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Coalgebraic Geometric Logic

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Abstract

Using the theory of coalgebra, we introduce a uniform framework for adding modalities to the language of propositional geometric logic. Models for this logic are based on coalgebras for an endofunctor T on some full subcategory of the category **Top** of topological spaces and continuous functions. We compare the notions of modal equivalence, behavioural equivalence and bisimulation on the resulting class of models, and we provide a final object for the corresponding category. Furthermore, we specify a method of lifting an endofunctor on **Set**, accompanied by a collection of predicate liftings, to an endofunctor on the category of topological spaces.

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1 Introduction

Propositional geometric logic arose at the interface of (pointfree) topology, logic and theoretical computer science as the logic of *finite observations* [1, 28]. Its language is constructed from a set of proposition letters by applying finite conjunctions and arbitrary disjunctions, these being the propositional operations preserving the property of finite observability. Through an interesting topological connection, formulas of geometric logic can be interpreted in the frame of open sets of a topological space. Central to this connection is the well-known adjunction between the category **Frm** of frames and frame morphisms and the category **Top** of topological spaces and continuous maps, which restricts to several interesting Stone-type dualities [15].

Coalgebraic logic is a framework in which generalised versions of modal logics are developed parametric in the signature of the language and a functor $\mathsf{T} : \mathbf{C} \to \mathbf{C}$ on some base category $\mathbf{C}$. With classical propositional logic as base logic, two natural choices for the base category are **Set**, the category of sets and functions, and **Stone**, the category of Stone spaces and continuous functions, i.e. the topological dual to the algebraic category of Boolean algebras.

[1] The presented material originates from the master's thesis of the second author, supervised by the first and third author [11].

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Coalgebraic logic for endofunctors on **Set** has been well investigated and still is an active area of research, see e.g. [8, 20]. In this setting, modal operators can be defined using the notion of relation lifting [22] or predicate lifting [23]. Coalgebraic logic in the category of Stone coalgebras has been studied in [19, 13, 9], and there is a fairly extensive literature on the design of a coalgebraic modal logic based on a general Stone-type duality (or adjunction), see for instance [7] and references therein.

In this paper we investigate some links between coalgebraic logic and geometric logic. That is, we shall use methods from coalgebraic logic to introduce modal operators to the language of geometric logic, with the intention of studying interpretations of these logics in certain topological coalgebras. Note that extensions of geometric logic with the basic modalities $\Box$ and $\Diamond$, which are closely related to the topological Vietoris construction, have received much attention in the literature, see [28] for some early history. A first step towards developing coalgebraic geometric logic was taken in [27], where a method is explored to lift a functor on **Set** to a functor on the category **KHaus** of compact Hausdorff spaces, and the connection is investigated between the lifted functor and a relation-lifting based "cover" modality.

Our aim here is to develop a framework for the coalgebraic geometric logics that arise if we extend geometric logic with modalities that are induced by appropriate *predicate liftings*. Guided by the connection between geometric logic and topological spaces, we choose the base category of our framework to be **Top** itself, or one of its full subcategories such as **Sob** (sober spaces), **KSob** (compact sober spaces) or **KHaus** (compact Hausdorff spaces). On this base category **C** we then consider an arbitrary endofunctor T which serves as the type of our topological coalgebras. Furthermore, we shall see that if we want our formulas to be interpreted as *open sets* of the coalgebra carrier, we need the predicate liftings that interpret the modalities of the language to satisfy some natural *openness* condition. Summarizing, we shall study the coalgebraic geometric logic induced by (1) a functor $\mathsf{T} : \mathbf{C} \to \mathbf{C}$, where **C** is a full subcategory of **Top**, and (2) a set Λ of open predicate liftings for T. As running examples we take the combination of the basic modalities for the Vietoris functor, and that of the monotone box and diamond modalities for various topological manifestations of the monotone neighborhood functor on **Set**. The structures providing the semantics for our coalgebraic geometric logics are the T-*models* consisting of a T-coalgebra together with a valuation mapping proposition letters to open sets in the coalgebra carrier.

The main results that we report on here are the following:

- In Section 4, we construct a final object in the category of T-models, where T is an endofunctor on **Top** which preserves sobriety and admits a Scott-continuous, characteristic geometric modal signature.
- After that, in Section 5 we adapt the method of [17], in order to lift a **Set**-functor together with a collection of predicate liftings to an endofunctor on **Top**. We obtain the Vietoris functor and monotone functor on **KHaus** as restrictions of such lifted functors.
- Finally, in Section 6 we transfer the notion of Λ-bisimilarity from [10, 2] to our setting, and we compare this to geometric modal equivalence, behavioural equivalence and Aczel-Mendler bisimilarity. Our main finding is that on the categories **Top**, **Sob** and **KSob**, the first three notions coincide, provided Λ and T meet some reasonable conditions.

We finish the paper with listing some questions for further research.

2 Preliminaries

We briefly fix notation and review some preliminaries.

Categories and functors

We use a **bold font** for categories. We assume familiarity with the following categories:
- **Set** is the category of sets and functions;
- **Top** is the category of topological spaces and continuous functions;
- **KHaus** and **Stone** are the full subcategories of **Top** whose objects are compact Hausdorff spaces and Stone spaces respectively;
- **BA** is the category of Boolean algebras and Boolean algebra homomorphisms.

Categories can be connected by functors. We use a sans serif font for functors. In particular, the following functors are regularly used in this paper:
- $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor sending a topological space to its underlying set. The functor U restricts to every subcategory of **Top**, in which case we shall abuse notation and also call it U;
- $\mathsf{P} : \mathbf{Set} \to \mathbf{Set}$ and $\breve{\mathsf{P}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ are the covariant and contravariant powerset functor respectively;
- $\mathsf{Q} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{BA}$ sends a set to its Boolean powerset algebra and a function to the inverse image map viewed as morphism in **BA**;
- $\Omega : \mathbf{Top} \to \mathbf{Set}$ sends a topological space to the set of opens.

More categories and functors will be defined along the way. We use the symbol $\equiv$ for categorical equivalence.

Coalgebra

Let $\mathbf{C}$ be a category and T an endofunctor on $\mathbf{C}$. A T-*coalgebra* is a pair (X, γ) where X is an object in $\mathbf{C}$ and $\gamma : X \to \mathsf{T}X$ is a morphism in $\mathbf{C}$. A T-*coalgebra morphism* between two T-coalgebras (X, γ) and (X', γ') is a morphism $f : X \to X'$ in $\mathbf{C}$ satisfying $\gamma' \circ f = \mathsf{T}f \circ \gamma$. The collection of T-coalgebras and T-coalgebra morphisms forms a category, which we shall denote by $\mathbf{Coalg}(\mathsf{T})$. The category $\mathbf{C}$ is called the *base category* of $\mathbf{Coalg}(\mathsf{T})$.

▶ **Example 1** (Kripke frames)**.** The category of Kripke frames and bounded morphisms is isomorphic to $\mathbf{Coalg}(\mathsf{P})$ [20].

▶ **Example 2** (Monotone neighbourhood frames)**.** Let $\mathsf{D} : \mathbf{Set} \to \mathbf{Set}$ be the functor given on objects by $\mathsf{D}X = \{W \subseteq \mathsf{P}X \mid \text{ if } a \in W \text{ and } a \subseteq b \text{ then } b \in W\}$, for X a set. For a morphism $f : X \to X'$ define $\mathsf{D}f : \mathsf{D}X \to \mathsf{D}X' : W \mapsto \{a' \in \mathsf{P}X' \mid f^{-1}(a') \in W\}$. Then the category of monotone frames a bounded morphisms is isomorphic to $\mathbf{Coalg}(\mathsf{D})$ [6, 12, 13].

Coalgebraic logic for Set-coalgebras

Let T be a **Set**-functor and Φ a set of proposition letters. A T-*model* is a triple (X, γ, V) where (X, γ) is a T-coalgebra and $V : \Phi \to \mathsf{P}X$ is a valuation of the proposition letters. An n-*ary predicate lifting for* T is a natural transformation $\lambda : \breve{\mathsf{P}}^n \to \breve{\mathsf{P}} \circ \mathsf{T}$, where $\breve{\mathsf{P}}^n$ denotes the n-fold product of the contravariant powerset functor. A predicate lifting is called *monotone* if for all sets X and subsets $a_1, \ldots, a_n, b \subseteq X$ we have $\lambda_X(a_1, \ldots, a_i, \ldots, a_n) \subseteq \lambda_X(a_1, \ldots, a_i \cup b, \ldots, a_n)$. For a set Λ of predicate liftings for T, define the *language* $\mathrm{ML}(\Lambda)$ by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n),$$

where $p \in \Phi$ and $\lambda \in \Lambda$ is n-ary. The *semantics* of $\varphi \in \mathrm{ML}(\Lambda)$ on a T-model $\mathfrak{X} = (X, \gamma, V)$ is given recursively by $[\![p]\!]^{\mathfrak{X}} = V(p)$, $[\![\varphi_1 \wedge \varphi_2]\!]^{\mathfrak{X}} = [\![\varphi_1]\!]^{\mathfrak{X}} \cap [\![\varphi_2]\!]^{\mathfrak{X}}$, $[\![\neg\varphi]\!]^{\mathfrak{X}} = X \setminus [\![\varphi]\!]^{\mathfrak{X}}$, and

$$[\![\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)]\!]^{\mathfrak{X}} = \gamma^{-1}(\lambda([\![\varphi_1]\!]^{\mathfrak{X}}, \dots, [\![\varphi_n]\!]^{\mathfrak{X}})),$$

where $p \in \Phi$ and λ ranges over Λ.

▶ **Example 3** (Kripke models)**.** Consider for P-models the predicate liftings $\lambda^{\Box}, \lambda^{\Diamond} : \breve{\mathsf{P}} \to \breve{\mathsf{P}} \circ \mathsf{P}$ given by $\lambda^{\Box}_X(a) = \{b \in \mathsf{P}X \mid b \subseteq a\}$ and $\lambda^{\Diamond}_X(a) = \{b \in \mathsf{P}X \mid b \cap a \neq \varnothing\}$. Then $\lambda^{\Box}$ and $\lambda^{\Diamond}$ yield the usual Kripke semantics of $\Box$ and $\Diamond$.

▶ **Example 4** (Monotone neighbourhood frames)**.** Monotone neighbourhood models are precisely D-models, where D is the functor defined in Example 2. The usual semantics for the box and diamond in this setting can be obtained from the predicate liftings given by

$$\lambda^{\Box}_X(a) = \{W \in \mathsf{D}X \mid a \in W\}, \qquad \lambda^{\Diamond}_X(a) = \{W \in \mathsf{D}X \mid X \setminus a \notin W\}. \tag{1}$$

We refer to [20] for many more examples of coalgebraic logic for **Set**-functors.

Frames and spaces

A *frame* is a complete lattice F in which for all $a \in F$ and $S \subseteq F$ we have $a \wedge \bigvee S = \bigvee\{a \wedge s \mid s \in S\}$. A *frame homomorphism* is a function between frames that preserves finite meets and arbitrary joins. For $a, b \in F$ we say that a is *well inside* b, notation: $a \preccurlyeq b$, if there is a $c \in F$ such that $c \wedge a = \bot$ and $c \vee b = \top$. An element $a \in F$ is called *regular* if $a = \bigvee\{b \in F \mid b \preccurlyeq a\}$ and a frame is called *regular* if all of its elements are regular. The *negation* of $a \in F$ is defined as ${\sim}a = \bigvee\{b \in F \mid a \wedge b = \bot\}$ and we have $a \preccurlyeq b$ iff ${\sim}a \vee b = \top$. A frame is said to be *compact* if $\bigvee S = \top$ implies that there is a finite subset $S' \subseteq S$ such that $\bigvee S' = \top$. Frames can be presented by generators and relations, and any presentation by generators and relations presents a unique frame. For details see [15, 28].

▶ Remark 5. We will regularly define a frame homomorphism $f : F \to F'$ from a frame F presented by $\langle G, R\rangle$ to some frame F'. It then suffices to give an assignment $f' : G \to F'$ such that whenever $x = x'$ is a relation in R, $f(x) = f(x')$ in F.

The collection of open sets of a topological space $\boldsymbol{\mathcal{X}}$ forms a frame, denoted $\mathsf{opn}\boldsymbol{\mathcal{X}}$. A continuous map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ induces $\mathsf{opn}f = f^{-1} : \mathsf{opn}\boldsymbol{\mathcal{X}}' \to \mathsf{opn}\boldsymbol{\mathcal{X}}$ and with this definition opn is a contravariant functor $\mathbf{Top} \to \mathbf{Frm}$. A frame is called *spatial* if it isomorphic to $\mathsf{opn}\boldsymbol{\mathcal{X}}$ for some topological space $\boldsymbol{\mathcal{X}}$.

A *point* of a frame F is a frame homomorphism $p : F \to 2$, where $2 = \{\top, \bot\}$ is the two-element frame. Let $\mathsf{pt}F$ be the collection of points of F endowed with the topology $\{\widetilde{a} \mid a \in F\}$, where $\widetilde{a} = \{p \in \mathsf{pt}F \mid p(a) = \top\}$. For a frame homomorphism $f : F \to F'$ define $\mathsf{pt}f : \mathsf{pt}F' \to \mathsf{pt}F$ by $p \mapsto p \circ f$. The assignment pt defines a contravariant functor $\mathbf{Frm} \to \mathbf{Top}$. A topological space that arises as the space of points of a lattice is called *sober*. The *sobrification* of a topological space $\boldsymbol{\mathcal{X}}$ is $\mathsf{pt}(\mathsf{opn}\boldsymbol{\mathcal{X}})$.

We denote by **Sob** and **KSob** the full subcategories of **Top** whose objects are sober spaces and compact sober spaces, respectively. Where **Frm** is the category of frames and frame homomorphisms, **SFrm**, **KSFrm** and **KRFrm** are the full subcategories of **Frm** whose objects are spatial frames, compact spatial frames and compact regular frames, respectively. The functor $\mathsf{Z} : \mathbf{Frm} \to \mathbf{Set}$ is the forgetful functor sending a frame to the underlying set, and restricts to every subcategory of **Frm**. Note that $\Omega = \mathsf{Z} \circ \mathsf{opn}$.

▶ **Fact 6.** The functor pt is a right adjoint to opn. This adjunction restricts to the duality $\mathbf{SFrm} \equiv \mathbf{Sob}^{\mathrm{op}}$, which in turn restricts to $\mathbf{KSFrm} \equiv \mathbf{KSob}^{\mathrm{op}}$ and $\mathbf{KRFrm} \equiv \mathbf{KHaus}^{\mathrm{op}}$.

For a more thorough exposition of frames and spaces, and a proof of the statements in Fact 6 we refer to section C1.2 of [16]. We explicitly mention one isomorphism which is part of this duality, for we will encounter it later on.

▶ Remark 7. Let $\mathcal{X}$ be a sober space. Then Fact 6 entails that there is an isomorphism $\mathcal{X} \to \mathsf{pt}(\mathsf{opn}\mathcal{X})$. This isomorphism sends x to p_x, where $p_x : \mathsf{opn}\mathcal{X} \to 2$ is the point given by $p_x(a) = \top$ iff $x \in a$, for all $x \in \mathcal{X}$ and $a \in \Omega\mathcal{X}$.

3 Logic for topological coalgebras

Although not all of our results can be proved for every full subcategory of **Top**, we will give the basic definitions in full generality. To this end, we let **C** be some full subcategory of **Top** and define coalgebraic logic over base category **C**. In particular **C** = **KHaus** and **C** = **Sob** will be of interest. Throughout this section T is an arbitrary endofunctor on **C**. Recall that Φ is an arbitrary but fixed set of proposition letters. We commence with defining the topological version of a predicate lifting, called an open predicate lifting.

▶ **Definition 8.** An *open predicate lifting for* T is a natural transformation

$$\lambda : \Omega^n \to \Omega \circ \mathsf{T}.$$

An open predicate lifting is called *monotone in its i-th argument* if for every $\mathcal{X} \in \mathbf{C}$ and all $a_1, \dots, a_n, b \in \Omega\mathcal{X}$ we have $\lambda_{\mathcal{X}}(a_1, \dots, a_i, \dots, a_n) \subseteq \lambda_{\mathcal{X}}(a_1, \dots, a_i \cup b, \dots, a_n)$, and *monotone* if it is monotone in every argument. It is called *Scott-continuous* in its i-th argument if for every $\mathcal{X}$ and every directed set $A \subseteq \Omega\mathcal{X}$ we have $\lambda_{\mathcal{X}}(a_1, \dots, \bigcup A, \dots, a_n) = \bigcup_{b \in A} \lambda_{\mathcal{X}}(a_1, \dots, b, \dots, a_n)$ and *Scott-continuous* if it is Scott-continuous in every argument.

A collection of open predicate liftings for T is called a *geometric modal signature* for T. A geometric modal signature for a functor T is called *monotone* if every open predicate lifting in it is monotone, *Scott-continuous* if every predicate lifting in it is Scott-continuous, and *characteristic* if for every topological space $\mathcal{X}$ in **C** the collection $\{\lambda_{\mathcal{X}}(a_1, \dots, a_n) \mid \lambda \in \Lambda \text{ } n\text{-ary}, a_i \in \Omega\mathcal{X}\}$ is a sub-base for the topology on $\mathsf{T}\mathcal{X}$.

▶ Remark 9. Using the fact that for any two (open) sets a, b the set $\{a, a \cup b\}$ is directed, it is easy to see that Scott-continuity implies monotonicity.

Scott-continuity will play a rôle in Section 4, where it is used to show that the collection of formulas modulo (semantic) equivalence is a set, rather than a proper class.

Let $\mathcal{S}$ be the Sierpinski space, i.e. the two-element set $2 = \{0, 1\}$ topologised by $\{\varnothing, \{1\}, 2\}$. For a topological space $\mathcal{X}$ and $a \subseteq \bigcup\mathcal{X}$ let $\chi_a : \mathcal{X} \to \mathcal{S}$ be the characteristic map (i.e. $\chi_a(x) = 1$ iff $x \in a$). Note that χ_a is continuous if and only if $a \in \Omega\mathcal{X}$. Analogously to predicate liftings for **Set**-functors [25, Proposition 43], one can classify n-ary predicate liftings as open subsets of $\mathsf{T}\mathcal{S}^n$. This elucidates the analogy with predicate liftings for **Set**-functors.

▶ **Proposition 10.** *Suppose $\mathcal{S} \in \mathbf{C}$, then there is a bijective correspondence between n-ary open predicate liftings and elements of $\Omega\mathsf{T}\mathcal{S}^n$. This correspondence is given as follows: To an open predicate lifting λ assign the set $\lambda_{\mathcal{S}^n}(\pi_1^{-1}(\{1\}), \dots, \pi_n^{-1}(\{1\})) \in \Omega\mathsf{T}\mathcal{S}^n$, where $\pi_i : \mathcal{S}^n \to \mathcal{S}$ be the i-th projection, and conversely, for $c \in \Omega\mathsf{T}\mathcal{S}^n$ define $\lambda^c : \Omega^n \to \Omega\mathsf{T}$ by $\lambda^c_{\mathcal{X}}(a_1, \dots, a_n) = (\mathsf{T}\langle\chi_{a_1}, \dots, \chi_{a_n}\rangle)^{-1}(c)$.*

▶ **Definition 11.** The *language* induced by a geometric modal signature Λ is the collection $\mathrm{GML}(\Lambda)$ of formulas defined by the grammar

$$\varphi ::= \top \mid p \mid \varphi_1 \wedge \varphi_2 \mid \bigvee_{i \in I} \varphi_i \mid \heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n),$$

where p ranges over the set Φ of proposition letters, I is some index set and $\lambda \in \Lambda$ is n-ary. Abbreviate $\bot := \bigvee \varnothing$. We call a formula in $\mathrm{GML}(\Lambda)$ *finitary* if it does not involve any infinite disjunctions.

The language $\mathrm{GML}(\Lambda)$ is interpreted in so-called geometric T-models.

▶ **Definition 12.** A *geometric* T*-model* is a triple $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ where $(\boldsymbol{\mathcal{X}}, \gamma)$ is a T-coalgebra and $V : \Phi \to \Omega\boldsymbol{\mathcal{X}}$ is a valuation of the proposition letters. A map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ is a *geometric* T*-model morphism* from $(\boldsymbol{\mathcal{X}}, \gamma, V)$ to $(\boldsymbol{\mathcal{X}}', \gamma', V')$ if f is a coalgebra morphism between the underlying coalgebras and $f^{-1} \circ V' = V$. The collection of geometric T-models and geometric T-model morphisms forms a category, which we denote by $\mathbf{Mod}(\mathsf{T})$.

The *semantics* of $\varphi \in \mathrm{GML}(\Lambda)$ on such a model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ is given recursively by

$$[\![\top]\!]^{\mathfrak{X}} = X, \quad [\![p]\!]^{\mathfrak{X}} = V(p), \quad [\![\varphi \wedge \psi]\!]^{\mathfrak{X}} = [\![\varphi]\!]^{\mathfrak{X}} \cap [\![\psi]\!]^{\mathfrak{X}}, \quad [\![\bigvee_{i \in I} \varphi_i]\!]^{\mathfrak{X}} = \bigcup_{i \in I} [\![\varphi_i]\!]^{\mathfrak{X}},$$
$$[\![\heartsuit^{\lambda}(\varphi_1, \ldots, \varphi_n)]\!]^{\mathfrak{X}} = \gamma^{-1}(\lambda_{\boldsymbol{\mathcal{X}}}([\![\varphi_1]\!]^{\mathfrak{X}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}})).$$

We write $\mathfrak{X}, x \Vdash \varphi$ iff $x \in [\![\varphi]\!]^{\mathfrak{X}}$. Two states x and x' are called *modally equivalent* if they satisfy the same formulas, notation: $x \equiv_{\Lambda} x'$.

The following proposition shows that morphisms preserve truth. Its proof is similar to the proof of theorem 6.17 in [26].

▶ **Proposition 13.** *Let Λ be a geometric modal signature for T. Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ be geometric T-models and let $f : \mathfrak{X} \to \mathfrak{X}'$ be a geometric T-model morphism. Then for all $\varphi \in \mathrm{GML}(\Lambda)$ and $x \in \boldsymbol{\mathcal{X}}$ we have $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}', f(x) \Vdash \varphi$.*

We state the notion of behavioural equivalence for future reference.

▶ **Definition 14.** Let $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ be two geometric T-models and $x \in \boldsymbol{\mathcal{X}}$, $x' \in \boldsymbol{\mathcal{X}}'$ two states. We say that x and x' are *behaviourally equivalent in* $\mathbf{Mod}(\mathsf{T})$ ($x \simeq_{\mathbf{Mod}(\mathsf{T})} x'$) if there exists a cospan $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xleftarrow{f'} \mathfrak{X}'$ in $\mathbf{Mod}(\mathsf{T})$ such that $f(x) = f'(x')$.

As an immediate consequence of Proposition 13 we find that behavioural equivalence implies modal equivalence. Let us give some concrete examples of functors.

▶ **Example 15** (Trivial functor). Let $\mathbf{2} = \{0, 1\}$ be topologised by $\{\varnothing, \{0, 1\}\}$ (the trivial topology). Define the functor $\mathsf{F} : \mathbf{Top} \to \mathbf{Top}$ by $\mathsf{F}\boldsymbol{\mathcal{X}} = \mathbf{2}$ for every $\boldsymbol{\mathcal{X}} \in \mathbf{Top}$ and $\mathsf{F}f = \mathrm{id}_{\mathbf{2}}$, the identity map on $\mathbf{2}$, for every continuous function f. This is clearly a functor. Consider the open predicate lifting $\lambda : \Omega \to \Omega \circ \mathsf{F}$ given by $\lambda_{\boldsymbol{\mathcal{X}}}(a) = \mathsf{U}\mathbf{2}$ for all $a \in \Omega\boldsymbol{\mathcal{X}}$. For a F-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ we then have $\mathfrak{X}, x \Vdash \heartsuit^{\lambda}\varphi$ iff $\gamma(x) \in \lambda([\![\varphi]\!]^{\mathfrak{X}})$ iff $[\![\varphi]\!]^{\mathfrak{X}} \in \Omega\boldsymbol{\mathcal{X}}$. So $\heartsuit^{\lambda} = \top$.

Next we have a look at the Vietoris functor on $\mathbf{KHaus}$. Coalgebras for this functor have also been studied in [3].

▶ **Example 16** (Vietoris functor). For a compact Hausdorff space $\boldsymbol{\mathcal{X}}$, let $\mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ be the collection of closed subsets of $\boldsymbol{\mathcal{X}}$ topologised by the subbase

$$\boxdot a := \{b \in \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid b \subseteq a\}, \quad \diamondsuit a := \{b \in \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \mid a \cap b \neq \varnothing\},$$

where a ranges over $\Omega\boldsymbol{\mathcal{X}}$. For a continuous map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ define $\mathsf{V}_{\mathsf{kh}}f : \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}} \to \mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}'$ by $\mathsf{V}_{\mathsf{kh}}f(a) = f[a]$. If $\boldsymbol{\mathcal{X}}$ is compact Hausdorff, then so is $\mathsf{V}_{\mathsf{kh}}\boldsymbol{\mathcal{X}}$ [21, Theorem 4.9], and if $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ is a continuous map between compact Hausdorff spaces, then $\mathsf{V}_{\mathsf{kh}}f$ is well defined and continuous [19, Lemma 3.8], so V_{kh} defines an endofunctor on $\mathbf{KHaus}$.

Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ be a $\mathsf{V_{kh}}$-model. The natural transformation $\lambda^{\Box}$ defined by

$$\lambda^{\Box}_{\mathcal{X}} : \Omega\mathcal{X} \to \Omega(\mathsf{V_{kh}}\mathcal{X}) : a \mapsto \{b \in \mathsf{V_{kh}}\mathcal{X} \mid b \subseteq a\},$$

where $\mathcal{X} \in \mathbf{Top}$, is such that $\mathfrak{X}, x \Vdash \heartsuit^{\lambda^{\Box}}\varphi$ iff $\mathfrak{X}, x \Vdash \Box\varphi$ (with the usual interpretation of $\Box$). Similarly $\lambda^{\Diamond}_{\mathcal{X}} : \Omega\mathcal{X} \to \Omega \circ \mathsf{V_{kh}}\mathcal{X}$, given by $\lambda^{\Diamond}_{\mathcal{X}}(a) = \Diamond a$, yields the usual semantics of the diamond modality.

The functor defined in the next example generalises the monotone functor on **Stone** [13].

▶ **Example 17** (Monotone functor)**.** For a compact Hausdorff space $\mathcal{X}$, let $\mathsf{D_{kh}}\mathcal{X}$ be the collection of sets $W \subseteq \mathsf{P}X$ such that $u \in W$ iff there exists a closed $c \subseteq u$ such that every open superset of c is in W. Endow $\mathsf{D_{kh}}\mathcal{X}$ with the topology generated by the subbase

$$\boxplus a := \{W \in \mathsf{D_{kh}}\mathcal{X} \mid a \in W\}, \quad \diamondplus a := \{W \in \mathsf{D_{kh}}\mathcal{X} \mid X \setminus a \notin W\},$$

where a ranges over $\Omega\mathcal{X}$. For continuous functions $f : \mathcal{X} \to \mathcal{X}'$ define $\mathsf{D_{kh}}f : \mathsf{D_{kh}}\mathcal{X} \to \mathsf{D_{kh}}\mathcal{X}' : W \mapsto \{a \in \mathsf{P}X \mid f^{-1}(a) \in W\}$. It is proven in the report version of the current paper [4] that this defines an endofunctor on **KHaus** which naturally extends the monotone functor on **Stone** [13, 9]. The open predicate liftings $\lambda^{\Box}, \lambda^{\Diamond} : \Omega \to \Omega\mathsf{T}$ defined by $\lambda^{\Box}_{\mathcal{X}}(a) = \boxplus a$ and $\lambda^{\Diamond}_{\mathcal{X}}(a) = \diamondplus a$ yield the usual box and diamond semantics of monotone modal logic [12].

In Section 6 it turns out to be useful to have a slightly stronger notion of open predicate liftings, called strong open predicate liftings, as this allows us to prove that behavioural equivalence implies so-called Λ-bisimilarity. Whereas the action of open predicate liftings is defined only on open subsets, a strong open predicate lifting acts on *every* subset of elements of a topological space. Recall that $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor.

▶ **Definition 18.** A *strong open predicate lifting* for $\mathsf{T} : \mathbf{C} \to \mathbf{C}$ is a natural transformation $\mu : (\breve{\mathsf{P}} \circ \mathsf{U})^n \to \breve{\mathsf{P}} \circ \mathsf{U} \circ \mathsf{T}$ such that for all $\mathcal{X} \in \mathbf{C}$ and $a_1, \dots, a_n \in \Omega\mathcal{X}$ the set $\lambda_{\mathcal{X}}(a_1, \dots, a_n)$ is open in $\mathsf{T}\mathcal{X}$. Monotonicity of strong open predicate liftings is defined in the standard way.

We call an open predicate lifting (from Definition 11) *strong* if it is the restriction of some strong open predicate lifting and *strongly monotone* if it is the restriction of a monotone strong open predicate lifting.

Evidently, every strong open predicate lifting restricts to an open predicate lifting, and it is only this weaker notion of open predicate lifting that has an effect on the semantics. Our notion of strong open predicate lifting is similar to the notion of a *topological predicate lifting* for endofunctors on **Stone**, which were introduced in [9].

▶ **Example 19.** The predicate lifting corresponding to the box modality from Example 16 is strong, for it is the restriction of $\mu : \mathsf{U} \to \mathsf{U} \circ \mathsf{V_{kh}}$ given by $\mu_{\mathcal{X}}(u) = \{b \in \mathsf{V_{kh}}\mathcal{X} \mid b \subseteq u\}$. Likewise, all other predicate liftings from Examples 15, 16 and 17 are strong as well.

We devote the remainder of this section to investigating strong open predicate liftings. Recall from Example 15 that $\mathbf{2}$ denotes the two-element set with the trivial topology. We claim that natural transformations $\mu : (\breve{\mathsf{P}} \circ \mathsf{U})^n \to \breve{\mathsf{P}} \circ \mathsf{U} \circ \mathsf{T}$ correspond one-to-one with elements of $\breve{\mathsf{P}}\mathsf{U}\mathsf{T}\mathbf{2}$, provided $\mathbf{2} \in \mathbf{C}$: To a natural transformation μ associate the set $\mu_{\mathbf{2}}(p_1^{-1}(\{1\}), \dots, p_n^{-1}(\{1\}))$, where $p_i : \mathbf{2}^n \to \mathbf{2}$ denotes the i-th projection. Conversely, for $c \in \breve{\mathsf{P}}\mathsf{U}\mathsf{T}\mathbf{2}$ define μ^c by $\mu^c_{\mathcal{X}}(a_1, \dots, a_n) = (\mathsf{T}\langle \chi'_{a_1}, \dots, \chi'_{a_n}\rangle)^{-1}(c)$, where $\mathcal{X}$ is a topological space, $a \subseteq \mathsf{U}\mathcal{X}$ and $\chi'_a : \mathcal{X} \to \mathbf{2}$ is the characteristic map. Note that χ'_a is continuous regardless of whether a is open or not, hence T acts on all χ'_a. Details of the bijection are left to the reader.

▶ **Proposition 20.** *Let* T *be an endofunctor on* $\mathbf{C}$ *and suppose that* $\mathbf{C}$ *contains the spaces* $\mathbf{2}$ *and* $\mathcal{S}$*. Let* $s : \mathcal{S} \to \mathbf{2}$ *be the identity map and let* $c \in \check{\mathsf{P}}\mathsf{UT}\mathbf{2}^n$*. The natural transformation* μ^c *is a strong open predicate lifting if and only if* $(\mathsf{T}s^n)^{-1}(c) \subseteq \mathsf{T}\mathcal{S}^n$ *is open.*

Proof. We give the proof for the case $n = 1$, the general case being similar. Left to right follows from the fact that $\{1\}$ is open in $\mathcal{S}$, hence $\mu^c_{\mathcal{S}}(\{1\}) = (\mathsf{T}\chi'_{\{1\}})^{-1}(c) = (\mathsf{T}s)^{-1}(c)$ must be open in $\mathsf{T}\mathcal{S}$. For the converse, let $\mathcal{X}$ be a topological space and $a \in \Omega\mathcal{X}$. We need to show that $\mu^c_{\mathcal{X}}(a)$ is open. Since a is open, the characteristic map $\chi_a : \mathcal{X} \to \mathcal{S}$ is continuous and hence $\chi'_a = s \circ \chi_a$. We have

$$\begin{aligned}
\mu^c_{\mathcal{X}}(a) &= (\mathsf{T}\chi'_a)^{-1}(c) && \text{(definition of } \mu^c) \\
&= (\mathsf{T}(s \circ \chi_a))^{-1}(c) && (\chi'_a = s \circ \chi_a) \\
&= (\mathsf{T}s \circ \mathsf{T}\chi_a)^{-1}(c) && \text{(definition of functors)} \\
&= (\mathsf{T}\chi_a)^{-1} \circ (\mathsf{T}s)^{-1}(c). && \text{(definition of inverse)}
\end{aligned}$$

Since $\mathsf{T}\chi_a$ is continuous and $(\mathsf{T}s)^{-1}(c)$ is assumed to be open in $\mathsf{T}\mathcal{S}$, the set $\mu^c_{\mathcal{X}}(a)$ is open in $\mathsf{T}\mathcal{X}$. ◀

The following proposition gives two sufficient conditions on T for its open predicate liftings to be strong. For a full subcategory $\mathbf{C}$ of $\mathbf{Top}$ let $\mathbf{preC}$ denote the category of topological spaces in $\mathbf{C}$ and (not necessarily continuous) functions.

▶ **Proposition 21.** *Let* T *be an endofunctor on* $\mathbf{C}$ *and suppose* $\mathbf{2}, \mathcal{S} \in \mathbf{C}$*.*

1. *If* T *preserves injective functions then every open predicate lifting for* T *is strong.*
2. *If* T *extends to* $\mathbf{preC}$*, then every open predicate lifting for* T *is strong.*

Proof. For the first item, let $c \in \Omega\mathsf{T}\mathcal{S}^n$ determine the n-ary open predicate lifting λ^c. Since s^n is injective, by assumption $\mathsf{T}s^n$ is as well, and hence $c = (\mathsf{UT}s^n)^{-1}((\mathsf{UT}s^n)[c])$. Proposition 20 now implies that $\mu^{(\mathsf{UT}s^n)[c]}$ is a strong open predicate lifting. It is easy to see that $\mu^{(\mathsf{UT}s^n)[c]}$ extends λ^c, hence the latter is strong.

For the second item we show that, under the assumption, T preserves injective functions. Let $f : \mathcal{X} \to \mathcal{Y}$ be an injective function in $\mathbf{C}$, then there exists a (not necessarily continuous) function $g : \mathcal{Y} \to \mathcal{X}$ satisfying $g \circ f = \mathrm{id}_{\mathcal{X}}$. Then $\mathsf{T}g \circ \mathsf{T}f = \mathsf{T}(g \circ f) = \mathsf{T}\,\mathrm{id}_{\mathcal{X}} = \mathrm{id}_{\mathsf{T}\mathcal{X}}$, so $\mathsf{T}f$ has a (set-theoretic) left-inverse, hence is injective. ◀

Monotone open predicate lifting for an endofunctor on $\mathbf{KHaus}$ are always strong:

▶ **Proposition 22.** *Let* T *be an endofunctor on* $\mathbf{KHaus}$ *and* Λ *a monotone geometric modal signature for* T*. Then* Λ *is strongly monotone.*

Proof. Let $\lambda \in \Lambda$. We need to show that λ is the restriction of some strong monotone predicate lifting. Define

$$\widetilde{\lambda}_{\mathcal{X}} : \check{\mathsf{P}}^n\mathsf{U}\mathcal{X} \to \check{\mathsf{P}}\mathsf{UT}\mathcal{X} : (b_1, \ldots, b_n) \mapsto \bigcap\{\lambda_{\mathcal{X}}(a_1, \ldots, a_n) \mid a_i \in \Omega\mathcal{X} \text{ and } a_i \supseteq b_i\}.$$

Monotonicity of $\lambda_{\mathcal{X}}$ ensures $\widetilde{\lambda}_{\mathcal{X}}(a) = \lambda_{\mathcal{X}}(a)$ for all $a \in \Omega\mathcal{X}$ and $\widetilde{\lambda}$ is monotone by construction. So we only need to show that $\widetilde{\lambda}$ is indeed a strong open predicate lifting, i.e. a natural transformation $\check{\mathsf{P}}^n\mathsf{U}\mathcal{X} \to \check{\mathsf{P}}\mathsf{UT}\mathcal{X}$. We assume λ to be unary, the general case being similar.

For a continuous map $f : \mathcal{X} \to \mathcal{X}'$ between compact Hausdorff spaces we need to show that $\widetilde{\lambda}_{\mathcal{X}} \circ f^{-1} = (\mathsf{T}f)^{-1} \circ \widetilde{\lambda}_{\mathcal{X}'}$. Since, by naturality of λ, the right hand side is equal to $\bigcap\{\lambda_{\mathcal{X}}(f^{-1}(a')) \mid a' \in \Omega\mathcal{X}' \text{ and } b' \subseteq a'\}$, it suffices to show

$$\bigcap\{\lambda_{\mathcal{X}}(c) \mid c \in \Omega\mathcal{X} \text{ and } f^{-1}(b') \subseteq c\} = \bigcap\{\lambda_{\mathcal{X}}(f^{-1}(a')) \mid a' \in \Omega\mathcal{X}' \text{ and } b' \subseteq a'\}. \tag{2}$$

If a' is an open superset of b' then clearly $f^{-1}(b') \subseteq f^{-1}(a')$. So every element in the intersection of the right hand side is contained in the one on the left hand side and therefore we have $\subseteq$ in (2). For the converse, suppose $c \in \Omega\boldsymbol{\mathcal{X}}$ and $f^{-1}(b') \subseteq c$. Then the set $a' = X' \setminus f[X \setminus c]$ is open, contains b', and satisfies $f^{-1}(b') \subseteq f^{-1}(a') \subseteq c$. Therefore $\lambda_{\boldsymbol{\mathcal{X}}}(f^{-1}(a'))$ is one of the elements in the intersection on the left hand side of (2). Since $\lambda_{\boldsymbol{\mathcal{X}}}(f^{-1}(a')) \subseteq \lambda_{\boldsymbol{\mathcal{X}}}(c)$ this shows "$\supseteq$" in (2). ◀

4 A final model

We construct a final model in $\mathbf{Mod}(\mathsf{T})$ for a functor T where either T is an endofunctor on $\mathbf{Sob}$, or T is an endofunctor on $\mathbf{Top}$ which preserves sobriety. This assumption need not be problematic: If a functor on $\mathbf{Top}$ does not preserve sobriety we can look at its sobrification. Topological functors which arise as lifts from set functors using the procedure in Section 5 automatically preserve sobriety.

▶ **Assumption.** Throughout this section, fix an endofunctor T on $\mathbf{Top}$ which preserves sobriety, and a Scott-continuous characteristic geometric modal signature Λ for T. Recall that Φ is a set of proposition letters.

▶ **Definition 23.** Call two formulas φ and ψ *equivalent* in $\mathbf{Mod}(\mathsf{T})$ with respect to Λ, notation: $\varphi \equiv_{\mathsf{T},\Lambda} \psi$, if $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}, x \Vdash \psi$ for all $\mathfrak{X} \in \mathbf{Mod}(\mathsf{T})$ and $x \in \mathfrak{X}$. Denote the equivalence class of φ in $\mathrm{GML}(\Lambda)$ by $[\varphi]$. Let $\mathbf{E} = \mathbf{E}(\mathsf{T}, \Lambda, \Phi)$ be the collection of formulas modulo $\equiv_{\mathsf{T},\Lambda}$.

Recall that a finitary formula is one which does not involve arbitary disjunctions.

▶ **Lemma 24** (Normal form). *Under the assumption, every formula is equivalent to a formula of the form $\bigvee_{i \in I} \varphi_i$, where all the φ_i are finitary formulas.*

Proof. The proof proceeds by induction on the complexity of the formula. Suppose $\varphi = \varphi_1 \vee \varphi_2$. By induction we may assume that $\varphi_1 \equiv_{\mathsf{T},\Lambda} \bigvee_{i \in I} \psi_i$ and $\varphi_2 \equiv_{\mathsf{T},\Lambda} \bigvee_{j \in J} \psi_j$, where all the ψ_i and ψ_j are finitary, and we have $\varphi \equiv_{\mathsf{T},\Lambda} \bigvee_{i \in I \cup J} \psi_i$, as desired. If $\varphi = \varphi_1 \wedge \varphi_2$, then $\varphi \equiv_{\mathsf{T},\Lambda} (\bigvee_{i \in I} \psi_i) \wedge (\bigvee_{j \in J} \psi_j) \equiv_{\mathsf{T},\Lambda} \bigvee_{(i,j) \in I \times J} \psi_i \wedge \psi_j$. Lastly, suppose $\varphi = \heartsuit^{\lambda}(\bigvee_{i \in I} \psi_i)$, where all the ψ_i are finitary. Then we have $\bigvee_{i \in I} \psi_i = \bigvee\{\bigvee_{i \in I'} \psi_i \mid I' \subseteq I \text{ finite}\}$ and by construction the set $\{[\![\bigvee_{i \in I'}]\!]^{\mathfrak{X}} \mid I' \subseteq I, I' \text{ finite}\}$ is directed for every T-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$. Hence by Scott-continuity of λ we obtain

$$\lambda_{\boldsymbol{\mathcal{X}}}([\![\bigvee_{i \in I} \psi_i]\!]^{\mathfrak{X}}) = \lambda_{\boldsymbol{\mathcal{X}}}(\bigcup\{[\![\bigvee_{i \in I'} \psi_i]\!]^{\mathfrak{X}} \mid I' \subseteq I \text{ finite}\}) = \bigcup\{\lambda_{\boldsymbol{\mathcal{X}}}([\![\bigvee_{i \in I'} \psi_i]\!]^{\mathfrak{X}} \mid I' \subseteq I \text{ finite}\}.$$

Therefore $\varphi \equiv_{\mathsf{T},\Lambda} \bigvee\{\heartsuit^{\lambda}(\bigvee_{i \in I} \psi_i) \mid I' \subseteq I \text{ finite}\}$, i.e. φ is equivalent to an arbitrary disjunction of finitary formulas. The case for n-ary modalities is similar. This proofs the lemma. ◀

▶ **Corollary 25.** *The collection* $\mathbf{E}$ *from Definition 23 is a set.*

Proof. This follows immediately from Lemma 24 and the fact that the collection of finitary formulas is a set. ◀

▶ **Definition 26.** Define disjunction and arbitrary conjunction on $\mathbf{E}$ by $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$ and $\bigvee_{i \in I}[\varphi_i] := [\bigvee_{i \in I} \varphi_i]$. It is easy to check that $\mathbf{E}$ is a frame.

Set $\mathsf{L} = \mathsf{opn} \circ \mathsf{T} \circ \mathsf{pt} : \mathbf{Frm} \to \mathbf{Frm}$. This functor restricts to an endofunctor on $\mathbf{SFrm}$ which is dual to the restriction of T to $\mathbf{Sob}$. Since Λ is characteristic, the frame $\mathsf{L}\mathbf{E}$ is generated by $\{\lambda_{\boldsymbol{\mathcal{X}}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \mid \lambda \in \Lambda, \varphi_i \in \mathrm{GML}(\Lambda)\}$. Define an L-algebra structure $\delta : \mathsf{L}\mathbf{E} \to \mathbf{E}$ on generators by

$$\delta : \mathsf{L}\mathbf{E} \to \mathbf{E} : \lambda_{\mathsf{pt}\mathbf{E}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \mapsto [\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)].$$

To show that δ is well defined it suffices to verify that the images of the generators of $\mathbf{E}$ satisfy the same relations that they satisfy in $\mathsf{L}\mathbf{E}$. We refer to the the report version of the current paper for details. The dual of $\mathbf{E}$ will be the topological space underlying the final model in $\mathbf{Mod}(\mathsf{T})$:

▶ **Definition 27.** Set $\boldsymbol{\mathcal{Z}} \coloneqq \mathsf{pt}\mathbf{E}$ and let $\zeta : \boldsymbol{\mathcal{Z}} \to \mathsf{T}\boldsymbol{\mathcal{Z}}$ be the composition

$$\mathsf{pt}\mathbf{E} \xrightarrow{\mathsf{pt}\delta} \mathsf{pt}(\mathsf{L}\mathbf{E}) = \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E}))) \xrightarrow{k^{-1}_{\mathsf{T}(\mathsf{pt}\mathbf{E})}} \mathsf{T}(\mathsf{pt}\mathbf{E}),$$

where $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})} : \mathsf{T}(\mathsf{pt}\mathbf{E}) \to \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E})))$ is the isomorphism given in Remark 7. Together with the valuation $V_{\boldsymbol{\mathcal{Z}}} : \Phi \to \Omega\boldsymbol{\mathcal{Z}} : p \mapsto \widetilde{[p]}$, the triple $\mathfrak{Z} = (\boldsymbol{\mathcal{Z}}, \zeta, V_{\boldsymbol{\mathcal{Z}}})$ forms a T-model.

For an object $\Gamma \in \boldsymbol{\mathcal{Z}}$, the element $(\mathsf{pt}\delta)(\Gamma)$ is the completely prime filter

$$F = \{\lambda(\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_n) \in \mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E}))) \mid [\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)] \in \Gamma\}$$

in $\mathsf{pt}(\mathsf{opn}(\mathsf{T}(\mathsf{pt}\mathbf{E})))$. The element $\zeta(\Gamma)$ is the unique element in $\mathsf{T}(\mathsf{pt}\mathbf{E})$ corresponding to F under the isomorphism $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$. By definition of $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$, this is the unique element in the intersection of $\{\lambda_{\mathsf{pt}\mathbf{E}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]}) \mid [\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)] \in \Gamma\}$. Moreover, it follows from the definition of $k_{\mathsf{T}(\mathsf{pt}\mathbf{E})}$ that $[\heartsuit^{\lambda}(\varphi_1, \dots, \varphi_n)] \notin \Gamma$ implies $\zeta(\Gamma) \notin \lambda_{\mathsf{pt}\mathbf{E}}(\widetilde{[\varphi_1]}, \dots, \widetilde{[\varphi_n]})$. The following lemma follows from the previous discussion and a straightforward induction. Both Lemma 28 and Proposition 29 are proven in detail in the report version of this paper.

▶ **Lemma 28** (Truth lemma). *For all $\Gamma \in \boldsymbol{\mathcal{Z}}$ we have $\mathfrak{Z}, \Gamma \Vdash \varphi$ iff $[\varphi] \in \Gamma$.*

▶ **Proposition 29.** *For every geometric T-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ the map $\mathrm{th}_{\mathfrak{X}} : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{Z}}$ given by $x \mapsto \{[\varphi] \in \mathbf{E} \mid \mathfrak{X}, x \Vdash \varphi\}$ is a T-model morphism.*

The developed theory results in the following theorem.

▶ **Theorem 30.** *Let T be a sobriety-preserving endofunctor on* **Top** *and Λ a Scott-continuous characteristic geometric modal signature for T. Then $\mathfrak{Z} = (\boldsymbol{\mathcal{Z}}, \zeta, V_{\boldsymbol{\mathcal{Z}}})$ is final in* $\mathbf{Mod}(\mathsf{T})$.

Proof. Proposition 29 states that for every geometric T-model $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ there exists a T-coalgebra morphism $\mathrm{th}_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{Z}$, so we only need to show that this morphism is unique. Let $f : \mathfrak{X} \to \mathfrak{Z}$ be any coalgebra morphism. Then by Proposition 13 and Lemma 28 we have $[\varphi] \in f(x)$ iff $\mathfrak{Z}, f(x) \Vdash \varphi$ iff $\mathfrak{X}, x \Vdash \varphi$ for all $x \in \boldsymbol{\mathcal{X}}$, hence $f = \mathrm{th}_{\mathfrak{X}}$. ◀

▶ **Theorem 31.** *Under the assumptions of Theorem 30, we have $\equiv_{\Lambda} = \simeq_{\mathbf{Mod}(\mathsf{T})}$.*

Proof. If x and x' are behaviourally equivalent, then they are modally equivalent by Proposition 13. Conversely, if they are modally equivalent, then $\mathrm{th}_{\mathfrak{X}}(x) = \mathrm{th}_{\mathfrak{X}'}(x')$ by construction, so they are behaviourally equivalent. ◀

▶ Remark 32. If T is an endofunctor on **Sob** rather than **Top**, the same procedure yields a final model in $\mathbf{Mod}(\mathsf{T})$. In particular, T need not be the restriction of a **Top**-endofunctor. However, if T is an endofunctor on **KSob** or **KHaus** the procedure above does not guarantee a final coalgebra in $\mathbf{Mod}(\mathsf{T})$; indeed the state space $\boldsymbol{\mathcal{Z}}$ of the final coalgebra $\mathfrak{Z}$ we construct need not be compact sober or compact Hausdorff. Of course, there may be a different way to attain similar results for **KSob** or **KHaus**. We leave this as an interesting open question. In Theorem 51 we prove an analog of Theorem 31 for endofunctors on **KSob**.

5 Lifting functors from Set to Top

In [18, Section 4] the authors give a method to lift a **Set**-functor $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$, together with a collection of predicate liftings Λ for T, to an endofunctor on **Stone**. We adapt their approach to obtain an endofunctor $\widehat{\mathsf{T}}_\Lambda$ on **Top**. In this section the notation $\bigvee^\uparrow$ is used for directed joins, i.e. joins over directed sets. To define the action of $\widehat{\mathsf{T}}_\Lambda$ on a topological space $\mathcal{X}$ we take the following steps:

Step 1. Construct a frame $\dot{\mathsf{F}}_\Lambda \mathcal{X}$ of the images of predicate liftings applied to the open sets of $\mathcal{X}$ (viewed simply as subsets of $\mathsf{T}(\mathsf{U}\mathcal{X})$);

Step 2. Quotient $\dot{\mathsf{F}}_\Lambda \mathcal{X}$ with a suitable relation that ensures $\bigvee^\uparrow{}_{b\in B}\lambda(b) = \lambda(\bigvee^\uparrow B)$ whenever λ is monotone;

Step 3. Employ the functor $\mathsf{pt} : \mathbf{Frm} \to \mathbf{Top}$ to obtain a (sober) topological space.

This is the content of Definitions 33, 35 and 37. Recall that $\mathsf{U} : \mathbf{Top} \to \mathbf{Set}$ is the forgetful functor and that Q is the contravariant functor sending a set to its Boolean powerset algebra.

▶ **Definition 33.** Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a functor and Λ a collection of predicate liftings for T. We define a contravariant functor $\dot{\mathsf{F}}_\Lambda : \mathbf{Top} \to \mathbf{Frm}$. For a topological space $\mathcal{X}$ let $\dot{\mathsf{F}}_\Lambda \mathcal{X}$ be the subframe of $\mathsf{Q}(\mathsf{T}(\mathsf{U}\mathcal{X}))$ generated by the set

$$\{\lambda_{\mathsf{U}\mathcal{X}}(a_1, \dots, a_n) \mid \lambda \in \Lambda \ n\text{-ary}, \ a_1, \dots, a_n \in \Omega\mathcal{X}\}.$$

That is, we close this set under finite intersections and arbitrary unions in $\mathsf{Q}(\mathsf{T}(\mathsf{U}\mathcal{X}))$. For a continuous map $f : \mathcal{X} \to \mathcal{X}'$ let $\dot{\mathsf{F}}_\Lambda f : \dot{\mathsf{F}}_\Lambda \mathcal{X}' \to \dot{\mathsf{F}}_\Lambda \mathcal{X}$ be the restriction of $\mathsf{Q}(\mathsf{T}(\mathsf{U}f))$ to $\dot{\mathsf{F}}_\Lambda \mathcal{X}'$.

▶ **Lemma 34.** *The assignment $\dot{\mathsf{F}}_\Lambda$ defines a contravariant functor.*

Proof. We need to show that $\dot{\mathsf{F}}_\Lambda$ is well defined on morphisms and that it is functorial. To show that the action of $\dot{\mathsf{F}}_\Lambda$ on morphisms is well-defined, it suffices to show that $(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\mathcal{X}}(a_1', \dots, a_n')) \in \dot{\mathsf{F}}_\Lambda(\mathcal{X})$ for all generators $\lambda_{\mathsf{U}\mathcal{X}}(a_1', \dots, a_n')$ of $\dot{\mathsf{F}}_\Lambda \mathcal{X}'$, because frame homomorphisms preserve finite meets and all joins. This holds by naturality of λ:

$$(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\mathcal{X}'}(a_1, \dots, a_n)) = (\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\mathcal{X}'}(a_1, \dots, a_n)) = \lambda_{\mathsf{U}\mathcal{X}}(f^{-1}(a_1), \dots, f^{-1}(a_n)).$$

By continuity of f we have $f^{-1}(a_i) \in \Omega\mathcal{X}$ so the latter is indeed in $\dot{\mathsf{F}}_\Lambda \mathcal{X}$. Functoriality of $\dot{\mathsf{F}}_\Lambda$ follows from functoriality of $\mathsf{Q} \circ \mathsf{T} \circ \mathsf{U}$. ◀

▶ **Definition 35.** Let Λ be a collection of predicate liftings for a set functor T. For $\mathcal{X} \in \mathbf{Top}$, let $\widehat{\mathsf{F}}_\Lambda \mathcal{X}$ be the quotient of $\dot{\mathsf{F}}_\Lambda \mathcal{X}$ with respect to the congruence $\sim$ generated by

$$\bigvee{}^\uparrow{}_{b\in B}\lambda(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \sim \lambda(a_1, \dots, a_{i-1}, \bigvee{}^\uparrow B, a_{i+1}, \dots, a_n)$$

for all $a_i \in \Omega\mathcal{X}$, $B \subseteq \Omega\mathcal{X}$ directed, and $\lambda \in \Lambda$ monotone in its i-th argument. Write $q_\mathcal{X} : \dot{\mathsf{F}}_\Lambda \mathcal{X} \to \widehat{\mathsf{F}}_\Lambda \mathcal{X}$ for the quotient map and $[x]$ for the equivalence class in $\widehat{\mathsf{F}}_\Lambda \mathcal{X}$ of an element $x \in \dot{\mathsf{F}}_\Lambda \mathcal{X}$. For a continuous function $f : \mathcal{X} \to \mathcal{X}'$ define $\widehat{\mathsf{F}}_\Lambda f : \widehat{\mathsf{F}}_\Lambda \mathcal{X}' \to \widehat{\mathsf{F}}_\Lambda \mathcal{X} : [\lambda_{\mathsf{U}\mathcal{X}}(a_1, \dots, a_n)] \mapsto [\dot{\mathsf{F}}_\Lambda(\lambda_{\mathsf{U}\mathcal{X}}(a_1, \dots, a_n))]$.

Quotienting by the congruence from Definition 35 ensures that the lifted versions of monotone predicate liftings are Scott-continuous, see Proposition 43 below.

▶ **Lemma 36.** *The assignment $\widehat{\mathsf{F}}_\Lambda$ defines a contravariant functor.*

Proof. We need to prove functoriality of $\widehat{\mathsf{F}}_\Lambda$ and that $\widehat{\mathsf{F}}_\Lambda f$ is well defined for every continuous map $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$. In order to show that $\widehat{\mathsf{F}}_\Lambda$ is well defined, it suffices to show that $\dot{\mathsf{F}}_\Lambda f$ is invariant under the congruence $\sim$. If $f : \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}'$ is a continuous, then

$$\begin{aligned}
&\textstyle\bigvee^{\uparrow}_{b' \in B}(\dot{\mathsf{F}}_\Lambda f)(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a'_1, \dots, a'_{i-1}, b', a'_{i+1}, \dots, a'_n)) \\
&\qquad = \textstyle\bigvee^{\uparrow}_{b' \in B}(\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}'}(a'_1, \dots, a'_{i-1}, b', a'_{i+1}, \dots, a'_n)) \\
&\qquad = \textstyle\bigvee^{\uparrow}_{b' \in B}\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(f^{-1}(a'_1), \dots, f^{-1}(a'_{i-1}), f^{-1}(b'), f^{-1}(a'_{i+1}), \dots, f^{-1}(a'_n)) \\
&\qquad \sim \lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(f^{-1}(a'_1), \dots, f^{-1}(a'_{i-1}), f^{-1}(\textstyle\bigvee^{\uparrow}B), f^{-1}(a'_{i+1}), \dots, f^{-1}(a'_n)) \\
&\qquad = \dot{\mathsf{F}}_\Lambda f(\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}(a'_1, \dots, a'_{i-1}, \textstyle\bigvee^{\uparrow}B, a'_{i+1}, \dots, a'_n))
\end{aligned}$$

so $\dot{\mathsf{F}}_\Lambda f$ is invariant under the congruence. In the $\sim$-step we use the fact that $\{f^{-1}(b') \mid b' \in B\}$ is directed in $\Omega\boldsymbol{\mathcal{X}}$. Functoriality of $\widehat{\mathsf{F}}_\Lambda f$ follows from functoriality of $\mathsf{Q} \circ \mathsf{T} \circ \mathsf{U}$. ◀

We are now ready to define the topological Kupke-Kurz-Pattinson lift of a functor on **Set** together with a collection of predicate liftings, to a functor on **Top**.

▶ **Definition 37.** Define the *topological Kupke-Kurz-Pattinson lift* (KKP lift for short) of T with respect to Λ to be the functor

$$\widehat{\mathsf{T}}_\Lambda = \mathsf{pt} \circ \widehat{\mathsf{F}}_\Lambda.$$

This is a functor **Top** → **Top** and since pt lands in **Sob** it restricts to an endofunctor on **Sob**.

Let us put our theory into action. For details see the report version of the current paper.

▶ **Example 38** (The monotone functor)**.** Recall the monotone functor D on **Set** and the corresponding set of predicate liftings $\Lambda = \{\lambda^\Box, \lambda^\Diamond\}$ from Examples 2 and 4. It can be seen that the topological KKP lift $\widehat{\mathsf{D}}_\Lambda$ of D with respect to Λ restricts to $\mathsf{D}_{\mathbf{kh}}$.

▶ **Example 39** (The Vietoris functor)**.** Likewise, one can show that, when restricted to **KHaus**, the topological KKP lift of P with respect to the usual box and diamond lifting coincides with the Vietoris functor from Example 16.

▶ **Example 40.** Not every endofunctor on **Top** can be obtained as the lift of a **Set**-functor with respect to a (cleverly) chosen set of predicate liftings in the sense of Definition 37. A trivial counterexample is the functor $\mathsf{F} : \mathbf{Top} \to \mathbf{Top}$ from Example 15. For every topological space $\boldsymbol{\mathcal{X}}$ we have $\mathsf{F}\boldsymbol{\mathcal{X}} = \mathbf{2}$, which is not a T_0 space, hence not a sober space. Therefore F does not preserve sobriety, while every lifted functor automatically preserves sobriety. Thus F is not the lift of a **Set**-functor.

We describe how to lift a predicate lifting to an open predicate lifting. Recall that $\mathsf{Z} : \mathbf{Frm} \to \mathbf{Set}$ is the forgetful functor which sends a frame to its underlying set.

▶ **Definition 41.** Let Λ be a collection of predicate liftings for a functor $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$. A predicate lifting $\lambda : \breve{\mathsf{P}}^n \to \breve{\mathsf{P}} \circ \mathsf{T}$ in Λ induces an open predicate lifting $\widehat{\lambda} : \Omega^n \to \Omega \circ \widehat{\mathsf{T}}$ for $\widehat{\mathsf{T}}$ via

$$\Omega^n\boldsymbol{\mathcal{X}} \xrightarrow{\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}} \mathsf{Z}(\dot{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}) \xrightarrow{\mathsf{Z}q_{\boldsymbol{\mathcal{X}}}} \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}) \xrightarrow{\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}}}} \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda\boldsymbol{\mathcal{X}})) = \Omega(\widehat{\mathsf{T}}\boldsymbol{\mathcal{X}}).$$

By $\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}$ we actually mean the restriction of $\lambda_{\mathsf{U}\boldsymbol{\mathcal{X}}}$ to $\Omega^n\boldsymbol{\mathcal{X}} \subseteq \breve{\mathsf{P}}(\mathsf{U}\boldsymbol{\mathcal{X}})$. The map $k_{\mathsf{F}\boldsymbol{\mathcal{X}}}$ is the frame homomorphism given by $a \mapsto \{p \in \mathsf{pt}(\mathsf{F}_\Lambda\boldsymbol{\mathcal{X}}) \mid p(a) = 1\}$. Then $\widehat{\Lambda} := \{\widehat{\lambda} \mid \lambda \in \Lambda\}$ is a geometric modal signature for $\widehat{\mathsf{T}}_\Lambda$.

▶ **Lemma 42.** *The assignment* $\widehat{\lambda}$ *is a natural transformation.*

Proof. For a continuous function $f : \mathcal{X} \to \mathcal{X}'$ the following diagram commutes in **Set**:

$$\begin{tikzcd}
\Omega^n \mathcal{X}' \arrow[r, "\lambda_{\mathsf{U}\mathcal{X}'}"] \arrow[d, "(f^{-1})^n"'] & \mathsf{Z}(\dot{\mathsf{F}}_\Lambda \mathcal{X}') \arrow[r, "\mathsf{Z}q_{\mathcal{X}'}"] \arrow[d, "(\mathsf{T}f)^{-1}"] & \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda \mathcal{X}') \arrow[r, "\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda \mathcal{X}'}"] \arrow[d, "(\mathsf{T}f)^{-1}"] & \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda \mathcal{X}')) \arrow[d, "\Omega(\mathsf{pt}((\mathsf{T}f)^{-1}))"] \\
\Omega^n \mathcal{X} \arrow[r, "\lambda_{\mathsf{U}\mathcal{X}}"'] & \mathsf{Z}(\dot{\mathsf{F}}_\Lambda \mathcal{X}) \arrow[r, "\mathsf{Z}q_{\mathcal{X}}"'] & \mathsf{Z}(\widehat{\mathsf{F}}_\Lambda \mathcal{X}) \arrow[r, "\mathsf{Z}k_{\widehat{\mathsf{F}}_\Lambda \mathcal{X}}"'] & \Omega(\mathsf{pt}(\widehat{\mathsf{F}}_\Lambda \mathcal{X}))
\end{tikzcd}$$

Commutativity of the left square follows from naturality of λ, commutativity of the middle square follows from the proof of Lemma 36 and commutativity of the right square can be seen as follows: let $a'_1, \dots, a'_n \in \Omega\mathcal{X}'$, then

$$\begin{aligned}
&\Omega(\mathsf{pt}((\mathsf{T}f)^{-1})) \circ \mathsf{Z}k_{\mathsf{F}_\Lambda \mathcal{X}'}(\lambda_{\mathsf{U}\mathcal{X}'}(a'_1, \dots, a'_n)) \\
&\qquad = \{q \in \mathsf{pt}(\mathsf{F}_\Lambda \mathcal{X}) \mid q \circ (\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\mathcal{X}'}(a'_1, \dots, a'_n)) = 1\} \\
&\qquad = \mathsf{Z}k_{\mathsf{F}_\Lambda \mathcal{X}}((\mathsf{T}f)^{-1}(\lambda_{\mathsf{U}\mathcal{X}'}(a'_1, \dots, a'_n))).
\end{aligned}$$

So $\widehat{\lambda}$ is an open predicate lifting. ◀

The nature of the definitions of $\widehat{\mathsf{T}}_\Lambda$ and $\widehat{\Lambda}$ yields the following desirable results.

▶ **Proposition 43.** **1.** *Let $\mathsf{T} : \mathbf{Set} \to \mathbf{Set}$ be a functor and Λ a collection of predicate liftings for T. Then $\widehat{\Lambda}$ is characteristic for $\widehat{\mathsf{T}}_\Lambda$.*
2. *If $\lambda \in \Lambda$ are monotone, then $\widehat{\lambda} \in \widehat{\Lambda}$ is Scott-continuous.*

Proof. Let $\mathcal{X}$ be a topological space. For the first item, we need to show that the collection

$$\{\widehat{\lambda}(a_1, \dots, a_n) \mid \lambda \in \Lambda \ n\text{-ary}, a_i \in \Omega\mathcal{X}\} \tag{3}$$

forms a subbase for the topology on $\widehat{\mathsf{T}}_\Lambda \mathcal{X}$. An arbitrary nonempty open set of $\widehat{\mathsf{T}}_\Lambda \mathcal{X}$ is of the form $\widetilde{x} = \{p \in \mathsf{pt}(\widehat{\mathsf{F}}_\Lambda \mathcal{X}) \mid p(x) = 1\}$, for $x \in \widehat{\mathsf{F}}_\Lambda \mathcal{X}$. An arbitrary element of $\widehat{\mathsf{F}}_\Lambda \mathcal{X}$ is the equivalence class of an arbitrary union of finite intersections of elements of the form $\lambda_{\mathsf{U}\mathcal{X}}(a_1, \dots, a_n)$, for $\lambda \in \Lambda$ and $a_1, \dots, a_n \in \Omega\mathcal{X}$. So we may write $x = \bigcup_{i \in I}(\bigcap_{j \in J_i}[\lambda^{i,j}_{\mathsf{U}\mathcal{X}}(a^{i,j}_1, \dots, a^{i,j}_{n_{i,j}})])$ for some index set I, finite index sets J_i, $\lambda^{i,j} \in \Lambda$ and open sets $a^{i,j}_k \in \Omega\mathcal{X}$. We get

$$\widetilde{x} = \bigcup_{i \in I}\Big(\widetilde{\bigcap_{j \in J_i} [\lambda^{i,j}_{\mathsf{U}\mathcal{X}}(a^{i,j}_1, \dots, a^{i,j}_{n_{i,j}})]}\Big) = \bigcup_{i \in I}\Big(\bigcap_{j \in J_i} \widehat{\lambda}^{i,j}_{\mathcal{X}}(a^{i,j}_1, \dots, a^{i,j}_{n_{i,j}})\Big).$$

The second equality follows from Definition 41. This shows that the open sets in (3) indeed form a subbase for the open sets of $\widehat{\mathsf{T}}_\Lambda \mathcal{X}$.

The second item follows immediately from the definitions. ◀

6 Bisimulations

This section is devoted to bisimulations and bisimilarity between coalgebraic geometric models. We compare two notions of bisimilarity, modal equivalence (Definition 12) and behavioural equivalence (Definition 14). Again, where **C** is be a full subcategory of **Top** and T an endofunctor on **C**, we give definitions and propositions in this generality where possible. When necessary, we restrict our scope to particular instances of **C**.

▶ **Definition 44.** Let $\mathfrak{X} = (\mathcal{X}, \gamma, V)$ and $\mathfrak{X}' = (\mathcal{X}', \gamma', V')$ be two geometric T-models. Let $B \subseteq \mathcal{X} \times \mathcal{X}$ be a relation such that, equipped with the subspace topology, it is in **C** and let $\pi : B \to \mathcal{X}, \pi' : B \to \mathcal{X}'$ be projections. Then B is called an *Aczel-Mendler bisimulation* between $\mathfrak{X}$ and $\mathfrak{X}'$ if for all $(x, x') \in B$ we have $x \in V(p)$ iff $x' \in V'(p)$, and there exists a transition map $\beta : B \to \mathsf{T}B$ that makes π and π' coalgebra morphisms. Two states $x \in \mathsf{U}\boldsymbol{\mathcal{X}}, x' \in \mathsf{U}\boldsymbol{\mathcal{X}}'$ are called *bisimilar* if there is some Aczel-Mendler bisimulation linking them, notation $x \leftrightarroweq x'$.

It follows from Proposition 13 that bisimilar states satisfy the same formulas. Furthermore, it easily follows by taking pushouts that Aczel-Mendler bisimilarity implies behavioural equivalence. If moreover T preserves weak pullbacks, the converse holds as well [24].

However, we do not wish to make this assumption on topological spaces, since few functors seem to preserve weak pullbacks. For example, the Vietoris functor does not preserve weak pullbacks [5, Corollary 4.3] and neither does the monotone functor from Definition 17. (To see the latter statement, consider the example given in Section 4 of [13] and equip the sets in use with the discrete topology.) Therefore we define Λ-bisimulations for **Top**-coalgebras as an alternative to Aczel-Mendler bisimulations. This notion is an adaptation of ideas in [2, 10]. Under some conditions on Λ, Λ-bisimilarity coincides with behavioural equivalence.

In the next definition we need the concept of coherent pairs: If X and X' are two sets and $B \subseteq X \times X'$ is a relation, then a pair $(a, a') \in \mathsf{P}X \times \mathsf{P}X'$ is called *B-coherent* if $B[a] \subseteq a'$ and $B^{-1}[a'] \subseteq a$. For details and properties see section 2 in [14].

► **Definition 45.** Let T be an endofunctor on $\mathbf{C}$, Λ a geometric modal signature for T and $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ and $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ two geometric T-models. A *Λ-bisimulation* between $\mathfrak{X}$ and $\mathfrak{X}'$ is a relation $B \subseteq \mathsf{U}\boldsymbol{\mathcal{X}} \times \mathsf{U}\boldsymbol{\mathcal{X}}'$ such that for all $(x, x') \in B$, all $p \in \Phi$ and all tuples of B-coherent pairs of opens $(a_i, a_i') \in \Omega\boldsymbol{\mathcal{X}} \times \Omega\boldsymbol{\mathcal{X}}'$ we have

$$x \in V(p) \quad \text{iff} \quad x' \in V'(p) \tag{4}$$

$$\gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n) \quad \text{iff} \quad \gamma'(x') \in \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \ldots, a_n'). \tag{5}$$

Two states are called Λ-bisimilar if there is a Λ-bisimulation linking them, notation: $x \leftrightarroweq_\Lambda x'$.

We give an alternative characterisation of (5) to elucidate the connection with [2].

► Remark 46. Let $B \subseteq \boldsymbol{\mathcal{X}} \times \boldsymbol{\mathcal{X}}'$ be a relation endowed with the subspace topology and let $\pi : B \to \boldsymbol{\mathcal{X}}$ and $\pi' : B \to \boldsymbol{\mathcal{X}}'$ be projections. Then $(a, a') \in \Omega\boldsymbol{\mathcal{X}} \times \Omega\boldsymbol{\mathcal{X}}'$ is B-coherent iff $\pi^{-1}(a) = (\pi')^{-1}(a')$.

Let P be the pullback of the cospan $\Omega\boldsymbol{\mathcal{X}} \xrightarrow{\Omega\pi} \Omega B \xleftarrow{\Omega\pi'} \Omega\boldsymbol{\mathcal{X}}'$ in **Frm** and let $p : P \to \boldsymbol{\mathcal{X}}$ and $p' : P \to \boldsymbol{\mathcal{X}}'$ be the corresponding projections. Then the B-coherent pairs are precisely $(p(x), p'(x))$, where x ranges over P. It follows from the definitions that equation (5) holds for all B-coherent pairs if and only if

$$\Omega\pi \circ \Omega\gamma \circ \lambda_{\boldsymbol{\mathcal{X}}} \circ p^n = \Omega\pi' \circ \Omega\gamma' \circ \lambda_{\boldsymbol{\mathcal{X}}'} \circ (p')^n,$$

where λ is n-ary.

As desired, Λ-bisimilar states satisfy the same formulas.

► **Proposition 47.** *Let T be an endofunctor on $\mathbf{C}$ and Λ a geometric modal signature for T. Then $\leftrightarroweq_\Lambda \subseteq \equiv_\Lambda$.*

Proof. Let B be a Λ-bisimulation between geometric T-models $\mathfrak{X}$ and $\mathfrak{X}'$, and suppose xBx'. Using induction on the complexity of the formula, we show that $\mathfrak{X}, x \Vdash \varphi$ iff $\mathfrak{X}', x' \Vdash \varphi$ for all $\varphi \in \mathrm{GML}(\Lambda)$. The propositional case is by definition, and $\wedge$ and $\bigvee$ are routine. Suppose $\mathfrak{X}, x \Vdash \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$, then $\gamma(x) \in \lambda_{\boldsymbol{\mathcal{X}}}([\![\varphi_1]\!]^{\mathfrak{X}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}})$. By the induction hypothesis $([\![\varphi_i]\!]^{\mathfrak{X}}, [\![\varphi_i]\!]^{\mathfrak{X}'})$ is B-coherent for all i. Then $\gamma'(x') \in \lambda_{\boldsymbol{\mathcal{X}}'}([\![\varphi_1]\!]^{\mathfrak{X}'}, \ldots, [\![\varphi_n]\!]^{\mathfrak{X}'})$ since B is a Λ-bisimulation, hence $\mathfrak{X}', x' \Vdash \heartsuit^\lambda(\varphi_1, \ldots, \varphi_n)$. The converse is proven symmetrically. ◄

The proof of the next proposition is similar to [2, Proposition 20].

▶ **Proposition 48.** *Let* T *be an endofunctor on* $\mathbf{C}$ *and* Λ *a geometric modal signature for* T*. Then* $\underline{\leftrightarrow} \subseteq \underline{\leftrightarrow}_\Lambda$*.*

We know by now that Λ-bisimilarity implies modal equivalence. Furthermore, if T is an endofunctor on $\mathbf{Top}$ which preserves sobriety, modal equivalence implies behavioural equivalence. In order to prove a converse, i.e. that behavioural equivalence implies Λ-bisimilarity, we need to assume that the geometric modal signature is strong.

Recall that two elements x, x' in two models are behaviourally equivalent in $\mathbf{Mod}(\mathsf{T})$, notation: $\simeq_{\mathbf{Mod}(\mathsf{T})}$, if there exist morphisms f, f' in $\mathbf{Mod}(\mathsf{T})$ such that $f(x) = f'(x')$.

▶ **Proposition 49.** *Let* Λ *a strongly monotone geometric modal signature for* $\mathsf{T} : \mathbf{C} \to \mathbf{C}$ *and let* $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ *and* $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ *be two geometric* T*-models. Then* $\simeq_{\mathbf{Mod}(\mathsf{T})} \subseteq \underline{\leftrightarrow}_\Lambda$*.*

Proof. Suppose x and x' are behaviourally equivalent. Then there are some geometric T-model $\mathfrak{Y} = (\boldsymbol{\mathcal{Y}}, \nu, V_{\boldsymbol{\mathcal{Y}}})$ and T-model morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ and $f' : \mathfrak{X}' \to \mathfrak{Y}$ such that $f(x) = f'(x')$. We will show that

$$B = \{(u, u') \in X \times X' \mid f(u) = f'(u')\}. \tag{6}$$

is a Λ-bisimulation B linking x and x'.

Clearly xBx'. It follows from Proposition 13 that u and u' satisfy precisely the same formulas whenever $(u, u') \in B$. Suppose $\lambda \in \Lambda$ is n-ary and for $1 \leq i \leq n$ let (a_i, a_i') be a B-coherent pair of opens. Suppose uBu' and $\gamma(u) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n)$. We will show that $\gamma'(u') \in \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \ldots, a_n')$. The converse direction is similar. By monotonicity and naturality of λ we obtain

$$\gamma(u) \in \lambda_{\boldsymbol{\mathcal{X}}}(a_1, \ldots, a_n) \subseteq \lambda_{\boldsymbol{\mathcal{X}}}(f^{-1}(f[a_1]), \ldots, f^{-1}(f[a_n])) = (\mathsf{T}f)^{-1}(\lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \ldots, f[a_n])),$$

so $(\mathsf{T}f)(\gamma(u)) \in \lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \ldots, f[a_n])$. (The $f[a_i]$ need not be open in $\boldsymbol{\mathcal{Y}}$, but since λ is strong, $\lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \ldots, f[a_n])$ is defined.) Because f and f' are coalgebra morphisms and $f(u) = f'(u')$ we have $(\mathsf{T}f)(\gamma(u)) = \nu(f(u)) = \nu(f'(u')) = (\mathsf{T}f')(\gamma'(u'))$. Finally, we get

$$\begin{aligned}
\gamma'(u') &\in (\mathsf{T}f')^{-1}(\lambda_{\boldsymbol{\mathcal{Y}}}(f[a_1], \ldots, f[a_n])) \\
&= \lambda_{\boldsymbol{\mathcal{X}}'}((f')^{-1}(f[a_1]), \ldots, (f')^{-1}(f[a_n])) && \text{(naturality of } \lambda) \\
&= \lambda_{\boldsymbol{\mathcal{X}}'}(B[a_1], \ldots, B[a_n]) && \text{(strong monotonicity of } \lambda) \\
&\subseteq \lambda_{\boldsymbol{\mathcal{X}}'}(a_1', \ldots, a_n'). && \text{(coherence of } (a_i, a_i'))
\end{aligned}$$ ◀

▶ Remark 50. If $\mathbf{C} = \mathbf{KHaus}$ in the proposition above, then Proposition 22 allows us to drop the assumption that Λ be strong.

Let T be an endofunctor on $\mathbf{Top}$ and let Λ be a geometric modal signature for T. The following diagram summarises the results from Propositions 47 and 49 and Theorem 31. The arrows indicate that one form of equivalence implies the other. Here (1) holds if T preserves weak pullbacks, (2) is true when Λ is Scott-continuous and characteristic and T preserves sobriety, and (3) holds when Λ is strongly monotone. Note that the converse of (2) always holds, because morphisms preserve truth (Proposition 13).

$$\begin{tikzcd}
\underline{\leftrightarrow} \arrow[r] & \underline{\leftrightarrow}_\Lambda \arrow[r] & \equiv_\Lambda \arrow[r, shift left, "(2)"] & \simeq_{\mathbf{Mod}(\mathsf{T})} \arrow[l, shift left] \arrow[lll, bend right, "(1)"'] \arrow[ll, bend left, "(3)"]
\end{tikzcd} \tag{7}$$

As stated in the introduction we are not only interested in endofunctors on **Top**, but also in endofunctors on full subcategories of **Top**, in particular **KHaus**.

The implications in the diagram hold for endofunctors on **Sob** as well (use Remark 32). Moreover, with some extra effort it can be made to work for endofunctors on **KSob** as well. In order to achieve this, we have to redo the proof for the bi-implication between modal equivalence and behavioural equivalence. This is the content of the following theorem.

▶ **Theorem 51.** *Let* T *be an endofunctor on* **KSob**, Λ *a Scott-continuous characteristic geometric modal signature for* T *and* $\mathfrak{X} = (\boldsymbol{\mathcal{X}}, \gamma, V)$ *and* $\mathfrak{X}' = (\boldsymbol{\mathcal{X}}', \gamma', V')$ *two geometric* T*-models. Then* $\equiv_\Lambda = \simeq_{\mathsf{Mod}(\mathsf{T})}$.

Proof. If x and x' are behaviourally equivalent then they are modally equivalent by Proposition 13. The converse direction can be proved using similar reasoning as in Section 4. The major difference is the following: We define an equivalence relation $\equiv_2$ on $\mathrm{GML}(\Lambda)$ by $\varphi \equiv_2 \psi$ iff $[\![\varphi]\!]^{\mathfrak{X}} = [\![\psi]\!]^{\mathfrak{X}}$ and $[\![\varphi]\!]^{\mathfrak{X}'} = [\![\psi]\!]^{\mathfrak{X}'}$. (Note that $\mathfrak{X}$ and $\mathfrak{X}'$ are now fixed.) That is, $\varphi \equiv_2 \psi$ iff φ and ψ are satisfied by precisely the same states in $\mathfrak{X}$ and $\mathfrak{X}'$ (compare Definition 23). The frame $\mathbf{E}_2 := \mathrm{GML}(\Lambda)/{\equiv_2}$ can then be shown to be a compact frame and hence $\boldsymbol{\mathcal{Z}}_2 := \mathsf{pt}\mathbf{E}_2$ is a compact sober space. The remainder of the proof is analogous to the proof of Theorem 31. A detailed proof can be found in [11, Theorem 3.34]. ◀

We summarise the results for **Top** and two of its full subcategories:

▶ **Theorem 52.** *Let* T *be an endofunctor on* **Top**, **Sob** *or* **KSob** *and* Λ *a Scott-continuous characteristic strongly monotone geometric modal signature for* T*. If* x *and* x' *are two states in two geometric* T*-models, then*

$$x \leftrightarrows_\Lambda x' \quad \text{iff} \quad x \equiv_\Lambda x' \quad \text{iff} \quad x \simeq_{\mathsf{Mod}(\mathsf{T})} x'.$$

For coalgebras over base category **KHaus** we have:

▶ **Theorem 53.** *Let* T *be an endofunctor on* **KHaus** *which is the restriction of an endofunctor* S *on* **Sob** *or* **KSob** *and let* Λ *be a Scott-continuous characteristic monotone geometric modal signature for* S *(hence for* T*). Then* $x \leftrightarrows_\Lambda x'$ *iff* $x \equiv_\Lambda x'$.

Both the Vietoris functor V_{kh} and the monotone functor D_{kh}, together with their respective open predicate liftings for box and diamond, satisfy the premises of this theorem.

7 Conclusion

We have started building a framework for coalgebraic *geometric* logic and we have investigated examples of concrete functors. There are still many unanswered and interesting questions. We outline possible directions for further research.

Modal equivalence versus behavioural equivalence From Theorem 52 we know that modal equivalence and behavioural equivalence coincide in **Mod**(T) if T is an endofunctor on **KSob**, **Sob** or an endofunctor on **Top** which preserves sobriety. A natural question is whether the same holds when T is an endofunctor on **KHaus**.

When does a lifted functor restrict to KHaus? We know of two examples, namely the powerset functor with the box and diamond lifting, and the monotone functor on **Set** with the box and diamond lifting, where the lifted functor on **Top** restricts to **KHaus**. It would be interesting to investigate whether there are explicit conditions guaranteeing that the lift of a functor restricts to **KHaus**. These conditions could be either for the **Set**-functor one starts with, or the collection of predicate liftings for this functor, or both.

Bisimulations In [2] the authors define Λ-bisimulations (which are inspired by [10]) between **Set**-coalgebas. In this paper we define Λ-bisimulations between **C**-coalgebras. A similar definition yields a notion of Λ-bisimulation between **Stone**-coalgebras, where the interpretants of the proposition letters are clopen sets, see [11, Definition 2.19]. This raises the question whether a more uniform treatment of Λ-bisimulations is possible, which encompasses all these cases.

Modalities and finite observations Geometric logic is generally introduced as the logic of finite observations, and this explains the choice of connectives ($\wedge$, $\bigvee$ and, in the first-order version, $\exists$). We would like to understand to which degree *modalities* can safely be added to the base language, without violating the (semantic) intuition of finite observability. Clearly there is a connection with the requirement of *Scott-continuity* (preservation of directed joins), and we would like to make this connection precise, specifically in the topological setting.

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