D}}$. Now, given syntactic equivalence assertions for closed expressions as syntactic equality, $\mathbb{A}$ admits an interpretation of open syntactic equivalence assertions as well.

Let σ be a valuation of variables $x, y, z, \ldots$ into the domain of $\mathbb{A}$. Now consider for instance the equivalence (assertion of equivalence) $t(x_1, \ldots, x_n) \equiv r(x_1, \ldots, x_n)$. Validity of the assertion in $\mathbb{A}$ is defined by:

$\mathbb{A} \models t(x_1, \ldots, x_n) \equiv r(x_1, \ldots, x_n)$ if and only if for all sequences of closed terms $t_1, \ldots, t_n$ over $\Sigma(\mathbb{A})$ it is the case that $t(t_1, \ldots, t_n) \equiv r(t_1, \ldots, t_n)$.

The equational/equivalence logic involving syntactic (underlined functions, for instance a two place syntactic function $\underline{f}(-,-)$) allows for syntactic equivalence assertions. It is not required that

$$x_1 = x_2 \wedge y_1 = y_2 \rightarrow \underline{f}(x_1, y_1) = \underline{f}(x_2, y_2)$$

while it is required to be the case that:

$$x_1 \equiv x_2 \wedge y_1 \equiv y_2 \rightarrow \underline{f}(x_1, y_1) \equiv \underline{f}(x_2, y_2)$$

for all pairs of terms t, r it is the case that $\mathbb{A} \models t \equiv r \rightarrow t = r$.

Now $\underline{\mathsf{Num}}(-)$ and $\underline{\mathsf{Denom}}(-)$ are defined by: $\underline{\mathsf{Num}}(t/r) \equiv t$ for all closed fracterms t/r and $\underline{\mathsf{Num}}(s) \equiv 0$ for all closed terms which are not fracterms s. Similarly $\underline{\mathsf{Denom}}(t/r) \equiv r$ for all closed fracterms t/r and $\underline{\mathsf{Denom}}(s) \equiv 0$ for all closed non-fracterms s. Now various familiar as well as unfamiliar assertions on fracterms allow formalization. Some examples, making use of inequality $u \neq 0$ and $u \neq v$ (for different constants $u, v \in \mathsf{C_D}$) both of which hold in $\mathbb{Q}^d_{0,\mathsf{C_D}}(\mathsf{Num}, \mathsf{Denom})$:

$$0 \neq \underline{\mathsf{Denom}}(x) \wedge \underline{\mathsf{Denom}}(x) \equiv \underline{\mathsf{Denom}}(y)$$

$$\rightarrow x + y = (\underline{\mathsf{Num}}(x) + \underline{\mathsf{Num}}(y))/\underline{\mathsf{Denom}}(x)$$

$$\underline{\mathsf{Num}}(x) = \underline{\mathsf{Denom}}(x) \wedge \underline{\mathsf{Num}}(x) \neq 0 \rightarrow x = 1$$

$$0 \neq \underline{\mathsf{Num}}(x) \rightarrow x = \underline{\mathsf{Num}}(x)/\underline{\mathsf{Denom}}(x)$$

$$\underline{\mathsf{Num}}((5+11)/(19-3)) = \underline{\mathsf{Denom}}((5+11)/(19-3))$$

$$\underline{\mathsf{Num}}(1/2) \not\equiv \underline{\mathsf{Num}}(2/4)$$

$$1 = \underline{\mathsf{Num}}(1/2) \neq \underline{\mathsf{Num}}(2/4) = 2$$

$$1 + 1 \equiv \underline{\mathsf{Num}}((1+1)/4) \not\equiv \underline{\mathsf{Num}}(2/4) \equiv 2$$

$$\underline{\mathsf{Denom}}(\underline{\mathsf{Num}}((1/2)/4)) = 2 + \underline{\mathsf{Num}}(\underline{\mathsf{Denom}}((1/2)/4))$$

It can be shown that $\underline{\mathsf{Num}}(-)$ is a new operator for $\mathbb{Q}^d_{0,\mathsf{C_D}}$. In particular $\underline{\mathsf{Num}}(-)$ cannot be eliminated from at least one kind of open expression: expressions consisting of a single free variable.

Proposition 6 *There is no open or closed expression t over $\Sigma_{\mathsf{urd},\perp,\mathsf{C_D}}$, (that is without $\underline{\mathsf{Num}}$ and without $\underline{\mathsf{Denom}}$), such that $\mathbb{Q}^d_{0,\mathsf{C_D}}(\mathsf{Num},\mathsf{Denom}) \models \underline{\mathsf{Num}}(x) = t$.*

Proof: If t is closed then $\underline{\mathsf{Num}}(x) = t$ implies, by substituting $(1+t)/1$ for x, that $1+t \equiv \underline{\mathsf{Num}}((1+t)/1) = t$ which implies that $t = \perp$. Then substituting $1/1$ for x yields $1 = \perp$ which is not the case. So we are left with the option that $\underline{\mathsf{Num}}(x) = t$ with t an open term. If say $t \equiv t(x,y,z)$ then $\underline{\mathsf{Num}}(x) = t(x,0,0)$. So it may be assumed that t has only one variable and $\underline{\mathsf{Num}}(x) = t(x)$. As $\underline{\mathsf{Num}}(u) = 0$ for all $u \in \mathsf{C_D}$ it follows that $t(u) = 0$ for all u. Suppose that $t(x)$ is a polynomial with integer coefficients then if it vanishes (in $\mathbb{Q}^d_{0,\mathsf{C_D}}$) on al natural numbers u it must be the zero polynomial so that in fact $\underline{\mathsf{Num}}(x) = 0$ for all x which is wrong as $\underline{\mathsf{Num}}(1/1) = 1$. In the general case the main result of [6] can be used from which it follows that $t(x) = p(x)+q(x)/r(x)$ with polynomials p, q, and r with integer coefficients. Suppose that $r(u) = 0$ for infinitely many u then $r(x)$ vanishes for all x and $t(x) = p(x)$, which has been shown to be impossible already. Thus assume that for all but finitely many u $r(u) \neq 0$ and thus $0 = t(u) = p(u) + q(u)/r(u) = (p(u)\cdot r(u))/r(u) + q(u)/r(u) = ((p(u)\cdot r(u)) + q(u))/r(u)$ and thus for these u, $(p(u)\cdot r(u))+q(u) = 0$. It follows that $(p(x)\cdot r(x))+q(x) = 0$ for all x so that $t(x) = p(x)\cdot(1 - r(x)/r(x))$ while for all x and for almost all u, $r(u) \neq 0$. So let $u \neq 0$ be such that $r(u) \neq 0$. Then consider $x \equiv u/1$. A contradiction arises: $u = \underline{\mathsf{Num}}(u/1) = p(u)\cdot(1 - r(u)/r(u)) = p(u)\cdot 0 = 0$. □

4 Foundational Specification of an Arithmetical Datatype

I will now assume that K is a class of agents. A class of agents K is considered to be endowed with a subclass K_a of authoritative members, who share essential opinions and beliefs. The application which I have in mind for this notion is that K includes all (or at least many) teachers of elementary mathematics as well as their students, and persons in close contact with these students. In this application a class of senior teachers would play the role of authoritative members.

Definition 4 *A conditional equational arithmetical datatype specification with signature $\Sigma \supseteq \Sigma_{\mathsf{Md}^d_{\mathsf{C_D}}}$ will be called foundational for a class K of agents (with authoritative members K_a) if:*

(*i*) *for each pair of closed* Σ *terms* t *and* r *either* $E \vdash t = r$ *or* $E \cup \{t = r\} \vdash 0 = 1$,

(*ii*) *there is no known (to any agent in* K_a*) proof of* $0 = 1$, *and*

(*iii*) *the specification has a dogmatic status for the agents in* K *in the following sense: (iiia) only if* $E \vdash t = r$, *and if such a proof is known to one or more agents in* K_a, *the belief that* $t = r$ *is justified for some agent in* K, *and (iiib) only if* $E \cup \{t = r\} \vdash 0 = 1$, *and if such a proof is known to one or more agents in* K_a, *the belief that* $t \neq r$ *is justified for some agent in* K.

According to [15], allowing the use of auxiliary functions, an equational specification with the property that for closed terms t and r, $E \cup \{t = r\} \vdash 0 = 1$ unless $E \vdash t = r$ can be obtained for a large class of datatypes, in fact for all co-semicomputable datatypes. It should be noticed that $\mathsf{Md} \cup \{0 = 2\} \nvdash 0 = 1$ so that Md fails to have said property, but $\mathsf{Md}^d_{\mathsf{C_D}}$ does.

4.1 $\mathsf{Md}^d_{\mathsf{C_D}}$ as a Foundational Specification of $\widetilde{\mathbb{Q}}^d_{0,\mathsf{C_D}}$

Now let K be the class of teachers of elementary arithmetic, and their students and friends, relatives and care takers of these students, with K_a the subclass of senior teachers. Admittedly the description of K is imprecise, and working towards precise descriptions of actual classes of agents lies outside the scope of this paper. So K is merely introduced as a thought experiment which is not meant to be turned into a point of departure for a rigorous empirical investigation.

Now, by way of a thought experiment $\mathsf{Md}^d_{\mathsf{C_D}}$, is said to have the status of a foundational specification of $\widetilde{\mathbb{Q}}^d_{0,\mathsf{C_D}}$ relative to the class of agents K.

4.2 Doing Away with the Numerator Paradox

Viewing $\mathsf{Md}^d_{\mathsf{C_D}}$ as a foundational specification its extension with Table 6 becomes unproblematic. The idea is that agents in K_a know how to avoid making wrong inferences from $\mathsf{Md}^d_{\mathsf{C_D}}(\mathsf{Num}, \mathsf{Denom})$ and teach other members of K how to avoid making such errors. There is no confusion about these matters possible: an inference is wrong if it leads to conflicts with the conclusions drawn from the specification $\mathsf{Md}^d_{\mathsf{C_D}}$. Indeed on the basis of its foundational status reasoning within $\mathsf{Md}^d_{\mathsf{C_D}}$ takes priority over reasoning in $\mathsf{Md}^d_{\mathsf{C_D}}(\mathsf{Num}, \mathsf{Denom})$ which in part lies outside the reasoning in $\mathsf{Md}^d_{\mathsf{C_D}}$.

Not only the numerator paradox may be dealt with in this manner. Adopting a version of naive set theory which includes the Russel paradox may be allowed in a similar manner, if it comes to a crunch, which won't happen with competent reasoning, when unexpected results are derived the foundational specification is used as criterion of truth, and reasoning patterns are designed which allow, or even guarantee, the avoidance of reasoning patterns that are known to lead to problems.

4.3 Immunization of Elementary Arithmetic Against (Potential) Conceptual Inconsistency

The thought experiment as detailed above can be used as an explanation of why the agents in class K_a may not be bothered to develop a watertight theory of fractions (or fracterms). They know that it is fairly easy to avoid reasoning errors when working with $\mathsf{Md}^{d}_{\mathsf{C_D}}(\mathsf{Num},\mathsf{Denom})$ by suggesting notions of syntax which are not made explicit and claiming that it is a sign of maturity and comprehension if students are able to avoid errors in the use of fracterm (fraction) decomposition operators.

Supposing that the numerator paradox is taken seriously at least three responses are possible, (i) to develop a consistent logic of fractions which resolves the paradox while allowing much of the fraction talk, (ii) to ignore the matter and to assume that no serious problems will result, while basing the principles of arithmetic on a foundational interpretation of one's favoured specification, and (iii) to combine both approaches: to develop a useful but limited logic of fractions on top of one's favourite specification of arithmetic and to claim a foundational status for the theory thus obtained.

In the practice of fractions, (i) is not so easy unless one agrees either that a fraction is fracterm or that a fraction is a quotient (or more generally: unless a definition of fractions that can be given in terms of fracterms is adopted), (ii) is quite feasible and seems to be an adequate rationalization of mainstream views on fractions, although that approach it has not been made explicit in any paper to the best of my knowledge, and (iii) is a matter of speculation, for which I have not yet seen any examples or preceding work.

5 Poly-Infix Operations

A vital advantage of the specification $\mathsf{Md}^{d}_{\mathsf{C_D}}$ over the specifications of the arithmetical abstract datatype of rationals in [16] and subsequent improvements of it in [10] and in [5] comes from the presence of decimal numbers in $\mathsf{Md}^d_{\mathsf{C_D}}$ which brings it closer to practice. The practice of arithmetic, however incorporates at least four other features which lie outside the traditional techniques and strengths of algebraic specification methods: (i) operator precedence, (ii) poly-infix operations, (iii) synonyms of operator symbols, and (iv) two-dimensional notation. Operator precedence is a relatively easy matter: assuming the following precedence: division $>$ multiplication $>$ subtraction (negation) $>$ addition it can be explained that $(2 \cdot 5) + 3$ and $2 \cdot 5 + 3$ are equivalent. One way to look at this is to say that whoever writes $2 \cdot 5 + 3$ actually means to write $(2 \cdot 5) + 3$ and that this decoration with additional brackets is supposed to precede further analysis. Against this perspective on operator precedence one may hold that in practice these steps are exclusively performed in the mind of the reader, thereby raising the question if such steps are at all applied. A different approach is to have both expressions as part of the syntax and to have axioms which imply $2 \cdot 5 + 3 = (2 \cdot 5) + 3$ or even $2 \cdot 5 + 3 \equiv (2 \cdot 5) + 3$. I will not discuss any of these details of operator precedence here. Two-dimensional notation (as in fractions written with a horizontal fraction bar) entirely escapes the algebraic specification format and admits no further clarification from that perspective. Design and analysis of two-dimensional notation interferes with synonyms of operator symbols. Dealing with synonyms is easy to the extent that different names for the same operator may be included in a signature and equations can be used to express semantic correspondence. However, one may insist that not only $\frac{1}{2} = 1/2$ but in fact $\frac{1}{2} \equiv 1/2$ thereby turning syntactic sameness into a non-trivial notion. Working with synonyms becomes more intricate when some operator symbols can be deleted, which may happen with the multiplication symbol, say in $2(17 - 5)$. I will not pay further attention to synonyms in this paper. An aspect which can be brought firmly within the scope of algebraic specifications, however, is the use of poly-infix operations. The phrase poly-infix operation was coined in [13].

5.1 Infinite Families of Operations

Following [13] addition and multiplication may be understood as infinite families of so-called poly infix operations. The idea is that say $0 + 8 + 17$ is an instance of the use of a 3-place addition function, which in the signature comes on top of the 2-place version of addition together with all other n-ary versions of addition for $n > 3$. The declaration $\_ + \_(\pi) : V \to V$ introduces

$\_ + \_$ as an infinite family of operations for each arity, with the idea that infix notation is generalized to mixfix notation. The 2-place function $\_ + \_(\pi) : V \to V$. is called the kernel of the function family. The introduction of a family of poly-infix operator symbols does not by itself include or assert any requirement regarding associativity. Similarly $\_ \cdot \_(\pi) : V \to V$ introduces a poly-infix family of multiplication operators with $\_ \cdot \_$ as kernel.

An advantage of the use of poly-infix operations is the initial absence of brackets. For instance the expression $1+2+3+4$ need not be explained by first introducing brackets as in $((1+2)+3)+4$ and then making an abstraction of syntax modulo equivalent bracketing patterns, or by introducing default brackets which are claimed to have been deleted in writing for the sake of simplicity of notation. Instead one may say that $1+2+3+4$ is an expression different from (i.e. not the same as) $(1+2)+(3+4)$ and different from $((1+2)+3)+4$. A value of the value of $1+2+3+4$ can be found by means of rewriting in both ways followed by successive evaluations of the binary addition operator.

5.2 Term Formation for Poly-Infix Operations

Term formation in the context of poly-infix functions requires some care. For instance redundant brackets around constants are usually deleted under the assumption that $(1) = 1$ while $(1) \not\equiv 1$. Further for a variable x, $((x)) = (x)$. The expression $X \equiv 0+1+2+3+(2+2)$ is an instance of 4-place poly-infix addition, with four arguments the last one of which results by an application of two-place addition. Let $Y \equiv (2 \cdot 5) + (6 \cdot 4 \cdot 3) + 0$ then Y is an instance of 3-place poly-infix addition and $X + Y$ is an instance of 8-place poly-infix addition while $(X) + Y$ is an instance of 4-place poly-infix addition.

The axioms stated in [13], as an infinite scheme, upon specialization to the case of addition, can now be stated succinctly as in Table 8. These axioms are meant to be used in combination with the axioms of Md^d.

As an application of these axioms one finds for instance: $1+2+3+4 = [1]+[2]+[3+4] = ([1]+[2])+[3+4] = (1+2)+3+4$. Here square brackets are used to indicate the parts of an expression which are matched with corresponding variables in a rewrite rule. In the above derivation equation LAA (left association for addition, Table 8) is applied from left to right and the matching is $(x \to 1, y \to 2, z \to 3+4)$. Another derivation is $x + y = [x] + [y] =_{\mathsf{eq:2}} [y] + [x] = [y] + x =_{\mathsf{eq:3}} [y + 0] + x = y + 0 + x =_{\mathsf{eq:LA}} (y + 0) + x =_{\mathsf{eq:3}} (y) + x =_{\mathsf{eq:2}} x + (y)$.

Table 8: Equations for poly-infix addition

$$
\begin{aligned}
(0) &= 0 \\
(u) &= u \text{ (for } u \in \mathsf{C_D}) \\
((x)) &= (x) \\
x + y + z &= (x + y) + z \qquad \text{(LAA)} \\
x - y + z &= (x - y) + z \\
x + y + z &= x + (y + z) \\
x + y - z &= x + (y - z) \\
x \cdot y \cdot z &= (x \cdot y) \cdot z \\
x \cdot y \cdot z &= x \cdot (y \cdot z)
\end{aligned}
$$

6 Concluding remarks

The work in this paper provides progress in the direction of making ideas, methods and techniques of the theory of datatypes and abstract datatypes available to an audience with a focus on elementary mathematics. As references to (abstract) datatypes and algebraic specification methods I mention: [22] and [20]. Systematic work on the arithmetical datatype $\mathbb{Q}_0$, though written in the tradition of mathematical logic, begins with [26]. Working with 0 as the value of $1/0$ has by now an extensive tradition in various parts of logic and computer science, from a methodological point of view it is a mere design decision for which alternatives exist. Working with $1/0 = \bot$, where $\bot$ is a symbolic number (see [27]), representing an absorptive element of an arithmetical datatype is studied in detail in [12]. Working with $1/0 = \infty$, where ∞ constitutes an unsigned infinite object so that $1/\infty = 0$ and $-\infty = \infty$ and in the presence of $\bot$ which satisfies $0 \cdot \infty = \infty + \infty = \bot$ leads to the topic of wheels which was initiated in [30] and has been worked out in full mathematical detail in [19]. Finally much work has been done in the setting of transrational arithmetic and transreal arithmetic (see e.g. [1]), where $1/0 = +\infty$ which is distinguished from $-\infty$ so that the ordering of numbers may be extended to the symbolic infinite values. A survey if these options is given in [2].

I hold that for applications in elementary arithmetic, and in particular for the formalization of school arithmetic, adopting $1/0 = \bot$ is more promising than adopting $1/0 = 0$. However, theoretical matters are more easily studied in the simpler context with $1/0 = 0$ which is closer to conventional algebra due to the absence of symbolic values.

Under the assumption that $1/0 = 0$, the specification $\mathsf{Md}^d_{\mathsf{C_D}}$ (Table 2) is a plausible endpoint of a development that has started with the specification given in [16]. $\mathsf{Md}^d_{\mathsf{C_D}}$ provides decimal notation and, upon hiding the decimal constants outside the signature of unital rings, the specification is most general, a notion which is introduced in this paper. The latter quality can be achieved only when making use at least one mechanism in excess of initial algebra specification with a finite set of conditional equations. It has been shown in Theorem 1 that working with a single auxiliary function suffices, but the resulting specification is unattractive. A new mechanism is proposed: schematic (conditional) equations where schemes involve metavariables that range over an infinite name space. A singe (conditional) equation scheme expands to an infinite collection of equations over an infinite signature. Making use of schematic equations, a proposal which seems to be new in the setting of algebraic specifications of arithmetical datatypes also suffices for specifying an abstract datatype of rational numbers in a most general way, thus leading to the specification $\mathsf{Md}^d_{\mathsf{C_D}}$, a specification which I consider to be quite satisfactory.

The notion of a specification being most general, the mechanism of schematic equations specification, its use in $\mathsf{Md}^d_{\mathsf{C_D}}$ and Theorem 1 are definitive in the sense that, viewed from the perspective of algebraic specification, there is not much room, or need for improvement. There is, however, an incentive to investigate adaptations of these results to arithmetical datatypes incorporating other values for $1/0$ than 0. Moreover, two themes were discussed in the paper for which obtaining definitive results, if possible at all, will require much more work: natural syntax for elementary arithmetic, and fracterm decomposition mechanisms.

The rewrite system for rationals with decimal notation as briefly mentioned above opens up an area of rewrite system design where the best options yet have to be found.

The development of “practical syntax”, that is an algebraic syntax which closely resembles the notations which are used in the daily practice of elementary arithmetic, leads to the notion of poly-infix operations, a topic which admits further elaboration. This paper only touches the surface of the subject of practical syntax for arithmetic.

6.1 Open Questions Concerning $\mathbb{Q}_0^d$, $\mathbb{Q}_{0,\mathsf{C_D}}^d$, and $\widetilde{\mathbb{Q}}_{0,\mathsf{C_D}}^d$

Remarkably much is not known about the datatype $\widetilde{\mathbb{Q}}_{0,\mathsf{C_D}}^d$ and its corresponding abstract datatype $\mathbb{Q}_{0,\mathsf{C_D}}^d$ and the (more familiar) reduct $\mathbb{Q}_0^d$ thereof. I conclude the paper with a listing of such open questions.

1. Is the conditional equational theory of $\mathbb{Q}_0^d$ decidable?
2. Is the equational theory of $\mathbb{Q}_0^d$ decidable?

   (One may notice that due to the undecidability results of Matijasevich validity of a schematic equation in $\mathbb{Q}_{0,\mathsf{C_D}}^d$ is undecidable.)
3. Is there a finite equational initial algebra specification for $\mathbb{Q}_0^d$ which, viewed as a term rewrite system is both weakly terminating and confluent?
4. Is there a finite schematic equational initial algebra specification for $\mathbb{Q}_{0,\mathsf{C_D}}^d$ which, viewed as a term rewrite system is both weakly ground terminating and ground confluent?
5. Is there an initial algebra specification of $\widetilde{\mathbb{Q}}_{0,\mathsf{C_D}}^d$ which consists of finitely many equational schemes and which, viewed as a TRS, is weakly terminating and ground confluent (preferably with the elements of the domain of $\widetilde{\mathbb{Q}}_{0,\mathsf{C_D}}^d$ as normal forms, or in fact with any other normal forms)?
6. What is the computational complexity of the word problem (equality between closed terms) of $\mathbb{Q}_{0,\mathsf{C_D}}^d$?
7. Is there a finite first order extension Γ of the specification in Table 2, perhaps with additional functions and sorts, which has the property that for each closed equation $t = r$ such that $\mathbb{Q}_{0,\mathsf{C_D}}^d \models t = r$ there is a proof of $t = r$ from Γ plus the equations of Table 2 with size linear in the size (as a term) of $t = r$.
8. In [5] it is shown that upon expanding the signature Σ_{Md} with constants $a_1, \ldots, a_n$, for each algebraic extension $\mathbb{Q}(a_1, \ldots, a_n)$ its expansion to a meadow $\mathbb{Q}_0^d(a_1, \ldots, a_n)$ has a finite initial algebra specification. The question is open if the converse is true: if $\mathbb{Q}_0^d(a_1, \ldots, a_n)$ has a finite initial algebra specification, must $a_1, \ldots, a_n$ be algebraic over $\mathbb{Q}$?

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