Complementarity systems in optimization

Schumacher, JM

DOI
10.1007/s10107-004-0544-5

Publication date
2004

Document Version
Accepted author manuscript

Published in
Mathematical programming

License
Unspecified

Mathematical Programming manuscript No.
(will be inserted by the editor)

J.M. Schumacher

Complementarity Systems in Optimization

the date of receipt and acceptance should be inserted later

Abstract. Complementarity systems consist of ordinary differential equations coupled to complementarity conditions. They form a class of nonsmooth dynamical systems that is of use in mechanical and electrical engineering as well as in optimization and in other fields. The paper illustrates how complementarity systems arise in mathematical programming by means of a number of examples of various nature. This is followed by a brief survey of the results that are available concerning existence, uniqueness, and generation of solutions. The emphasis in this paper is on linear complementarity systems.

1. Introduction

Historically, variational inequality problems and complementarity problems have been studied in parallel research efforts. The close relation between the two problems was already clarified in 1971 [32]. While infinite-dimensional settings have been typically used in connection with variational inequalities, complementarity problems are often studied in a finite-dimensional framework; see [17] for a survey of the development of the two strands of research. In the context of variational inequalities, problems of evolution have been considered from an early stage on [35]. Much detailed material, including computational methods, can be found for instance in [15, 21]. Remarkably, a parallel systematic development of evolutionary complementarity problems in finite dimensions has been much less pronounced.

There has been substantial work though in specific areas where combinations of differential equations with complementarity conditions arise. Such combinations come up naturally for instance in applications of the maximum principle to optimal control problems with state and/or control constraints. These problems have been studied since the very beginnings of optimal control theory [45]. The two-point boundary value problems that arise in this context are however not typical for evolutionary models, and their analysis may be burdened by high-index complications as discussed later in this paper. In the early eighties, in the context of mechanical systems, Lötstedt [36, 37] developed complementarity formulations and numerical methods for problems of impact and friction, while Moreau [38] used a "sweeping process" formulation for inelastic collisions. Around the same time, Van Bokhoven [6] used complementarity as a convenient way of expressing piecewise linear characteristics of elements of electrical networks; his main focus was on steady-state solutions but he did also consider transients. Also in the 1980s, connections between linear complementarity problems and reflected Brownian

J.M. Schumacher: Department of Econometrics and Operations Research, Tilburg University, P. O. Box 90153, 5000 LE Tilburg, the Netherlands, e-mail: `jms@uvt.nl`.

motion were studied by Harrison and Reiman [23, 22]. Chen and Mandelbaum [12] related this subject to deterministic problems formulated in terms of Leontiev economies. Further work on complementarity formulations of mechanical problems with impact and friction can be found for instance in [43, 1]; see also [18] for further references.

In the area of dynamical systems, a major contribution to the study of differential equations with discontinuous right hand sides appeared in the classical book by Filippov [19]. The reformulation of "Filippov systems" (or "relay systems") in terms of complementarity conditions was pioneered by Stewart [53]. In his book, Filippov made extensive use of the theory of differential inclusions (see for instance [2] for a general exposition of this subject). Differential inclusions form a very broad class of dynamical systems which do not necessarily possess a complementarity structure. A proposal for a framework that is based more strictly on variational inequalities was made by Dupuis and Nagurney in 1993 [14]. These authors defined "projected dynamical systems" which allow the modeling of certain types of problems where both dynamics and inequality constraints play a role; see subsection 2.4 below for further discussion.

A general way of coupling ordinary differential equations to complementarity conditions was proposed in 1996 by Van der Schaft and Schumacher [47], who extended their work in a later paper [48]. These papers were stimulated by interest in "multimode" dynamical systems as examples of dynamical systems with both continuous and discrete state variables (hybrid systems). Inspiration came in part from early work by Brockett [8]. The proposal of [47, 48] leans strongly on ideas from input/output systems theory [31, 40, 52, 55], and in particular the formulation chosen in the cited papers can be seen as a natural extension of earlier work within mathematical systems theory (for instance [51, 34]) on systems with equality constraints (differential-algebraic systems).

The formulation proposed in [47, 48] is the following. Start from a nonlinear input/output system, with k inputs and k outputs:

$$\frac{dx}{dt}(t) = f(x(t), u(t)) \tag{1a}$$

$$y(t) = h(x(t), u(t)) \tag{1b}$$

where $x(t)$ is an n-dimensional state variable, $u(t) \in \mathbb{R}^k$ is the input vector and $y(t) \in \mathbb{R}^k$ is the output vector.[1] Add to this system the standard complementarity relation:

$$0 \leq y(t) \perp u(t) \geq 0. \tag{1c}$$

Implicit in the above relation is the choice of an "active index set" $\alpha(t) \subset \{1, \ldots, k\}$ which is such that $y_i(t) = 0$ for $i \in \alpha(t)$ and $u_i(t) = 0$ for $i \notin \alpha(t)$. Any such index set is said to represent a *mode*. In a fixed mode, the system above behaves as the dynamical system described by the differential equation (1a) and the algebraic relations (1b) together with the equalities that follow from the choice of the active index set in (1c). A "change of mode" occurs when continuation within a given mode would violate the nonnegativity constraints associated with this mode. Examples of questions that arise within the "multimode" line of thinking are the following: (i) is there a unique solution

[1] The formulation in (1) can be made more general by allowing the functions f and h to depend directly on time; this extension will be taken for granted in what follows.

within each mode; (ii) when a change of mode must take place, is it always possible to switch to a new mode that has the current state as a starting point and that allows a valid evolution on some time interval; (iii) if the answer to the previous question is negative, is it possible to formulate an appropriate jump rule for the state variable? Answers to these questions depend on the input/output system (1a–1b).

As a matter of terminology, systems of the form (1) will be referred to in this paper as *complementarity systems*. The term *dynamic complementarity problem*, which has sometimes been employed to denote combinations of ordinary differential equations and complementarity conditions, will be reserved below for a certain algebraic problem that is motivated by the initial value problem for complementarity systems, following terminology in [48]. A subclass of particular interest arises when the functions f and h in (1) are required to be linear; the resulting *linear complementarity systems* [29] are described by relations of the form

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t) \tag{2a}$$

$$y(t) = Cx(t) + Du(t) \tag{2b}$$

$$0 \leq y(t) \perp u(t) \geq 0 \tag{2c}$$

where A, B, C, and D are linear mappings. In some applications it is natural to allow an external input (forcing term) in a complementarity system. The equations (1a) and (1b) are then replaced by equations of the form

$$\frac{dx}{dt}(t) = f(x(t), u(t), v(t)) \tag{3a}$$

$$y(t) = h(x(t), u(t), v(t)) \tag{3b}$$

where $v(t)$ denotes the forcing term; the equation (1c) is unchanged. In linear complementarity systems we require that the forcing term also enters linearly, so that the system (2) is replaced by

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t) + Ev(t) \tag{4a}$$

$$y(t) = Cx(t) + Du(t) + Fv(t) \tag{4b}$$

$$0 \leq y(t) \perp u(t) \geq 0 \tag{4c}$$

where A, B, C, D, E, and F are linear mappings. A useful generalization of (1) is obtained when the complementarity relation (1c) is replaced by the relation

$$\mathcal{C} \ni y(t) \perp u(t) \in \mathcal{C}^* \tag{5}$$

where $\mathcal{C}$ is a cone in $\mathbb{R}^k$ and $\mathcal{C}^*$ is the dual cone defined by $\mathcal{C}^* = \{u \mid \langle y, u \rangle \geq 0 \text{ for all } y \in \mathcal{C}\}$. In particular, this format allows the incorporation of both equality and inequality constraints. Very recently, Pang and Stewart [41] have proposed the notion of a *differential variational inequality* (DVI) which is based on conditions of the form

$$\frac{dx}{dt}(t) = f(x(t), u(t)) \tag{6a}$$

$$u(t) \in \text{SOL}(K, F(x(t), \cdot)) \tag{6b}$$

where SOL(K, Φ) is defined as the solution set of the variational inequality

$$\langle u' - u, \Phi(u) \rangle \geq 0, \quad u' \in K \tag{6c}$$

in which K is a nonempty closed convex set. If in particular K is a convex cone, then the DVI becomes a cone complementarity system, as is seen most easily by rewriting the equations (6) in the form

$$\frac{dx}{dt}(t) = f(x(t), u(t)) \tag{7a}$$

$$y(t) = F(x(t), u(t)) \tag{7b}$$

$$u(t) \in K \tag{7c}$$

$$\langle u' - u, y \rangle \geq 0, \quad u' \in K. \tag{7d}$$

Variational inequalities of evolution (VIE) have been studied mostly in a PDE context; see for instance [35,15,21]. Spatial boundary conditions play an essential role in this setting, so that a translation to a finite-dimensional context is somewhat hazardous. One may argue though that a finite-dimensional analog may be written as follows:

$$\frac{dx}{dt}(t) = f(x(t)) + u(t) \tag{8a}$$

$$y(t) = x(t) \tag{8b}$$

$$y(t) \in K \tag{8c}$$

$$\langle y' - y, u \rangle \geq 0, \quad y' \in K \tag{8d}$$

where K is a nonempty closed convex set. The above form may be readily generalized to the formulation

$$\frac{dx}{dt}(t) = f(x(t), u(t)) \tag{9a}$$

$$y(t) = F(x(t), u(t)) \tag{9b}$$

$$y(t) \in K \tag{9c}$$

$$\langle y' - y, u \rangle \geq 0, \quad y' \in K \tag{9d}$$

which is in a sense dual to the DVI formulation (cf. (7)). Again, if K is a convex cone, then the VIE reduces to a cone complementarity system.

The focus of this paper is on complementarity systems, i.e. systems of the form (1), and even more specifically on linear complementarity systems (2). It will be shown through a number of examples that formulations in terms of cone complementarity systems come up in a number of applications in mathematical programming. A number of results will be reviewed that have been obtained for complementarity systems, in particular pertaining to existence and uniqueness of solutions and to the computation of solutions by means of time-stepping methods.

Existence and uniqueness of solutions is a basic issue in the formulation of any dynamical system. From the point of view of mathematical programming, conditions that ensure existence and uniqueness of solutions are of interest because they can serve as a soundness test on a proposed model. In the frequently occurring situation where one expects a model to produce unique solutions when initial and/or final conditions (and exogenous forcing functions) are given, non-uniqueness of solutions indicates in general that too few equations have been written down in the model, and non-existence indicates that too many equations have been formulated. In models with inequality constraints however the issue is more subtle since also sign conditions play a role, which makes the availability of well-posedness tests all the more valuable.

Compared to the alternative formulations that have been mentioned above, a particular feature of complementarity systems is that they emphasize relations to input/output systems theory. Ideas from mathematical systems theory are used below both as a tool for classification of complementarity systems and as support for results on well-posedness and simulation.

Of course complementarity conditions may also be coupled to *difference* equations rather than to differential equations. In a modeling framework in which one is interested in the interplay between dynamics on the one hand and state space partitioning induced by complementarity conditions on the other hand, it may not seem attractive to entertain an *a priori* restriction on the relevant time instants. There are situations, however, where a discrete-time formulation appears naturally; an instance is provided by the so-called max-plus systems, as discussed for instance in [3]. It may be noted that in typical applications of the max-plus framework, such as in manufacturing systems, the "time" index refers to a production cycle counter rather than to physical time. See [27] for a discussion of the relation between various discrete-time modeling frameworks that involve inequalities and logical connectives. There is an interesting interplay that arises between continuous-time and discrete-time complementarity systems in the context of time-stepping methods for numerical approximation of solutions (cf. section 6).

2. Examples

In this section a number of examples of complementarity systems are given. Further discussion of these examples will take place in later sections.

2.1. Unilateral constraints in mechanical systems

For purposes of comparison with the examples related to optimization, let us recall the equations of motion that are associated with mechanical systems subject to inequality constraints. Equilibrium problems for such systems have motivated much of the early work on optimization problems with constraints by Fourier and by Farkas (see the historical survey in [50]). Complementarity conditions play a natural role in this context, and therefore it is no surprise that complementarity systems arise in the description of the related out-of-equilibrium situations. The following formulation can be found in textbooks on analytical mechanics [42, 33].

Let the system under consideration be described by n coordinates collected in the configuration vector $q(t)$, and let $p(t)$ denote the corresponding vector of momenta. The Hamiltonian $H(q,p)$ denotes total energy, generally given as the sum of a kinetic energy $\frac{1}{2}p^T M^{-1}(q)p$, where $M(q)$ denotes the mass matrix, and a potential energy $V(q)$. Suppose that the system is subject to unilateral constraints of the form $C(q(t)) \geq 0$ which should be satisfied for all times t; also suppose that there is an external force $v(t)$ acting on the system. One may then write down equations of motion as follows:

$$\frac{dq}{dt}(t) = \frac{\partial H}{\partial p}(q(t), p(t)) \tag{10a}$$

$$\frac{dp}{dt}(t) = -\frac{\partial H}{\partial q}(q(t), p(t)) + \frac{\partial C^T}{\partial q}(q(t))u(t) + v(t) \tag{10b}$$

$$y(t) = C(q(t)) \tag{10c}$$

$$0 \leq y(t) \perp u(t) \geq 0 \tag{10d}$$

where $\partial H/\partial p$ and $\partial H/\partial q$ denote column vectors of partial derivatives, $u(t) \geq 0$ is the vector of Lagrange multipliers producing the constraint force vector given by $(\partial C/\partial q)^T(q(t))u(t)$, and the expression $\partial C^T/\partial q$ denotes an $n \times k$ matrix whose i-th column is given by $\partial C_i/\partial q$. The perpendicularity condition expresses in particular that the i-th component of $u(t)$ can only be non-zero when the i-th constraint is active, that is, $y_i(t) = C_i(q(t)) = 0$. The appearance of the reaction force in the above form, with $u_i(t) \geq 0$, can be derived from the classical principle requiring that the reaction forces do not exert any work along virtual displacements that are compatible with the constraints.

2.2. Dynamic optimization with state constraints

Complementarity conditions are a familiar appearance in the context of the Karush-Kuhn-Tucker conditions for unilaterally constrained optimization problems; therefore, complementarity systems are expected to show up in dynamic optimization problems subject to state constraints. Such a relation can be developed in full generality, but let us consider here a problem with linear dynamics and quadratic cost: minimize

$$\tfrac{1}{2}\int_0^T (x(t)^T Q x(t) + u(t)^T u(t))dt \tag{11}$$

subject to

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{12a}$$
$$Cx(t) \geq 0 \tag{12b}$$

where A, B, and C are matrices of appropriate sizes, and Q is a nonnegative definite matrix. A standard application of the maximum principle leads to the system

$$\frac{dx}{dt}(t) \;=\; Ax(t) + Bu(t), \quad x(0) = x_0 \tag{13a}$$
$$\frac{d\lambda}{dt}(t) \;=\; Qx(t) - A^T\lambda(t) - C^T\eta(t), \quad \lambda(T) = 0 \tag{13b}$$
$$u(t) \;=\; \text{arg max}\left[-\tfrac{1}{2}u^T u + \lambda^T(t)Bu\right] \tag{13c}$$
$$0 \;\leq\; Cx(t) \perp \eta(t) \;\geq\; 0. \tag{13d}$$

Solving for u from (13c) produces the equations

$$\frac{d}{dt}\begin{bmatrix} x \\ \lambda \end{bmatrix}(t) \;=\; \begin{bmatrix} A & BB^T \\ Q & -A^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}(t) + \begin{bmatrix} 0 \\ -C^T \end{bmatrix}\eta(t) \tag{14a}$$
$$y(t) \;=\; \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}(t) \tag{14b}$$
$$0 \;\leq\; y(t) \;\perp\; \eta(t) \;\geq 0. \tag{14c}$$

This is a linear complementarity system. In the general case (nonlinear dynamics and/or non-quadratic cost), we obtain a nonlinear complementarity system. The system has to be solved subject to boundary conditions given partly at $t = 0$ and partly at $t = T$.

2.3. Dynamic optimization with control constraints

Dynamic optimization problems in which the decision variable is subject to inequality constraints often give rise to necessary equations for optimality in which the relation of the control variable to the state variable and the adjoint variable is described in a piecewise manner, for instance by means of a maximization over a finite number of alternatives. In such cases it is possible to obtain a representation of the necessary conditions for optimality in the form of a complementarity system, on the basis of the fact that a maximization over a finite set can be described in complementarity terms. A special case arises when the Hamiltonian is linear in the control variable. In this situation the relation between the control variable on the one hand and the state and adjoint variables on the other hand can no longer be expressed as a function. Consider for instance the well-known Vidale-Wolfe advertizing model:

$$\text{maximize} \quad \int_0^T (x(t) - cu(t))dt \tag{15a}$$

$$\text{subject to} \quad \frac{dx}{dt}(t) = -ax(t) + (1 - x(t))u(t) \tag{15b}$$

$$x(0) = x_0 \in (0,1) \tag{15c}$$

$$0 \leq u(t) \leq 1 \tag{15d}$$

where a and c are positive constants, $x(t)$ represents market share, and $u(t)$ represents marketing effort. The Hamiltonian is

$$H(x, u, \lambda) = x - cu + \lambda(-ax + (1 - x)u) \tag{16}$$

where λ denotes the adjoint variable. The condition

$$u(t) = \text{arg max}_{0 \leq u \leq 1} H(x(t), u, \lambda(t)) \tag{17}$$

can be satisfied in the following ways:

$$\begin{aligned} &u(t) = 0, && -c + \lambda(t)(1 - x(t)) < 0 \\ &u(t) = 1, && -c + \lambda(t)(1 - x(t)) > 0 \\ &0 \leq u(t) \leq 1, && -c + \lambda(t)(1 - x(t)) = 0. \end{aligned} \tag{18}$$

If we introduce

$$y(t) = -c + \lambda(t)(1 - x(t)) \tag{19}$$

we get a relation between $u(t)$ and $y(t)$ that might be described in terms of the following propositional logic formula:

$$\big((y < 0) \wedge (u = 0)\big) \vee \big((y = 0) \wedge (0 \leq u \leq 1)\big) \vee \big((y > 0) \wedge (u = 1)\big). \tag{20}$$

Such a relation is known in engineering as a characteristic of *relay type*. One might describe $u(t)$ as a multivalued function of $y(t)$. However, it is also possible to give a representation in complementarity form, by noting that the characteristic (20) can be described by the algebraic relations

$$v_1 = 1 - u, \quad v_2 = u, \quad z_1 - z_2 = y \tag{21}$$

and the complementarity conditions

$$0 \leq z_1 \perp v_1 \geq 0, \quad 0 \leq z_2 \perp v_2 \geq 0. \tag{22}$$

These two complementarity relations in principle give rise to four different "modes." However, subject to (21) the mode $v_1 = v_2 = 0$ is excluded. The three remaining modes correspond exactly to the three modes of the relay-type characteristic (20). For the standard interpretation to apply in which a complementarity system is an input-output dynamical system of the form (1a–1b) in which inputs and outputs have been connected by complementarity conditions, the variables v_1 and z_1 or z_2 should be categorized as inputs and v_2 and z_2 or z_1 as outputs, or vice versa; taking v_1 and v_2 as inputs and z_1 and z_2 as outputs or vice versa does not lead to a representation in standard form.

2.4. Projected dynamical systems

In [14], Dupuis and Nagurney have proposed a general way of embedding a given static equilibrium problem into a dynamic system. Dupuis and Nagurney assume that the static equilibrium problem can be formulated in terms of a variational inequality

$$\langle F(\bar{x}),\, x-\bar{x}\,\rangle \;\geq\; 0 \quad \text{for all } x \in K \tag{23}$$

where K is a closed convex subset of $\mathbb{R}^k$, F is a function F from K to $\mathbb{R}^k$, $\langle\cdot\,,\cdot\rangle$ denotes the standard inner product on $\mathbb{R}^k$, and $\bar{x} \in K$ is the equilibrium point. With the variational problem they associate a nonsmooth dynamical system that is defined by $(dx/dt)(t) = -F(x(t))$ in the interior of K, but that behaves differently on the boundary of K in such a way as to make sure that solutions will not leave the convex set K. They then prove that the stationary points of the so defined dynamical system coincide with the solutions of the variational equality. Dupuis and Nagurney mention a number of interesting applications in their paper, including oligopolistic markets, continuous-time tâtonnement processes, and traffic networks. Specifically, the dynamical system considered by Dupuis and Nagurney is defined by

$$\frac{dx}{dt}(t) \;=\; \pi(x(t), -F(x(t))), \quad x(0) = x_0 \in K \tag{24}$$

where $\pi(x,v)$ is defined by

$$\pi(x,v) \;=\; \lim_{\delta\to 0}\frac{P(x+\delta v)-x}{\delta} \tag{25}$$

and P is the projection onto K, that is,

$$P(x) \;=\; \arg\min_{z\in K}\|x-z\|. \tag{26}$$

Assume now that the set K is given, as is typically the case, by an expression of the form

$$K = \{x \mid h_i(x) \geq 0,\ i = 1,\ldots,N\}$$

where $h_1,\ldots,h_N$ are smooth functions. Let $H(x)$ denote the gradient matrix defined by the functions $h_i(x)$; that is to say, the (i,j)-th element of $H(x)$ is

$$(H(x))_{ij} \;=\; \frac{\partial h_i}{\partial x_j}(x). \tag{27}$$

For $x \in K$, let $\alpha(x)$ be the set of "active" indices, that is,

$$\alpha(x) \;=\; \{i \mid h_i(x) = 0\}. \tag{28}$$

Denote by $H_{\alpha(x)\bullet}$ the matrix formed by the rows of $H(x)$ whose indices are active; it will be assumed that this matrix has full row rank for all x on the boundary of K

("independent constraints"). Now, compare the projected dynamical system (24) to the complementarity system defined by

$$\frac{dx}{dt}(t) = -F(x(t)) + H^T(x(t))u(t) \tag{29a}$$

$$y(t) = h(x(t)) \tag{29b}$$

$$0 \;\leq\; y(t) \perp u(t) \;\geq\; 0 \tag{29c}$$

where $h(x)$ is a vector defined in the obvious manner by $(h(x))_i = h_i(x)$. It can be shown (see [30]) that the continuous solutions of (29) coincide with the solutions of (24). This result can be viewed as a dynamic extension of the well-known fact that, under mild conditions, a finite-dimensional variational inequality may be written as a nonlinear complementarity problem with added algebraic equations. Moreover, the added equations now receive an interpretation as corresponding to the constant solutions of the differential equation (29a). Note that the equations (29) may also be considered when the constraint set K defined by the functions $h_i(x)$ is not convex.

2.5. A Leontiev economy

A model for a continuous-time Leontiev economy may be set up as follows. Let $x_i(t)$, $u_i(t)$, and $v_i(t)$ respectively denote the inventory, production rate, and net exogenous demand associated with commodity i at time t. Furthermore, let q_{ij} denote the amount of commodity i required for the production of one unit of commodity j. A balance equation for the evolution of the inventory may then be written in the form

$$\frac{dx}{dt}(t) = (I - Q)u(t) - v(t) \tag{30a}$$

where Q is the matrix formed from the elements q_{ij}. It is natural to impose that inventory should be nonnegative; but this is by no means sufficient to determine a solution uniquely. However, if we furthermore impose that the economy is efficient in the sense that it produces the lowest amounts of commodities that are sufficient to meet demand, then commodities are not produced when there is still a positive inventory, and are otherwise produced in just sufficient amounts to prevent inventory from becoming negative [46, 12]. In other words, the complementarity relation

$$0 \leq x(t) \perp u(t) \geq 0 \tag{30b}$$

must hold for all t. The system (30) is in the form of a linear complementarity system (4) with $A = 0$, $B = I - Q$, $C = I$, $D = 0$, $E = -I$, and $F = 0$.

Actually, in [46, 12] the integral version of equation (30a) is used, so as to allow for solutions of more general form. In the language of distribution theory, the cited papers consider for instance demand patterns v that feature an infinite number of Dirac pulses in a finite time interval. This level of generality is motivated by applications in which the demand patterns are in fact sample paths of stochastic processes. The applications relate to interpretations of the system (30) as expressing a mapping from $v(\cdot)$ to $x(\cdot)$ which turns a given process v into a process x that is restricted to the nonnegative cone in $\mathbb{R}^n$. By letting v be a Brownian motion, one obtains in this way a "reflected Brownian motion" with a reflection rule that is expressed by the matrix Q; however the recipe can also be applied to more general Lévy processes. Reflected Brownian motions are used for instance as diffusion approximations to heavily loaded queueing networks [22]. Chen and Mandelbaum [12] refer to the (integral version of the) problem (30) as the *dynamic complementarity problem*, but as noted before we use that term in this paper in a different sense.

2.6. Optimal stopping

Consider a continuous-time Markov chain with n discrete states numbered $1, \ldots, n$. When state i is current at time t, there is a probability $\lambda_{ij}\Delta t + o(\Delta t)$ that at time $t + \Delta t$ the state has moved to position j. Suppose that a decision maker has the opportunity to stop the process at any time t prior to some given terminal time T in order to receive a payment $g_i(t)$ if the process is in state i at time t; alternatively, in case the process is not terminated early, the decision maker receives f_i at time T where i is the state reached at time T. In order to solve the problem of maximizing the expected payoff, introduce the value function $V_i(t)$. In position i at time t, the decision maker chooses between stopping with reward $g_i(t)$ or continuing with expected payoff

$$(1 - \sum_{j \neq i} \lambda_{ij}\Delta t)V_i(t) + \frac{dV_i}{dt}(t)\Delta t + \sum_{j \neq i} \lambda_{ij}\Delta t\, V_j(t) + o(\Delta t).$$

This leads to the following differential equation for the vector function $V(t)$:

$$0 = \max\left(g(t) - V(t), \frac{dV}{dt}(t) + AV(t)\right) \tag{31}$$

where A is the matrix defined by

$$a_{ij} = \lambda_{ij} \quad (i \neq j), \qquad a_{ii} = -\sum_{j \neq i} \lambda_{ij}. \tag{32}$$

The equation (31) may be rewritten in the form of a complementarity system with forcing term as

$$-\frac{dV}{dt}(t) = AV(t) + u(t) \tag{33a}$$

$$y(t) = V(t) - g(t) \tag{33b}$$

$$0 \leq y(t) \perp u(t) \geq 0. \tag{33c}$$

This equation comes with a terminal condition $V(T) = f$, and it should be solved in reverse time. The stopping region is determined by the states i and times t where the constraint $y_i(t) \geq 0$ is active.

The optimal stopping problem is important in finance because it relates to the pricing problem for American-type options. In financial applications, the horizon is typically finite and the time dependence of the solution is essential. The pricing problem is usually formulated in a continuous space rather than a discrete space as above; however, discretization is necessary for numerical work and can often be viewed as the replacement of a diffusion model by a Markov chain model. In fact a method for computing prices of American options in multifactor models (high-dimensional state spaces) can be based on such an approximation [5].

2.7. Optimal retention

Consider a continuous-time Markov chain with $n+1$ states $0,\dots,n$, interpreted as the number of "tokens" held by a decision maker. (Various specific interpretations of the tokens can be given; one may think for instance of rare goldfish.) Transitions between states are controlled partly by chance, with state-dependent probabilities, and partly by the decision maker. The agent may at each moment, as long as the number of tokens held is nonzero, decide to sell a token for an immediate revenue of one unit. At the planning horizon T, there is moreover a final payoff that depends on the then prevalent state. The agent's goal is to maximize expected total revenue. Because of the state-dependence of the transition probabilities and the final payoff, it may not be optimal to move to state 0 and so the question is how many tokens should be optimally retained by the decision maker.

Introduce the value function $V_i(t)$, where i denotes the current state and t denotes current time, and denote transition intensities by λ_{ij} as above. The expression for the evolution of value when the agent takes no action is the same as in the previous example, whereas the effect of selling a token for the decision maker is to move from the value $V_i(t)$ to $V_{i-1}(t)+1$; the latter option is only available though when $i \geq 0$. For $i \geq 1$, introduce

$$y_i(t) = V_i(t) - V_{i-1}(t) \tag{34}$$

and

$$u_i(t) = -\frac{dV_i}{dt}(t) - (AV)_i(t) \tag{35}$$

where the matrix A is defined as in (32). Moreover, define the matrices $B \in \mathbb{R}^{(n+1)\times n}$ and $C \in \mathbb{R}^{n\times(n+1)}$ by

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -1 \end{bmatrix}. \tag{36}$$

The evolution equation for the value function may then be written in the complementarity form

$$-\frac{dV}{dt}(t) = AV(t) + Bu(t) \tag{37a}$$

$$y(t) = CV(t) - 1\!\!1 \tag{37b}$$

$$0 \leq y(t) \perp u(t) \geq 0 \tag{37c}$$

where $\mathbb{1}$ denotes a vector all of whose entries are 1. As above, the equation comes with a terminal condition determined by the final payoff and should be solved backward in time.

2.8. *Optimal depletion*

Consider a nonrenewable resource (for instance copper in a given mine) that can be extracted and sold for a stochastically varying price. Assume that there are n different price levels and that transitions between these price levels take place according to a Markov chain model with transition intensities λ_{ij}. Obviously, if the price at a given time is low, it may be attractive for the resource owner to suspend production in anticipation of a price increase. In order to formulate an optimization problem, introduce the following quantities: $\tau(t)$ is the amount of resource still available at time t, $u(t)$ is the rate of extraction at time t (normalized to lie between 0 and 1), $p_i(\tau)$ is the net revenue of extraction when the Markov chain is in state i and τ units are left, and finally r is a discount factor. The dependence of the net revenue p on τ may arise for instance because the cost of extraction increases at low levels of the stock of resource. We look for a control strategy u that optimizes the discounted infinite-horizon criterion

$$E\int_0^\infty e^{-rt}u(t)p_{i(t)}(\tau(t))dt$$

subject to the constraints

$$\tau(t) \geq 0, \quad 0 \leq u(t) \leq 1.$$

Given the nature of the problem, it is natural to look for a value function of the form

$$V_i(t,\tau) = e^{-rt}F_i(\tau). \tag{38}$$

We write heuristically

$$V_i(t,\tau) = \max_{0\leq u\leq 1}\Big[e^{-rt}up_i(\tau)\Delta t + \big(1-\sum_{j\neq i}\lambda_{ij}\Delta t\big)V_i(t+\Delta t, \tau - u\Delta t) \\ + \sum_{j\neq i}\lambda_{ij}\Delta t\, V_j(t+\Delta t,\tau-u\Delta t) + o(\Delta t)\Big] \tag{39}$$

which leads to the equation

$$0 = \max_{0\leq u\leq 1}\Big[e^{-rt}up_i(\tau) - \sum_{j\neq i}\lambda_{ij}V_i(t,\tau) + \frac{\partial V_i}{\partial t}(t,\tau) \\ - u\frac{\partial V_i}{\partial \tau}(t,\tau) + \sum_{j\neq i}\lambda_{ij}V_j(t,\tau)\Big]. \tag{40}$$

After substituting (38), we find

$$0 = \max_{0\leq u\leq 1}\Big[(p_i(\tau) - \frac{dF_i}{d\tau}(\tau))u\Big] - rF_i(\tau) - \big(\sum_{j\neq i}\lambda_{ij}\big)F_i(\tau) + \sum_{j\neq i}\lambda_{ij}F_j(\tau). \tag{41}$$

The term that is maximized over u is linear in u, so the maximum is always reached at $u = 0$ (no extraction) or $u = 1$ (extraction at full capacity). Introducing the transition rate matrix A defined by (32) and passing to vector form, we obtain

$$0 = \max\big(0, p(\tau) - \frac{dF}{d\tau}(\tau)\big) + (A - rI)F(\tau). \tag{42}$$

Introducing $y(\tau) = (rI - A)F(\tau)$ and $u(\tau) = (dF/d\tau)(\tau) - p(\tau) + (rI - A)F(\tau)$, we may rewrite the above equation in complementarity form:

$$\frac{dF}{d\tau}(\tau) = -(rI - A)F(\tau) + p(\tau) + u(\tau) \tag{43a}$$

$$y(\tau) = (rI - A)F(\tau) \tag{43b}$$

$$0 \leq y(\tau) \perp u(\tau) \geq 0. \tag{43c}$$

The variable τ does *not* denote time in this problem. Nevertheless the equations that we obtain do fit the mold of a complementarity system. The system (43) is to be solved for nonnegative values of τ with the "initial condition" $F(0) = 0$.

2.9. A user/resource model

Many models for network usage can be described in terms of users who have access to several resources. For instance, users may be origin-destination pairs in a traffic network model, and in this case resources are the links between crossings. In the context of production planning, users may be products and resources may be machines. We might even model the submission system of scientific papers by defining authors to be users and journals to be resources. The use of a given resource generates a certain cost for the user, for instance in terms of incurred delay; this cost depends in general on the load that is placed on the resource by all users. A typical purpose of modeling is to describe the behavior of users in determining their demand for services from the resources available to them.

To set up a general model in mathematical terms, suppose that we have p users and m resources. Introduce the following quantities:

- $l_{ij}(t)$ = load per unit of time placed by user i on resource j at time t;
- $q_{ij}(t)$ = cost incurred at time t by user i when applying to resource j;
- $d_i(t)$ = total demand of user i at time t;
- $a_i(t)$ = cost accepted by user i at time t.

The above quantities are summarized in a *load matrix* $L(t) \in \mathbb{R}_+^{p\times m}$ (load is taken to be nonnegative), a *cost matrix* $Q(t) \in \mathbb{R}^{p\times m}$, a *demand vector* $d(t) \in \mathbb{R}^p$, and an *accepted cost vector* $a(t) \in \mathbb{R}^p$. Moreover we introduce a *state vector* $x(t) \in \mathbb{R}^n$ in terms of which the dynamics of the system is described, and which moreover determines the cost matrix:

$$\frac{dx}{dt}(t) = f(x(t), L(t)) \tag{44a}$$

$$Q(t) = h(x(t), L(t)). \tag{44b}$$

To describe the behavior of users, we assume that the Wardrop principle holds at every time instant t. In other words, given a demand level, each user distributes its load over resources in such a way that all resources that are used generate the same cost (this is the accepted cost), and there is no resource that is not used and that would generate a lesser cost. This behavioral principle, together with the nonnegativity of the load, can be expressed in matrix terms by

$$0 \leq L(t) \perp Q(t) - a(t) \cdot 1\!\!1^T \geq 0 \tag{44c}$$

where the “perp” relation is understood in the sense of the inner product $\langle A, B \rangle = \mathrm{tr}(A^T B)$ for $A, B \in \mathbb{R}^{p \times m}$. To close the model, we furthermore need the accounting relation

$$L(t)1\!\!1 = d(t) \tag{44d}$$

as well as a “constitutive relation” between the demand and the accepted cost which we take to be of the form

$$R(d, a) = 0 \tag{44e}$$

where R is a mapping from $\mathbb{R}^p \times \mathbb{R}^p$ to $\mathbb{R}^p$. The system (44) can be rendered as a cone complementarity system (1a–1b–5) by means of the identifications

$$\begin{aligned} u &= (L, a) \\ y &= \big(h(x, L) - a \cdot 1\!\!1^T, R(L1\!\!1, a)\big) \\ \mathcal{C} &= \mathbb{R}_+^{p \times m} \times \{0\} \subset \mathbb{R}^{p \times (m+1)}. \end{aligned} \tag{45}$$

As a specific case, consider a situation where the resources consist of m noninteracting queues and the state variables are the queue lengths. Ignoring the situations in which buffers are empty or full (which in fact could be naturally modeled in a complementarity framework), we write simple queue dynamics

$$\frac{dx_j}{dt}(t) = (1\!\!1^T L)_j - c_j \tag{46}$$

where c_j is a constant that represents the processing speed of queue j. A possible expression for cost is

$$Q_{ij} = k_j x_j + m_{ij}$$

where k_j is a proportionality constant, and the constants m_{ij} represent a fixed cost that may be user-specific. Finally assume that demand is constant, say, $d(t) = d_0$ irrespective of the actual cost $a(t)$. We then arrive at the following dynamic model:

$$\dot{x}(t) = L^T(t)1\!\!1 - c \tag{47a}$$

$$Q(t) = 1\!\!1 \cdot (Kx(t))^T + M \tag{47b}$$

$$L(t)1\!\!1 = d_0 \tag{47c}$$

$$0 \leq L(t) \perp Q(t) - a(t) \cdot 1\!\!1^T \geq 0. \tag{47d}$$

This is a linear (actually affine) cone complementarity system. The constant terms can be treated as external inputs, analogously to (4).

3. The notion of solution and the index

The specification (1) is not complete without an accompanying notion of solution. In the classical theory of "smooth" dynamical systems, many function spaces and associated notions of solution are in use to accommodate particular applications and circumstances. In the nonsmooth case, the presence of events gives rise to even more alternative solution concepts. In this section we mention a few possible choices, focusing mainly on initial value problems rather than on two-point boundary value problems as discussed in subsections 2.2 and 2.3.

One possible avenue is based on reformulation as a differential inclusion:

$$\frac{dx}{dt}(t) \in \{f(x(t),u) \mid 0 \leq u \perp h(x(t),u) \geq 0\}. \tag{48}$$

The standard notion of solution for differential inclusions reads as follows (see for instance [2]). An absolutely continuous function $x(t)$ is said to be a solution of the differential inclusion

$$\frac{dx}{dt}(t) \in F(x(t)) \tag{49}$$

on an interval $[t_0, t_1]$, if the relation (49) holds for almost all $t \in [t_0, t_1]$. In particular, this allows for solutions that are not differentiable everywhere.

In the applications discussed above one would usually expect to obtain unique solutions. However, differential inclusions of the general form $(dx/dt)(t) = F(x(t))$, where $F(\cdot)$ is a set-valued function, typically have many solutions unless the set-valued function F is drastically restrained. Unfortunately, the standard notion of solution allows nonunique solutions to appear in inconspicuous ways. Examples of nonuniqueness arising in switching systems have been provided by Filippov [19, p. 116] and by Pogromsky et al. [44]. In the latter paper, the authors study the differential inclusion

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}(t) \in \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}(t) - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\operatorname{sgn} x_1(t) \tag{50}$$

where sgn is the set-valued function defined by

$$\operatorname{sgn} y = \begin{cases} \{-1\} & \text{if } y < 0 \\ [-1,1] & \text{if } y = 0 \\ \{1\} & \text{if } y > 0. \end{cases}$$

The equation may be rewritten in complementarity form by the method indicated in subsection 2.3. The above differential inclusion, with initial condition $x(0) = 0$ and considered under the standard solution concept for differential inclusions as recalled above, is shown in [44] to allow an infinite number of solutions. Apart from the obvious solution $x(t) = 0$, these are solutions that show a "reverse accumulation of events," i. e. $t = 0$ is an accumulation point of the points of nondifferentiability of nonzero solutions. It does not seem easy to provide general conditions that exclude this phenomenon.

An alternative to the solution concept of differential inclusions is the notion of *forward solutions*, which is natural in the context of “multimode systems.” Broadly speaking, if one considers a system that is capable of evolution in several different modes, one may envisage solutions that, starting from a given initial point, first evolve in one of the modes for some time, then switch to another mode in which further evolution takes place, and so on. It is assumed that following every switch there is a positive time during which evolution takes place according to a fixed mode. So reverse accumulations of event times are excluded by definition, although forward accumulations may in general be admitted. Solution definitions of this type are popular in the literature on hybrid systems (systems that have both continuous and discrete state variables, see for instance [49]); in fact in this literature one often excludes forward accumulations as well, which leads to the notion of “non-Zeno solutions.” In the context of complementarity systems, the concept of a forward solution may be defined formally as follows.

Definition 1. *A triple of functions* $(x(\cdot), y(\cdot), u(\cdot))$ *defined on an interval* $[0, T)$ *is said to constitute a* forward solution *of the complementarity system (1) on* $[0, T)$ *if* $x(\cdot)$ *is continuous and there exists an increasing sequence of time instants* $0 = t_0, t_1, t_2 \ldots$ *converging to* T *such that the following conditions hold: the functions* $(x(\cdot), y(\cdot), u(\cdot))$ *are real-analytic on all open intervals* (t_j, t_{j+1})*, and the equations (1) are satisfied for all* $t \in [0, T) \setminus \{t_0, t_1, \ldots\}$.

In the above definition, the sequence $t_0, t_1, \ldots$ may be finite or countably infinite; in the first case, convergence to T simply means that the last element of the sequence is equal to T, whereas in the latter case it has the usual meaning $\lim_{j \to \infty} t_j = T$. The requirement of real-analyticity helps in finding algebraic conditions for feasible solutions, as illustrated below. The notion of a forward solution may be extended by replacing the sequence of event times $\{t_0, t_1, \ldots\}$ by a set of event times $\mathcal{E}$ that is only required to be well-ordered with respect to the usual ordering of the real numbers (i.e. every nonempty subset of $\mathcal{E}$ has a least element, which implies that the notion of a “next event” is always well-defined). Under this extension, it is possible to continue solutions beyond accumulations of event times.

The usual notions of solutions for ordinary differential equations are time-reversible in the sense that if $x(\cdot)$ is a solution to the ODE $(dx/dt)(t) = f(x(t))$, then $x(-t)$ is a solution to the ODE $(dx/dt)(t) = -f(x(t))$; this property however does *not* carry over to forward solutions, as a result of the fact that forward accumulations of events are admitted but reverse accumulations are not. The notion of forward solution therefore represents a breach with a tradition in dynamical systems theory that treats the forward and backward directions of time on an equal footing. It should be noted though that nonsmooth systems differ from the more traditionally studied smooth systems in that merging of solutions may take place (for instance in mechanical systems with friction), which is a phenomenon that is associated with a particular direction of time. Also, in a situation such as the one discussed in subsection 2.8 (the depletion example), negative values of the variable playing the role of “time” actually do not have a meaning at all.

From a functional-analytic perspective, it is natural to look at the equations (1a–1b) as defining a mapping from the function $u(\cdot)$ defined on some interval $[0, T]$ to a function $y(\cdot)$ defined on the same interval. The mapping from u to y is parametrized by the initial condition x_0 and so we might write $y = \mathcal{F}(x_0; u)$. If the (generalized) function spaces for u and y can be taken such that the integral $\int_0^T \langle u(t), y(t)\rangle dt$ has a meaning and both spaces contain well-defined nonnegative cones, then instead of (1) one may consider the problem

$$y = \mathcal{F}(x_0; u), \quad y \geq 0, \quad u \geq 0, \quad \int_0^T \langle u(t), y(t)\rangle \, dt = 0 \tag{51}$$

which can be considered as a parametric infinite-dimensional nonlinear complementarity problem. In particular, a natural choice is to take $L_2^k(0,T)$ as function space both for u and for y. We then have the following notion of solution.

Definition 2. *A triple of functions* $(x(\cdot), y(\cdot), u(\cdot)) \in L_2^{k+n+k}(0,T)$ *is said to constitute an* L_2-solution *of the complementarity system (1) on the interval* $[0,T]$ *if the state trajectory* $x(\cdot)$ *is absolutely continuous, the equations (1a) and (1b) hold almost everywhere, the inequalities* $u(t) \geq 0$ *and* $y(t) \geq 0$ *hold almost everywhere, and moreover*

$$\int_0^T \langle u(t), y(t)\rangle \, dt = 0. \tag{52}$$

Under the nonnegativity conditions on u and y, the condition (52) is equivalent to the condition $\langle u(t), y(t)\rangle = 0$ almost everywhere. The L_2 solution concept does not contain a time preference, and it allows accumulations of event times at interior points of the interval of definition.

Both the forward solution concept and the L_2 concept require continuity of state trajectories. This requirement is too strong in some cases. For instance, the equations of mechanical systems subject to impacts may be written in complementarity form, with positions and velocities as state variables. The simplest way of modeling the behavior at impact times is to let velocities experience an instantaneous jump, which means that some components of the state variable are not continuous across event times. In such cases, one is therefore led to allowing jumps of the state variables at event times. Jumps typically are needed when the number of inherent integration steps from "input" u to "output" y is two or more; for instance, in the case of mechanical systems where "inputs" are forces and "outputs" are displacements, the number of integration steps is two as is already evident from the physical dimensions (mg/s^2 vs. m). The specification of jumps is a major issue in the formulation of a solution concept for higher-index complementarity systems. In the case of mechanical systems, a natural rule to use for inelastic and frictionless collisions is projection to the cone of feasible velocities in the metric induced by the kinetic energy [38]. In the case of linear complementarity systems, a general rule may be derived from linear systems theory; see Def. 5 below.

The notion of "index" is well-known in the theory of differential-algebraic equations, and analogous notions have been developed in mathematical systems theory as well. A system of the form

$$\frac{dx}{dt}(t) = f(x(t), u(t)) \tag{53a}$$

$$0 = h(x(t), u(t)) \tag{53b}$$

specifies a differential-algebraic equation, or more specifically, a “semi-explicit” DAE. Obviously there is a close connection with the complementarity systems, as well as with input/output dynamical systems (cf. (1)). Differential-algebraic equations are often classified in terms of a quantity called the *index*, which measures the extent to which the algebraic constraint (53b) restricts the dynamical system (53a). The notion of the index can be formalized in several ways, which in the general context of nonlinear and possibly time-dependent DAEs sometimes lead to different results; see for instance [7]. Moreover, even in simple cases, conventions differ between mathematical systems theory and the DAE literature. Here we shall identify the index with the system-theoretic notion of *relative degree*, which aims to express the “number of integration steps” between inputs and outputs; see for instance [39, 40].

Definition 3. *In a nonlinear input/output system of the form (1a–1b), the* relative degree *of the i-th output y_i is the number of times one has to differentiate y_i to get a result that depends explicitly on the inputs u. The system is said to have* constant uniform relative degree at x_0 *if the relative degrees of all outputs are the same and are constant in a neighborhood of x_0.*

The simplest application of this notion is in the case of constant-coefficient single-input-single-output linear systems, i. e. systems of the form

$$\frac{dx}{dt}(t) = Ax(t) + Bu(t) \tag{54a}$$

$$y(t) = Cx(t) + Du(t) \tag{54b}$$

where A is a square matrix, B is a column vector, C is a row vector, and D is a number. The system above gives rise to the sequence of *Markov parameters* defined by

$$(M_0, M_1, M_2, M_3, \dots) := (D, CB, CAB, CA^2B, \dots). \tag{55}$$

In terms of the Markov parameters, the relative degree can be described as the first index i for which M_i is nonzero. Equivalently, the relative degree can be defined as the smallest value of i for which the rational function $s^i[C(sI - A)^{-1}B + D]$ does not vanish at $s = \infty$. The rational function

$$G(s) = C(sI - A)^{-1}B + D \qquad (s \in \mathbb{C}) \tag{56}$$

is called the *transfer function* of the linear input/output system (54). The same term is used when y and u are vectors rather than scalars; in this case $G(s)$ as defined above is a matrix-valued function of the complex variable s. The transfer function is a basic object of study in linear systems theory.

One may wonder whether high-index systems actually arise in situations where one might use complementarity modeling. In this connection the following proposition, which refers to dynamic optimization under state constraints (subsection 2.2), is relevant.

Proposition 1. *Suppose that the dimensions of the vectors y and u in the linear input/output system (12) are equal to 1. The index of the associated system (14a–14b) is then equal to twice the index of the original system (12).*

Proof. The transfer function of the system (14a–14b) may be written, using the Schur complement rule, as

$$\begin{aligned}
&\begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} sI-A & -BB^T \\ -Q & sI+A^T \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -C^T \end{bmatrix} = \\
&\qquad = -C\left[(sI-A) - BB^T(sI+A^T)^{-1}Q\right]^{-1} BB^T(sI+A^T)^{-1}C^T = \\
&\qquad = G(s)(I-\Phi(s))^{-1}G^T(-s) \qquad (57)
\end{aligned}$$

where $G(s) = C(sI-A)^{-1}B$ is the transfer function of the original system (12) and

$$\Phi(s) = B^T(sI+A^T)^{-1}Q(sI-A)^{-1}B. \qquad (58)$$

The formula (57–58) is valid in the vector case. In the scalar case, it follows immediately from (57) that the index of (14a–14b) is equal to two times the index of (12).

Since the system (12) can have any nonnegative index, the proposition shows that complementarity systems of arbitrarily high index may occur naturally. However, the example refers to a two-point boundary problem and in such situations high index may have a different impact than in initial value problems. In particular, the smoothness of a solution of a complementarity system depends in general on the initial condition, which in the case of mixed boundary conditions is not determined *a priori*. Examples suggest that the smoothest possible solutions of (14) are of particular interest for optimization purposes. Especially for higher-index situations there is no general theory available yet, however; see [24] for a survey.

4. Existence and uniqueness of solutions

Algebraic conditions for the existence of a forward solutions can be found by noting that from (1) a number of relations can be obtained between the successive time derivatives of $y(\cdot)$ evaluated at $t = 0$ on the one hand, and on the other hand the same quantities derived from $u(\cdot)$. By repeatedly differentiating (1b) and using (1a), we find

$$\begin{aligned}
y(t) \;&=\; h(x(t), u(t)), \\
\frac{dy}{dt}(t) \;&=\; \frac{\partial h}{\partial x}(x(t), u(t)) f(x(t), u(t)) + \frac{\partial h}{\partial u}(x(t), u(t)) \frac{du}{dt}(t) \\
&=: F_1(x(t), u(t), \frac{du}{dt}(t)),
\end{aligned}$$

and in general

$$y^{(j)}(t) \;=\; F_j(x(t), u(t), \ldots, u^{(j)}(t)) \qquad (59)$$

where F_j is a function from $\mathbb{R}^{n+(j+1)k}$ to $\mathbb{R}^k$. To be precise, the functions F_j can be specified recursively by

$$F_0(x, u^0) = h(x, u^0) \tag{60}$$

$$F_{j+1}(x, u^0, \ldots, u^j) = \frac{\partial F_j}{\partial x} f(x, u^0) + \sum_{i=0}^{j} \frac{\partial F_j}{\partial u^i} u^{i+1}. \tag{61}$$

From the complementarity conditions (1c), it follows moreover that for each index i either

$$(y_i(0), \frac{dy_i}{dt}(0), \ldots) \; = \; 0 \;\text{ and }\; (u_i(0), \frac{du_i}{dt}(0), \ldots) \; \succeq \; 0 \tag{62}$$

or

$$(y_i(0), \frac{dy_i}{dt}(0), \ldots) \; \succeq \; 0 \;\text{ and }\; (u_i(0), \frac{du_i}{dt}(0), \ldots) \; = \; 0 \tag{63}$$

(or both), where we use the symbol $\succeq$ to denote lexicographic nonnegativity. (A sequence $(a_0, a_1, \ldots)$ of real numbers is said to be *lexicographically nonnegative* if either all a_i are zero or the first nonzero element is positive.) This suggests the formulation of the following problem, which was called the "dynamic complementarity problem" in [48].

Problem DCP. Let a point $x_0 \in \mathbb{R}^n$ be given, and let f and h be smooth functions from $\mathbb{R}^{n+k}$ to $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively. Find sequences $(y^0, y^1, \ldots)$ and $(u^0, u^1, \ldots)$ of k-vectors such that for all j we have

$$y^j \; = \; F_j(x_0, u^0, \ldots, u^j) \tag{64a}$$

where the functions F_j are defined in (60), and for each index $i \in \{1, \ldots, k\}$ at least one of the following is true:

$$(y_i^0, y_i^1, \ldots) \; = \; 0 \;\text{ and }\; (u_i^0, u_i^1, \ldots) \; \succeq \; 0 \tag{64b}$$

$$(y_i^0, y_i^1, \ldots) \; \succeq \; 0 \;\text{ and }\; (u_i^0, u_i^1, \ldots) \; = \; 0. \tag{64c}$$

We shall also consider truncated versions where j only takes on the values from 0 up to some integer ℓ; the corresponding problem will be denoted by DCP(ℓ). The problem DCP(0) is a parametric nonlinear complementarity problem, so the DCP may be seen as a "prolonged" version of the NLCP.

It follows from the triangular structure of the equations (64) that if the pair of sequences $((y^0, \ldots, y^\ell), (u^0, \ldots, u^\ell))$ is a solution of DCP(ℓ), then, for any $\ell' < \ell$, the pair $((y^0, \ldots, y^{\ell'}), (u^0, \ldots, u^{\ell'}))$ is a solution of DCP(ℓ'). We call this the *nesting property* of solutions. We define the *active index set at stage* ℓ, denoted by α_ℓ, as the set of indices i for which $(u_i^0, \ldots, u_i^\ell) \succ 0$ in *all* solutions of DCP(ℓ), so that necessarily $y_i^j = 0$ for all j in any solution of DCP (if one exists). Likewise we define the *inactive index set at stage* ℓ, denoted by β_ℓ, as the set of indices i for which $(y_i^0, \ldots, y_i^\ell) \succ 0$ in *all* solutions of DCP(ℓ), so that necessarily $u_i^j = 0$ for all j in any solution of DCP. Finally we define γ_ℓ as the complementary index set $\{1, \ldots, k\} \setminus (\alpha_\ell \cup \beta_\ell)$. It follows from the nesting property of solutions that the index sets α_ℓ and β_ℓ are nondecreasing as functions of ℓ. Since both sequences are obviously bounded above, there must exist an index ℓ^* such that $\alpha_\ell = \alpha_{\ell^*}$ and $\beta_\ell = \beta_{\ell^*}$ for all $\ell \geq \ell^*$. We finally note that all index sets defined here of course depend on x_0; we suppress this dependence however to alleviate the notation.

It turns out that under fairly mild assumptions the DCP can be reduced to a series of *linear* complementarity problems. In the context of mechanical systems this idea is due to Lötstedt [37]. Let us assume that the dynamics (1a–1b) can be written in the *affine* form

$$\begin{aligned} \frac{dx}{dt}(t) &= f(x(t)) + \textstyle\sum_{i=1}^k g_i(x(t))u_i(t) \\ y(t) &= h(x(t)). \end{aligned} \tag{65}$$

Many nonlinear systems appear naturally in this form; see for instance [40]. The following theorem has been proved in [48]. In the theorem, the notation $L_f h$ denotes the *Lie derivative* of h along the vector field given by f; that is, $L_f h(x) = (\partial h/\partial x)(x) f(x)$. The k*-th Lie derivative* $L_f^k h$ is defined for $k = 2, 3, \ldots$ by $L_f^k h = L_f(L_f^{k-1} h)$ with $L_f^1 h := L_f h$.

Theorem 1. *Consider the system of equations (65) together with the complementarity conditions (1c), and suppose that the system (65) has constant uniform relative degree ρ at a point $x_0 \in \mathbb{R}^n$. Suppose that x_0 is such that*

$$(h(x_0), \ldots, L_f^{\rho-1} h(x_0)) \succeq 0 \tag{66}$$

(with componentwise interpretation of the lexicographic inequality), and that the matrix $L_g L_f^{\rho-1} h(x_0)$ is a P-matrix. For such x_0, the dynamic complementarity problem DCP(ℓ) has for each ℓ a solution $((y^0, \ldots, y^\ell), (u^0, \ldots, u^\ell))$ which can be found by solving a sequence of LCPs. Moreover this solution is unique, except for the values of u_i^j with $i \notin \beta_\ell$ and $j > \ell - \rho$.

The lack of uniqueness that may be present in the solution of the dynamic complementarity problem does not affect the uniqueness of the corresponding solution of the complementarity system. This is stated in the following theorem which is also from [48].

Theorem 2. *Assume that the functions f, g_i, and h appearing in (65) are analytic. Under the conditions of Thm. 1, there exists an $\varepsilon > 0$ such that (65) together with the complementarity condition (1c) has a smooth solution with initial condition x_0 on $[0, \varepsilon]$. Moreover, this solution is unique and corresponds to any mode α such that $\alpha_{\ell^*} \subset \alpha \subset \alpha_{\ell^*} \cup \gamma_{\ell^*}$.*

In particular, suppose that in (65) the relative degree at each point x_0 in a neighborhood of the *feasible set* $\{x \mid h(x) \geq 0\}$ is 1, and that the matrix with columns $(\partial h/\partial x)(x_0) g_i(x_0)$ is a P-matrix for all such x_0. Then the theorem above guarantees that whenever one of the constraints $h_i(x) \geq 0$ becomes active the solution can be continued, be it possibly in a different mode. Note that this situation applies to the complementarity reformulation of projected dynamical systems (subsection 2.4). Neither convexity nor boundedness of the feasible set needs to be assumed. The solution is of “forward” type and is actually constructed in a piecewise manner. At this level of generality, however, existence of a solution for arbitrarily long horizons cannot be guaranteed; an obstruction might occur either by escape to infinity (as may already happen in the case of ordinary differential equations) or by accumulation of event times.

In higher-index situations, it may happen that equation (66) is not satisfied at some x_0, even though x_0 is feasible in the sense that $h(x_0) \geq 0$. It is easily seen that in such cases the initial condition x_0 is not *consistent* in the sense that there is a forward solution of (65–1c) starting from x_0. As a consequence, the continuity requirement has to be abandoned and one has to allow the possibility of jumps. For nonlinear systems, it is difficult (perhaps impossible) to give a general jump rule. However, in the case of linear systems, a theory is available that suggests a particular choice. This is discussed in the next section.

5. Linear complementarity systems

In this section we focus on linear complementarity systems (2). As has been shown in the examples section, complementarity systems arise naturally with linear dynamics in a number of applications. In focusing attention on linear complementarity systems we also follow a pattern in mathematical input/output systems theory, where many ideas have been first developed and tested in the context of linear and finite-dimensional systems before they are extended to nonlinear and/or infinite-dimensional settings.

The system (2) may well be of higher index (in particular this happens when $D = 0$ and $CB = 0$), and as has been noted above in such cases the forward solution concept of Def. 1, which requires continuity of solutions, is no longer satisfactory. The formulation of a general jump rule for linear complementarity systems can be based on a decomposition that goes back in essence to Weierstrass [20, Thm. XII.3]. In terms of the linear input/output system (2a–2b), the decomposition can be stated as in the theorem below, which is a summary of results that are described in more detail in [55, Ch. 7,8].

Theorem 3. *Suppose that the transfer function (56) corresponding to a given matrix quadruple (A, B, C, D) is invertible. Under this condition, the following two subspaces of the state space $\mathbb{R}^n$ constitute a direct sum decomposition of $\mathbb{R}^n$:*

$V(A, B, C, D) = V^n,$ where V^n is the index-n element of the sequence

$$V^0 = \mathbb{R}^n \tag{67a}$$

$$V^{i+1} = \{x \in V^i \mid \exists u \in \mathbb{R}^k \text{ s. t. } Ax + Bu \in V^i,\ Cx + Du = 0\} \tag{67b}$$

$T(A, B, C, D) = T^n,$ where T^n is the index-n element of the sequence

$$T^0 = \{0\} \tag{68a}$$

$$T^{i+1} = \{x \in \mathbb{R}^n \mid \exists \tilde{x} \in T^i,\ \tilde{u} \in \mathbb{R}^k \text{ s. t. } x = A\tilde{x} + B\tilde{u},\ C\tilde{x} + D\tilde{u} = 0\}. \tag{68b}$$

A vector $x_0 \in \mathbb{R}^n$ belongs to $V(A,B,C,D)$ if and only if there exists a real-analytic state trajectory $x(\cdot)$ such that $x(0)=x_0$ and for all $t \geq 0$

$$\begin{bmatrix} \frac{dx}{dt}(t) - Ax(t) \\ Cx(t) \end{bmatrix} \in \mathrm{im} \begin{bmatrix} B \\ D \end{bmatrix}. \tag{69}$$

Moreover, under the invertibility assumption on the transfer matrix $G(s)$, there is exactly one solution of (69) for each initial condition $x_0 \in V(A,B,C,D)$.

The theorem guarantees existence and uniqueness, under certain circumstances, of solutions to the set of differential-algebraic equations $(dx/dt)(t) = Ax(t) + Bu(t)$, $Cx(t) + Du(t) = 0$. The uniqueness holds only for the state trajectory $x(\cdot)$ and not without further qualification for the input trajectory $u(\cdot)$, because it may happen that the matrix $[B^T \; D^T]^T$ has a nontrivial nullspace. In the context of linear complementarity systems, the role of the matrices "C" and "D" is played in each mode by the constraints that are active in this mode; the inputs corresponding to inactive indices must be zero, so that we always have a situation in which the number of nonzero inputs is equal to the number of outputs that are constrained to zero, as in the theorem. A consistent initial condition must be in the space "V" corresponding to a given mode.

Of course there are also inequality constraints to be satisfied. These can be formulated by means of a Dynamic Complementarity Problem as in the general nonlinear case; however, for linear complementarity systems a more compact alternative is available in the form of the so-called Rational Complementarity Problem (RCP) [28]. The RCP is quite similar to the LCP except that it is formulated over the field of rational functions with real coefficients rather than over the field of the reals, and so it requires a linear ordering of the set of rational functions. In fact there are many ways in which the field of rational functions can be made into an ordered field; in the context of complementarity systems, however, the following definition is in particular relevant. In the notation below we use the symbol s to denote a generic complex number and σ to denote a specific real number.

Definition 4. *An element $f(s)$ of $\mathbb{R}(s)$ is said to be* nonnegative (at infinity), *and we write $f \succeq 0$, if there exists $\sigma_0 \in \mathbb{R}$ such that $f(\sigma) \geq 0$ for all $\sigma \geq \sigma_0$.*

In terms of its Laurent expansion at infinity $f(s) = f_k s^k + \cdots + f_1 s + f_0 + f_{-1} s^{-1} + \cdots$, a rational function is nonnegative at infinity if and only if the sequence of coefficients $(f_k, \ldots, f_1, f_0, f_{-1}, \ldots)$ is lexicographically nonnegative. For simplicity we therefore use the same symbol for nonnegativity of rational functions in the above sense and for lexicographic nonnegativity of sequences. It is easily verified that the order defined on the field $\mathbb{R}(s)$ of rational functions with real coefficients by setting $f \succeq g$ when $f - g \succeq 0$ is in fact a total order. Extending the usual conventions in the case of the real field, we shall say that a rational vector is nonnegative if and only if all of its entries are nonnegative, and we shall write $f(s) \perp g(s)$, where $f(s)$ and $g(s)$ are rational vectors, if for each index i at least one of the component functions $f_i(s)$ and $g_i(s)$ is identically zero. With these conventions, the rational complementarity problem with data $(q(s), M(s))$ is given by

$$\begin{aligned} y(s) &= q(s) + M(s)u(s) \\ 0 \preceq y(s) &\perp u(s) \succeq 0. \end{aligned} \tag{70}$$

We shall say that $\alpha \subset \{1,\dots,k\}$ is an *active index set* for the RCP with data $q(s)$ and $M(s)$, and we write $\alpha \in$ RCP$(q(s), M(s))$, when the rational equations (70) have a solution $(y(s), u(s))$ in which $y_i(s)$ is identically zero for $i \in \alpha$, and $u_i(s)$ is identically zero for $i \notin \alpha$. Given a linear system (2a–2b), we also write RCP($x_0; A, B, C, D$) instead of RCP($q(s), M(s)$) when $q(s)$ and $M(s)$ are defined by

$$q(s) = C(sI - A)^{-1}x_0, \quad M(s) = C(sI - A)^{-1}B + D. \tag{71}$$

We can now define a notion of solution for linear complementarity systems of arbitrarily high index.

Definition 5. *A triple of functions $(x(\cdot), y(\cdot), u(\cdot))$ defined on an interval $[0, T)$ is said to constitute a* Weierstrass solution *of the linear complementarity system (2) on $[0, T)$ with initial condition x_0 if there exists an increasing sequence of time instants $0 = t_0, t_1, t_2 \dots$ converging to T such that the following conditions hold:*

(i) the functions $(x(\cdot), y(\cdot), u(\cdot))$ are real-analytic on all open intervals (t_j, t_{j+1}),
(ii) the equations (1) are satisfied for all $t \in [0, T) \setminus \{t_0, t_1, \dots\}$
(iii) for all $j \geq 0$,

$$x(t_j^+) - x(t_j^-) \in T(A, B_{\bullet\alpha}, C_{\alpha\bullet}, D_{\alpha\alpha}) \tag{72}$$

with

$$\alpha \in \text{RCP}(x(t_j^-); A, B, C, D) \tag{73}$$

where $x(t_j^+) = \lim_{t \downarrow t_j} x(t)$ for $j \geq 0$, $x(t_j^-) = \lim_{t \uparrow t_j} x(t)$ for $j > 0$, and $x(t_0^-) = x_0$.

In (72), the subspace $T(A, B_{\bullet\alpha}, C_{\alpha\bullet}, D_{\alpha\alpha})$ is as defined in (68), and we use the standard convention that $M_{\alpha\beta}$ denotes the submatrix of a matrix M defined by the row index set α and the column index set β, with row or column index set replaced by a dot if all rows or columns are used.

Existence and uniqueness of Weierstrass solutions for linear complementarity systems has been proved in a number of cases. In the results below the following terminology is used. Given a linear complementarity system (2) and an initial condition x_0, we define $T(x_0)$ as the infimum of the set of all $\tau \geq 0$ such that the system (2) does not allow a Weierstrass solution on $[0, \tau)$. If this set is empty, we set $T(x_0) = \infty$. The number $T(x_0)$ is called the *length of the maximal interval of existence* of a Weierstrass solution starting from x_0. The theorem below states conditions for well-posedness result in terms of the rational complementarity problem [28, Thm. 5.10, Thm. 5.16].

Theorem 4. *Consider the linear complementarity system (2). Suppose that all principal minors of the transfer matrix of the linear system (2a–2b) are nonzero, and that the problem RCP($x_0; A, B, C, D$) is uniquely solvable for all x_0. Under these conditions, the length of the maximal interval of existence of a Weierstrass solution starting* *from any initial condition x_0 is positive, and the solution on $[0, T(x_0))$ is unique. If $T(x_0)$ is finite, then it is an accumulation point of event times of the solution defined on $[0, T(x_0))$.*

Unique solvability of the rational complementarity problem can be connected to unique solvability of a family of standard linear complementarity problems in the following way [28, Thm. 4.1, Cor. 4.10].

Theorem 5. *For given rational vector $q(s) \in \mathbb{R}^k(s)$ and rational matrix $M(s) \in \mathbb{R}^{k\times k}(s)$, the problem RCP$(q(s), M(s))$ is uniquely solvable if and only if there exists $\sigma_0 \in \mathbb{R}$ such that for all fixed $\sigma > \sigma_0$ the problem LCP$(q(\sigma), M(\sigma))$ is uniquely solvable.*

In particular, the following corollary is obtained.

Corollary 1. *If there exists $\sigma_0 \in \mathbb{R}$ such that $G(\sigma)$ is a P-matrix for all $\sigma > \sigma_0$, where $G(s)$ is the transfer function of the linear system (2a–2b), then the conclusions of Thm. 4 hold for the linear complementarity system (2).*

Noting that the set of P-matrices is open, one obtains the result in the theorem below. This theorem can also be obtained a special case of Thm. 6.3 in [29], which itself is not readily derived from Thm. 4 (it is based on consideration of DCP rather than of RCP). Recall that the leading Markov parameter of a linear system of the form (2a–2b) is by definition the first nonzero term in the Laurent expansion around infinity of the transfer matrix $G(s) = C(sI - A)^{-1}B + D$; in other words, it is the first nonzero matrix in the sequence D, CB, CAB, CA^2B, $\ldots$.

Theorem 6. *If the leading Markov parameter of the linear system (2a–2b) is a P-matrix, then the conclusions of Thm. 4 hold for the linear complementarity system (2).*

The above results can be extended to situations in which there is forcing term (external input), under appropriate smoothness requirements on the input function [26]. The smoothness assumption used in [26] is that external inputs are *piecewise Bohl*, where Bohl functions are defined as function that have rational Laplace transforms, such as polynomials, exponentials, sinusoids, and combinations of these. While it is likely that this assumption can be relaxed, the class of piecewise Bohl functions already covers many cases of interest.

By way of illustration, we apply these results to the examples of linear complementarity systems that were discussed before.

Optimal stopping. The zeroth-order Markov parameter of the linear system appearing in (33) vanishes; the first-order Markov parameter is the identity matrix. Obviously therefore, the leading Markov parameter is a P-matrix.

Optimal retention. The leading Markov parameter of the linear system appearing in (37) is

$$CB = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \tag{74}$$

which is a P-matrix.

Optimal depletion. In this case the leading Markov parameter is $rI - A$. The matrix A is the infinitesimal generator of a continuous-time Markov chain and therefore all of its eigenvalues have nonpositive real parts. Under the assumption that $r > 0$, it follows that all eigenvalues of $rI - A$ have strictly positive real parts. Since moreover $rI - A$ has nonpositive off-diagonal elements, it follows that $rI - A$ is a P-matrix [4, Thm. 6.2.3].

Leontiev economy. The leading Markov parameter is $CB = I - Q$. Standard assumptions on the matrix Q are that it is nonnegative and that its spectral radius is less than 1 (an "open" Leontiev economy [12]). Under these conditions, the matrix $I - Q$ is a P-matrix by the same result as cited before [4, Thm. 6.2.3]. It should be noted that we assume here sufficient smoothness of the forcing term v; more strict requirements may be needed for well-posedness when highly irregular inputs are considered [12]. If the spectral radius of Q is allowed to take the value 1, it is easy to find situations in which solutions are non-unique for zero demand, or do not exist for positive demand; take for instance the single-commodity economy with $Q = 1$.

Dynamic optimization with state constraints. The zeroth-order and the first-order Markov parameter of the linear system appearing in (14) both vanish. The second-order Markov parameter is equal to $-CBB^TC^T$. In the simplest case, namely the one in which the matrix CB is invertible, this means that the leading Markov parameter of the transfer function associated to (14) is *negative* definite. A suggestion arises that the system (14) is *not* well-posed in the sense defined above, and simple examples confirm that this may indeed happen. However, it should be noted that (14) should not be read as an initial-value problem, but rather as a problem with mixed boundary conditions. Even when there is no uniqueness of solutions in an forward sense, existence and uniqueness may still hold when mixed boundary conditions are imposed. Negative definiteness of the leading Markov parameter of (14) implies that well-posedness holds in a backward sense, so that solutions forward in time cannot merge but may split. Such splitting of solutions can be useful to meet boundary conditions at the terminal time.

User/resource model. The user/resource model of subsection 2.9 falls in the category of cone complementarity systems and is therefore not within the scope of the well-posedness discussion that was presented here. Conditions for well-posedness of linear cone complementarity systems were presented recently by Heemels et al. [26] (see also [10]) and by Pang and Stewart [41]. Both Thm. 2 in [26] and Thm. 12 in [41] can be used to establish existence of solutions (in slightly different senses) of the affine cone complementarity system (47). One of the conditions of Thm. 2 in [26] is not satisfied;

however, this condition (the full column rank condition in “Assumption 1”) is used only to establish the uniqueness of input trajectories. The uniqueness of state trajectories, in the sense of L_2, still follows.

6. Numerical approximation

If the well-posedness of a complementarity system (1) has been established, the question arises how one can approximate the solution of this system when an initial condition and possibly an exogenous input function are given. Several approaches are possible.

Event-based methods. This approach is based on the observation that the behavior of a complementary system in each separate mode is given by a differential-algebraic system, and that evolution in a given mode continues as long as certain inequality constraints are satisfied. Therefore, the evolution of the system may tracked by standard methods for DAEs together with a monitoring function for the inequalities associated to the current mode. When violation of these constraints is detected, an event must have taken place. The time at which the event has taken place is found by a backtracking procedure; after it has been determined in which mode continuation may take place and possible whether a state jump takes place, the integration procedure is restarted in the new mode and from the new initial state.

Regularization. In this approach, the complementarity characteristic is replaced by an approximating smooth characteristic, for instance $y = \varepsilon(\exp(-u/\varepsilon)-1)$ (many choices are possible). The resulting smooth system which replaces the original complementarity system is then solved by standard techniques. In a sequence of successive approximations, the accuracy of the solution to the approximating smooth system should be increased and the regularization parameter should be decreased.

Time stepping. This name is sometimes given to methods that replace the continuous-time complementarity system by a discrete-time version which requires the solution of a complementarity problem at each step.[2] Typically, the discrete-time version of the complementarity system is obtained by applying a (partially) implicit Euler method. In these methods, no attempt is undertaken to locate the time points at which events take place; the time step may in fact be constant in the most basic versions.

For each method that may be proposed, a key issue is to provide conditions under which the method will generate solutions that converge to the true solution when the approximation parameters tend to zero. For event-based methods, standard results from the theory of the numerical solution of DAEs may be applied. The convergence issue is more problematic in the case of the two other approaches. Regularization methods are dependent upon the type of smoothing approximation that is being used. Time-stepping methods are in a sense more intrinsic, although there is more than one way to convert a continuous-time complementarity system into a discrete-time version. Applying

[2] The terminology may be somewhat misleading inasmuch as time discretization is used by other numerical methods as well.

a straightforward Euler method with general implicitness parameter $\theta \in [0,1]$ and fixed time step Δt to the system (1), one obtains the following system of equations:

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = (1-\theta) f(x(t), u(t)) + \theta f(x(t+\Delta t), u(t+\Delta t)) \tag{75}$$

$$y(t+\Delta t) = h(x(t+\Delta t), u(t+\Delta t)) \tag{76}$$

$$0 \leq y(t+\Delta t) \perp u(t+\Delta t) \geq 0 \tag{77}$$

where it is assumed that the system is to be solved in forward time and where the complementarity condition is imposed at $t+\Delta t$ to ensure the satisfaction of this constraint at all points in the chosen time grid. The complementarity problem that must be solved at each time step therefore takes the following form (where for brevity we write x instead of $x(t)$, x' instead of $x(t+\Delta t)$, and so on):

$$x' - \theta \Delta t\, f(x', u') = x + (1-\theta)\Delta t\, f(x, u) \tag{78a}$$

$$0 \leq h(x', u') \perp u' \geq 0. \tag{78b}$$

This could be described as a parametric mixed nonlinear complementarity problem. In the case of linear complementarity systems, a more explicit form can be written down. For a system of the form (2), the equations (78) lead to the update rule

$$x' = (I - \theta \Delta t\, A)^{-1} \big[(I + (1-\theta)\Delta t\, A)x + \Delta t\, B[(1-\theta)u + \theta u'] \big] \tag{79}$$

where u' satisfies

$$y' = Hx + Ju + G(1/(\theta \Delta t))u' \tag{80a}$$

$$0 \leq y' \perp u' \geq 0 \tag{80b}$$

with

$$H = C(I - \theta \Delta t\, A)^{-1}(I + (1-\theta)\Delta t\, A) \tag{80c}$$

$$J = (1-\theta)\Delta t\, C(I - \theta \Delta t\, A)^{-1} B. \tag{80d}$$

In (80a), $G(\cdot)$ is the transfer function as defined in (56); if $\theta = 0$, then $G(1/(\theta \Delta t))$ is replaced by the matrix D. Note that the existence of the matrix inverses in the above equations is guaranteed for sufficiently small Δt. The problem (80) is, for given x and u, a standard LCP. In particular, for non-explicit methods ($\theta \neq 0$), if $G(s)$ is a P-matrix for all sufficiently large real s (note that this is the well-posedness condition of Cor. 1), then for sufficiently small time step Δt the problem (80) can be solved for all x and u, and the update rule in (79) is well-defined.

However, this fact by itself is in no way a guarantee that the state path computed by (79–80) will converge to a solution of (2) when the time step Δt tends to zero. For example, consider the linear complementarity system defined by the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = 0 \tag{81}$$

(note that this is a high-index system). Take the initial condition $x_0 = (0, -1, 0)$ and consider the state path obtained by the fully implicit method (take $\theta = 1$ in (79–80)) together with interpolation between the discrete-time state values. It can be verified easily by direct computation (see [9]) that for any fixed $t > 0$ the value of the thus obtained state path at t is unbounded as Δt tends to zero. In particular, the state path does not converge to the Weierstrass solution, which is identically zero for all $t > 0$ following a state jump at $t = 0$.

One therefore has to look for conditions under which convergence of time-stepping schemes will take place. Obviously there are many different situations that one might consider, even within the limited context of Euler schemes with a fixed step size. There are implicitness choices to be made in the algorithm (where to use current values and where to use new values; this also relates to jump rules), and convergence may be considered in various function spaces and in various topologies depending on the problem at hand. Stewart [54] has proved a convergence result for a partly implicit Euler method designed for rigid body problems with impact and friction, and used this to prove existence of solutions to the classical Painlevé problem. There is in fact a large body of work in the modeling of impact and friction in mechanical systems and the numerical solution of the associated evolutionary problems; for a recent survey of various approaches, see [16]. In an infinite-dimensional context (variational inequalities of evolution for equations of parabolic type), convergence results under a coercivity assumption are provided by Glowinski et al. [21, Ch. 6], but this covers only part of the many situations that may be considered with PDEs. For L_2-solutions of linear complementarity systems, one can show the following result concerning the fully implicit Euler scheme (see [9]).

Theorem 7. *Consider the linear complementarity system (2) with initial condition x_0, and suppose that the following conditions hold:*

(i) *the system (2) has a unique L_2-solution with initial condition x_0 on an interval $[0, T]$;*
(ii) *for all sufficiently small $\Delta t > 0$, the problem* LCP($C(I - \Delta t\, A)^{-1}x$,$G(1/\Delta t)$) *is uniquely solvable for all x;*
(iii) *for each k, the successive values of $u(k\Delta t)$ generated by the fully implicit Euler method remain bounded as Δt tends to zero.*

Under these conditions, the state path obtained by constant interpolation from the successive values computed by the implicit Euler method converges in the sense of $L_2(0, T)$ to the unique solution of (2) on $[0, T]$.

Actually in the cited paper there is a stronger version of the above theorem which allows an initial jump; convergence is proved in an appropriately modified (generalized) function space with a correspondingly modified topology. This makes the theorem suited for applications in electrical networks with diodes, where inconsistent initial states can arise when a switch is turned; such inconsistencies can be resolved in a spark, which may be modeled mathematically as a state jump.

Condition (iii) in the above theorem is somewhat unpleasant since it refers to the behavior of the approximating algorithm itself; a natural question is under what conditions one can prove the boundedness property. Sufficient conditions have been provided in [9] on the basis of the so-called *passivity* property. A four-tuple of matrices (A, B, C, D) is said to be passive if there exists a symmetric nonnegative definite matrix K such that

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leq 0. \tag{82}$$

In case there is no feedthrough matrix ($D = 0$), this condition simplifies to: there exists a symmetric nonnegative matrix K such that

$$KB = C^T \quad \text{and} \quad A^T K + KA \leq 0. \tag{83}$$

This is close to the coercivity condition used by Glowinski et al. [21], but a direct comparison is difficult because the inequality constraints are incorporated in the variational inequality formulation of [21] in a different manner than in the formalism of complementarity systems. The notion of passivity is classical in the study of electrical networks, but it also applies to the user/resource model of subsection 2.9; this is seen from the already noted fact that the model (47) is covered by the analysis of [26].

7. Conclusions

By means of a number of examples, it has been shown that complementary systems come up in the consideration of static optimization problems (via the Dupuis-Nagurney imbedding) as well as dynamic ones, in deterministic as well as in stochastic problems, and in finite-horizon as well as in infinite-horizon problems. It has been argued that, although complementarity systems may be seen as special instances of a more general modeling format for nonsmooth systems, their prominence in applications as well as their particular structure warrant a separate study. The nature of complementarity systems as nonsmooth dynamical systems generates many important issues, beginning with the definition of what is to be understood by a “solution.” We have reviewed results concerning basic issues such as existence and uniqueness of solutions in a suitable sense and the construction of numerical approximations methods. Special attention has been paid to the class of linear complementarity systems. While there are other formalisms which can be used to address part of, or more than, the range of problems covered in terms of complementarity systems, the representation in the latter form especially emphasizes connections to mathematical systems theory, thus making available the wealth of results obtained in this field of science over the past decades for purposes of classification and further investigation of the properties of complementarity systems.

Clearly there are some unanswered questions that suggest topics for future research. We do not have a general and well-motivated jump rule for high-index nonlinear complementarity systems, except for mechanical systems where one can project onto the cone of feasible velocities using the metric defined by the kinetic energy. It would be of interest to know to what extent it is possible to formulate a general rule. Another topic of interest is to bring the theories of evolutionary variational inequalities and of complementarity systems closer together, perhaps in a semigroup framework as is available in the case of input/output systems [13]. A largely open issue concerns conditions for the absence of accumulation of events in higher-index systems, or conditions for convergence of state variables when such accumulations do occur. Some results are available for low-index systems (see [25, 11]), but similar conditions are harder to come by in high-index cases. The lack of results in this direction has been an impediment to obtaining conditions for existence of solutions on semi-infinite intervals, even for linear complementarity systems. A further interesting topic for research is the control of systems described by complementarity dynamics. Here one may think for instance of collections of agents that solve optimization problems under possibly time-varying constraints, while the circumstances under which the optimization takes place can be influenced by an external manipulator. Such situations may be considered in economical as well as in ecological studies.

Acknowledgements. The depletion example in subsection 2.8 has come up in discussions with Greg Hunter. I have had the pleasure of exploring many of the ideas on complementarity systems described in this paper over the past years in close collaboration with in particular Kanat Çamlıbel, Maurice Heemels, and Arjan van der Schaft. I would like to thank the reviewers of the present paper for their insightful remarks. This research was supported in part by EU project IST-2001-37172 (SICONOS).

References

1. M. Anitescu and F.A. Potra. Formulating dynamic multi-rigid-body contact problems with friction as solvable Linear Complementarity Problems. *Nonlinear Dynamics*, 14:231–247, 1997.
2. J.P. Aubin and A. Cellina. *Differential Inclusions*. Springer, Berlin, 1984.
3. F. Baccelli, G. Cohen, G.J. Olsder, and J.-P. Quadrat. *Synchronization and Linearity. An Algebra for Discrete Event Systems*. Wiley, New York, 1992.
4. A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. Acad. Press, New York, 1979.
5. S. J. Berridge and J. M. Schumacher. An irregular grid method for high-dimensional free-boundary problems in finance. *Future Generation Computer Systems*, 20:353–362, 2004.
6. W. M. G. van Bokhoven. *Piecewise Linear Modelling and Analysis*. Kluwer, Deventer, the Netherlands, 1981.
7. K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. Classics in Applied Mathematics, 14. SIAM, Philadelphia, 1996. Extended reprint of original edition, 1989.
8. R. W. Brockett. Smooth multimode control systems. In L. Hunt and C. Martin, editors, *Proc. 1983 Berkeley-Ames Conference on Nonlinear Problems in Control and Fluid Mechanics*, pages 103–110, Brookline, 1984. Math. Sci. Press.
9. M. K. Çamlıbel, W. P. M. H. Heemels, and J. M. Schumacher. Consistency of a time-stepping method for a class of piecewise-linear networks. *IEEE Transactions on Circuits and Systems I*, 49:349–357, 2002.
10. M. K. Çamlıbel, W. P. M. H. Heemels, A. J. van der Schaft, and J. M. Schumacher. Switched networks and complementarity. *IEEE Transactions on Circuits and Systems I*, 50:1036–1046, 2003.
11. M. K. Çamlıbel and J. M. Schumacher. On the Zeno behavior of linear complementarity systems. In *Proc. 40th IEEE Conf. Dec. Contr.* (Orlando, Fl., Dec. 2001), pages 346–351. 2001.
12. H. Chen and A. Mandelbaum. Leontief systems, RBV's and RBM's. In M. H. A. Davis and R. J. Elliott, editors, *Applied Stochastic Analysis*, Stochastics Monographs, vol. 5, pages 1–43. Gordon & Breach, New York, 1991.
13. R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer, New York, 1995.
14. P. Dupuis and A. Nagurney. Dynamical systems and variational inequalities. *Annals of Operations Research*, 44:9–42, 1993.
15. G. Duvaut and J. L. Lions. *Inequalities in Mechanics and Physics*. Grundlehren der mathematischen Wissenschaften 219. Springer, Berlin, 1976. Translation of French original, 1972.
16. B. Brogliato (ed.). *Impacts in Mechanical Systems. Analysis and Modelling*. Lecture Notes in Physics, vol. 551. Springer, Berlin, 2000.
17. F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems* (Vol. I, II). Springer, New York, 2003.

18. M. C. Ferris and J.-S. Pang. Engineering and economic applications of complementarity problems. *SIAM Review*, 39:669–713, 1997.
19. A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer, Dordrecht, 1988.
20. F. R. Gantmacher. *The Theory of Matrices (Vol. II)*. Chelsea, New York, 1959.
21. R. Glowinski, J. L. Lions, and R. Trémolières. *Numerical Analysis of Variational Inequalities*. North-Holland, Amsterdam, 1981. Translation of French original, 1976.
22. J. M. Harrison. *Brownian Motion and Stochastic Flow Systems*. Wiley, New York, 1985.
23. J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. *Annals of Probability*, 9:302–308, 1981.
24. R. F. Hartl, S. P. Sethi, and R. G. Vickson. A survey of the maximum principles for optimal control problems with state constraints. *SIAM Review*, 37:181–218, 1995.
25. W. P. M. H. Heemels, M. K. Çamlıbel, and J. M. Schumacher. On the dynamic analysis of piecewise-linear networks. *IEEE Transactions on Circuits and Systems I*, 49:315–327, 2002.
26. W. P. M. H. Heemels, M. K. Çamlıbel, A. J. van der Schaft, and J. M. Schumacher. Modelling, well-posedness, and stability of switched electrical networks. In O. Maler and A. Pnueli, editors, *Hybrid Systems: Computation and Control* (Proc. 6th Int. Workshop, HSCC03, Prague, April 2003), pages 249–266. Springer, Berlin, 2003.
27. W. P. M. H. Heemels, B. De Schutter, and A. Bemporad. Equivalence of hybrid dynamical models. *Automatica*, 37(7):1085–1091, July 2001.
28. W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland. The rational complementarity problem. *Lin. Alg. Appl.*, 294:93–135, 1999.
29. W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland. Linear complementarity systems. *SIAM J. Appl. Math.*, 60:1234–1269, 2000.
30. W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland. Projected dynamical systems in a complementarity formalism. *Operations Research Letters*, 27:83–91, 2000.
31. T. Kailath. *Linear Systems*. Prentice-Hall, Englewood Cliffs, N.J., 1980.
32. S. Karamardian. Generalized complementarity problem. *Journal of Optimization Theory and Applications*, 8:161–167, 1971.
33. C. W. Kilmister and J. E. Reeve. *Rational Mechanics*. Longmans, London, 1966.
34. M. Kuijper and J. M. Schumacher. Input/output structure of differential-algebraic systems. *IEEE Transactions on Automatic Control*, AC-38:404–414, 1993.
35. J. L. Lions and G. Stampacchia. Variational inequalities. *Communications on Pure and Applied Mathematics*, 20:493–519, 1967.
36. P. Lötstedt. Coulomb friction in two-dimensional rigid body systems. *Zeitschrift für Angewandte Mathematik und Mechanik*, 61:605–615, 1981.
37. P. Lötstedt. Mechanical systems of rigid bodies subject to unilateral constraints. *SIAM Journal of Applied Mathematics*, 42:281–296, 1982.
38. J. J. Moreau. Liaisons unilatérales sans frottement et chocs inélastiques. *C. R. Acad. Sc. Paris*, 296:1473–1476, 1983.
39. H. Nijmeijer and J. M. Schumacher. On the inherent integration structure of nonlinear systems. *IMA J. Math. Contr. Inf.*, 2:87–107, 1985.
40. H. Nijmeijer and A. J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer-Verlag, Berlin, 1990.
41. J.-S. Pang and D. E. Stewart. Differential variational inequalities. Preprint, 2003.
42. J. Pérès. *Mécanique Générale*. Masson & Cie., Paris, 1953.
43. F. Pfeiffer and C. Glocker. *Multibody Dynamics with Unilateral Contact*. Wiley, New York, 1996.
44. A. Yu. Pogromsky, W. P. M. H. Heemels, and H. Nijmeijer. On solution concepts and well-posedness of linear relay systems. *Automatica*, 39:2139–2147, 2003.
45. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The Mathematical Theory of Optimal Processes*. Interscience, New York, 1962.
46. M. I. Reiman. Open queueing networks in heavy traffic. *Mathematics of Operations Research*, 9:441–458, 1984.
47. A. J. van der Schaft and J. M. Schumacher. The complementary-slackness class of hybrid systems. *Mathematics of Control, Signals and Systems*, 9:266–301, 1996.
48. A. J. van der Schaft and J. M. Schumacher. Complementarity modeling of hybrid systems. *IEEE Trans. Automat. Contr.*, AC-43:483–490, 1998.
49. A. J. van der Schaft and J. M. Schumacher. *An Introduction to Hybrid Dynamical Systems*. Springer, London, 2000.
50. A. Schrijver. *Theory of Linear and Integer Programming*. Wiley, Chichester, 1986.
51. J. M. Schumacher. State representations of linear systems with output constraints. *Mathematics of Control, Signals and Systems*, 3:61–80, 1990.

52. E. D. Sontag. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Springer Verlag, New York, 1990.
53. D. E. Stewart. A high accuracy method for solving ODEs with discontinuous right-hand side. *Numerische Mathematik*, 58:299–328, 1990.
54. D. E. Stewart. Convergence of a time-stepping scheme for rigid body dynamics and resolution of Painlevé's problem. *Archive for Rational Mechanics and Analysis*, 145:215–260, 1998.
55. H. L. Trentelman, A. A. Stoorvogel, and M. L. J. Hautus. *Control Theory for Linear Systems*. Springer, London, 2001.