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ABSTRACT

This paper introduces the τ-value for risk capital allocation. First, the existence of this value is shown. Second, the τ-value capital allocation rule is shown to satisfy six desirable properties. Finally, a characterization of this value for risk capital allocation problems is provided based on two additional properties.

1. Introduction

Financial institutions need to withhold a level of capital as buffer, which is called risk capital. An integral part of enterprise risk management is to allocate this risk capital to divisions or constituents. Capital allocations are used for, e.g., performance measurement, loan pricing, or general investment [12]. The key issue then is what is a "fair" allocation, and in this paper we introduce a new capital allocation rule based on cooperative game theory.

In practice, risk capital is often determined by Choquet risk measures, such as the expected shortfall that is used in Basel III regulations for banks. Choquet risk measures satisfy a continuity property [11], which means roughly that small changes in the underlying distribution do not have a substantial impact on the risk capital of the firm. The firm may wish to consider capital allocation rules that satisfy a continuity property as well, which means that the allocated capital to divisions is also continuous in changes of the underlying distributions.

In the academic literature on risk capital allocation problems, one of the most popular capital allocation rules is the Aumann–Shapley value. The Aumann–Shapley value has been proposed based on a suitability property [14], and arguments based on finance [13] and game theory [8,18]. However, the Aumann–Shapley value has been criticized for its lack of continuity [2,19], which makes risk capital allocations very volatile for small changes in the underlying probability distributions. In particular, if the underlying distribution of the risk is approximated via past observations, a small change in the data can have a substantial impact on the allocated capital to a division [4,19]. As an alternative to the Aumann–Shapley value that satisfies this continuity property, we propose the *τ-value for risk capital allocations*.

The τ-value is proposed by Tijs [15] for the class of transferable utility (TU) games, which is a popular class of cooperative games where the utility of a coalition can be expressed by one number. Suppose that the divisions of the firm can state their aspired risk capital allocation, which is called the utopia allocation. If the firm can allocate the risk capital according to the utopia allocation, then the utopia allocation coincides with the τ-value capital allocation rule. However, generally, allocating the utopia allocation leads to an under-allocation of the risk capital of the firm. Based on a specific choice of the utopia allocation, the divisions can also determine a worst-case allocation, which is constructed by allocating the utopia allocations to all other divisions in a coalition. The worst-case allocation generally leads to an over-allocation of the risk capital of the firm. The τ-value capital allocation rule can be seen as a compromise between the utopia allocation and the worst-case allocation, and is given by the convex combination of both allocations such that the risk capital of the firm is allocated to the divisions.

When the risk capital is by a coherent risk measure [1] such as the expected shortfall, we show in this paper that the τ-value is well-defined for risk capital allocation problems. Moreover, we show that it satisfies six desirable properties for capital allocation rules. To the best of our knowledge, there is no well-known capital allocation rule in the literature that also satisfies all these properties. Moreover, we propose a characterization based on two properties for capital allocation rules.

This paper is set out as follows. Section 2 defines the τ-value, and shows that it is well-defined for risk capital allocation problems. Section 3 shows that the τ-value satisfies six properties for capital allocation rules. Section 4 provides a characterization, and Section 5 concludes with a remark. The proofs are delegated to Appendix.

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2. Definition of the τ-value capital allocation rule

A risk is represented by a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The realization of X can be interpreted as the (net) loss faced at a pre-specified future time. The class of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is defined by $\mathcal{X}$.

A risk measure is a mapping $\rho : \mathcal{X} \mapsto \mathbb{R}$, where we interpret $\rho(X)$ as risk capital that a firm needs to withhold if the firm faces risk $X \in \mathcal{X}$. We assume that the risk measure ρ is coherent, i.e., ρ satisfies the following four properties [1]:

- *Sub-additivity*: For all $X, Y \in \mathcal{X}$, we have $\rho(X+Y) \leq \rho(X) + \rho(Y)$.
- *Monotonicity*: For all $X, Y \in \mathcal{X}$ such that $X \geq Y$, we have $\rho(X) \geq \rho(Y)$.
- *Positive Homogeneity*: For every $X \in \mathcal{X}$ and every $c \geq 0$, we have $\rho(cX) = c\rho(X)$.
- *Translation Invariance*: For every $X \in \mathcal{X}$ and every $c \in \mathbb{R}$, we have $\rho(X+c) = \rho(X) + c$.

In Artzner et al. [1], random variables are interpreted as gains, whereas we define risk measures on random variables that are interpreted as losses. This affects the definition of *Monotonicity* and *Translation Invariance*. Note that *Positive Homogeneity* of ρ implies the normalization $\rho(0) = 0$.

Define $N = \{1 \ldots, n\}$ as the set of all divisions within the firm. A risk capital allocation problem is a tuple $((X_i)_{i\in N}, \rho)$, where $X_i \in \mathcal{X}$ for all $i \in N$ and ρ is a coherent risk measure. Moreover, we assume that $|\rho(X_i)| < \infty$ for all $i \in N$. The class of such risk capital allocation problems is defined as $\mathcal{R}$. A capital allocation rule $(K_i)_{i\in N}$ assigns to every problem $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$ a vector in $\mathbb{R}^n$ such that $\sum_{i=1}^n K_i(R) = \rho(\sum_{i=1}^n X_i)$.

A well-known solution concept for risk capital allocation problems is the Aumann–Shapley value (also called the Euler rule). Using cooperative game theory, the Aumann–Shapley value is studied in [6,8,10], and given by

$$AS_i(R) = \frac{\partial}{\partial \lambda_i}\rho\left(\sum_{j=1}^n \lambda_j X_j\right)\Bigg|_{\lambda=(1,\ldots,1)} \quad \text{for all } i \in N \text{ and } R \in \mathcal{R}', \quad (1)$$

where $\mathcal{R}' \subset \mathcal{R}$ is the class of risk capital allocation problems for which the partial derivatives in (1) exist. Denault [8] shows that this is indeed a capital allocation rule, i.e., $\sum_{i=1}^n AS_i(R) = \rho(\sum_{i=1}^n X_i)$. For a game-theoretic generalization of the Aumann–Shapley value to the class $\mathcal{R}$, we refer to [5].

Next, we introduce the τ-value capital allocation rule. For evaluating outside options of divisions in risk capital allocation problems, a realistic instability is that a division prefers to become independent. In this way, we construct the *utopia* allocation. This is the best-case capital allocation that a division can claim. We define the utopia point by $(M_i(R))_{i\in N}$, where

$$M_i(R) = \rho\left(\sum_{j=1}^n X_j\right) - \rho\left(\sum_{j\in N\backslash\{i\}} X_j\right) \quad \text{for all } i \in N.$$

In general, it is reasonable for division $i \in N$ to expect allocated capital of at least $M_i(R)$. If the division claims less than $M_i(R)$, it is to the advantage of the coalition $N\backslash\{i\}$ to throw i out of the firm. Hence, $M_i(R)$ can be seen as a lower bound for the allocated capital or as a utopia allocation for division i. The vector $(M_i(R))_{i\in N}$ can indeed be used as lower bound for capital allocation, which is shown in the following proposition.

Proposition 2.1. *For all $R \in \mathcal{R}$, it holds that $\rho(\sum_{i=1}^n X_i) \geq \sum_{i=1}^n M_i(R)$.*

Proposition 2.1 states that the utopia allocation leads to an under-allocation of the risk capital. With the aid of this utopia allocation, we now derive an upper bound for the capital allocations. Consider division $i \in N$, who knows the utopia allocations of the other divisions and who also knows that coalitions of divisions in the firm can be formed. Division i can form a coalition with the divisions in $S \subseteq N\backslash\{i\}$, that is interpreted as if the aggregate losses of the members of this coalition are the losses of a hypothetical firm that must withhold risk capital $\rho(\sum_{j\in S\cup\{i\}} X_j)$. For the formation of a coalition with division i, assume that division i is willing to allocate the utopia allocation to all other members of the coalition. Then, the remaining risk capital is allocated to division i, and this is interpreted as an upper bound of the allocated capital to division i. Division i selects the coalition with the smallest upper bound of the capital allocation, which leads to the following definition of $(m_i(R))_{i\in N}$:

$$m_i(R) = \min_{S\subseteq N\backslash\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - \sum_{j\in S} M_j(R), \qquad \text{for all } i \in N.$$

We will refer to $(m_i(R))_{i\in N}$ as the worst-case allocation.

Proposition 2.2. *For all $R \in \mathcal{R}$ and $i \in N$, it holds that $m_i(R) \geq M_i(R)$.*

Proposition 2.3. *For all $R \in \mathcal{R}$, it holds that $\sum_{i=1}^n m_i(R) \geq \rho(\sum_{i=1}^n X_i)$.*

Note that the results in Propositions 2.2 and 2.3 are well-known to hold for convex games [9]. The game (N, c), defined by $c(S) = \rho(\sum_{i\in S} X_i)$, $S \subseteq N$, is however not convex (see Theorem 5 of [8]). Jointly, the three properties shown in Propositions 2.1–2.3 imply that the game (N, c) is quasi-balanced [17]. The game (N, c) is further studied in [7].

Propositions 2.2 and 2.3 state that allocating the worst-case capital allocation leads to an over-allocation of the risk capital, and every division prefers the utopia allocation over the worst-case allocation. We are now ready to define the τ-value capital allocation rule, which is a compromise between the utopia allocation and the worst-case allocation.

Definition 1. For all $R \in \mathcal{R}$, the τ-value capital allocation rule is given by $\tau_i(R) = (1-\alpha)M_i(R) + \alpha m_i(R)$ for all $i \in N$, where $\alpha \in \mathbb{R}$ is chosen such that $\sum_{i=1}^n [(1-\alpha)M_i(R) + \alpha m_i(R)] = \rho\left(\sum_{i=1}^n X_i\right)$.

We readily find that if $\sum_{i=1}^n M_i(R) < \sum_{i=1}^n m_i(R)$, then

$$\alpha = \frac{\rho\left(\sum_{i=1}^n X_i\right) - \sum_{i=1}^n M_i(R)}{\sum_{i=1}^n (m_i(R) - M_i(R))}. \quad (2)$$

From Propositions 2.1 and 2.3, we find that if $\sum_{i=1}^n M_i(R) < \sum_{i=1}^n m_i(R)$, then $\alpha \in [0, 1]$. Moreover, if $\sum_{i=1}^n M_i(R) = \sum_{i=1}^n m_i(R)$, then $\tau(R) = M(R) = m(R)$, and α can take any value.

Example 2.4. Let $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$ such that $N = \{1, 2, 3\}$, $(X_i)_{i\in N}$ is multivariate Gaussian and mutually independent, $X_1 \sim N(0, 1)$, $X_2 \sim N(0, 4)$ and $X_3 \sim N(0, 9)$, and ρ is an α-expected shortfall (also known as the Conditional Value-at-Risk) with $\alpha \in (0, 1)$. From a practical point of view, the use of an α-expected shortfall might be a reasonable choice in many applications, as it is used Basel III regulations for banks and the Swiss Solvency Test for Swiss insurers. For Gaussian random variables, the α-expected shortfall is given by $\rho(X) = \mu + \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\sigma$, where $X \sim N(\mu, \sigma^2)$, Φ is the cumulative distribution function of $Z \sim N(0, 1)$ and $\phi := \Phi'$ is its density function. The α-expected shortfall is well-known to be a coherent (e.g., [3]). For more information on the α-expected shortfall, we refer to, e.g., [21].

We readily get $\rho(\sum_{i=1}^{n}\lambda_i X_i) = \rho(X_1)\sqrt{\lambda_1^2+4\lambda_2^2+9\lambda_3^2} = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\sqrt{\lambda_1^2+4\lambda_2^2+9\lambda_3^2}$. Thus, taking partial derivatives and substituting $\lambda = (1, 1, 1)$ yield that the Aumann–Shapley value is given by: $K_i^{AS} = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}\frac{1}{\sqrt{var(X_1+X_2+X_3)}}var(X_i)$ for $i \in N$, which yields for $\alpha = 99\%$ the allocation: $K_1^{AS} \approx 0.71$, $K_2^{AS} \approx 2.85$ and $K_3^{AS} \approx 6.41$.

The utopia allocation is given by $(M_i(R))_{i\in N}$, where $M_1(R) = 0.36$, $M_2(R) = 1.54$ and $M_3(R) = 4.01$. Moreover, $(m_i(R))_{i\in N}$ is given by $m_1(R) = 2.67$ (with the minimum attained at $S = \{\emptyset\}$), $m_2(R) = 5.25$ (with the minimum attained at $S = \{1, 3\}$ or $S = \{1\}$) and $m_3(R) = 7.72$ (with the minimum attained at $S = \{1, 2\}$ or $S = \{1\}$). We find $\alpha \approx 42\%$, and thus $\tau_1(R) \approx 1.32$, $\tau_2(R) \approx 3.09$ and $\tau_1(R) \approx 5.56$. In this example, the τ-value capital allocation rule exhibits less spread in allocation than the Aumann–Shapley value, where the spread is the largest allocation minus the smallest allocation.

3. Properties

Based on Denault [8] and Van Gulick et al. [19], we define the following six properties of a capital allocation rule $K : \mathcal{R} \mapsto \mathbb{R}^n$:

- *Translation Invariance*: For all $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$, it holds that if $\widetilde{R} = ((\widetilde{X}_i)_{i\in N}, \rho) \in \mathcal{R}$ where, for some $c \in \mathbb{R}$ and $j \in N$, $\widetilde{X}_j = X_j + c$ and $\widetilde{X}_i = X_i$ for all $i \in N\backslash\{j\}$, then $K_j(\widetilde{R}) = K_j(R) + c$ and $K_i(\widetilde{R}) = K_i(R)$ for all $i \in N\backslash\{j\}$.
- *Scale Invariance*: For all $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$, it holds that if $\widetilde{R} = ((\widetilde{X}_i)_{i\in N}, \rho) \in \mathcal{R}$ where $(\widetilde{X}_i)_{i\in N} = (c \cdot X_i)_{i\in N}$ for some $c > 0$, then $K(\widetilde{R}) = c \cdot K(R)$.
- *Monotonicity*: For all $R \in \mathcal{R}$ and $i \in N$ such that $\rho\left(\sum_{j=1}^{n} X_j\right) \geq \rho\left(\sum_{j\in N\backslash\{i\}} X_j\right)$, we have $K_i(R) \geq 0$.
- *Riskless Portfolio*: For all $R \in \mathcal{R}$ and $i \in N$ such that $X_i = c$ for some $c \in \mathbb{R}$, it holds $K_i(R) = c$.
- *No Diversification*: For all $R \in \mathcal{R}$, it holds that if $\sum_{i=1}^{n}\rho(X_i) = \rho(\sum_{i=1}^{n} X_i)$, then $K_i(R) = \rho(X_i)$ for all $i \in N$.
- *Continuity*: For every sequence $\{((X_i^k)_{i\in N}, \rho), k \in \mathbb{N}\} \subset \mathcal{R}$ such that ρ is continuous and $X_i^k \to X_i$ for all $i \in N$, it holds that $\lim_{k\to\infty} K((X_i^k)_{i\in N}, \rho) = K((X_i)_{i\in N}, \rho)$.

Denault [8] introduces a weaker definition of *Monotonicity*, as he focusses on "fuzzy" games. A risk measure ρ is called continuous if $\lim_{k\to\infty}\rho(X^k) = \rho(X)$ for every sequence $(X^k : k \in \mathbb{N}) \subset \mathcal{X}$ such that $X^k \to X$ in distribution. Choquet integrals are well-known to be continuous (see Proposition 4.11 of [11]), and thus coherent risk measures that are Choquet integrals are a canonical example of continuous risk measures. Since distortion risk measures as defined in [20] are special cases of Choquet integrals, distortion risk measures are continuous. The α-expected shortfall as introduced in Example 2.4 is continuous, as it is a distortion risk measure (e.g., [3]).

To the best of our knowledge, none of the proposed capital allocation rules in the literature satisfy all the above six properties. For instance, the proportional rule does not satisfy *Translation Invariance* and *Riskless Portfolio*. This is a consequence of the fact that the proportional rule does not take into account the division-specific marginal contributions to the total risk capital of the firm by risk aggregation, but instead is a function of $(\rho(X_i))_{i\in N}$ and $\rho(\sum_{i=1}^{n} X_i)$ only. Denault [8] shows that the Aumann–Shapley value satisfies *Translation Invariance* and *Scale Invariance* on the class of risk capital allocation problems such that the Aumann–Shapley value exists. The Aumann–Shapley value does however not satisfy *Continuity*, as shown in [19]. The Excess Based Allocation does not satisfy *Monotonicity* [19]. We next show that the τ-value capital allocation rule satisfies all these six properties.

Theorem 3.1. *The τ-value capital allocation rule satisfies Translation Invariance, Scale Invariance, Monotonicity, Riskless Portfolio, No Diversification and Continuity.*

4. Characterization

We define the following two properties for a capital allocation rule $K : \mathcal{R} \mapsto \mathbb{R}^n$:

- *Minimum Obligation*: for all $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$, it holds that $K(R) = M(R) + K(\widetilde{R})$, where $\widetilde{R} = ((\widetilde{X}_i)_{i\in N}, \rho)$ and $(\widetilde{X}_i)_{i\in N} = (X_i - M_i(R))_{i\in N}$.
- *Restricted Proportionality*: for all $R \in \mathcal{R}$ such that $M_i(R) = 0$ for all $i \in N$, it holds that $K(R)$ is a multiple of $m(R)$.

These two properties are inspired by [16]. Tijs [16] studies transferable utility games, where the values are interpreted as gains. In this paper, when capital allocations are interpreted as costs, the utopia allocation $M(R)$ is interpreted as a lower bound on the allocated capital for every division. The *Minimum Obligation* property is then interpreted as an additivity property, where the risk capital allocation problem is reduced by already first allocating the vector $M(R)$ to the divisions.

Theorem 4.1. *The unique capital allocation rule on $\mathcal{R}$ that satisfies Minimum Obligation and Restricted Proportionality is the τ-value capital allocation rule.*

Theorem 4.1 can be modified by replacing the property *Minimum Obligation* by *Translation Invariance*. While the property *Translation Invariance* is stronger than *Minimum Obligation*, the property *Translation Invariance* is better-known in the literature on risk capital allocation [8].

5. Concluding remark

The definition of the τ-value capital allocation rule hinges on a definition of the vector $(m_i(R))_{i\in N}$, that is a worst-case capital allocation for every division. Alternatively, a rule could be defined similarly as the τ-value capital allocation rule, but based on the vectors $(\hat{m}_i(R))_{i\in N}$ and $(\hat{M}_i(R))_{i\in N}$ such that $\hat{m}_i(R) = \rho(X_i)$ and $\hat{M}_i(R) = M_i(R)$ for $i \in N$. This rule is left for further study.

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Appendix. Proofs

Proof of Proposition 2.1. This follows directly from:

$$\begin{aligned}\sum_{i=1}^{n} M_i(R) &= \sum_{i=1}^{n}\left[\rho\left(\sum_{j=1}^{n} X_j\right) - \rho\left(\sum_{j\in N\backslash\{i\}} X_j\right)\right]\\ &= n\rho\left(\sum_{j=1}^{n} X_j\right) - \sum_{i=1}^{n}\rho\left(\sum_{j\in N\backslash\{i\}} X_j\right)\\ &\geq n\rho\left(\sum_{j=1}^{n} X_j\right) - \rho\left(\sum_{i=1}^{n}\sum_{j\in N\backslash\{i\}} X_j\right)\\ &= n\rho\left(\sum_{j=1}^{n} X_j\right) - \rho\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n} X_j - X_i\right)\right)\end{aligned}$$

$$= n\rho\left(\sum_{j=1}^{n} X_j\right) - \rho\left(n\sum_{j=1}^{n} X_j - \sum_{i=1}^{n} X_i\right)$$
$$= n\rho\left(\sum_{j=1}^{n} X_j\right) - \rho\left((n-1)\sum_{j=1}^{n} X_j\right)$$
$$= n\rho\left(\sum_{j=1}^{n} X_j\right) - (n-1)\rho\left(\sum_{j=1}^{n} X_j\right) = \rho\left(\sum_{j=1}^{n} X_j\right),$$

where the inequality is due to *Sub-additivity* of ρ and sixth equality is due to *Positive Homogeneity* of ρ. □

Proof of Proposition 2.2. Fix $i \in N$ and $S \subseteq N\backslash\{i\}$. Then, we get

$$\rho\left(\sum_{k\in S\cup\{i\}} X_k\right) + \sum_{j\in S}\rho\left(\sum_{k\in N\backslash\{j\}} X_k\right) + \rho\left(\sum_{k\in N\backslash\{i\}} X_k\right)$$
$$\geq \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) + \rho\left(\sum_{j\in S}\sum_{k\in N\backslash\{j\}} X_k\right) + \rho\left(\sum_{k\in N\backslash\{i\}} X_k\right)$$
$$= \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) + \rho\left(\sum_{j\in S}(\sum_{k=1}^{n} X_k - X_j)\right) + \rho\left(\sum_{k=1}^{n} X_k - X_i\right)$$
$$= \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) + \rho\left(|S|\sum_{k=1}^{n} X_k - \sum_{j\in S} X_j\right) + \rho\left(\sum_{k=1}^{n} X_k - X_i\right)$$
$$\geq \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) + \rho\left((|S|+1)\sum_{k=1}^{n} X_k - \sum_{j\in S\cup\{i\}} X_j\right)$$
$$\geq \rho\left((|S|+1)\sum_{k=1}^{n} X_k\right) = (|S|+1)\rho\left(\sum_{k=1}^{n} X_k\right),$$

where $|S|$ is the cardinality of set S. Here, all inequalities are due to *Sub-additivity* of ρ. We get

$$M_i(R) = \rho\left(\sum_{k=1}^{n} X_k\right) - \rho\left(\sum_{k\in N\backslash\{i\}} X_k\right)$$
$$\leq \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) - |S|\rho\left(\sum_{k=1}^{n} X_k\right) + \sum_{j\in S}\rho\left(\sum_{k\in N\backslash\{j\}} X_k\right)$$
$$= \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) - \sum_{j\in S}\left[\rho\left(\sum_{k=1}^{n} X_k\right) + \rho\left(\sum_{k\in N\backslash\{j\}} X_k\right)\right]$$
$$= \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) - \sum_{j\in S} M_j(R).$$

Since this holds for arbitrary $S \subseteq N\backslash\{i\}$, we get

$$M_i(R) \leq \min_{S\subseteq N\backslash\{i\}} \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) - \sum_{j\in S} M_j(R) = m_i(R). \quad \square$$

Proof of Proposition 2.3. For all $i \in N$ and $S \subseteq N\backslash\{i\}$, it holds that

$$\rho\left(\sum_{k\in S\cup\{i\}} X_k\right) - \sum_{j\in S} M_j(R)$$
$$= \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - \sum_{j\in S}\left[\rho\left(\sum_{k=1}^{n} X_k\right) - \rho\left(\sum_{k\in N\backslash\{j\}} X_k\right)\right]$$
$$= \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - |S|\rho\left(\sum_{k=1}^{n} X_k\right) + \sum_{j\in S}\rho\left(\sum_{k\in N\backslash\{j\}} X_k\right)$$
$$\geq \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - |S|\rho\left(\sum_{k=1}^{n} X_k\right) + \rho\left(\sum_{j\in S}\sum_{k\in N\backslash\{j\}} X_k\right)$$
$$= \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - |S|\rho\left(\sum_{k=1}^{n} X_k\right) + \rho\left(\sum_{j\in S}(\sum_{k=1}^{n} X_k - X_j)\right)$$
$$= \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - |S|\rho\left(\sum_{k=1}^{n} X_k\right) + \rho\left(|S|\sum_{k=1}^{n} X_k - \sum_{j\in S} X_j\right)$$
$$\geq \rho\left(|S|\sum_{k=1}^{n} X_k + X_i\right) - |S|\rho\left(\sum_{k=1}^{n} X_k\right),$$

where the inequalities are a result of *Sub-additivity* of the risk measure ρ. Note that the right hand side only depends on S via $|S|$. Define $s_i^* \in \{0, \ldots, n-1\}$ as

$$s_i^* := \underset{s_i\in\{0,\ldots,n-1\}}{\arg\min}\ \rho\left(s_i\sum_{k=1}^{n} X_k + X_i\right) - s_i\rho\left(\sum_{k=1}^{n} X_k\right),$$

and it thus holds that $m_i(R) \geq \rho\left(s_i^*\sum_{k=1}^{n} X_k + X_i\right) - s_i^*\rho\left(\sum_{k=1}^{n} X_k\right)$. Then, we find

$$\sum_{i=1}^{n} m_i(R) \geq \sum_{i=1}^{n}\left[\rho\left(s_i^*\sum_{k=1}^{n} X_k + X_i\right) - s_i^*\rho\left(\sum_{k=1}^{n} X_k\right)\right]$$
$$\geq \rho\left(\sum_{i=1}^{n}\left[s_i^*\sum_{k=1}^{n} X_k + X_i\right]\right) - \rho\left(\sum_{k=1}^{n} X_k\right)\sum_{i=1}^{n} s_i^*$$
$$= \rho\left([\sum_{k=1}^{n} X_k][\sum_{i=1}^{n} s_i^*] + \sum_{k=1}^{n} X_k\right) - \rho\left(\sum_{k=1}^{n} X_k\right)\sum_{i=1}^{n} s_i^*$$
$$= \rho\left([\sum_{k=1}^{n} X_k][1 + \sum_{i=1}^{n} s_i^*]\right) - \rho\left(\sum_{k=1}^{n} X_k\right)\sum_{i=1}^{n} s_i^*$$
$$= [1 + \sum_{i=1}^{n} s_i^*]\rho\left(\sum_{k=1}^{n} X_k\right) - (\sum_{i=1}^{n} s_i^*)\cdot\rho\left(\sum_{k=1}^{n} X_k\right)$$
$$= \rho\left(\sum_{k=1}^{n} X_k\right). \quad \square$$

Proof of Theorem 3.1. *Translation Invariance*: let $j \in N$, $c \in \mathbb{R}$, $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$ and $\widetilde{R} = ((\widetilde{X}_i)_{i\in N}, \rho) \in \mathcal{R}$ such that $\widetilde{X}_j = X_j + c$ and $\widetilde{X}_i = X_i$ for all $i \in N\backslash\{j\}$. Since the risk measure ρ is *Translation Invariant*, it holds that

$$\rho(\sum_{i\in S}\widetilde{X}_i) = \rho(\sum_{i\in S} X_i) + c\cdot\mathbf{1}_S(j),$$

where $\mathbf{1}_S(j) = 1$ if $j \in S$ and $\mathbf{1}_S(j) = 0$ otherwise. From this, we directly get $M_j(\widetilde{R}) = M_j(R) + c$ and $M_i(\widetilde{R}) = M_i(R)$ for all $i \in N\backslash\{j\}$. Moreover, if $i \in N\backslash\{j\}$, then

$$\rho\left(\sum_{k\in S\cup\{i\}}\widetilde{X}_k\right) - \sum_{k\in S} M_k(\widetilde{R}) = \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) - \sum_{k\in S} M_k(R)$$

for all $S \subseteq N\backslash\{i\}$, and thus $m_i(\widetilde{R}) = m_i(R)$. Also, since the risk measure ρ is *Translation Invariant*, it holds that

$$\rho\left(\sum_{k\in S\cup\{j\}} \widetilde{X}_k\right) - \sum_{j\in S} M_j(\widetilde{R}) = \rho\left(\sum_{k\in S\cup\{j\}} X_k\right) + c - \sum_{j\in S} M_j(R)$$

for all $S \subseteq N\backslash\{i\}$, and thus $m_j(\widetilde{R}) = m_j(R) + c$. Hence, for α defined in (2), we find that α is the same in $\widetilde{R}$ and R. Hence, we immediately find $\tau_j(\widetilde{R}) = \tau_j(R) + c$ and $\tau_i(\widetilde{R}) = \tau_i(R)$ for all $i \in N\backslash\{j\}$.

Scale Invariance: let $c > 0$, $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$ and $\widetilde{R} = ((\widetilde{X}_i)_{i\in N}, \rho) \in \mathcal{R}$ such that $(\widetilde{X}_i)_{i\in N} = (c \cdot X_i)_{i\in N}$. Since the risk measure ρ is *Positive Homogeneous*, it holds that

$$\rho(\sum_{i\in S} \widetilde{X}_i) = \rho(\sum_{i\in S} cX_i) = c \cdot \rho(\sum_{i\in S} X_i),$$

for all $S \subseteq N$. From this, we directly get $M_i(\widetilde{R}) = c \cdot M_i(R)$ for all $i \in N$. Moreover, for $i \in N$,

$$\rho\left(\sum_{k\in S\cup\{i\}} \widetilde{X}_k\right) - \sum_{k\in S} M_k(\widetilde{R}) = c \cdot \rho\left(\sum_{k\in S\cup\{i\}} X_k\right) - c\sum_{k\in S} M_k(R)$$

for all $S \subseteq N\backslash\{i\}$, and thus $m_i(\widetilde{R}) = c \cdot m_i(R)$. Thus, for α defined in (2), we find that α is the same in $\widetilde{R}$ and R. Hence, we immediately find $\tau(\widetilde{R}) = c \cdot \tau(R)$.

Monotonicity: this follows directly from $M_i(R) \leq m_i(R)$ and $\sum_{i=1}^n M_i(R) \leq \rho(\sum_{i=1}^n X_i) \leq \sum_{i=1}^n m_i(R)$ (Propositions 2.1–2.3) and the condition $M_i(R) \geq 0$.

Riskless Portfolio: let $c > 0$, $R \in \mathcal{R}$, and $i \in N$ such that $X_i = c$. Then, since the risk measure ρ is *Positive Homogeneous*, it holds that

$$\begin{aligned}M_i(R) &= \rho(\sum_{j=1}^n X_j) - \rho(\sum_{j\in N\backslash\{i\}} X_j) = \rho(X_i + \sum_{j\in N\backslash\{i\}} X_j) - \rho(\sum_{j\in N\backslash\{i\}} X_j)\\ &= c + \rho(\sum_{j\in N\backslash\{i\}} X_j) - \rho(\sum_{j\in N\backslash\{i\}} X_j) = c.\end{aligned}$$

Moreover, if $S = \{\emptyset\}$, then $\rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - \sum_{j\in S} M_j(R) = \rho(X_i) = c$, and so $m_i(R) \leq c$. Also,

$$\begin{aligned}m_i(R) &= \min_{S\subseteq N\backslash\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - \sum_{j\in S} M_j(R)\\ &= \min_{S\subseteq N\backslash\{i\}} c + \rho\left(\sum_{j\in S} X_j\right)\\ &\quad - \sum_{j\in S}\left[\rho\left(\sum_{k=1}^n X_k\right) - \rho\left(\sum_{k\in N\backslash\{j\}} X_k\right)\right]\\ &\geq \min_{S\subseteq N\backslash\{i\}} c + \rho\left(\sum_{j\in S} X_j\right) - |S|\rho\left(\sum_{k=1}^n X_k\right)\\ &\quad + \rho\left(\sum_{j\in S}\sum_{k\in N\backslash\{j\}} X_k\right)\\ &= \min_{S\subseteq N\backslash\{i\}} c + \rho\left(\sum_{j\in S} X_j\right) - |S|\rho\left(\sum_{k=1}^n X_k\right)\\ &\quad + \rho\left(\sum_{j\in S}(\sum_{k=1}^n X_k - X_j)\right)\\ &\geq \min_{S\subseteq N\backslash\{i\}} c + \rho\left(\sum_{j\in S} X_j\right) - |S|\rho\left(\sum_{k=1}^n X_k\right)\\ &\quad + \rho\left(|S|\sum_{k=1}^n X_k - \sum_{j\in S} X_j\right)\\ &\geq \min_{S\subseteq N\backslash\{i\}} c - |S|\rho\left(\sum_{k=1}^n X_k\right) + \rho\left(|S|\sum_{k=1}^n X_k\right) = c,\end{aligned}$$

where all inequalities are due to *Sub-additivity* of ρ. Thus, $m_i(R) = c$, and combining this with $M_i(R) = c$ yields $\tau_i(R) = c$.

No Diversification: let $R \in \mathcal{R}$ such that $\sum_{i=1}^n \rho(X_i) = \rho(\sum_{i=1}^n X_i)$. First, we get by *Sub-additivity* of ρ:

$$\begin{aligned}\rho\left(\sum_{j\in N\backslash\{i\}} X_j\right) &\geq \rho\left(\sum_{j=1}^n X_j\right) - \rho(X_i) = \sum_{j=1}^n \rho(X_j) - \rho(X_i)\\ &= \sum_{j\in N\backslash\{i\}} \rho(X_j) \geq \rho\left(\sum_{j\in N\backslash\{i\}} X_j\right),\end{aligned}$$

and thus $\rho(\sum_{j\in N\backslash\{i\}} X_j) = \sum_{j\in N\backslash\{i\}} \rho(X_j)$. Repeating this yields directly $\sum_{i\in S} \rho(X_i) = \rho(\sum_{i\in S} X_i)$ for all $S \subseteq N$. Then,

$$\begin{aligned}M_i(R) &= \rho\left(\sum_{j=1}^n X_j\right) - \rho\left(\sum_{j\in N\backslash\{i\}} X_j\right)\\ &= \sum_{j=1}^n \rho(X_j) - \sum_{j\in N\backslash\{i\}} \rho(X_j) = \rho(X_i),\end{aligned}$$

for all $i \in N$. Moreover,

$$\begin{aligned}m_i(R) &= \min_{S\subseteq N\backslash\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - \sum_{j\in S} M_j(R)\\ &= \min_{S\subseteq N\backslash\{i\}} \sum_{j\in S\cup\{i\}} \rho\left(X_j\right) - \sum_{j\in S} \rho(X_j) = \rho(X_i),\end{aligned}$$

for all $i \in N$. Hence, $m_i(R) = M_i(R) = \rho(X_i)$ for all $i \in N$, and thus $\tau_i(R) = \rho(X_i)$.

Continuity: let there be a sequence $(X_j^k : k \in \mathbb{N}\} \subset \mathcal{X}$ such that $X_j^k \to X_j$, for all $j \in N$, and let ρ be continuous. Then, since ρ is continuous and $\sum_{j\in S} X_j^k \to \sum_{j\in S} X_j$, it holds that $\lim_{k\to\infty} \rho\left(\sum_{j\in S} X_j^k\right) = \rho\left(\sum_{j\in S} X_j\right)$ for all $S \subseteq N$. For $M_i((X_i^k)_{i\in N}, \rho) = \rho\left(\sum_{j\in N} X_j^k\right) - \rho\left(\sum_{j\in N\backslash\{i\}} X_j^k\right)$, it holds

$$\begin{aligned}\lim_{k\to\infty} M_i((X_i^k)_{i\in N}, \rho) &= \lim_{k\to\infty} \rho\left(\sum_{j\in N} X_j^k\right) - \lim_{k\to\infty} \rho\left(\sum_{j\in N\backslash\{i\}} X_j^k\right)\\ &= \rho\left(\sum_{j\in N} X_j\right) - \rho\left(\sum_{j\in N\backslash\{i\}} X_j\right)\\ &= M_i((X_i)_{i\in N}, \rho).\end{aligned}$$

Likewise, for $m_i((X_i^k)_{i\in N}, \rho) = \min_{S\subseteq N\backslash\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} X_j^k\right) - \sum_{j\in S\cup\{i\}} M_j((X_i^k)_{i\in N}, \rho)$, it holds $\lim_{k\to\infty} m_i((X_i^k)_{i\in N}, \rho) = m_i((X_i)_{i\in N}, \rho)$. Hence, since $\lim_{k\to\infty} M_i((X_i^k)_{i\in N}, \rho) = M_i((X_i)_{i\in N}, \rho)$, $\lim_{k\to\infty} m_i((X_i^k)_{i\in N}, \rho) = m_i((X_i)_{i\in N}, \rho)$, and $\lim_{k\to\infty} \rho\left(\sum_{j\in N} X_j^k\right) = \rho\left(\sum_{j\in N} X_j\right)$, the τ-value capital allocation rule satisfies *Continuity*. □

Proof of Theorem 4.1. The τ-value capital allocation rule satisfies *Minimum Obligation* because the τ-value capital allocation rule satisfies *Translation Invariance* (Theorem 3.1). If $M(R) = 0$, then it follows directly from Definition 1 that $\tau(R) = \alpha m(R)$, and thus the τ-value capital allocation rule satisfies *Restricted Proportionality*.

We next show that any capital allocation rule satisfying *Minimum Obligation* and *Restricted Proportionality* must be the τ-value capital allocation rule. Suppose now that $K : \mathcal{R} \mapsto \mathbb{R}^n$ satisfies these two properties. Take $R = ((X_i)_{i\in N}, \rho) \in \mathcal{R}$. Since K satisfies *Minimum Obligation*, we can write $K(R) = M(R) + K(\widetilde{R})$, where $\widetilde{R} = ((\widetilde{X}_i)_{i\in N}, \rho)$ and $(\widetilde{X}_i)_{i\in N} = (X_i - M_i(R))_{i\in N}$. For the problem $\widetilde{R}$, the utopia allocation is given by

$$\begin{aligned} M_i(\widetilde{R}) &= \rho(\sum_{j=1}^{n} \widetilde{X}_j) - \rho(\sum_{j\in N\setminus\{i\}} \widetilde{X}_j) = \rho(\sum_{j=1}^{n}(X_j - M_j(R))) \\ &\quad - \rho(\sum_{j\in N\setminus\{i\}} (X_j - M_j(R))) \\ &= \rho(\sum_{j=1}^{n} X_j) - \sum_{j=1}^{n} M_j(R) - \rho(\sum_{j\in N\setminus\{i\}} X_j) - \sum_{j\in N\setminus\{i\}} M_j(R) \\ &= \rho(\sum_{j=1}^{n} X_j) - \rho(\sum_{j\in N\setminus\{i\}} X_j) - M_i(R) \\ &= M_i(R) - M_i(R) = 0, \end{aligned}$$

for all $i \in N$, where we use *Translation Invariance* of the risk measure ρ. Thus, since K also satisfies *Restricted Proportionality*, we can write $K(\widetilde{R}) = \beta m(\widetilde{R})$ for some $\beta \in \mathbb{R}$, where $(m_i(\widetilde{R}))_{i\in N}$ is given by

$$\begin{aligned} m_i(\widetilde{R}) &= \min_{S\subseteq N\setminus\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} \widetilde{X}_j\right) - \sum_{j\in S} M_j(\widetilde{R}) \\ &= \min_{S\subseteq N\setminus\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} (X_j - M_j(R))\right) \\ &= \min_{S\subseteq N\setminus\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - \sum_{j\in S\cup\{i\}} M_j(R) \\ &= \min_{S\subseteq N\setminus\{i\}} \rho\left(\sum_{j\in S\cup\{i\}} X_j\right) - \sum_{j\in S} M_j(R) - M_i(R) \\ &= m_i(R) - M_i(R), \end{aligned}$$

for all $i \in N$, where the third equality is due to *Translation Invariance* of the risk measure ρ. Hence, $K(\widetilde{R}) = \beta[m(R) - M(R)]$ for some $\beta \in \mathbb{R}$, and $K(R) = M(R)+K(\widetilde{R}) = M(R)+\beta[m(R)-M(R)]$. Since K is a capital allocation rule, it must hold $\beta = \alpha$ whenever $M(R) \neq m(R)$, where α is defined in (2). Hence, K is equal to the τ-value capital allocation rule. □

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