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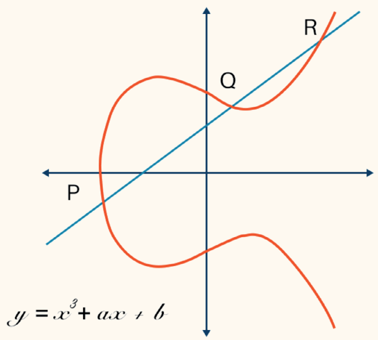

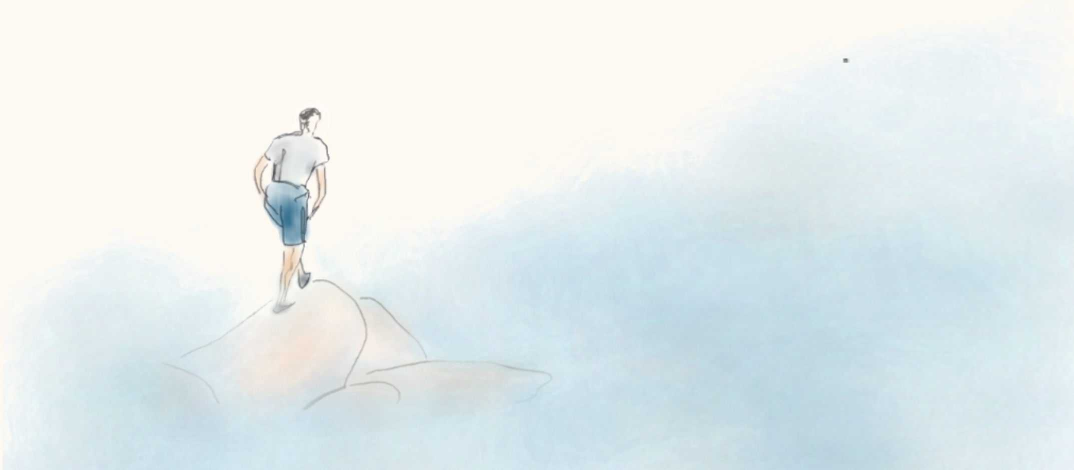

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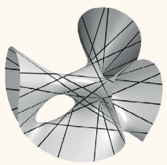

Algebraic Cycles on Cubic Hypersurfaces and Fano Scheme of Lines

ACADEMISCH PROEFSCHRIFT

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Renjie Lyu

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To my parents

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Contents

Chapter 1

Introduction

In algebraic geometry, the study of algebraic cycles is crucial to understand the structure of algebraic varieties. An algebraic cycle of an algebraic variety X is the linear combination of subvarieties of X. The *Chow group* of X is simply the formal abelian group generated by all the algebraic cycles modulo an equivalence relation called rational equivalence.

Let X be an irreducible proper k-variety. The Chow group of zero-cycles on X has a degree map

$$\begin{aligned}\deg : \mathrm{CH}_0(X) &\to \mathbb{Z}\\ \sum_i n_i P_i &\mapsto \sum n_i [k(P_i) : k].\end{aligned}$$

The Chow group $\mathrm{CH}_0(X)$ is called *universally trivial* if for any field extension $k \subset L$, the group $\mathrm{CH}_0(X)$ is isomorphic to $\mathbb{Z}$ via the induced degree map. This universal property on the Chow group of zero-cycles is an important (stable) birational invariant (see [5]) that is extensively used to study (stable) rationality problems, e.g. [7, 19, 56, 65, 72].

In the article [58], Shen introduced the notion called *universal generation*, which can be viewed as a variant of the universal triviality of the CH_0 group. The universal generation concerns the universal property of homomorphisms on Chow groups. Let Y and Z be smooth projective varieties defined over a field k. Let $k(Y)$ be the function field of Y. An algebraic cycle $\gamma \in \mathrm{CH}_r(Z_{k(Y)})$ of dimension r is *universally generating* if a spreading $\Gamma \in \mathrm{CH}_{r+\dim Y}(Z \times Y)$ of γ induces the *universally surjective* homomorphism

$$[\Gamma]_* : \mathrm{CH}_0(Y) \to \mathrm{CH}_r(Z).$$

Namely the induced map $(\Gamma_L)_* : \mathrm{CH}_0(Y_L) \to \mathrm{CH}_r(Z_L)$ is surjective for any field extension $k \subset L$.

In this thesis, we study the universal generation of the *cylinder homomorphism* of cubic hypersurfaces. A cubic hypersurface X is the algebraic variety defined by one homogeneous polynomial of degree 3. The *Fano scheme of lines* on X, denote by $F(X)$, is the parameter space of lines contained in X. More details of the geometry of this parameter space is reviewed in Section 2.1. The universal family of lines is the incidence scheme

$$P := \{([\ell], x) \in F(X) \times X \mid x \in \ell \subset X\} \tag{1.0.1}$$

with the natural projections $p : P \to F(X)$ and $q : P \to X$. The *cylinder homomorphism* associated to the family P (cf. Definition 3.1) is the group homomorphism

$$\Psi_P := q_* p^* : \mathrm{CH}_{*-1}(F(X)) \to \mathrm{CH}_*(X) \tag{1.0.2}$$

that sends any cycle γ to the cycle $[\bigcup_{s\in\gamma} L_s]$ of the union of lines over γ. The following statement shows that the universal family of lines P restricts to a universally generating 1-cycles in $\mathrm{CH}_1(X_{k(F)})$.

Theorem 1.1. *[58, Thm. 1.7] Let X be a smooth cubic hypersurface defined over a field k. Assume the Fano scheme of lines $F(X)$ has a degree one 0-cycle and $\dim X \geq 3$, then the cylinder homomorphism for one-cycles*

$$\Psi_{P_L} : \mathrm{CH}_0(F(X)_L) \to \mathrm{CH}_1(X_L) \tag{1.0.3}$$

is surjective for any field extension $k \subset L$.

For cubic hypersurfaces of low dimensions, this theorem is crucial to study the universal triviality of the CH_0 group. For instance, if X is a smooth complex cubic 4-fold, this theorem implies that $\mathrm{CH}_0(X)$ is universally trivial if and only if the canonical Beauville-Bogomolov bilinear form on $F(X)$ is algebraic, see [58, Thm. 1.4].

Theorem 1.1 motivates the first part of this thesis. Our first main result is the following generalization of Theorem 1.1 for higher dimensional cycles

Theorem 1.2. *(cf. Corollary 3.14) Let X be a smooth cubic hypersurface defined over a field k. Assume that X contains a line defined over k and $\dim X \geq 3$, then the cylinder homomorphism*

$$\Psi_{P_L} : \mathrm{CH}_{r-1}(F(X)_L) \to \mathrm{CH}_r(X_L),\ r \geq 1$$

is surjective for any field extension $k \subset L$ except for the divisor class group of cubic threefolds.

When the field k is algebraically closed, Mboro used a different approach to obtain the same surjectivity result for 2-cycles on X (see [48]). Moreover, he applied it to study the *unramified cohomology* of complex cubic 5-folds. In Section 3.4 we present other applications of our main result to the integral Hodge conjecture (Corollary 3.16) and the integral Tate conjecture (Corollary 3.19).

Another motivation of this thesis is the cylinder homomorphism for topological cycles. Let $k = \mathbb{C}$, using the vanishing cycles, Shimada [60] proved that the topological cylinder homomorphism

$$\Psi_P : H_{n-2}(F(X), \mathbb{Z}) \to H_n(X, \mathbb{Z})$$

is surjective for a smooth cubic X of dimension n. See Theorem 3.1 for the full statement. In particular, when $n = 3$ or 4, the map Ψ_P is even an isomorphism due to Clemens-Griffiths [17] and Beauville-Donagi [8]. The similar conclusion for Chow groups is difficult to prove since the Chow groups are usually too large to work with. Few other examples were studied by Lewis [45, 46]

The main result is a consequence of the following theorem. The key result in the theorem is two key formulae of algebraic cycles related by the cylinder homomorphism. Using the two formulae, one can recover a given cycle on a cubic hypersurface by cycles on its Fano scheme of lines.

Theorem 1.3. *(= Theorem 3.9) Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface defined over a field k, and let $F(X)$ be the Fano scheme of lines. Denote by $H_X \in \mathrm{CH}^1(X)$ the hyperplane section class of X. Assume that X admits a degree one 1-cycle in $\mathrm{CH}_1(X)$. Then for any r-cycle $\Gamma \in \mathrm{CH}_r(X)$ with $r \geq 1$, there exist $(r-1)$-cycles $\gamma, \gamma' \in \mathrm{CH}_{r-1}(F(X))$ satisfying the following*

$$2 \cdot \Gamma + \Psi_P(\gamma) \in \mathbb{Z} \cdot H_X^{n-r}; \tag{1.0.4}$$
$$(2\deg\Gamma - 3) \cdot \Gamma + \Psi_P(\gamma') \in \mathbb{Z} \cdot H_X^{n-r}, \tag{1.0.5}$$

where $\deg \Gamma$ is defined to be the intersection number $\Gamma \cdot H_X^r$.

The construction of the key formulae follows the method in [57] and [58] to establish the similar formulae for one-cycles. The method of the construction builds upon a key result of Voisin in [73] (see Theorem 3.3) that concerns a crucial birational map

$$\Phi : X^{[2]} \dashrightarrow \mathbb{P}(T_{\mathbb{P}^{n+1}}|_X) \tag{1.0.6}$$

initiated by Galkin-Shinder [28]. Here $X^{[2]}$ is the second punctual Hilbert scheme of the cubic hypersurface X, and $T_{\mathbb{P}^{n+1}}$ denotes the tangent bundle of $\mathbb{P}^{n+1}$. This birational map will be reviewed in details in Section 2.4.

We notice that Mboro [48] also use the same idea to prove the formula (1.0.5) for $\Gamma \subset X$ being a smooth subvariety in general position. Our proof in Theorem 1.3 could circumvent this technical constraint.

Galkin and Shinder showed that the birational map (1.0.6) gives rise to a remarkable relation

$$\mathbb{L}^2 \cdot [F(X)] + [X] \cdot [\mathbb{P}^n] = [X^{[2]}], \text{ see } 2.10 \tag{1.0.7}$$

in the Grothendieck ring of varieties. They applied this relation to study the rationality problem of cubic hypersurfaces, see [28]. In addition, by the relation one can read off the topological and geometric information(e.g. Euler numbers, Hodge structures) of the Fano scheme of lines(cf. Theorem 4.13). Inspired by the relation (1.0.7), Laterveer proved the following motivic counterparts.

Theorem 1.4. *[42, Thm. 5] Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. Let $F(X)$ be the Fano scheme of lines on X, and let $X^{[2]}$ be the second punctual Hilbert scheme of X. There is an isomorphism of Chow motives*

$$\mathfrak{h}(F(X))(-2) \oplus \bigoplus_{i=0}^{n} \mathfrak{h}(X)(-i) \simeq \mathfrak{h}(X^{[2]}), \tag{1.0.8}$$

This isomorphism shows that one can characterize the motives of the Fano scheme of lines by the motives of the cubic hypersurface. For example, Laterveer applied it to study the Kimura finite-dimensionality for the motives of the Fano scheme of lines.

Although morphisms in the category of motives are a priori with rational coefficients, we speculate that the formula (1.0.8) turns out to be an integral isomorphism. In Chapter 4, we confirm our speculation by presenting a formula of decomposition of the diagonal of $X^{[2]}$. We need some notations to state the result. Let us view the graph closure

$$\Phi_* := [\overline{\Gamma}_\Phi] \in \mathrm{CH}^{2n}(X^{[2]} \times \mathbb{P}(T_{\mathbb{P}^{n+1}}|_X))$$

of the birational map (1.0.6) as an algebraic correspondence of degree 0 from $X^{[2]}$ to $\mathbb{P}(T_{\mathbb{P}^{n+1}}|_X)$. Let $P_2 \subset X^{[2]}$ be the indeterminacy locus of the birational map Φ. It is an algebraic correspondence of degree -2 from $X^{[2]}$ to $F(X)$. The main statement is the following

Theorem 1.5. *(cf. Theorem 4.12) Use the notations above. Let $\Phi^* := [\overline{\Gamma}_{\Phi^{-1}}]$ be the transpose of Φ_*, and Let ${}^\top[P_2]$ be the transpose of $[P_2]$. Then there is a decomposition of the diagonal*

$$\Delta_{X^{[2]}} = \Phi^* \circ \Phi_* + {}^\top[P_2] \circ [P_2] \tag{1.0.9}$$

in $\mathrm{CH}^{2n}(X^{[2]} \times X^{[2]})$.

The formula can reprove the isomorphism (1.0.8). In addition, the decomposition formula has integral coefficients. Hence the information of the integral Chow and cohomology groups of the Fano scheme of lines is enclosed by the isomorphism, e.g. see Corollary 4.14.

L. Fu pointed out that the birational map (1.0.6) turns out to be a *standard flip* in the sense of birational geometry. The main result of Section 4.2 indeed verify this claim, see also [11, §4]. Under this general setting, the integral isomorphism of (1.0.7) is a consequence that follows from the work [39, §3.2].

Notation and conventions:

i Let E be a vector bundle over a scheme X. By the locally free sheaf $\mathscr{E}$ associated to E we mean the dual of the sheaf of sections of E. For the projective bundle $\mathbb{P}(E)$ of E, we adopt the definition

$$\mathbb{P}(E) = \mathbb{P}(\mathscr{E}) := \mathrm{Proj}(\mathrm{Sym}^{\bullet} \mathscr{E}).$$

ii Let $N_{X/Y}$ be the normal bundle of a regular embedding $X \hookrightarrow Y$ of schemes. The normal sheaf of X in Y, denoted by $\mathcal{N}_{X/Y}$, is the sheaf of section of the normal bundle $N_{X/Y}$. We write $\mathbb{P}(N)$ for the projective normal bundle $\mathbb{P}(\mathcal{N}_{X/Y}^{\vee})$

Chapter 2

Preliminaries

2.1 Fano scheme of cubic hypersurfaces

In this section, we recall some foundations on the Fano scheme of lines of a cubic hypersurface. These results can be found in [2] and [20].

Let k be a field, and let V be a k-linear space of dimension $n+1$. Denote by

$$G := \mathbf{Gr}(r+1, V)$$

the Grassmannian of $(r+1)$-dimensional subspaces of V. Very often we use the notation $\mathbf{Gr}(r+1, n+1)$ if there is no need to specify the underlying V. Let V_G denote the trivial vector bundle $V \times_k G$ over G. By the universal property of Grassmannians, there exists a rank $r+1$ tautological subbundle $\mathcal{S} \subset V_G$ of $(r+1)$-dimensional subspaces of V. Let

$$0 \to \mathcal{S} \to V_G \to \mathcal{Q} \to 0. \tag{2.1.1}$$

be the fundamental sequence of vector bundles. The Plücker embedding of G into the projective space $\mathbb{P}(\wedge^{r+1} V)$ is obtained via the canonical epimorphism of $\mathcal{O}_G$-modules

$$\wedge^{r+1} V^\vee \otimes \mathcal{O}_G \to \wedge^{r+1} \mathcal{S}^\vee.$$

We denote by

$$\mathcal{O}_G(1) := \wedge^{r+1} \mathcal{S}^\vee$$

the Plücker polarization of G.

Definition 2.1. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d over the field k. The *Fano scheme of lines* $F(X)$ of X is the k-scheme parameterizing lines of $\mathbb{P}^n$ that contained in X.

Remark 1. The Hilbert polynomial of an r-dimensional subspaces of $\mathbb{P}^n$ contained in X is $P(t) = \binom{t+r}{r}$. Hence the Fano scheme of lines on X can be regarded as the Hilbert scheme of closed subschemes of X with the Hilbert polynomial $P(t) = t+1$.

Let $\mathbb{P}^N := \mathbb{P}(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^\vee)$ be the parameter space of degree d hypersurfaces in the projective space $\mathbb{P}^n$. The *universal family of Fano scheme of lines* over $\mathbb{P}^N$ is the incidence subscheme

$$\Phi := \{(X, [L]) \in \mathbb{P}^N \times \mathbf{Gr}(2, n+1) \mid L \subset X\}.$$

Let $\varphi : \Phi \to \mathbb{P}^N$ and $\gamma : \Phi \to \mathbf{Gr}(2, n+1)$ be natural projections. The fibre of φ at a hypersurface $[X] \in \mathbb{P}^N$ is the Fano scheme of lines on X. The fibre of γ at a line $[L] \in \mathbf{Gr}(2, n+1)$ consists of the space

$$\mathrm{Ker}(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to \Gamma(L, \mathcal{O}_L(d)))$$

of homogeneous polynomials that vanish along L. Hence the universal family of Fano scheme Φ is smooth of dimension $N + 2n - d - 3$.

By the fibre dimension theorem (see [36, §II, Exe. 3.22]), the Fano scheme of lines of a hypersurface of degree d has expected dimension $2n - d - 3$. Moreover, the locus of Fano scheme of lines having expected dimension is Zariski open in $\mathbb{P}^N$. There exists smooth hypersurface of large degree such that the dimension of the Fano scheme of lines exceeds $2n - d - 3$. For example, the Fano scheme of lines of the Fermat hypersurface

$$x_0^d + \cdots + x_n^d = 0$$

has dimension $n - 3$ if $d \geq n$, see [20, 2.5]. For more surveys on the topics of lines on hypersurfaces, we refer the interested readers to [10]. Nevertheless, for low degree hypersurfaces, it is conjectured that

Conjecture 2.1 (Debarre-de Jong)**.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \leq n$. Then the dimension of the Fano scheme of lines on X equals $2n - d - 3$.

In particular, the conjecture is true for $d \leq 3$.

Lemma 2.1. *The Fano scheme of lines of any smooth hypersurface of degree $d \leq 3$ is smooth of dimension $2n - 3 - d$.*

Proof. Let X be a smooth hypersurface X of degree $d \leq 3$. Suppose that $[L]$ is any closed point of the Fano scheme F of X. It suffices to show that the tangent space $T_{[L]}F$ has dimension $2n-d-3$. Let $\mathcal{I}$ be the ideal sheaf of L in X. By deformation theory (see [25, Corollary 6.4.11]), the first-order deformation of the Hilbert scheme $F(X)$ at $[L]$ is

$$T_{[L]}F(X) \cong \mathrm{Hom}_L(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_L) = H^0(L, N_{L/X})$$

By Grothendieck splitting theorem [33], the normal bundle $N_{L/X}$ is isomorphic to a direct sum

$$\bigoplus_{i=1}^{n-2} \mathcal{O}_L(a_i).$$

The numerical conditions on the integers $\{a_i\}$ are determined by the following exact sequence of normal bundles

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^n} \to N_{X/\mathbb{P}^n}|_L \to 0.$$

The normal bundle $N_{L/\mathbb{P}^n}$ is isomorphic to $\bigoplus_{i=1}^{n-1} \mathcal{O}_L(1)$. Hence the integers $\{a_i\}$ satisfies $a_i \leq 1$ for all i. Moreover, the degree of $N_{L/X}$ is

$$\sum_{i=1}^{n-2} a_i = \deg N_{L/\mathbb{P}^n} - \deg N_{X/\mathbb{P}^n}|_L = n - d - 1.$$

It implies $a_i \geq -1$ for all i since $d \leq 3$. The conditions on $\{a_i\}$ shows that

$$H^1(L, N_{L/X}) = 0.$$

Therefore, the dimension of the tangent space $H^0(L, N_{L/X})$ equals

$$\begin{aligned}\chi(N_{L/X}) &= \chi(N_{L/\mathbb{P}^n}) - \chi(N_{X/\mathbb{P}^n}|_L) \\ &= \chi(\mathcal{O}_L(1)^{\oplus n-1}) - \chi(\mathcal{O}_L(3)) \\ &= 2n - d - 3.\end{aligned}$$

□

It is useful to interpret the Fano scheme of lines as a *local complete intersection* of the Grassmannian of lines as follows.

Proposition 2.2. *Let $X \subset \mathbb{P}^n$ be a smooth cubic hypersurface, and let $G := \mathbf{Gr}(2, n+1)$ be the Grassmannian of lines On $\mathbb{P}^n$. Denote by $\mathcal{S}$ the rank 2 tautological vector bundle over G. The Fano scheme of lines $F(X)$ is the zeros of a regular section of the vector bundle $\operatorname{Sym}^3(\mathcal{S}^\vee)$. In particular, the normal bundle $N(F, G)$ is isomorphic to $\operatorname{Sym}^3(\mathcal{S}^\vee)|_F$*

Proof. The fundamental sequence induces a surjective map $\operatorname{Sym}^3 V_G^\vee \twoheadrightarrow \operatorname{Sym}^3 \mathcal{S}^\vee$. The cubic equation $f \in \operatorname{Sym}^3 V^\vee$ defining X induces a section s of $\operatorname{Sym}^3 \mathcal{S}^\vee$ via the surjection. The zero set of s is exactly the lines contained in L. The regularity of s essentially follows from $\dim F(X) = 2(n-3)$, see [2] for the details. □

Corollary 2.3. *Let F be the Fano scheme of lines of a smooth cubic hypersurface of dimension $n-1$. Denote by $\mathcal{O}_F(1)$ the induced Plücker polarization. The canonical sheaf of F is*

$$\omega_F \cong \mathcal{O}_F(5-n).$$

Proof. The natural embedding $F \hookrightarrow G$ yields the canonical exact sequence of the normal bundle

$$0 \to T_F \to T_G|_F \to N(F, G) \to 0.$$

The adjunction formula shows

$$\omega_F \cong \omega_G|_F \otimes \wedge^4 N(F, G).$$

Recall that $\mathcal{S}$ is the tautological bundle of G and $\mathcal{Q}$ the quotient bundle in the fundamental sequence (2.1.1). The tangent bundle of G is isomorphic to $\mathcal{S}^\vee \otimes \mathcal{Q}$. Therefore, the canonical sheaf is given by

$$\omega_G \cong \wedge^{2n-2} T_G^\vee \cong (\wedge^2 \mathcal{S})^{\otimes n-1} \otimes (\wedge^{n-1} \mathcal{Q}^\vee)^{\otimes 2}.$$

It again follows from the fundamental sequence (2.1.1) that $\wedge^2 \mathcal{S} \cong \wedge^{n-1} \mathcal{Q}^\vee$, which implies

$$\omega_G \cong \wedge^2 \mathcal{S}^{\otimes n+1} \cong \mathcal{O}_G(-n-1).$$

Through the previous proposition we know the normal bundle $N(F, G)$ is isomorphic to $\operatorname{Sym}^3(\mathcal{S}^\vee|_F)$. Therefore we obtain

$$\omega_F \cong \mathcal{O}_G(-n-1)|_F \otimes (\wedge^2 \mathcal{S}^\vee|_F)^{\otimes 6} \cong \mathcal{O}_F(5-n).$$

□

Example 1. 1. A non-singular cubic surface over an algebraically closed field contains exactly 27 distinct lines. This classical result is important in enumerative geometry.

2. Suppose X is a smooth cubic threefold. The Fano scheme of lines of X is a surface of general type, which was first studied by G.Fano [24].

3. When X is smooth complex cubic fourfold, Beauvillea and Donagi [8] showed that $F(X)$ is a hyper-Kähler manifold that is deformation equivalent to $S^{[2]}$ for some $K3$ surface S.

4. For $n \geq 5$, the $F(X)$ of a cubic X of dimension n has anti-ample canonical divisor. If the field k is algebraically closed, the Fano scheme of lines is a rationally connected variety (see [41, §IV]).

2.2 Algebraic cycles and motives

2.2.1 Algebraic cycles and Chow groups

Let X be a variety over k of dimension d. An *algebraic i-cycle* a finite linear combination

$$Z := \sum_{\alpha} n_\alpha Z_\alpha, n_\alpha \in \mathbb{Z}.$$

of irreducible closed subvarieties Z_α of dimension i. The group of i-cycles is the free abelian group $\mathcal{Z}_i(X)$ generated by i-cycles.

Rational equivalence. Let W be an $i+1$-dimensional subvariety of X, and let $\varphi \in k(W)^*$ be a non-zero rational function. One defines the i-cycle

$$\mathrm{div}(\varphi) := \sum_{V \subset W} \mathrm{ord}_V(\varphi)[V]$$

where V runs over all codimension one subvarieties of W. We say an i-cycle Z on X is rationally equivalent to zero, denoted by $Z \sim 0$, if there is finitely many $(i+1)$-dimensional subvarities Y_α of X with $\varphi_\alpha \in k(Y_\alpha)^*$ such that

$$Z = \sum_{\alpha} div(\varphi_\alpha).$$

All the algebraic i-cycles that rationally equivalent to zero generates a subgroup $Z_i^{rat}(X)$ of $Z_i(X)$.

Definition 2.2. The *Chow group* of algebraic i-cycles on X is defined to be the quotient group

$$\mathrm{CH}_i(X) := Z_i(X)/Z_i^{rat}(X).$$

Very often it is convenient to employ the notation $\mathrm{CH}^{d-i}(X) := \mathrm{CH}_i(X)$ indexed by the codimension of cycles.

Proper push-forward. Let $f : X \to Y$ be a proper morphism. Let $V \subset X$ be an r-dimensional irreducible closed subvariety. The image $W = f(V)$ is irreducible and closed. The function field $k(V)$ is a finite extension of $k(W)$ if V and W have the same dimension. Then define

$$f_*[V] := \begin{cases} [k(V) : k(W)][W], & \text{if } \dim W = r, \\ 0, & \text{if } \dim W < r. \end{cases}$$

This extend linearly to a group homomorphism $f_* : \mathscr{Z}_r(X) \to \mathscr{Z}_r(Y)$. One can further show f_* preserves the rational equivalence (see [27, Prop. 1.4]). Hence it descends to a homomorphism

$$f_* : \mathrm{CH}_r(X) \to \mathrm{CH}_r(Y)$$

on the Chow groups.

Flat pull-back. Let $f : X \to Y$ be a flat morphism. Let V be a subvariety of Y of codimension r. The scheme-theoretic inverse image $f^{-1}(V)$ is of pure codimension r in X. We set

$$f^*[V] = [f^{-1}(V)],$$

which extend linearly to a group homomorphism $f^* : \mathscr{Z}^r(Y) \to \mathscr{Z}^r(X)$. It is known that f^* also descends to a homomorphism

$$f^* : \mathrm{CH}^r(X) \to \mathrm{CH}^r(Y),$$

see [27, Thm. 1.7] for the proof.

Intersection product. Suppose X is a smooth variety. Let V, W be subvarieties of X. Since X is smooth the diagonal embedding

$$\Delta : X \hookrightarrow X \times X$$

is regular. Then the *refined Gysin homomorphism* (cf. Appendix A.2)

$$\Delta^* : \mathrm{CH}^d(X \times X) \to \mathrm{CH}^d(X)$$

in the sense of Fulton's intersection theory [27] defines the intersection product

$$[Y] \cdot [Z] = \Delta^*[Y \times Z].$$

In this manner the Chow group of X is equipped with a well-defined intersection product

$$\begin{aligned} \mathrm{CH}^i(X) \otimes \mathrm{CH}^j(X) &\to \mathrm{CH}^{i+j}(X \times X) \to \mathrm{CH}^{i+j}(X) \\ ([V], [W]) &\mapsto [V \times W] \mapsto V \cdot W. \end{aligned}$$

2.2.2 Chow motives

In this section we review the constructions and other fundamental concepts of pure Chow motives.

The construction of pure motives is divided into three steps. It starts with the category $\mathscr{V}_k$ of smooth projective k-varieties. One first enlarges $\mathscr{V}_k$ to a pre-additive category $C_\sim \mathscr{V}_k$ by admitting *algebraic correspondences* as morphisms. Then taking the Karoubian of $C_\sim \mathscr{V}_k$ leads to the category of effective motives $\mathrm{Mot}^{\mathrm{eff}}(k)$. At last, by twisting the morphisms in $\mathrm{Mot}^{\mathrm{eff}}(k)$, one obtain the category of pure motives. The outline of the construction in this section basically follows from [52, §2].

Correspondences. Assume X is of pure dimension d. The group of *correspondences* of degree r from X to Y is

$$\mathrm{Corr}^r(X, Y) := \mathrm{CH}^{d+r}(X \times Y) \otimes \mathbb{Q}.$$

The composition of correspondences is defined by

$$\begin{aligned} \mathrm{Corr}^r(X, Y) \times \mathrm{Corr}^s(Y, Z) &\to \mathrm{Corr}^{r+s}(X, Z) \\ (f, g) &\mapsto g \circ f := p_{XZ*}(p_{XY}^* f \cdot p_{YZ}^* g) \end{aligned}$$

where pr_{XY} indicates the projection $pr_{XY} : X \times Y \times Z \to X \times Y$. Any correspondence $\Gamma \in \mathrm{Corr}^r(X, Y)$ of degree r induces a group homomorphism on Chow groups

$$\Gamma_* : \mathrm{CH}^i(X)_{\mathbb{Q}} \to \mathrm{CH}^{i+r}(Y)_{\mathbb{Q}}$$
$$\alpha \mapsto \Gamma_*(\alpha) := pr_{Y*}(pr_X^* \alpha \cdot \Gamma)$$

Assume Y is of pure dimension e. The transpose ${}^{\top}\Gamma \in \mathrm{Corr}^{r+d-e}(Y, X)$ of $\Gamma \in \mathrm{Corr}^r(X, Y)$ represents the permutation of Γ in $Y \times X$. Denote by Γ^* the group homomorphism of the transpose

$$\Gamma^* := {}^{\top}\Gamma_* : \mathrm{CH}^i(Y)_{\mathbb{Q}} \to \mathrm{CH}^{i+r+d-e}(X)_{\mathbb{Q}}$$
$$\beta \mapsto \Gamma^*(\beta) := pr_{X*}(pr_Y^* \beta \cdot {}^{\top}\Gamma).$$

In the thesis the following Lieberman's lemma will be applied in many places.

Lemma 2.4 (Lieberman). *Let $f \in \mathrm{Corr}(X, Y)$, $\alpha \in \mathrm{Corr}(X, X')$ and $\beta \in \mathrm{Corr}(Y, Y')$. Then we have*

$$(\alpha \times \beta)_*(f) = \beta \circ f \circ {}^{\top}\alpha.$$

Here $\alpha \times \beta$ is a correspondence in $\mathrm{Corr}(X \times Y, X' \times Y')$. Similarly, if $g \in \mathrm{Corr}(Y, X)$, for the other direction there is

$$(\alpha \times \beta)^*(g) = {}^{\top}\beta \circ g \circ \alpha.$$

Proof. See [40, p.73]. □

Effective motives. We say a correspondence $p \in \mathrm{Corr}^0(X, X)$ is a projector of X if $p \circ p = p$. The category of effective Chow motive $\mathrm{Mot}^{\mathrm{eff}}(k)$ consists of

- pairs (X, p) with $X \in \mathscr{V}_k$ and p is a projector;

- the morphisms $f : (X, p) \to (Y, q)$ that has the form:

$$\{f = q \circ f' \circ p \mid f' \in \mathrm{Corr}^0(X, Y)\}.$$

Note that an equivalent way to describe the morphism is that they are the degree 0 correspondences f satisfying $p \circ f = f = f \circ q$.

To any smooth projective k-variety X, one can associate a natural effective motive

$$\mathfrak{h}(X) := (X, \Delta_X),$$

where Δ_X is the correspondence of the diagonal. Let $\phi : X \to Y$ be any morphism of smooth projective varieties with $\dim X = d$ and $\dim Y = e$. The graph Γ_ϕ of ϕ gives rise to correspondences

$$\phi_* := \Gamma_\phi \in \mathrm{Corr}^{e-d}(X, Y);$$
$$\phi^* := {}^{\top}\Gamma_\phi \in \mathrm{Corr}^0(Y, X).$$

In this way there is the contravariant functor $\mathfrak{h} : \mathscr{V}_k \to \mathrm{Mot}^{\mathrm{eff}}(k)$ given by

$$X \mapsto \mathfrak{h}(X)$$
$$\phi \in \mathrm{Hom}_{\mathscr{V}_k}(X, Y) \mapsto \phi^* \in \mathrm{Hom}_{\mathrm{Mot}}(\mathfrak{h}(Y), \mathfrak{h}(X)).$$

The category $\mathrm{Mot}^{\mathrm{eff}}(k)$ is a *pseudo-abelian*. By pseudo-abelian we mean every projector has a kernel. In fact, given any projector q on (X, p), one can show the kernel of q is the effective Chow motive $(X, p - q)$. Meanwhile, the image of q is (X, q). It turns out that (X, p) splits into a direct sum (see below for the definition)

$$(X, p) \cong (X, p - q) \oplus (X, q).$$

Pure Chow motives. A pure Chow motive is a triple (X, p, m) where (X, p) is an effective motive and m is an integer. A morphism $f : (X, p, m) \to (Y, q, n)$ is a degree $n - m$ correspondence $f \in \mathrm{Corr}^{n-m}(X, Y)$ satisfies $f = f \circ p = q \circ f$.

1. *Tensor products.* If $M := (X, p, m)$ and $N := (Y, q, n)$ are two motives, the tensor product of M and N is

$$M \otimes N := (X \times Y, p \times q, m + n).$$

2. *Direct sums.* Suppose $m = n$. Then the sum of motives (X, p, m) and (Y, q, m) is

$$(X, p, m) \oplus (Y, q, m) = (X \sqcup Y, p + q, m).$$

3. *Tate motive and Lefschetz motive.* Let e be a k-point of $\mathbb{P}^1$. The Lefschetz motive is

$$\mathbf{Q}(-1) := (\mathbb{P}^1, \mathbb{P}^1 \times e, 0). \tag{2.2.1}$$

The dual of Lefschetz motive is the Tate motive

$$\mathbf{Q}(1) := (\mathrm{Spec}(k), \mathrm{id}, 1), \tag{2.2.2}$$

where id denotes the trivial diagonal of a single point.

4. *Chow group and cohomology of motives.* Let $M = (X, p, m)$ be a pure Chow motive, the i-th Chow group of M is defined to be

$$\mathrm{CH}^i(M) := \mathrm{im}(p_* : \mathrm{CH}^{i+m}(X)_{\mathbb{Q}} \to \mathrm{CH}^{i+m}(X)_{\mathbb{Q}}).$$

Suppose that H is any Weil-cohomology theory, e.g. Betti, de-Rham, étale, the i-th cohomology group $H^i(M)$ is

$$H^i(M) := \mathrm{im}(p_* : H^{i+2m}(X) \to H^{i+2m}(X)).$$

Remark 2. The Lefschetz motive plays a role as the top degree cohomology of any smooth projective variety. First of all, it is easy to see

$$H^i(\mathbf{Q}(-1)) = \begin{cases} 0, & i \neq 2 \\ H^0(e)(-1), & i = 2. \end{cases}$$

Moreover, $\mathbf{Q}(-1)$ is isomorphic to the motive $(\mathrm{Spec}(k), id, -1)$. Then for any $X \in \mathscr{V}_k$ of dimension d, we have

$$\mathbf{Q}(-d) := \mathbf{Q}(-1)^{\otimes d} \cong (\mathrm{Spec}(k), id, -d) \cong (X, X \times e, 0).$$

The motive $(X, X \times e, 0)$ controls the piece of the top degree cohomology of X.

Lemma 2.5. *Let $M := (X, p, m)$ and $N := (Y, q, n)$ be Chow motives. Suppose the homomorphisms $\alpha : M \to N$ and $\beta : N \to M$ satisfy $\alpha \circ \beta = \mathrm{id}_N = q$. Then $\beta \circ \alpha$ is a projector of M, and N is isomorphic to the Chow motive $(X, \beta \circ \alpha, m)$. In particular, M splits into a direct sum*

$$N \oplus (X, p - \beta \circ \alpha, m) \cong M. \tag{2.2.3}$$

Proof. The correspondence $\beta \circ \alpha$ is a projector because

$$\beta \circ \alpha \circ \beta \circ \alpha = \beta \circ q \circ \alpha = \beta \circ \alpha.$$

Under the hypothesis, the correspondence β satisfies $\beta \circ \alpha \circ \beta = \beta \circ q = \beta$. Hence it induces an homomorphism $\beta : N \to (X, \beta \circ \alpha, m)$ with the natural inverse $\alpha : (X, \beta \circ \alpha, m) \to N$. □

Example 2. Assume $f : X \to Y$ is a generically finite morphism for $X, Y \in \mathscr{V}_k$. By Lieberman's Lemma 2.4

$$f_* \circ f^* = (f \times f)_*(\Delta_X) = \deg f \cdot \Delta_Y.$$

Apply the above lemma to homomorphisms $\frac{1}{\deg f} \cdot f_* : \mathfrak{h}(X) \to \mathfrak{h}(Y)$ and $f^* : \mathfrak{h}(Y) \to \mathfrak{h}(X)$, we obtain $\mathfrak{h}(Y) \cong (X, \frac{1}{\deg f} \cdot f^* \circ f_*, 0)$ and

$$\mathfrak{h}(Y) \oplus (X, \Delta_X - \frac{1}{\deg f} \cdot f^* \circ f_*, 0) \cong \mathfrak{h}(X).$$

Manin's identity principle

Suppose that a morphism $f : M \to N$ of Chow motives is zero, the induced homomorphisms $\mathrm{ch}(f)$ on Chow groups must be zero. However, the converse is not true. For example, let C be a general smooth projective curve of genus ≥ 3. Suppose $x, y \in C$ are distinct closed points. Then the 0-cycle

$$(x, x) - (x, y) - (y, x) + (y, y)$$

acts trivially on $\mathrm{CH}^*(C)$ though it is not trivial in $\mathrm{CH}_0(C \times C)$, see [9, Proposition 4.2]. However, if we require the T-valued homomorphism

$$\mathrm{ch}(f_T) : \mathrm{CH}(T \times X) \to \mathrm{CH}(T \times Y)$$

is zero for any $T \in \mathscr{V}_k$, then the morphism f is zero. It is simply because we can take T to be X then it yields

$$0 = \mathrm{ch}(f_X)(\Delta_X) = f$$

as correspondences. This is the so-called Manin's identity principle:

Theorem 2.6. *[47, p. 450] Let $f, g \in \mathrm{Corr}(X, Y)$ for $X, Y \in \mathscr{V}_k$. Then the following are equivalent:*

1. *$f = g$;*
2. *$\mathrm{ch}(f_T) = \mathrm{ch}(g_T)$ for all $T \in \mathscr{V}_k$;*
3. *$\mathrm{ch}(f_X) = \mathrm{ch}(g_X)$.*

Example 3. 1. *Projective bundle.* Let $\mathscr{E}$ be a locally free sheaf of rank $r+1$ over $S \in \mathscr{V}_k$. The motive of the projective bundle $\pi : \mathbb{P}(\mathscr{E}) \to S$ is isomorphic to

$$\mathfrak{h}(\mathbb{P}(\mathscr{E})) \cong \bigoplus_{i=0}^{r} \mathfrak{h}(S) \otimes \mathbf{Q}(-i) \tag{2.2.4}$$

Denote by $\xi \in \mathrm{CH}^1(\mathbb{P}(\mathscr{E}))$ the first Chern class of the canonical line bundle $\mathcal{O}_{\mathscr{E}}(1)$ on $\mathbb{P}(\mathscr{E})$. The Chow ring $\mathrm{CH}^*(\mathbb{P}(\mathscr{E}))$ is a free $\mathrm{CH}^*(S)$-module with basis $\{1, \xi, \ldots, \xi^r\}$. The isomorphism

$$\sum_{i=0}^{r} \xi^i \cdot \pi^*(\_) : \bigoplus_{i=0}^{r} \mathrm{CH}^*(S)[\xi^i] \xrightarrow{\sim} \mathrm{CH}^*(\mathbb{P}(\mathscr{E})) \tag{2.2.5}$$

is given by correspondences $\Delta_{\mathbb{P}(\mathscr{E})_*}\xi^i \circ {}^{\top}\Gamma_\pi \in \mathrm{Corr}^i(S, \mathbb{P}(\mathscr{E}))$ on the i-th summand. The isomorphism (2.2.5) universally holds under any base change. By the Manin's identity principle, this yields an isomorphism of Chow motives

$$\sum_{i=0}^{r} \Delta_{\mathbb{P}(\mathscr{E})_*}\xi^i \circ \pi^* : \bigoplus_{i=0}^{r} \mathfrak{h}(S) \otimes \mathbf{Q}(-i) \xrightarrow{\sim} \mathfrak{h}(\mathbb{P}(\mathscr{E}))$$

2. *Blowup.* Let X be a smooth projective variety and $S \subset X$ a smooth closed subvariety of codiemnsion c. Let $\widetilde{X} := \mathrm{Bl}_S X$ be the blow-up of X along the smooth center S. Consider the blow-up diagram

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "g"] & \widetilde{X} \arrow[d, "f"] \\ S \arrow[r, hook, "i"] & X \end{tikzcd}$$

with the exceptional divisor E. There is an isomorphism of motives

$$\mathfrak{h}(\widetilde{X}) \cong \mathfrak{h}(X) \oplus \bigoplus_{i=1}^{c-1} \mathfrak{h}(S) \otimes \mathbf{Q}(-i). \tag{2.2.6}$$

Let $N_{S/X}$ be the normal bundle of S in X. The exceptional divisor E is isomorphic to the projective bundle $\mathbb{P}(N_{S/X})$. Denote by ξ the first Chern class of the canonical line bundle $\mathcal{O}_E(1)$ on E. By the localization exact sequence, the Chow group of $\widetilde{X}$ is isomorphic to

$$f^*(\_) + \sum_{i=0}^{c-2} j_*(\xi^i \cdot g^*(\_)) : \mathrm{CH}^k(X) \oplus \bigoplus_{i=0}^{c-2} \mathrm{CH}^{k-i-1}(S)[\xi^i] \xrightarrow{\sim} \mathrm{CH}^k(\widetilde{X})$$

The homomorphism $j_*(\xi^{i-1} \cdot g^*(\_))$ on Chow groups is induced by the correspondence

$$j_* \circ (\Delta_E)_* \xi^{i-1} \circ g^* : \mathfrak{h}(S) \otimes \mathbf{Q}(-i) \to \mathfrak{h}(\widetilde{X}).$$

The format of the blow-up diagram remains the same after base change. Therefore Manin's principle identity implies the isomorphism of Chow motives

$$f^* + \sum_{i=1}^{c-1} j_* \circ (\Delta_E)_* \xi^{i-1} \circ g^* : \mathfrak{h}(X) \oplus \bigoplus_{i=1}^{c-1} \mathfrak{h}(S) \otimes \mathbf{Q}(-i) \xrightarrow{\sim} \mathfrak{h}(\widetilde{X}).$$

2.3 Hodge Conjectures and Tate conjectures

2.3.1 Hodge structures

Definition 2.3. Let R be a subring of real number $\mathbb{R}$, and let V_R be an R-module of finite type. *An (pure) R-Hodge structure of weight k* on V_R is a decomposition

$$V_R \otimes_R \mathbb{C} = \bigoplus_{p+q=k} V_{\mathbb{C}}^{p,q}$$

of complex subspaces such that $V_{\mathbb{C}}^{p,q} = \overline{V_{\mathbb{C}}^{q,p}}$ for all p, q.

Let V_R and W_R be two R-Hodge structures of weight k. *A morphism of Hodge structures* is an R-module homomorphism

$$f : V_R \to W_R$$

that sends $V^{p,q}$ to $W^{p,q}$.

More generally, we call V_R carries a graded R-Hodge structure if V_R is a finite direct sum $\bigoplus_{k\in\mathbb{Z}} V_R^{(k)}$, where $V_R^{(k)}$ carries a pure R-Hodge structure of weight k

Example 4 (Tate twist). The Tate twist $R(k)$ is given by the underlying rank one R-module

$$R(k) := (2\pi i)^k R \subset \mathbb{C},$$

with the Hodge structure of type $(-k, -k)$, namely

$$R(k) \otimes_R \mathbb{C} = (R(k) \otimes_R \mathbb{C})^{-k,-k}.$$

The k-th Tate twist of an R-Hodge structure V_R is defined to be $V(k) := V \otimes_R R(k)$, which decrease the weight by $2k$.

Example 5. Let X be a compact Kähler manifold. The k-th de Rham cohomology group $H^k_{\mathrm{dR}}(X)$ admits a Hodge decomposition

$$H^k_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}. \tag{2.3.1}$$

Here $H^{p,q}$ is the space of harmonic (p, q)-forms. The complex conjugate

$$\overline{\mathcal{A}_X^{p,q}} = \mathcal{A}_X^{q,p}$$

on the sheaf of differentiable (p, q)-forms deduces $\overline{H^{p,q}} = H^{q,p}$. Then the de Rham isomorphism

$$H^k_{\mathrm{dR}}(X) \cong H^k_{\text{sing}}(X; R) \otimes \mathbb{R}$$

shows that the singular cohomology $H^k_{\text{sing}}(X; R)$ carries a pure R-Hodge structure of weight k.

Let X be a smooth complex projective variety. The complex space $X(\mathbb{C})^{an}$ endowed with the analytic topology is a compact Kähler manifold. The Kähler metric inherits from the *Fubini-Study metric* on the ambient projective space. Suppose $R \subset \mathbb{R}$. By the R-Hodge structures of X we mean the Hodge structure on the singular cohomology $H^*(X(\mathbb{C})^{an}; R)$.

2.3.2 Integral Hodge conjectures

Let X be a smooth complex projective variety, and let $i : Z \hookrightarrow X$ be an irreducible closed subvariety. Suppose Z is smooth of codimension p. By the *tubular neighborhood* argument [14, Prop. 6.25] and the *Thom isomorphism* [14, Thm. 6.17], there is the *Gysin map*

$$i_* : H^0(Z;\mathbb{Z}(-p)) \xrightarrow{\sim} H^{2p}(X, X\setminus Z;\mathbb{Z}) \to H^{2p}(X;\mathbb{Z}). \tag{2.3.2}$$

Let $\mathbb{1}_Z \in H^0(Z;\mathbb{Z}(-p)) \cong \mathbb{Z}$ be the identity element. The *cycle class* of Z in X, denote by $cl^p_X(Z)$, is defined to be the fundamental class $[Z] := i_*(\mathbb{1}_Z)$.

If the subvariety Z is singular, one can take the open complement $V := X \setminus Z_{\text{sing}}$ of the singular locus of Z. Using the *cohomological purity* (cf. [68, Lem. 11.13]), there is a refined Gysin map

$$H^0(V\cap Z;\mathbb{Z}(-p)) \to H^{2p}(V, V\setminus Z;\mathbb{Z}) \xrightarrow{\sim} H^{2p}(X, X\setminus Z;\mathbb{Z}) \to H^{2p}(X,\mathbb{Z}).$$

Through the refined Gysin map, one defines the cycle class $cl^p_X(Z)$ to be the fundamental class $[V\cap Z]$.

The map cl^p_X linearly extends to the whole group of cycles of codimension p on X. It can be proved that the cycle class $cl^p_X(Z)$ is zero if Z is rationally equivalent to zero (see [69, Lem.9.18]). Therefore it defines a *cycle class map*

$$cl^p_X : \mathrm{CH}^p(X) \to H^{2p}(X;\mathbb{Z}). \tag{2.3.3}$$

A rational(resp. integral) *Hodge class of type* (p,p) is a cohomology class in $H^{2p}(X;\mathbb{Q})\cap H^{p,p}(X)$(resp. $H^{2p}(X;\mathbb{Z})\cap H^{p,p}(X)$). For any algebraic cycle Z of codimension p, the cycle class $cl^p_X(Z)$ is an Hodge class of type (p,p). The well-known Hodge conjecture asserts that all the rational Hodge classes are represented by rational combinations of algebraic cycles. In this thesis, we concern the integral version of the conjecture:

Integral Hodge Conjecture. *For a smooth complex projective variety X. The cycle class map*

$$cl^p_{X,\mathbb{Z}} : \mathrm{CH}^p(X) \to H^{2p}(X;\mathbb{Z})\cap H^{p,p}(X) \tag{2.3.4}$$

is surjective for any p.

For divisor classes the Hodge conjecture is always true due to the Lefschetz theorem on $(1,1)$-classes [30, pp. 163–164]. The Hard Lefschetz theorem [69, Thm. 1.23] affirms the Hodge conjecture for 1-cycles, unlike it, the integral Hodge conjecture is known to be false in general. Since Atiyah-Hirzebruch [4] many counterexamples has been found through [6], [64] and [12]. However, there also exist non-trivial examples (e.g. uniruled or Calabi-Yau 3-folds) for which the integral Hodge conjecture for 1-cycles are true, see [70], [67] and [51].

In Chapter 3, we apply our main result 3.14 on the cylinder homomorphism to prove the integral Hodge conjecture for 1-cycles on certain Fano variety of lines (see Corollary 3.16).

2.3.3 Integral Tate conjectures

Using the étale cohomology, one can define the cycle class map for varieties over an arbitrary field. Here are some notations. Let k be an algebraically closed field. Fix an integer $n > 0$ with $\mathrm{char}(k) \nmid n$. Let μ_n be the sheaf of n-th roots of unity. For any étale sheaf $\mathscr{F} \in Ab(X_{\mathrm{et}})$ of $\mathbb{Z}/n\mathbb{Z}$-modules, the r-th *Tate twist* of $\mathscr{F}$ is

$$\mathscr{F}(r) := \begin{cases} \mathscr{F} \otimes \mu_n^{\otimes r}, r > 0, \\ \mathscr{F}, r = 0, \\ \mathscr{F} \otimes \mathcal{H}om(\mu_n^{\otimes -r}, \mathbb{Z}/n\mathbb{Z}), r < 0. \end{cases}$$

By any choice of a primitive n-th root of unity, the sheaves $\mu_n^{\otimes r}$ are isomorphic to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$.

Let X be a k-variety and $i : Z \hookrightarrow X$ be a closed immersion. To define the cycle class of Z, the *cohomology with supports on* Z, denoted by $H^*_Z(X, -)$, plays the same role as the relative cohomology in the analytic case. The following cohomological purity due to Gabber is essential.

Theorem 2.7. *[50, Thm. 16.1] Let (Z, X) be a smooth pair of codimension p, namely X is a non-singular k-variety and the closed subvariety Z is non-singular of codimension p. For any locally constant étale sheaf $\mathscr{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules, there is a canonical isomorphism*

$$H^r(Z, \mathscr{F}|_Z) \cong H^{r+2p}_Z(X, \mathscr{F}(p))$$

for all $r \geq 0$.

Apply the theorem to the sheaf $\mu_n^{\otimes p}$ yields the Gysin map

$$i_* : H^0(Z, \mu_n) \xrightarrow{\sim} H^{2p}_Z(X, \mu_n^{\otimes p}) \to H^{2p}(X, \mu_n^{\otimes p}).$$

Again the cycle class $cl_X(Z)$ is defined to be the fundamental class $[Z] := i_*(\mathbb{1}_Z)$. If Z is singular, the Theorem 2.7 essentially allows us to employ the same manner as the singular cohomology to obtain a refined Gysin map.

Let X be a smooth variety over a field k that not necessarily algebraically closed. Let k_s be the separable closure of k. Denote by G_k the absolute Galois group $\mathrm{Gal}(k_s/k)$ and $\bar{X}$ the base change $X \times_k k_s$. Fix a prime number ℓ with $\ell \neq \mathrm{char}(k)$. The scalar action of G_k on $\bar{X}$ induces a natural Galois action on the ℓ-adic cohomology groups

$$H^{2p}(\bar{X}, \mathbb{Z}_\ell(p)) := \varprojlim_n H^{2p}_{\mathrm{et}}(\bar{X}, \mu_{\ell^n}^{\otimes p});$$
$$H^{2p}(\bar{X}, \mathbb{Q}_\ell(p)) := (\varprojlim_n H^{2p}_{\mathrm{et}}(\bar{X}, \mu_{\ell^n}^{\otimes p})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Passing to the inverse limit we have the cycle class map

$$\begin{aligned} cl^p_{X,\mathbb{Z}_\ell} : \mathrm{CH}^p(X) \otimes \mathbb{Z}_\ell &\to H^{2p}(\bar{X}, \mathbb{Z}_\ell(p)) \\ Z &\mapsto cl^p_{\bar{X}}(\bar{Z}) \end{aligned} \tag{2.3.5}$$

The cycle class of Z is invariant under the Galois action. The Tate conjecture asserts the converse. Namely, any cohomology class with $\mathbb{Q}_\ell$-coefficients fixed by the Galois action is represented by an algebraic cycle.

We may simply state the integral Tate conjecture by saying the cycle class map (2.3.5) surjects onto the subspace fixed by the Galois group G_k. But in [55] Schoen showed that such statement already fails even for 0-cycles on the conic $x_0^2 + x_1^2 + x_2^2 = 0$ defined over $\mathbb{Q}$ and $\ell = 2$. Instead he considered the integral Tate conjecture for algebraic cycles over the separable closure.

To be precise, let X be a smooth projective k-variety and $\bar{X}$ the separable closure of X. A closed subvariety Z of codimension p in $\bar{X}$ is defined by finitely many polynomials. There exists a intermediate field k' with finite degree over k such that Z is a variety defined over k'. Then the cycle class of Z is fixed by the Galois group $\mathrm{Gal}(k_s/k')$. Then there is a cycle class map

$$cl^p_{\bar{X},\mathbb{Z}_\ell}, : \mathrm{CH}^p(\bar{X}) \otimes \mathbb{Z}_\ell \to \varinjlim_{k \subset k'} H^{2p}(\bar{X}, \mathbb{Z}_\ell(p))^{G_{k'}} \tag{2.3.6}$$

where k' runs over the intermediate fields of $k \subset k_s$ that finite over k. In this thesis we employ this cycle class map for the integral Tate conjecture.

Integral Tate Conjecture. *Suppose the field k is finitely generated over its prime field. Let X be a smooth projective k-variety. Let ℓ be a prime number different from* $\mathrm{char}(k)$. *The cycle class map* (2.3.6) *is surjective.*

This statement of the integral Tate conjecture is more convinced at least for 1-cycles over finite fields. A strong evidence provided by Schoen is the following result

Theorem 2.8. *[55, Thm. 0.5] Let X be a smooth projective variety over a finite field k. Assume the Tate conjecture is true for divisors on surfaces over finite extensions of k, then the cycle class map* (2.3.6) *is surjective for* 1*-cycles, i.e.* $p = \dim X - 1$.

Our main result 3.14 on the cylinder homomorphism can also be applied to prove the integral Tate conjecture for 1-cycles on certain Fano variety of lines, see Corollary 3.19.

Theorem 2.9 (Comparison Theorem)**.** *[26, Thm. 11.6] Let $f : X \to S$ be morphism of schemes and $\mathscr{K}^\bullet$ be complex in the derived category $D_+(X, tor)$. Then the functorial morphism*

$$(Rf_!(\mathscr{K}^\bullet))_{an} \to Rf_{an!}(\mathscr{K}^\bullet_{an})$$

is an isomorphism in $D(X_{an})$. In particular, if $S = \mathrm{Spec}\,\mathbb{C}$ and $\mathscr{F}$ is an torsion sheaf, we have

$$H^i_c(X, \mathscr{F}) \cong H^i_c(X_{an}, \mathscr{F}_{an}).$$

2.4 Grothendieck ring of varieties

The geometry and topology of cubic hypersurfaces and the Fano scheme of lines are intimately related in many aspects. In the article [28], Galkin-Shinder established an interesting relation (2.4.5) between the two objects in the *Grothendieck ring of varieties*. The main purpose of Chapter 4 is to explore the motivic analogy of the relation. In this section we review the main statement 2.10 of the relation. In the next section, we move to its motivic nature.

Definition 2.4. Let k be a field. The Grothendieck ring of varieties $K_0(\mathrm{Var}/k)$ is the abelian group generated by isomorphism classes $[X]$ of quasi-projective k-varieties X that subjects to the scissor relations

$$[X] = [U] + [Z] \tag{2.4.1}$$

for any closed subvariety $Z \subset X$ with the open complement $U = X \setminus Z$. The ring structure on $K_0(\mathrm{Var}/k)$ is given by the multiplication

$$[X] \cdot [Y] = [X \times Y]. \tag{2.4.2}$$

Obviously the class of a single point $1 := [\mathrm{Spec}(k)]$ is the identity of the ring. The relation (2.4.1) verifies

$$[X] = [F] \cdot [Z] \tag{2.4.3}$$

for any Zariski locally trivial fibration $X \to Z$ with the fibre F. And if X is a smooth variety with a smooth closed subvariety $Z \subset X$ of codimension c. The blow-up of X along Z yields the relation

$$[\mathrm{Bl}_Z X] - [\mathbb{P}^{c-1}] \cdot [Z] = [X] - [Z]. \tag{2.4.4}$$

Let $X \subset \mathbb{P}^{n+1}_k$ be a smooth cubic hypersurface over a field k, and let $F := F(X)$ be the Fano scheme of lines on X. Denote by $X^{[2]}$ the Hilbert scheme of two points on X. The *Lefschetz class*

$$\mathbb{L} := [\mathbb{A}^1]$$

is the class of the affine line $\mathbb{A}^1$ in $K_0(\mathrm{Var}/k)$. Then the following statement is crucial.

Theorem 2.10. *[28, Theorem 5.1] With notations as above. We have the following relation in* $K_0(\mathrm{Var}/k)$*:*

$$\mathbb{L}^2 \cdot [F(X)] + [X] \cdot [\mathbb{P}^n] = [X^{[2]}]. \tag{2.4.5}$$

The main ingredient used to establish the relation is a birational map defined on $X^{[2]}$. As the birational map is a recurring key object in our thesis, we describe its details and present the proof of the above theorem in the rest of the section.

Let us denote by P_X the incidence subscheme

$$P_X := \{([L], x) \in \mathbf{Gr}(2, n+2) \times X \mid x \in L\}. \tag{2.4.6}$$

The space of lines in $\mathbb{P}^{n+1}$ passing through a point x can be identified with the projective space of the tangent space $T_x\mathbb{P}^{n+1}$ of $\mathbb{P}^{n+1}$ at x. Hence P_X is isomorphic to the projective bundle $\mathbb{P}(T_{\mathbb{P}^{n+1}}|_X)$.

For the Hilbert scheme of two points on X there is a birational map

$$\Phi : X^{[2]} \dashrightarrow P_X \tag{2.4.7}$$

defined as follows: Any closed point $[\xi] \in X^{[2]}$ represents a 2-cluster ξ having one of the following forms:

1. two distinct k-rational points of X;
2. a single k-rational point with a tangential direction;
3. a pair of Galois conjugate points.

For each case there is a unique k-line L_ξ passing through ξ. If ξ is general, i.e. the line L_ξ does not lie entirely in X, the intersection $L_\xi \cap X$ determines a unique residue point $z \in X$ since the degree of the hypersurface X is three. Therefore the birational map is given by

$$\Phi([\xi]) = ([L_\xi], z).$$

Conversely, let $([L], x)$ represents a point in P_X. If L is not contained in X, the intersection $L \cap X$ is x together with a 2-cluster $\xi \in X^{[2]}$. The inverse rational map is thus defined by

$$\Phi^{-1}([L], x) = [\xi].$$

Proof. Recall the tautological rank two vector bundle $\mathcal{S}$ in (2.1.1). The projective bundle $\mathbb{P}(\mathcal{S})$ over F is the universal family of lines

$$P := \{([\ell], x) \in F \times X \mid x \in \ell\}.$$

Through the above discussion, we see that Φ is not defined for the 2-clusters supported on the L_ξ that contained in X. Therefore the indeterminacy locus is the relative Hilbert scheme of two points on P over F. The fibres of P over F are projective lines. Hence relative Hilbert scheme

$$\mathrm{Hilb}^2(P/F)$$

is a $\mathbb{P}^2$-bundle over F. Meanwhile, the inverse map Φ^{-1} is not defined along $P \subset P_X$. Then the birational map is an isomorphism on the open subsets

$$X^{[2]} \setminus \mathrm{Hilb}^2(P/F) \xrightarrow{\sim} P_X \setminus P.$$

The scissor relation (2.4.1) implies that

$$[X^{[2]}] - [\mathrm{Hilb}^2(P/F)] = [P_X] - [P]$$

in $K_0(\mathrm{Var}/k)$. It follows from the fibration relation (2.4.3) that

$$\begin{aligned}[X^{[2]}] &= [X] \cdot [\mathbb{P}^n] + [F] \cdot ([\mathbb{P}^2] - [\mathbb{P}^1]) \\ &= [X] \cdot [\mathbb{P}^n] + [F] \cdot \mathbb{L}^2.\end{aligned}$$

The last equality holds since $[\mathbb{P}^2] - [\mathbb{P}^1] = [\mathbb{A}^2]$. □

2.4.1 Motivic X-F(X) relation

To discuss the parallel relation in terms of the Chow motives. It is helpful to begin with the motivic *Euler characteristic with compact support*

$$\chi^c_{\mathrm{Mot}} : K_0(\mathrm{Var}/k) \to K_0(\mathrm{Mot}(k)) \tag{2.4.8}$$

when the field k is of characteristic zero. Here $K_0(\mathrm{Mot}(k))$ is the Grothendieck ring of pure Chow motives. It is the abelian group generated by isomorphism classes of pure Chow motives subjects to the relations

$$[M] + [N] = [M \oplus N],\ \forall M, N \in \mathrm{Mot}(k).$$

The ring structure on $K_0(\mathrm{Mot}(k))$ is given by the tensor product

$$[M] \cdot [N] = [M \otimes N].$$

The motivic Euler characteristic with compact support χ^c_{Mot} is a ring homomorphism satisfying

$$\chi^c_{\mathrm{Mot}}([X]) = [\mathfrak{h}(X)]$$

for any smooth projective variety X. Different methods can be employed to construct χ^c_{Mot}, see [35], [29] and [13]. Among them the most convenient way is due to Bittner, which is a consequence of the following alternative presentation of the Grothendieck ring of varieties.

Theorem 2.11. *[13] Let k be a field of $\mathrm{char}(k) = 0$. The Grothendieck ring of varieties $K_0(\mathrm{Var}/k)$ is the quotient of free abelian group of the set of isomorphism classes of smooth projective k-varieties, by relations generated by*

1. $[\emptyset] = 0$,
2. $[Bl_Z X] - [\mathbb{P}^{c-1}] \cdot [Z] = [X] - [Z]$, *the blow-up relation* (2.4.4).

The corresponding blow-up formula (A.6) holds in the category of pure Chow motives. Therefore, if $\mathrm{char}(k) = 0$, the map that assigns the Chow motive to any smooth projective variety is well-defined.

Recall the Lefschetz class $\mathbb{L}$ and the Lefschetz motive $\mathbf{Q}(-1)$. Follows from the scissor relation (2.4.1) we have

$$\chi^c_{\mathrm{Mot}}(\mathbb{L}) = [\mathfrak{h}(\mathbb{P}^1)] - [\mathfrak{h}(\mathrm{Spec}(k))].$$

Let e be a rational k-point of the projective line $\mathbb{P}^1$. The canonical decomposition of the motive of $\mathbb{P}^1$

$$\mathfrak{h}(\mathbb{P}^1) \cong (\mathbb{P}^1, \mathbb{P}^1 \times e, 0) \oplus (\mathbb{P}^1, e \times \mathbb{P}^1, 0) \cong (\mathbb{P}^1, \mathbb{P}^1 \times e, 0) \oplus \mathfrak{h}(\mathrm{Spec}(k)).$$

implies $\chi^c_{\mathrm{Mot}}(\mathbb{L}) = [(\mathbb{P}^1, \mathbb{P}^1 \times e, 0)] = [\mathbf{Q}(-1)]$.

Similarly it is easy to see the relation on the class of the projective space $[\mathbb{P}^d] = \sum_{i=0}^d \mathbb{L}^i$ is compatible with the the decomposition $\mathfrak{h}(\mathbb{P}^d) \equiv \bigoplus_{i=0}^d \mathbf{Q}(-i)$ via the realization homomorphism.

The realization homomorphism χ^c_{Mot} transfers the relation (2.4.5) to the tautological formula

$$[\mathfrak{h}(F(X))(-2)] + \sum_{i=0}^{d} [\mathfrak{h}(X)(-i)] = [\mathfrak{h}(X^{[2]})]. \tag{2.4.9}$$

in $K_0(\mathrm{Mot}(k))$ at least when $\mathrm{char}(k) = 0$. It is known that the category of Chow motives is not semi-simple. In fact, the category of motive defined by an adequate equivalence relation $\sim$ is semi-simple if and only if $\sim$ is the numerical equivalence, see [38]. Hence the expected underlying isomorphism

$$\mathfrak{h}(F(X))(-2) \oplus \bigoplus_{i=0}^{\dim X} \mathfrak{h}(X)(-i) \simeq \mathfrak{h}(X^{[2]}). \tag{2.4.10}$$

in $\mathrm{Mot}(k)$ cannot be deduced directly from (2.4.9).

However, Laterveer later justified the expected isomorphism (2.4.10) in the article [42]. His proof was carried out on the level of Chow groups. Then the Manin's idenitity principle 2.6 deduces the isomorphism on the motives. This method is independent of the characteristic of the base field.

In Chapter 4, we improve Laterveer's result by presenting an integral decomposition of the diagonal of $X^{[2]}$ (Theorem 4.12). The main purpose of the improvement is to show that the formula (2.4.10) is an integral isomorphism. By that we mean the algebraic correspondences of the isomorphism and its inverse are algebraic cycles with $\mathbb{Z}$-coefficient.

Suppose that the field k is a subfield of the complex numbers. The Hodge realization

$$\begin{aligned} \mu_{\mathrm{HS}} : K_0(\mathrm{Var}/k) &\to K_0(\mathrm{HS}_{\mathbb{Q}}) \\ [X] &\mapsto \sum_i (-1)^k [H^k(X_{\mathbb{C}}, \mathbb{Q})] \end{aligned} \tag{2.4.11}$$

is another important realization homomorphism. By the notion $\mathrm{HS}_{\mathbb{Q}}$ we mean the additive category of polarisable $\mathbb{Q}$-Hodge structures. The objects in $\mathrm{HS}_{\mathbb{Q}}$ are graded pure Hodge structures

$$H := \bigoplus_{n\in\mathbb{Z}} H^n,\ H^n \text{ pure Hodge stucture of weight } n.$$

The Grothendieck ring $K_0(\mathrm{HS}_{\mathbb{Q}})$ is the abelian group generated by isomorphism classes $[H]$ of graded Hodge structures modulo relations

$$[H] \oplus [H'] = [H \oplus H'].$$

For any quasi-projective complex variety X, we may consider Deligne's mixed Hodge structure [22] on the cohomology $H^*(X, \mathbb{Q})$. If X is smooth and projective, the mixed Hodge structure degenerates to the classical Hodge structure on $H^k(X, \mathbb{Q})$. Therefore, using Bittner's theorem 2.11, it is sufficient to describe the Hodge realization μ_{HS} on the set of smooth projective varieties. The blow-up formula (cf. (A.6)) on the level of cohomology

$$H^k(\widetilde{X}, \mathbb{Q}) \cong H^k(X, \mathbb{Q}) \bigoplus_{i=1}^{c-1} H^{k-2i}(S, \mathbb{Q})(-i)$$

is an isomorphism of pure Hodge structures of weight k. Hence the assignment

$$[X] \mapsto \sum_i (-1)^k [H^k(X_{\mathbb{C}}, \mathbb{Q})]$$

descends to a ring homomorphism.

It is easy to see that the Hodge realization μ_{HS} factors through χ^c_{Mot} via the following ring homomorphism

$$\begin{aligned} K_0(\mathrm{Mot}(k)) &\to K_0(\mathrm{HS}_{\mathbb{Q}}) \\ M := (X, p, m) &\mapsto \sum_k (-1)^{k+2m} [H^k(M)] \end{aligned}$$

Under the settings of Theorem 2.10, the map μ_{HS} implies the isomorphism of polarisable Hodge structures

$$H^*(F(X), \mathbb{Q})(-2) \oplus \bigoplus_{i=0}^{n} H^*(X, \mathbb{Q})(-i) \simeq H^*(X^{[2]}, \mathbb{Q}) \tag{2.4.12}$$

for a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$. The above isomorphism holds in $\mathrm{HS}_{\mathbb{Q}}$ instead of $K_0(\mathrm{HS}_{\mathbb{Q}})$ since the category of polarisable $\mathbb{Q}$-Hodge structures is semi-simple ([54, Corollary 2.12]).

Through this formula, one can determine some topological invariants and geometric structures of the Fano variety of lines by the cubic hypersurfaces, e.g. see Example 6. Moreover, as an application of our result (Theorem 4.12), the motivic relation (2.4.10) indeed shows (2.4.12) is an isomorphism of polarisable $\mathbb{Z}$-Hodge structures.

Remark 3. The Hodge realization μ_{HS} can also be defined with no signs. We prefer the alternating sum because it implicitly respects the scissor relation. To be precise, let

(X, Z) be a smooth pair of codimension p that described in Section 2.3.2. Let $U = X \setminus Z$ be the open complement. There is a long exact sequence of relative cohomology groups

$$\cdots \to H^k(X, U; \mathbb{Q}) \to H^k(X; \mathbb{Q}) \to H^k(U; \mathbb{Q}) \to H^{k+1}(X, U; \mathbb{Q}) \to \cdots$$

By the same reason the cohomology $H^k(X, U; \mathbb{Q})$ is isomorphic to $H^{k-2p}(Z; \mathbb{Q})$ such that the composition

$$H^{k-2p}(Z; \mathbb{Q}) \to H^k(X; \mathbb{Q})$$

is the *Gysin homomorphism*. Hence the long exact sequence implies

$$\sum_k (-1)^k [H^k(X)] = \sum_k (-1)^k [H^k(Z)] + \sum_k (-1)^k [H^k(U)].$$

The form of alternating sum seems to coincide with the scissor relation (2.4.1).

Chapter 3

Cylinder homomorphisms of cubic hypersurfaces

3.1 Introduction

Definition 3.1. Let X be a smooth projective variety of dimension n. Assume that $\mathcal{Z} := \{Z_s\}_{s\in S}$ is a flat family of algebraic k-cycles on X parametrized by a smooth variety S. The *cylinder homomorphism* associated to the family $\mathcal{Z} \to S$ is the group homomorphism

$$\begin{aligned}\Psi_{\mathcal{Z}} : \mathrm{CH}_{*-k}(S) &\to \mathrm{CH}_*(X)\\ \gamma &\mapsto [\bigcup_{s\in\gamma} Z_s]\end{aligned}$$

induced by $\mathcal{Z}$. To be precise, to each cycle γ the map $\Psi_{\mathcal{Z}}$ assigns the cycle of the union of Z_s over γ. For suitable (co)homology functor H, one could define the cohomological cylinder homomorphism

$$\Psi_{\mathcal{Z}} : H_{*-2k}(S) \to H_*(X)$$

in the same manner.

The study of cylinder homomorphisms begins with the *intermediate Jacobian*. As a generalization of the Jacobian of curves, the intermediate Jacobian of a smooth complex projective variety V is a complex torus $J(V)$ defined by the Hodge structures. In particular, if V is a Fano threefold, $J(V)$ is an abelian variety [31].

Suppose that a connected smooth complex variety T parameterizes a flat family of algebraic curves $\mathcal{C} := \{C_t\}_{t\in T}$ on V. Due to Griffiths [32], there is an homomorphism of complex tori

$$\Phi_T : \mathrm{Alb}(T) \to J(V),$$

where $\mathrm{Alb}(T)$ denotes the *Albanese variety* of T. The map Φ_T is thus determined, up to a translation, by the induced homomorphism on the first homology groups. The induced homomorphism identifies to the cylinder homomorphism with respect to the family $\mathcal{C}$ as follows

$$\begin{array}{ccc} H_1(\mathrm{Alb}(T),\mathbb{Z}) & \xrightarrow{\Phi_{T*}} & H_1(J(V),\mathbb{Z}) \\ \| & & \| \\ H_1(T,\mathbb{Z}) & \xrightarrow{\alpha\mapsto[\bigcup_{t\in\alpha} C_t]} & H_3(V,\mathbb{Z}). \end{array} \tag{3.1.1}$$

The celebrated work on cubic threefolds by Clemens-Griffiths [17] shows the importance of the cylinder homomorphism. For a non-singular complex cubic threefold X, let $T = F(X)$ be the Fano surface of lines. Consider the family of lines on X, the cylinder homomorphism

$$H_1(F(X), \mathbb{Z}) \to H_3(X, \mathbb{Z})$$

is an isomorphism of integral Hodge structures, see [17, Theorem 4.5]. Hence the Able-Jacobi mapping

$$\Phi_{F(X)} : \mathrm{Alb}(F(X)) \to J(V)$$

is an isomorphism of abelian varieties. This crucial fact is used to prove the irrationality of any smooth cubic threefold [17]. Other remarkable examples that the cylinder homomorphism (3.1.1) is isomorphic are found in [74] for lines on a quartic double solid and [44] for conics on a general quartic threefold.

Beyond threefolds, the cylinder homomorphism also play important roles. In [59, 60, 61], Shimada thoroughly investigated the cohomological cylinder homomorphism for lines on a hypersurface. The main result of his works partially initiates our study. We present it in the following.

Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree a over the complex numbers $\mathbb{C}$. Recall Section 2.1, if X is generic, the Fano variety of lines $F(X)$ is smooth of dimension $2n - a - 1$. Let P denote the universal family of lines

$$\begin{tikzcd}[column sep=small] & P := \{([\ell], x) \in F(X) \times X \mid x \in \ell\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ F(X) & & X. \end{tikzcd} \tag{3.1.2}$$

By Lefschetz hyperplane theorem [69, Theorem 1.23], the non-trivial (co)homology groups of a hypersurface concentrate at its middle degree. Hence we consider the cylinder homomorphism associated to P

$$\Psi_P := q_* p^* : H_{n-2}(F(X), \mathbb{Z}) \to H_n(X, \mathbb{Z}).$$

Theorem 3.1. *[60] Suppose that $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a generic hypersurface of degree a with $n \geq 3$ and $n \geq a$. Let G be the grassimannian of lines in $\mathbb{P}^{n+1}_{\mathbb{C}}$. Then $\Psi_P \otimes \mathbb{Q}$ is surjective. Moreover,*

1. *if n is odd, then is an isomorphism modulo torsion.*
2. *If n is even, then*
   (a) $\mathrm{Ker}\,\Psi_P \otimes \mathbb{Q}$ *is contained in the image of the natural restriction map* $H_{n+2a}(G, \mathbb{Q}) \to H_{n-2}(F(X), \mathbb{Q})$, *and* $\dim \mathrm{Ker}\,\Psi_P \otimes \mathbb{Q} \leq (n-2)/4$,
   (b) if $n/2 + 2 \geq a$, then Ψ_P is surjective.

From the above theorem, it is natural to raise the question whether the corresponding cylinder homomorphism on Chow groups (with integer coefficients) is surjective. Very few examples are known toward this question. In [45] Lewis justified the case of degree n hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$. Under some numerical conditions, he also studied the cylinder homomorphisms associated to a family of linear subspaces contained in a hypersurface, see [46].

In this chapter, we aim to study the cylinder homomorphism on Chow groups for lines on a cubic hypersurface. The main result we obtain is

Theorem 3.2. *For any smooth cubic hypersurface X defined over a field k, assume that X contains a k-line and $\dim X \geq 3$, then the cylinder homomorphism*

$$\Psi_P : \mathrm{CH}_{r-1}(F(X)) \to \mathrm{CH}_r(X),\ r \geq 1$$

is surjective except for the divisor class group of cubic threefolds. In particular, for any field extension L/k, the cylinder homomorphism Ψ_{P_L} under base change remains surjective.

Remark 4. Let $\bar{k}$ be the algebraic closure of k. Notice that after base change $X_{\bar{k}}$ always contains $\bar{k}$-lines. Hence one may state the above result by saying, up to a finite extension of k, the cylinder homomorphism is surjective. Nevertheless, we prefer the assumption in the statement. It can be reflected through the proof of Lemma 3.12.

The Chow groups of both cubic hypersurface and its Fano variety of lines are complicated. There is no obvious reason to justify the surjectivity of the cylinder homomorphism. For example, consider the cylinder homomorphism

$$\mathrm{CH}_0(F(X)) \to \mathrm{CH}_1(X)$$

for 1-cycles on a smooth cubic hypersurface X. The map is surjective if and only if $\mathrm{CH}_1(X)$ is generated by lines on X. Suppose that X is defined over an algebraically closed field. It is until Paranjape [53] who first showed that

$$\mathrm{CH}_1(X) \cong \mathbb{Z},\ \text{for}\ \dim X \geq 5$$

which is generated by lines on X. Using the technique of rational curves Kollár [41], Tian and Zong improved Paranjape's result. They proved that for any non-singular cubic X with $\dim X \geq 3$, the Chow group of 1-cycles is generated by lines on X, see [63, Theorem 1.7].

The result for one-cycles in Theorem 3.2 is a contribution of Shen [57, 58]. The conclusion is valid for arbitrary base field, which strengthens Paranjape and Tian-Zong's results. The assumption on algebraically closed field is necessary for their results since they essentially applied the Tsen's theorem.

Recall the illustration in the main introduction. Theorem 3.2 is a consequence of Theorem 3.9, where the main outputs are two formulae of algebraic cycles. The main ingredients in the proof of Theorem 3.9 are the the birational map (2.4.7) and Voisin's result Proposition 3.3.

The outline of the chapter is as follows. In Section 3.2 we review the birational map (2.4.7) and the key result Proposition 3.3. Then we settle several lemmas to step towards the main Proposition 3.8 and Theorem 3.9. In Section 3.3 we prove Proposition 3.8, Theorem 3.9 and Lemma 3.12, which deduce the surjectivty property of the cylinder map. In Section 3.4 we applied our result (Corollary 3.14) to the integral Hodge conjecture and integral Tate conjecture of Fano scheme of lines of cubic fourfolds.

3.2 The Hilbert square of cubic hypersurfaces

Let X/k be a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}_k$, let $F := F(X)$ be the Fano scheme of lines on X, and let P be the universal family of lines (cf. (3.1.2)).

Recall in Section 2.4 the incidence scheme P_X is defined to be

$$P_X := \{([\ell], x) \in \mathbf{Gr}(2, n+2) \times X \mid x \in \ell\}.$$

We also describe the birational map

$$\Phi : X^{[2]} \dashrightarrow P_X$$

from the second punctual Hilbert scheme $X^{[2]}$ to P_X. The indeterminacy locus of the birational map Φ is the relative Hilbert scheme of two points $\mathrm{Hilb}^2(P/F)$ of P over F. The indeterminacy locus of the inverse map Φ^{-1} is universal family of lines P. We simplify write P_2 to denote $\mathrm{Hilb}^2(P/F)$.

Let $\mathscr{E}$ be the rank two universal quotient sheaf on F. Very often we use the following equivalent notions:

1. the universal family of lines P equals the $\mathbb{P}^1$-bundle $p : \mathbb{P}(\mathscr{E}) \to F$,
2. the relative Hilbert scheme of two points P_2 is identified with the relative second symmetric product $p_2 : \mathbb{P}(\mathrm{Sym}^2 \mathscr{E}) \to F$

In the following proposition, Voisin presented an explicit resolution of the birational map which enjoys additional good properties.

Proposition 3.3. *[73, Proposition 2.8]*

1. *The birational map $\Phi : X^{[2]} \dashrightarrow P_X$ can be desingularized by the blowing up $\tau : \widetilde{X^{[2]}} \to X^{[2]}$ along the smooth center $P_2 \subset X^{[2]}$.*
2. *The resulting morphism $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$ is isomorphic to the blowing up $\widetilde{P_X} \to P_X$ along the smooth center $P \subset P_X$.*
3. *The exceptional divisors of the blowing ups τ and $\widetilde{\Phi}$ are identified via the isomorphism $\widetilde{X^{[2]}} \cong \widetilde{P_X}$.*

Denote by $\mathcal{E}$ the exceptional divisor of the two blow-ups. We set the following diagrams and notations to summarize the results in the above proposition.

$$\begin{tikzcd}
 & & \mathcal{E} \arrow[lldd, "\pi_2"'] \arrow[d, hook, "j"] \arrow[rrdd, "\pi_1"] & & \\
 & & \widetilde{X^{[2]}} \arrow[ld, "\tau"'] \arrow[rd, "\widetilde{\Phi}"] & & \\
P_2 \arrow[r, hook, "i_2"] & X^{[2]} \arrow[rr, dashed, "\Phi"] & & P_X & P. \arrow[l, hook', "i_1"']
\end{tikzcd} \tag{3.2.1}$$

Moreover, the exceptional divisor $\mathcal{E}$ is isomorphic to the fibre product $P_2 \times_F P$. We simply have the cartesian diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & P \arrow[d, "p"] \\
P_2 \arrow[r, "\pi_F"] & F.
\end{tikzcd} \tag{3.2.2}$$

The main contents appeared in the proof of the preceding proposition constitute the basic settings to present and carry out our main results. So we explain them in the next paragraphs.

The way one define the birational map Φ yields a proper morphism

$$\varphi : X^{[2]} \to \mathbf{Gr}(2, n+2)$$

that sends any 2-cluster $\xi \in X^{[2]}$ to the unique line in $\mathbb{P}^{n+1}$ passing through ξ. By abuse of notation, we use $\mathscr{E}$ to denote the rank two universal quotient sheaf on $\mathbf{Gr}(2, n+2)$. Then the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathscr{E})$ can be regarded as the incidence(Flag) variety

$$\mathbf{Gr}(1, 2, n+2) := \{([\ell], x) \in \mathbf{Gr}(2, n+2) \times \mathbb{P}^{n+1} \mid x \in \ell\},$$

Let $\mathcal{Q} \to X^{[2]}$ be the base change of $\pi_G : \mathbb{P}(\mathscr{E}) \to \mathbf{Gr}(2, n+2)$ under the map φ, and let $\alpha : \mathcal{Q} \to \mathbb{P}(\mathscr{E}) \to \mathbb{P}^{n+1}$ be the composition. Then the divisor $\alpha^{-1}(X)$ on $\mathcal{Q}$ consists of two components. The first component parametrizes the pairs (z, ξ) of a generic 2-cluster $\xi \in X^{[2]}$ and the residue point $z \in L_\xi \cap X$ where L_ξ is the line generated by ξ. The blow-up $\widetilde{X^{[2]}}$ can be identified with this component. Another component of $\alpha^{-1}(X)$ parametrizes the pairs (x, ξ) of a 2-cluster $\xi \in X^{[2]}$ and $x \in \operatorname{Supp} \xi$. It is exactly isomorphic to the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal. For more rigorous argument, we refer the readers to [73, Proposition 2.8]. To summarize, we have the following cartesian diagrams

$$\begin{tikzcd}
\widetilde{X \times X} \bigcup \widetilde{X^{[2]}} \arrow[r, "\Psi \cup \widetilde{\Phi}"] \arrow[d, hook] & P_X \arrow[r, "\pi_X"] \arrow[d, hook, "i'"] & X \arrow[d, hook, "i_X"] \\
\mathcal{Q} \arrow[r] \arrow[d] & \mathbb{P}(\mathscr{E}) \arrow[r, "\pi_P"] \arrow[d, "\pi_G"] & \mathbb{P}^{n+1} \\
X^{[2]} \arrow[r, "\varphi"] & \mathbf{Gr}(2, n+2). &
\end{tikzcd} \tag{3.2.3}$$

Lemma 3.4. *Use the notations in in diagrams* (3.2.1), (3.2.2). *For any algebraic cycle* $\Xi \in \mathrm{CH}_k(X^{[2]})$ *of dimension* k, *we have the* $(k-2)$*-dimensional algebraic cycle* $\gamma = \pi_{F*} {i_2}^* \Xi \in \mathrm{CH}_{k-2}(F)$ *such that*

$$\pi_{X*} \widetilde{\Phi}_* (\mathcal{E} \cdot \tau^* \Xi) = \Psi_P(\gamma) := q_* p^* \gamma$$

in $\mathrm{CH}_{k-1}(X)$.

Proof. Recall $j : \mathcal{E} \hookrightarrow \widetilde{X^{[2]}}$ is the inclusion in diagram (3.2.1). By projection formula

$$\mathcal{E} \cdot \tau^* \Xi = j_* j^* \tau^* \Xi.$$

The commutative diagrams (3.2.1) and (3.2.2) implies that

$$\begin{aligned} \pi_{X*} \widetilde{\Phi}_* j_* j^* \tau^* \Xi &= q_* \pi_{1*} \pi_2^* {i_2}^* \Xi \\ &= q_* p^* \pi_{F*} {i_2}^* \Xi. \end{aligned}$$

□

Let us think of the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal as the universal family $\sigma : \widetilde{X \times X} \to X^{[2]}$ of length two closed subschemes in X. The natural map

$$\Psi : \widetilde{X \times X} \to P_X$$

obtained from the diagram (3.2.1) is given by sending a point (x,ξ) to $(x,L_\xi)\in P_X$ where the line L_ξ is generated by ξ. Then it is easy to verify the following commutative diagram

$$\begin{tikzcd} X^{[2]} & \widetilde{X\times X} \arrow[l, "\sigma"'] \arrow[r, "\Psi"] \arrow[d, "\rho"] & P_X \arrow[d, "\pi_X"] \\ & X\times X \arrow[r, "p_1"] & X, \end{tikzcd} \tag{3.2.4}$$

where p_1 is the projection to the first factor.

Lemma 3.5. *We work in the settings of diagrams* (3.2.1), (3.2.3) *and* (3.2.4). *Given any algebraic cycle* $W\in \mathrm{CH}^i(X^{[2]})$, *we have*

$$\pi_{X*}\widetilde{\Phi}_*\tau^*W = i_X^*\pi_{P*}{\pi_G}^*\varphi_*W-\pi_{X*}\Psi_*\sigma^*W. \tag{3.2.5}$$

Basically the formula says that one can compute the action of the correspondence of the birational map Φ in terms of the blow-up $\widetilde{X^{[2]}}$ and the Grassmannian $\mathbf{Gr}(2,n+2)$.

Proof. In the settings of diagrams (3.2.3) and (3.2.4), the action of correspondences $[\widetilde{X\times X}]+[\widetilde{X^{[2]}}]$ on the Chow group of $X^{[2]}$ can be represented by

$$\Psi_*\circ\sigma^*+\widetilde{\Phi}_*\circ\tau^*.$$

Notice that the cartesian diagram

$$\begin{tikzcd} \widetilde{X\times X}\bigcup\widetilde{X^{[2]}} \arrow[r, "\Psi\cup\widetilde{\Phi}"] \arrow[d, "\sigma\cup\tau"'] & P_X \arrow[d, "\pi_G\circ i'"] \\ X^{[2]} \arrow[r, "\varphi"] & \mathbf{Gr}(2,n+2) \end{tikzcd}$$

is a fibre product square of local complete intersection morphisms which satisfies the settings in Corollary A.5. Hence the formula (A.3.2) of commutative Gysin homomorphisms implies

$$\Psi_*\circ\sigma^*+\widetilde{\Phi}_*\circ\tau^*=(\pi_G\circ i')^*\circ\varphi_*$$

Therefore the action of the component $\widetilde{X^{[2]}}$ is equal to

$$\widetilde{\Phi}_*\circ\tau^*=i'^*\circ\pi_G^*\circ\varphi_*-\Psi_*\circ\sigma^*.$$

By taking the push-forward map π_{X*} we obtain the desired formula (3.2.5). □

Recall the blowing up $\rho:\widetilde{X\times X}\to X\times X$ and the universal family $\sigma:\widetilde{X\times X}\to X^{[2]}$. Let $\mu: X\times X \dashrightarrow X^{[2]}$ be the natural rational map defined to be $\sigma\circ\rho^{-1}$. It forms the commutative diagram:

$$\begin{tikzcd} E_{\Delta,X} \arrow[r, hook, "j_E"] \arrow[d, "\pi_\Delta"] & \widetilde{X\times X} \arrow[r, "\sigma"] \arrow[d, "\rho"] & X^{[2]} \\ X \arrow[r, hook, "\Delta_X"] & X\times X, \arrow[ur, dashed, "\mu"'] & \end{tikzcd} \tag{3.2.6}$$

where $E_{\Delta,X}$ is the exceptional divisor. We end this section with the following lemma.

Lemma 3.6. *Assume that Γ is a symmetric codimension r algebraic cycle on $X \times X$ that is not contained in the diagonal Δ_X, i.e. no irreducible component of Γ is entirely supported in Δ_X. Then there exists a codimension r algebraic cycle Σ in $X^{[2]}$ such that $\mu^*\Sigma := \rho_*\sigma^*\Sigma = \Gamma$.*

Proof. By symmetric we mean the cycle Γ is the sum $Z_1 + Z_2$, where $Z_1 = \sum n_i Z_{1,i}$ is a linear combination of irreducible subvarieties $Z_{1,i}$ of $X \times X$ invariant under the natural involution i, and Z_2 has the form $Z_2' + i(Z_2')$. The proof for $r = n$ is presented by Voisin in [73, Corollary 2.4]. Our argument is similar.

Since $Z_{1,i}$ does not lie in Δ_X, we take $\widetilde{Z_{1,i}}$ to be the strict transform of $Z_{1,i}$ in $\widetilde{X \times X}$. . We define the algebraic cycle Σ on $X^{[2]}$ by

$$\sum_i n_i[\sigma(\widetilde{Z_{1,i}})] + \sigma_*\rho^* Z_2'.$$

Notice that the morphism σ is finite hence proper. The image $\sigma(\widetilde{Z_{1,i}})$ are closed and irreducible in $X^{[2]}$.

Now we verify that $\mu^*\Sigma$ recovers Γ. One the one hand, the morphism σ is also flat. Then for any closed subvariety $W \subset \widetilde{X \times X}$ invariant under the involution i, we have $\sigma^*[\sigma(W)] = \sigma^{-1}(\sigma(W)) = W$. On the other hand, the morphism σ is a quotient map by the induced involution i. Hence for any cycle α on $\widetilde{X \times X}$ there is $\sigma^*\sigma_*\alpha = \alpha + i(\alpha)$. Under the two observations we obtain

$$\begin{aligned}\mu^*\Sigma &= \sum_i n_i \rho_*\sigma^*[\sigma(\widetilde{Z_{1,i}})] + \rho_*\sigma^*\sigma_*\rho^* Z_2' \\ &= \sum_i n_i \rho_*[\widetilde{Z_{1,i}}] + \rho_*\rho^* Z_2' + \rho_*\rho^* i(Z_2') \\ &= \sum n_i Z_{1,i} + Z_2' + i(Z_2'),\end{aligned}$$

□

Remark 5. The statement and the proof of the previous lemma of course hold for any smooth projective varieties. Do not confuse $[\sigma(\widetilde{Z_{1,i}})]$ with the class $\sigma_*[\widetilde{Z_{1,i}}]$ under the push-forward. Otherwise we will have $\sigma^*\sigma_*[\widetilde{Z_{1,i}}] = 2[\widetilde{Z_{1,i}}]$.

3.3 Surjectivity of the cylinder homomorphism

For simplicity of notations, for any given cycle classes $\alpha, \beta \in \mathrm{CH}^*(X)$, we denote the hat tensor $\alpha\hat{\otimes}\beta$ to be the algebraic cycle $\sigma_*\rho^*(\alpha \otimes \beta)$ in $\mathrm{CH}^*(X^{[2]})$. Recall the definition of the notation $\alpha\hat{\otimes}\beta$, there is

$$\sigma^*(\alpha\hat{\otimes}\beta) = \rho^*(\alpha \times \beta + \beta \times \alpha).$$

Lemma 3.7. *Recall that $\mathcal{E}$ is the exceptional divisor of the blowing up $\tau : \widetilde{X^{[2]}} \to X^{[2]}$. Let h_X be hyperplane section class of X, let $h_{\mathcal{Q}} \in \mathrm{Pic}(\mathcal{Q})$ polarization through the morphism $\alpha : \mathcal{Q} \to \mathbb{P}^{n+1}$, see (3.2.3), and let $\delta \in \mathrm{Pic}(\widetilde{X^{[2]}})$ the half diagonal divisor of $X^{[2]}$ such that $\sigma^*\delta$ is the canonical exceptional divisor E_Δ on $\widetilde{X \times X}$. Then we have*

$$\mathcal{E} = -h_{\mathcal{Q}}|_{\widetilde{X^{[2]}}} - \tau^*(2h_X\hat{\otimes}X - 3\delta), \tag{3.3.1}$$

and

$$g := c_1(\varphi^*\mathscr{E}) = -h_X\hat{\otimes}1 + \delta \tag{3.3.2}$$

in $\mathrm{Pic}(\widetilde{X^{[2]}})$.

Proof. See [58, Lemma 4.3] $\square$

Proposition 3.8. *Let X be a smooth cubic hypersurface. Denote by h_X the hyperplane section class of X. Let $\Gamma := \sum n_i \Gamma_i$ be an algebraic cycle in $\mathrm{CH}_{r>1}(X)$ with irreducible components Γ_i. Set $e_i := \Gamma_i \cdot h_X^{n-r}$ to be the degree of each Γ_i. Assume that there exists a degree one 1-cycle $\mathfrak{l}$ in $\mathrm{CH}_1(X)$. With the notations in settings of diagrams* (3.2.4) *and* (3.2.6)*,*

1. *there exists an algebraic cycle Γ' of dimension $r+1$ on $X^{[2]}$ such that*

$$\begin{aligned} \pi_*\Psi_*\sigma^*\Gamma' &= 0; \\ \pi_*\Psi_*\sigma^*(h_X \hat{\otimes} X \cdot \Gamma') &= \Gamma; \\ \pi_*\Psi_*\sigma^*(\delta \cdot \Gamma') &= 0, \end{aligned}$$

2. *and there exits an algebraic cycle Γ'' of dimension $2r$ on $X^{[2]}$ such that*

$$\pi_*\Psi_*\sigma^*(\Gamma'' \cdot (h_X \hat{\otimes} 1)^k \cdot \delta^l) = \begin{cases} 0, & \forall\, 0 \le k, l \le r-1; \\ \sum n_i e_i \cdot \Gamma_i, & k = r, l = 0; \\ (-1)^{r+1}\Gamma, & k = 0, l = r. \end{cases}$$

Proof. We first prove the case for Γ being a cycle with a single component. Then the general case will follow.

Now let Γ be an irreducible closed subvariety of X. Set a symmetric algebraic cycle $\Gamma \times \mathfrak{l} + \mathfrak{l} \times \Gamma$ on $X \times X$. By the proof of Lemma 3.6, one can produce an $(r+1)$-cycle Γ' on $X^{[2]}$ such that $\rho_*\sigma^*\Gamma' = (\Gamma \times \mathfrak{l} + \mathfrak{l} \times \Gamma)$. Then from the cartesian diagram (3.2.4) we have

$$\pi_*\Psi_*\sigma^*\Gamma' = p_{1*}\rho_*\sigma^*\Gamma' = p_{1*}(\Gamma \times \mathfrak{l} + \mathfrak{l} \times \Gamma) = 0.$$

In the same manner we have

$$\begin{aligned} \pi_*\Psi_*\sigma^*(h_X \hat{\otimes} X \cdot \Gamma') &= p_{1*}\rho_*(\rho^*(h_X \times X + X \times h_X) \cdot \sigma^*\Gamma') \\ &= p_{1*}((h_X \times X + X \times h_X) \cdot \rho_*\sigma^*\Gamma') \text{ (projection formula)} \\ &= p_{1*}((h_X \times X + X \times h_X) \cdot (\Gamma \times \mathfrak{l} + \mathfrak{l} \times \Gamma)) \end{aligned}$$

Notice that $p_{1*}((h_X \cdot \Gamma) \times \mathfrak{l}) = 0$ since $\dim \Gamma > 1$. Therefore we obtain

$$\begin{aligned} p_{1*}((h_X \times X + X \times h_X) \cdot (\Gamma \times \mathfrak{l} + \mathfrak{l} \times \Gamma)) &= \deg(h_X \cdot \mathfrak{l}) \cdot \Gamma \\ &= \Gamma. \end{aligned}$$

Hence the second formula holds. Recall that for the half divisor divisor δ on the Hilbert square $X^{[2]}$ we have $\sigma^*\delta = E_{\Delta,X}$. It thus follows that

$$\begin{aligned} \pi_*\Psi_*\sigma^*(\delta \cdot \Gamma') &= p_{1*}\rho_*(E_{\Delta,X} \cdot \sigma^*\Gamma') \\ &= p_{1*}\rho_*(j_{E*}{j_E}^*\sigma^*\Gamma') \text{ (projection formula)} \end{aligned}$$

Notice that $\sigma^*\Gamma' = \rho^*(\Gamma \times \mathfrak{l} + \mathfrak{l} \times \Gamma)$. Then through the diagram (3.2.6), we further get

$$\begin{aligned} p_{1*}\rho_*(j_{E*}{j_E}^*\sigma^*\Gamma') &= \pi_{\Delta*}({j_E}^*\sigma^*\Gamma') \\ &= \pi_{\Delta*}{j_E}^*\rho^*(\Gamma \times \mathfrak{l} + \mathfrak{l} \times \Gamma) \\ &= 2\pi_{\Delta*}{\pi_\Delta}^*(\Gamma \cdot \mathfrak{l}) = 0, \end{aligned}$$

which complete the first assertion. In general, for Γ being $\sum n_i\Gamma_i$, to each Γ_i we just associate the Γ_i' satisfying the equations in the first assertion. Then $\Gamma' := \sum n_i\Gamma_i'$ is the desired cycle class on $X^{[2]}$ for the first assertion.

For the second assertion, again assume Γ is irreducible. We consider the closed subscheme $\Gamma \times \Gamma \subset X \times X$. By Lemma 3.6 there is a $2r$-dimensional algebraic cycle $\Gamma'' \in \mathrm{CH}_{2r}(X^{[2]})$ such that $\rho_*\sigma^*\Gamma'' = [\Gamma \times \Gamma]$. In fact, the construction in the proof of Lemma 3.6 indicates that the cycle class $\sigma^*\Gamma''$ is represented by the strict transform $\widetilde{\Gamma \times \Gamma}$. By the cartesian diagram (3.2.6), we have

$$\begin{aligned}\pi_*\Psi_*\sigma^*(\Gamma'' \cdot (h_X\hat{\otimes}X)^k \cdot \delta^l) &= p_{1*}\rho_*(\rho^*(h_X \times X + X \times h_X)^k \cdot \sigma^*\Gamma'' \cdot E^l_{\Delta,X}) \\ &= p_{1*}(\sum_{s=0}^{k}\binom{k}{s} h_X^s \times h_X^{k-s} \cdot \rho_*([\widetilde{\Gamma \times \Gamma}] \cdot E^l_{\Delta,X})).\end{aligned}$$

In the following we calculate the equations in assertion (2) case by case.

1. Start with the case $l = 0$. Notice that $\rho_*[\widetilde{\Gamma \times \Gamma}] = [\Gamma \times \Gamma]$. Then it follows that

$$\begin{aligned}\pi_*\Psi_*\sigma^*(\Gamma'' \cdot (h_X\hat{\otimes}1)^k) &= p_{1*}(\sum_{s=0}^{k}\binom{k}{s} h_X^s \times h_X^{k-s} \cdot \rho_*[\widetilde{\Gamma \times \Gamma}]) \\ &= \sum_{s=0}^{k}\binom{k}{s} p_{1*}(h_X^s \cdot \Gamma)(h_X^{k-s} \cdot \Gamma) \\ &= \begin{cases} 0, & k < r; \\ \deg\Gamma \cdot \Gamma, & k = r. \end{cases}\end{aligned}$$

2. Suppose $1 \le l \le r-1$. Under this condition we claim that

$$\rho_*([\widetilde{\Gamma \times \Gamma}] \cdot E^l_{\Delta,X}) = 0$$

Let Q be the excess normal bundle

$$0 \to \mathcal{O}_E(-1) \to \pi_\Delta^* N_{X/X\times X} \to Q \to 0. \tag{3.3.3}$$

on the exceptional divisor $E_{\Delta,X}$. The key point here is the Blow-up formula, see [27, Theorem 6.7]

$$\rho^*([\Gamma \times \Gamma]) = [\widetilde{\Gamma \times \Gamma}] + j_{E*}\{c(Q) \cap \pi_\Delta^* s(\Gamma, \Gamma \times \Gamma)\}_{2r},$$

where $s(\Gamma, \Gamma \times \Gamma)$ is the Segre class of the closed subvariety Γ in $\Gamma \times \Gamma$. For the notation of the serge class and the blow-up formula, we refer to Appendix A. We simply write s_i for the i-th Segre class $s_i(\Gamma, \Gamma \times \Gamma)$. We obtain

$$\begin{aligned}\rho_*([\widetilde{\Gamma \times \Gamma}] \cdot E^l_{\Delta,X}) &= [\Gamma \times \Gamma] \cdot \rho_* E^l_{\Delta,X} \\ &\quad - \Delta_{X*}\pi_{\Delta*}(\{c(Q) \cap \pi_\Delta^* s(\Gamma, \Gamma \times \Gamma)\}_{2r} \cdot j_E^* E^l_{\Delta,X}).\end{aligned} \tag{3.3.4}$$

On the one hand, for any integer $i \in \mathbb{Z}_{\ge 0}$, the i-th self-intersection of $E_{\Delta,X}$ is

$$E^i_{\Delta,X} = j_{E*}j_E^* E^{i-1}_{\Delta,X} = j_{E*}c_1(\mathcal{O}_E(-1))^{i-1}.$$

The exceptional divisor $E_{\Delta,X}$ is a projective $\mathbb{P}^{n-1}$-bundle over X. The push-forward image $\rho_* E^i_{\Delta,X}$ is equal to

$$\begin{aligned}\rho_* j_{E*} c_1(\mathcal{O}_E(-1))^{i-1} &= \Delta_{X*}\pi_{\Delta*}c_1(\mathcal{O}_E(-1))^{i-1} \\ &= \begin{cases} 0, & i \leq n-1; \\ (-1)^{n-1}\Delta_X, & i = n \end{cases}\end{aligned} \tag{3.3.5}$$

On the other hand, the second term in (3.3.4) can be rephrased to be

$$\sum_{i+t=n-r-1} \iota_{\Delta*}(s_i \cdot \pi_{\Delta*}(c_t(Q)c_1(\mathcal{O}_E(-1))^l)).$$

By the exact sequence of the excess normal bundle (3.3.3), it is not hard to show

$$\pi_{\Delta*}(c_t(Q)c_1(\mathcal{O}(-1))^l)) = 0, \text{ for } t + l \leq n-2.$$

Therefore the numerical condition of this case asserts the vanishing of the equation (3.3.4)

3. In the last $k = 0, l = r$. Again use the blow-up formula (3.3.4) we have

$$\begin{aligned}\pi_*\Psi_*\sigma^*(\Gamma'' \cdot \delta^r) =& p_{1*}\rho_*([\widetilde{\Gamma \times \Gamma}] \cdot E^r_{\Delta,X}) \\ =& p_{1*}([\Gamma \times \Gamma] \cdot \rho_* E^r_{\Delta,X} \\ & -\Delta_{X*}\pi_{\Delta*}(\{c(Q) \cap \pi^*_\Delta s(\Gamma, \Gamma \times \Gamma)\}_{2r} \cdot j^*_E E^r_{\Delta,X})).\end{aligned}$$

If $r \leq n-1$, by the computation (3.3.5) the first term above again vanishes. For the same reason, the second term is equal to

$$\begin{aligned}&= (-1)^{r+1} \sum_{i+t=n-r-1} \Delta_{X*}(s_i \cdot \pi_{\Delta*}(c_t(Q)c_1(\mathcal{O}_E(1))^r)) \\ &= (-1)^{r+1}\Delta_{X*}(s_0 \cdot c_0(X) \cdot \pi_{\Delta*}c_1(\mathcal{O}_E(1))^{n-1})) \\ &= (-1)^{r+1}\Gamma.\end{aligned}$$

If $r = n$, then $\Gamma = [X]$. It is straightforward to see

$$p_{1*}\rho_*(\sigma^*\Gamma'' \cdot E^r_\Delta) = p_{1*}([X \times X] \cdot \rho_* E^n_\Delta) = (-1)^{r+1}\Gamma.$$

Hence the last formula follows. Now let $\Gamma := \sum n_i\Gamma_i$. Similar to the first assertion, to each component Γ_i we associates the corresponding cycle class Γ''_i. Take $\Gamma'' := \sum n_i\Gamma''_i$. Through the previous computations, one can easily verify that Γ'' satisfies the equations in the second assertion.

□

Remark 6. As mentioned in Remark 4, instead of the assumption of the existence of 1-cycle we cam say up to a finite extension of k, the assertions in Proposition 3.8 are true. In general, there exist cases for cubic hypersufraces over a non-closed field containing no lines. For instance, cubic hypersurfaces over number fields or finite field with small orders, see [21].

Using the above proposition, we can state the following main theorem.

Theorem 3.9. *Let X be a smooth cubic hypersurface, let h_X be the hyperplane section class. Let $\Gamma \in \mathrm{CH}_r(X)$ be an algebraic cycle of dimension $r > 1$ with degree e. Suppose that there exists a degree one 1-cycle $\mathfrak{l}$ in $\mathrm{CH}_1(X)$. Then we have*

(i)

$$2\Gamma + \Psi_P(\gamma) \equiv 0 \operatorname{mod} \mathbb{Z} \cdot h_X^{n-r},$$

in $\mathrm{CH}_r(X)$;

(ii) There exists a cycle $\gamma' \in \mathrm{CH}_{r-1}(F)$ such that

$$(2e-3)\Gamma + \Psi_P(\gamma') \equiv 0 \operatorname{mod} \mathbb{Z} \cdot h_X^{n-r},$$

in $\mathrm{CH}_r(X)$.

Proof. For the algebraic cycle Γ, let the notation $\Gamma' \in \mathrm{CH}_{r+1}(X^{[2]})$ denote the same cycle used in Proposition 3.8. Recall Lemma 3.4 we have

$$\pi_* \widetilde{\Phi}_* (\mathcal{E} \cdot \tau^* \Gamma') = \Psi_P(\gamma),$$

where $\gamma = \pi_{F*} i_2{}^* \Gamma' \in \mathrm{CH}_{r-1}(F)$. We claim that the algebraic cycle $\pi_* \widetilde{\Phi}_* (\mathcal{E} \cdot \tau^* \Gamma')$ is equal to

$$-2\Gamma + \text{ a multiple of } h_X^{n-r}$$

which asserts the formula (i).

As a result of Lemma 3.7, the cycle class of the exceptional divisor $\mathcal{E}$ is represented by

$$\mathcal{E} = -h_{\mathcal{Q}}|_{\widetilde{X^{[2]}}} - \tau^*(2h_X \hat{\otimes} X - 3\delta).$$

It follows that

$$\begin{aligned} \pi_* \widetilde{\Phi}_* (\mathcal{E} \cdot \tau^* \Gamma') &= -\pi_* \widetilde{\Phi}_* (h_{\mathcal{Q}}|_{\widetilde{X^{[2]}}} \cdot \tau^* \Gamma') - \pi_* \widetilde{\Phi}_* (\tau^*(2h_X \hat{\otimes} X - 3\delta) \cdot \tau^* \Gamma') \\ &= -h_X \cdot \pi_* \widetilde{\Phi}_* \tau^* \Gamma' - \pi_* \widetilde{\Phi}_* \tau^* ((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma'). \end{aligned}$$

By Lemma 3.5 we have the first term

$$\pi_* \widetilde{\Phi}_* \tau^* \Gamma' = i_X^* \pi_{P*} \pi_G{}^* \varphi_* \Gamma' - \pi_* \Psi_* \sigma^* \Gamma'$$

Notice that the result of Proposition 3.8 shows that $\pi_* \Psi_* \sigma^* \Gamma' = 0$. In addition, by dimension counting the class $i_X^* \pi_{P*} \pi_G{}^* \varphi_* \Gamma'$ which comes from $\mathbb{P}^{n+1}$ has to be a certain multiple of the class h_X^{n-r-1}. Hence

$$-h_X \cdot \pi_* \widetilde{\Phi}_* \tau^* \Gamma' = \text{ a multiple of } h_X^{n-r}.$$

Similarly it follows Lemma 3.5 and Proposition 3.8 that

$$\begin{aligned} \pi_* \widetilde{\Phi}_* \tau^* ((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma') &= i_X^* \pi_{P*} \pi_G{}^* \varphi_* ((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma') \\ &\quad - \pi_* \Psi_* \sigma^* ((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma') \\ &= \text{ a multiple of } h_X^{n-r} - \pi_* \Psi_* \sigma^* ((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma') \\ &= \text{ a multiple of } h_X^{n-r} - 2\Gamma. \end{aligned}$$

Therefore it yields the required relation

$$2\Gamma + \Psi_P(\gamma) \equiv 0 \bmod \mathbb{Z} \cdot h_X^{n-r}. \tag{3.3.6}$$

For the second cycle relation, we first prove it by assuming Γ is an irreducible closed subvariety, then deduce the conclusion for general algebraic cycles.

Let $g = c_1(\varphi^*\mathscr{E})$ be the divisor of the polarization on $X^{[2]}$ induced by the Plücker polarization on the Grassmannian $G(2, n+2)$. It is known from Lemma 3.7 that

$$g = -h_X \hat{\otimes} X + \delta.$$

Let the notation $\Gamma'' \in \mathrm{CH}_{2r}(X^{[2]})$ denote the same algebraic cycle used in Proposition 3.8. In order to get the formula (ii), we compute the a priori cycle class

$$\pi_* \Phi_*(\mathcal{E} \cdot \tau^*(\Gamma'' \cdot g^{r-1})).$$

Notice that the cycle $\Gamma'' \cdot g^{r-1}$ has dimension $r+1$. By Lemma 3.4 we have

$$\pi_* \tilde{\Phi}_*(\mathcal{E} \cdot \tau^*(\Gamma'' \cdot g^{r-1})) = \Psi_P(\gamma'),$$

where $\gamma' = \pi_{F*} i_2{}^*(\Gamma'' \cdot g^{r-1}) \in \mathrm{CH}_{r-1}(F)$. In the following we show that

$$\pi_* \tilde{\Phi}_*(\mathcal{E} \cdot \tau^*(\Gamma'' \cdot g^{r-1})) = \text{ a multiple of } h_X^{n-r} + (-1)^r (2 \deg \Gamma - 3)\Gamma.$$

Again the cycle class form (3.3.1) of the exceptional divisor $\mathcal{E}$ implies

$$\begin{aligned} \pi_* \tilde{\Phi}_*(\mathcal{E} \cdot \tau^*(\Gamma'' \cdot g^{r-1})) = &- h_X \cdot \pi_* \widetilde{\Phi}_* \tau^*(\Gamma'' \cdot g^{r-1}) \\ &- \pi_* \widetilde{\Phi}_* \tau^*((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma'' \cdot g^{r-1}). \end{aligned}$$

Again Lemma 3.5 shows that the first term above has the form

$$\begin{aligned} \pi_* \widetilde{\Phi}_* \tau^*(\Gamma'' \cdot g^{r-1}) &= i_X^* \pi_{P*} \pi_G{}^* \varphi_*(\Gamma'' \cdot g^{r-1}) - \pi_* \Psi_* \sigma^*(\Gamma'' \cdot g^{r-1}) \\ &= \text{ a multiple of } h_X^{n-r-1} - \pi_* \Psi_* \sigma^*(\Gamma'' \cdot g^{r-1}), \end{aligned}$$

and by the computations in Proposition 3.8 we get

$$\pi_* \Psi_* \sigma^*(\Gamma'' \cdot g^{r-1}) = \pi_* \Psi_* \sigma^*(\Gamma'' \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} (-h_X \hat{\otimes} 1)^k \cdot \delta^{r-k-1}) = 0$$

Use the same argument. The second term turns out to be

$$\begin{aligned} \pi_* \widetilde{\Phi}_* \tau^*((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma'' \cdot g^{r-1}) &= i_X^* \pi_{P*} \pi_G{}^* \varphi_*((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma'' \cdot g^{r-1}) \\ &\quad - \pi_* \Psi_* \sigma^*((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma'' \cdot g^{r-1}) \\ &= \text{ a multiple of } h_X^{n-r} \\ &\quad - \pi_* \Psi_* \sigma^*((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma'' \cdot g^{r-1}) \end{aligned}$$

By the result of Proposition 3.8, it is easy to see that

$$\begin{aligned} &\pi_* \Psi_* \sigma^*((2h_X \hat{\otimes} X - 3\delta) \cdot \Gamma'' \cdot \sum_{k=0}^{r-1} \binom{r-1}{k} (-h_X \hat{\otimes} X)^k \cdot \delta^{r-k-1}) \\ &= (-1)^r 2 \deg \Gamma \cdot \Gamma + 3(-1)^{r+1} \Gamma. \end{aligned}$$

Therefore we conclude that

$$(2\deg\Gamma - 3)\Gamma \equiv \Psi_P(\gamma') \bmod \mathbb{Z}\cdot h_X^{n-r} \tag{3.3.7}$$

for an irreducible closed subvariety Γ. Here we may replace γ' by $(-1)^r\gamma'$

Now we extend the result to a general r-dimensional cycle Γ. Suppose that $\Gamma := \sum n_i\Gamma_i$ with irreducible components Γ_i and degrees e_i. Recall $e = \deg\Gamma$. For each Γ_i, we consider the cycle class $2(e-e_i)\Gamma_i$. The first cycle-relation (3.3.6) asserts that

$$2(e-e_i)\Gamma_i \in \Psi_P\,\mathrm{CH}_{r-1}(F) + \mathbb{Z}\cdot h_X^{n-r}.$$

Add up with the formula (3.3.7) for Γ_i implies

$$(2e-3)\Gamma_i \in \Psi_P\,\mathrm{CH}_{r-1}(F) + \mathbb{Z}\cdot h_X^{n-r}.$$

Add them together shows that there exist a cycle $\gamma' \in \mathrm{CH}_{r-1}(F)$ such that

$$(2e-3)\Gamma + \Psi_P(\gamma') \equiv 0 \bmod \mathbb{Z}\cdot h_X^{n-r}.$$

□

Corollary 3.10. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface of dimension $n \geq 3$ over a field k, and let $F(X)$ be the Fano variety of lines. Denote by h_X the hyperplane section class of X. Assume that there exists a degree one 1-cycle in $\mathrm{CH}_1(X)$. Then the quotient map induced by the cylinder homomorphism*

$$\Psi_P : \mathrm{CH}_{r-1}(F(X)) \to \mathrm{CH}_r(X)/\mathbb{Z}\cdot h_X^{n-r}$$

is surjective for any degree $r \geq 1$.

Proof. Let Γ be any cycle class in $\mathrm{CH}_{r>1}(X)$. The coefficients before the term Γ in the two cycle relations of Theorem 3.9 are coprime, i.e. $(2\deg\Gamma - 3,\ 2) = 1$. Hence, up to a multiple of the hyperplane class h_X^{n-r}, the class Γ is the image of some cycle class on $F(X)$ via the cylinder homomorphism. Notice that Theorem 3.9 does not cover the case of 1-cycles. However, the result for 1-cycles has been proved by Shen, see [58, Proposition 4.2]. □

The results show that, up to the class of hyperplane intersections, any algebraic cycle on X is explicitly recovered by cycles on $F(X)$ via the cylinder homomorphism. In the rest of the chapter, we show that the intersections of hyperplanes is also contained in the image of cylinder homomorphism. In Lemma 3.12, we first prove the claim for algebraically closed fields. Then the following lemma helps to generalize the results to non-closed fields.

Lemma 3.11. *[18, p.599] Let V be an smooth k-variety of dimension d. For any non-empty Zariski open subset $U \subset V$, any zero-cycle on V is rationally equivalent to a zero-cycle supported on U.*

Proof. For the reader's conveniences, we gave the English translation of the proof in [18].

It suffices to prove the case of a closed point $p \in V$ supported on a closed subset Z. Let $(\mathcal{O}_{V,p}, m_p)$ be the local ring at the point p. The local ring is Cohan-Macaulay since V is smooth. Hence, we can find a regular sequence $(g, f_1, \ldots, f_{d-1})$ in m_p such that g vanishes on F. Then the integral curve C defined by the equations $f_1, \ldots, f_{d-1}$ is regular at p and not contained in Z. Let $D \to C$ be the normalization of C. The inverse image of the point $p \in C$ is a single closed point $q \in D$. The composition of maps $\pi : D \to V$ is proper, which induces a homomorphism on Chow groups of zero-cycles. Hence, the 0-cycle p is rationally equivalent to $\pi_*(q)$. The inverse image $W = \pi^{-1}(Z)$ is a finite subset of closed points of D. We take $R := \mathcal{O}_{W,D}$ to be the localization at W, which is a semi-local ring. It is known that the Picard group of a semi-local ring is trivial. Then the divisor q of D is a principal divisor, i.e. there exists an element f in the fraction field $K(R) = K(D)$ such that $\operatorname{div}(f) = q$ on $\operatorname{Spec}(R)$. It implies that $\operatorname{div}(f) = z + q$ on D where z is a zero-cycle supported on $D \setminus W$. Therefore, p is rationally equivalent to a zero-cycle on V away from F. □

Lemma 3.12. *Let X be a smooth hypersurface of dimension $n \geq 3$ over a field k, and let $H_X \in CH^1(X)$ be the class of hyperplane section. Assume that X contains a line L defined over k. Then for any $i > 1$, the class H_X^i is contained in the image of the cylinder homomorphism*

$$q_*p^* : \mathrm{CH}^{n+i-3}(F(X)) \to \mathrm{CH}^i(X).$$

Here $p : P \to F$ and $q : P \to X$ are natural projections.

Proof. Assume k is algebraically closed. Let x be a closed point of X. We denote by

$$C_x := \{[\ell] \in F(X) \mid x \in \ell\}$$

the closed subsvariety of $F(X)$ parameterizing lines in X passing through x. Let ℓ be a line on X. We denote by

$$S_\ell := \{\ell' \in F(X) \mid \ell' \cap \ell \neq \emptyset \text{ for a generic } \ell \subset X\}.$$

the closed subvariety of $F(X)$ parameterizing lines in X meeting with ℓ. We claim that

$$q_*p^*[C_x] = 2H_X^2, \text{ and } q_*p^*[S_\ell] = 5H_X.$$

We may assume $x = [1, 0, \ldots, 0] \in X$. Then the cubic equation defining X has the form

$$X_0^2 L(X_1, \ldots, X_{n+1}) + X_0 Q(X_1, \ldots, X_{n+1}) + C(X_1, \ldots, X_{n+1})$$

with homogeneous polynomial L(resp. Q, C) of degree 1(resp. 2, 3). So the lines in X passing through x is parametrized by the set of points in $\mathbb{P}^n$ cut out by the equations $L = Q = C = 0$. If x is a generic point, the dimension of C_x is $n-3$. It is easy to verify that the closed subvariety $q_*p^*C_x$ of X is cut out by the equations $L = Q = 0$. Therefore we obtain $q_*p^*[C_x] = 2H_X^2 \in \mathrm{CH}^2(X)$.

If ℓ is a general line, the dimension of S_ℓ is $n-2$. Therefore the algebraic cycle $q_*p^*[S_\ell]$ is equal to $m \cdot H_X$ for some integer m. The integer m can be determined by a result of Shen [57, Lemma 3.10]. One consequence of the result says that there are exactly 5 lines on X meeting two distinct general lines on X. Let $\tilde{\ell}$ be another general line on X. Then it is direct to see

$$\begin{aligned} m &= q_*p^*[S_\ell] \cdot \tilde{\ell} = [S_\ell] \cdot p_*q^*\tilde{\ell} \\ &= \#\{\text{lines meeting } \ell \text{ and } \tilde{\ell}\} \\ &= 5. \end{aligned}$$

Therefore we have $q_*p^*[S_\ell] = 5H_X$.

For hyperplane section classes of higher codimensions, we take the above closed subvarieties in complete intersections of X. Let X_{n-i} be the i-th complete intersection of generic hyperplanes of X. We define

$$C_{x,X_{n-i}} := \{[\ell] \in F(X) \mid x \in \ell \subset X_{n-i}\}$$

to be the closed subvarieties which parametrizes the lines in X_{n-i} passing through a generic point $x \in X_{n-i}$. Hence $C_{x,X_{n-i}}$ is contained in $F(X_{n-i})$, and we have

$$q_*p^*[C_{x,X_{n-i}}] = 2j_*H^2_{X_{n-i}} = 2H_X^{i+2},\ j : X_{n-i} \hookrightarrow X. \tag{3.3.8}$$

Similarly let X_{n-j} be the j-th complete intersection of generic hyperplanes of X. We define

$$S_{\ell,X_{n-j}} := \{[\ell'] \in F(X) \mid \ell' \subset X_{n-j}, \ell' \cap \ell \neq \emptyset \text{ for a generic } \ell \subset X_{n-j}\}$$

to be the closed subvarieties which parametrizes the lines on X_{n-j} meeting a generic line ℓ on X_{n-j}. Hence, we have

$$q_*p^*[S_{\ell,X_{n-j}}] = 5j_*H_{X_{n-j}} = 5H_X^{j+1},\ \text{for } n-j \geq 3. \tag{3.3.9}$$

Therefore, for any $i > 1$, the image of the cycle

$$3[C_{x,X_{n+2-i}}] - [S_{\ell,X_{n+1-i}}] \in \mathrm{CH}^{n+i-3}(F(X))$$

equals H_X^i.

Now let us prove it for arbitrary field. In fact, through the above argument, we can see that if x is a generic point in X, the subvariety C_x has dimension $n-3$ and the cycle $q_*p^*[C_x]$ remains equal to $2H_X^2$ since the equations L, Q, C intersect properly. Similarly, if ℓ is a general line in X, the intersection number $q_*p^*[S_\ell] \cdot \tilde{\ell}$ is invariant under base change. Hence we still have $q_*p^*[S_\ell] = 5H_X$.

By the assumption, let L be the k-line on X. Choose any k-rational point $p \in L$. It follows from Lemma 3.11 that p is rationally equivalent to a generic zero-cycle z of degree one. We define the algebraic cycle $[C_p] := [C_z] = p_*q^*z \in CH_{n-3}(X)$. Since the support of z consists of generic points in X and degree $z = 1$ we obtain $q_*p^*[C_p] = 2H_X^2$.

We regard L as a closed point of $F(X)$. Use Lemma 3.11 again, L is rationally equivalent to $\sum n_i\ell_i$, a sum of general lines. Let us denote the cycle $[S_L]$ to be $\sum n_i S_\ell$. Then $p_*q^*[S_L] = 5H_X$. □

Lemma 3.13. *Let $X \subset \mathbb{P}(V)$ be a smooth hypersurface of dimension $n \geq 3$, let $F(X)$ be the Fano scheme of lines, and let $\mathscr{E}$ be the rank two universal quotient sheaf on $F(X)$. Denote by*

$$g = c_1(\mathscr{E}), c = c_2(\mathscr{E})$$

the Chern classes of $\mathscr{E}$. Then we have

$$\Psi_P(g^{n-3}) = 6[X].$$

As a consequence, let Y be any generic hyperplane section of a smooth cubic X of dimension $n \geq 4$, then

$$\Psi_P(g^{n-4} \cap [F(Y)]) = 6H_X$$

where H_X is the hyperplane section class of X.

Proof. There is a simple a priori observation related to the conclusion: there are 6 distinct lines passing through a generic closed point of a smooth cubic threefold X.

By abuse of notations, we also use $\mathscr{E}$(resp. g, c) to denote the rank two universal quotient sheaf(resp. Chern classes) on the Grassmannian of lines $\mathbf{Gr}(2, V)$. Recall the diagram (3.2.3). The projective bundle $\mathbb{P}(\mathscr{E})$ over $\mathbf{Gr}(2, V)$ can be viewed as the flag scheme

$$\mathbf{Gr}(1, 2, V) := \{([\ell], x) \in \mathbf{Gr}(2, V) \times \mathbb{P}(V) \mid x \in \ell\}.$$

Let $\pi_G : \mathbf{Gr}(1, 2, V) \to \mathbf{Gr}(2, V)$ and $q : \mathbf{Gr}(1, 2, V) \to \mathbb{P}(V)$ be natural projections. Denote by $\mathcal{S}$ be the tautological rank two vector bundle associated to $\mathscr{E}$. There exists the canonical filtration of vector bundles

$$q^*\mathcal{O}_{\mathbb{P}(V)}(-1) \subset \pi_G^*\mathcal{S} \subset V_{\mathbf{Gr}(1,2,V)}$$

on the flag scheme $\mathbf{Gr}(1, 2, V)$. Then $q : \mathbf{Gr}(1, 2, V) \to \mathbb{P}(V)$ is the projective bundle of the vector bundle $V/\mathcal{O}_{\mathbb{P}(V)}(-1)$

By the canonical filtration, the tautological line bundle $\mathcal{O}_q(-1)$ of the projection q is isomorphic to $\pi_G^*\mathcal{S}/q^*\mathcal{O}(-1)$. Moreover, since $q^*\mathcal{O}(1) \cong \mathcal{O}_{\mathscr{E}}(1)$ we obtain the exact sequence

$$0 \to \mathcal{O}_q(1) \to \pi_G^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1) \to 0$$

on $\mathbf{Gr}(1, 2, V)$. Denote by ζ the first Chern class of the lines bundle $\mathcal{O}_q(1)$. From the exact sequence we have

$$\pi_G^* g = \zeta + q^*H, c = \zeta \cdot q^*H. \tag{3.3.10}$$

where H is the hyperplane class on $\mathbb{P}(V)$.

The cycle class g^{n-3} lies in $\mathrm{CH}_{n-1}(F(X))$. Hence the cylinder homomorphism sends g^{n-3} to $m[X]$, a multiple of the fundamental class. To determine the m it suffices to calculate $i_{X*}\Psi_P(g^{n-4} \cdot c) = i_{X*}(m[X]) = 3m \cdot H$ where $i_X : X \hookrightarrow \mathbb{P}(V)$ is the inclusion. Consider the natural commutative diagram

$$\begin{tikzcd}
 & P_X \arrow[r, "\pi_X"] \arrow[d, hook] & X \arrow[d, hook, "i_X"] \\
P \arrow[ur, hook, "i_1"] \arrow[r, hook] \arrow[d, "p"] & \mathbf{Gr}(1,2,V) \arrow[r, "q"] \arrow[d, "\pi_G"] & \mathbb{P}(V) \\
F(X) \arrow[r, hook] & \mathbf{Gr}(2,V).
\end{tikzcd}$$

that is similar to 3.2.3. It is easy to see that

$$\begin{aligned} i_{X*}\Psi_P(g^{n-3}) &= i_{X*}\pi_{X*}i_{1*}(\pi_G^*(g^{n-3})|_P) \\ &= q_*(\pi_G^*(g^{n-3}) \cap [P]) \end{aligned}$$

The cycle class of the Fano scheme of lines $F(X)$ in $\mathbf{Gr}(2, V)$ is equal to

$$9c(2g^2 + c),$$

see [2, Proposition 1.6]. Hence we need to compute the cycle class

$$q_*\pi_G^*(18g^{n-1} \cdot c + 9g^{n-3} \cdot c^2).$$

By the expressions of the Chern classes (3.3.10), the above cycle class is equal to

$$\begin{aligned} &q_*(18(\zeta + q^*H)^{n-1} \cdot q^*H \cdot \zeta + 9(\zeta + q^*H)^{n-3} \cdot q^*H^2 \cdot \zeta^2) \\ =&18q_*\zeta^n \cdot H + (18n-9)q_*\zeta^{n-1} \cdot H^2 + \cdots . \end{aligned}$$

Since $q : \mathbb{P}(V_{\mathbf{Gr}(1,2,V)}/q^*\mathcal{O}(-1)) \to \mathbb{P}(V)$ is a $\mathbb{P}^n$-bundle we know $q_*\zeta^i$ vanishes for $i < n$ and $q_*\zeta^n = [X]$. Therefore, we obtain

$$i_{X*}\Psi_P(g^{n-3}) = 18H$$

and $\Psi_P(g^{n-3}) = 6[X]$.

Assume $n \geq 4$. A generic hyperplane section Y of X is smooth. Then the cycle $\Psi_P(g^{n-4} \cap [F(Y)])$ supported on Y is equal to $6[Y] = 6H_X$. The last assertion follows. □

Corollary 3.14. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface of dimension $n \geq 3$ over a field k, and let F be the Fano variety of lines. Assume that X has a line L defined over k. Then the cylinder homomorphism*

$$\Psi_P : \mathrm{CH}_{r-1}(F) \to \mathrm{CH}_r(X)$$

is surjective for all integer $r < n-1$. Moreover, if X is not a cubic threefold, the assertion also holds for divisor classes, i.e. $r = n-1$.

Proof. Corollary 3.10 shows that the cylinder homomorphism is surjective up to the classes of the intersection of hyperplanes. By Lemma 3.12 and Lemma 3.13, we conclude that the classes of the intersection of hyperplanes are contained in the image of cylinder homomorphisms except for the case of the divisor classes of cubic threefolds. □

Remark 7. For a generic cubic threefold X, the Néron-Severi group $NS(F(X))$ is rank one, see [37, p. 154]. Hence it is generated by the primitive class S_ℓ that represents the lines on X passing through a fixed line ℓ. Then the image of cylinder homomorphism does not generate the whole group of divisor classes in the case.

3.4 Integral Hodge conjectures and Tate conjectures for 1-cycles

Let X be a smooth complex cubic 4-fold. According to Beauville-Donagi [8], the Fano variety of lines $F(X)$ is irreducible holomorphic symplectic of $K3$ type. In the section, we apply the main result 3.14 to prove the integral Hodge conjecture for 1-cycles on $F(X)$ (Corollary 3.16). The proof is deduced from the integral Hodge conjecture for 2-cycles on X by Voisin [71].

We notice that Mongardi and Ottem proved a stronger statement in [51]. They showed that the integral Hodge conjecture holds for 1-cycles on irreducible holomorphic symplectic varieties of $K3$ type and of generalized Kummer type. Their result can reprove the integral Hodge conjecture for 2-cycles on X.

On the arithmetic side, let X be a smooth cubic 4-fold defined over a finitely generated field k with $\mathrm{char}(k) \neq 2, 3$. Charles-Pirutka [16] proved the integral Tate conjecture holds for 2-cycles on X. Using the étale Abel-Jacobi isomorphism(Lemma 3.17), the same strategy we use in the analytic case asserts the integral Tate conjecture for 1-cycles on $F(X)$ (Corollary 3.19).

We begin with the Abel-Jacobi isomorphism due to Beauville and Donagi [8, Prop. 6]

Proposition 3.15. *Let X be a smooth complex cubic fourfold, and let $F(X)$ be the Fano variety of lines of X. Let P be the universal family of lines (cf.* (3.1.2)*) with the natural projectionS $p : P \to F(X)$ and $q : P \to X$. Then the Abel-Jacobi mapping*

$$p_*q^* : H^4(X, \mathbb{Z}) \to H^2(F(X), \mathbb{Z})$$

is an isomorphism of Hodge structures.

Remark 8. The proposition [8, Prop. 6] is stronger. It further proves the map q^*p_* is an Hodge isometry on the primitive parts.

Corollary 3.16. *Let X be a smooth complex cubic fourfold. The integral Hodge conjecture holds for* 1*-cycles on the Fano variety of lines $F(X)$.*

Proof. Let $\alpha \in H^6(F(X), \mathbb{Z}) \cap H^{3,3}(F(X))$ be an integral Hodge class of type $(3, 3)$. The cylinder homomorphism

$$\Psi_P = q_*p^* : H^6(F(X), \mathbb{Z}) \to H^4(X, \mathbb{Z})$$

is a morphism of Hodge structures. Then $q_*p^*\alpha \in H^4(X, \mathbb{Z}) \cap H^{2,2}(X, \mathbb{Z})$ is a Hodge class of type $(2, 2)$. Voisin [71] proved the integral Hodge conjecture for 2-cycles on X. Hence there exists a 2-cycle $\gamma \in \mathrm{CH}^2(X)$ such that the cohomology class $[\gamma]$ equals $q_*p^*\alpha$. By our main result 3.14 on surjectivity of cylinder homomorphisms, there exists an algebraic cycle $\Gamma \in \mathrm{CH}_1(F(X))$ such that $q_*p^*\Gamma = \gamma$.

Then we have $[\Gamma] - \alpha \in \mathrm{Ker}(q_*p^*)$. By the Poincaré duality, the cylinder homomorphism q_*p^* is dual to the Abel-Jacobi isomorphism

$$p_*q^* : H^4(X, \mathbb{Z}) \to H^2(F(X), \mathbb{Z}).$$

Therefore the map q_*p^* is an isomorphism of Hodge structures and the integral Hodge class $\alpha = [\Gamma]$ is algebraic. □

In higher dimensions, Shimada's result 3.1 indicates that our main result also intimately connect the Hodge conjectures for cubics and the Fano varieties of lines. Suppose that n is even. Let X be a complex smooth cubic of dimension n. It follows from Lemma 2.1 that the Fano variety of lines $F(X)$ is a smooth of dimension $2n - 4$. Recall Theorem 3.1. The kernel of the cylinder homomorphism

$$\Psi \otimes \mathbb{Q} : H^{3n-6}(F(X), \mathbb{Q}) \to H^n(X, \mathbb{Q})$$

consists of algebraic classes. Then the surjectivity result concludes that the Hodge conjecture is true for $(\frac{n}{2} - 1)$-cycles on $F(X)$ if the Hodge conjecture holds for n-cycles on X, e.g. the smooth cubic 8-fold [62]. Moreover, it is interesting to know whether the kernel of the $\mathbb{Z}$-linear cylinder homomorphism Ψ is generated by algebraic classes.

Tate Conjecture.

Let $F(X)$ be the Fano variety of lines on a smooth cubic fourfold X/k. If k is a number field or a finite field, the Tate conjecture for divisors on $F(X)$ is affirmative by [3] and [15]. By Deligne's Hard Lefschetz theorem on ℓ-adic cohomology, the Tate conjecture for 1-cycles on F holds true. Our goal is to generalize the conclusion to the integral Tate conjecture 2.3.6.

A smooth hypersurface X defined over a field k of positive characteristic can always be lifted to characteristic zero. To be precise, let R be any complete local ring of mixed characteristics with the residue field k, e.g. the Witt ring of k. The projective space $\mathbb{P}^N_R := \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^n_R}(d))^\vee)$ is the universal parameter space of degree d hypersurfaces with R-coefficients. Let U be the open subset of $\mathbb{P}^N_R$ that parametrizes smooth hypersurfaces. The smooth hypersurface X corresponds to a unique k-point in U. Since U is smooth over $\operatorname{Spec} R$, the generic point of U is of characteristic zero.

Lemma 3.17. *Let k be a finite or number field. Let X be a smooth cubic fourfold over k and $F := F(X)$ be the Fano scheme of lines. Let ℓ be a prime number different from $\operatorname{char} k$. Then the cylinder homomorphism on the étale cohomology*

$$q_*p^* : H^6_{\text{et}}(F_{\bar{k}}, \mathbb{Z}_\ell(3)) \to H^4_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2)) \tag{3.4.1}$$

is an isomorphism of $\operatorname{Gal}(\bar{k}/k)$*-modules.*

Proof. Suppose k is a finite field. Let R be any complete local ring of mixed characteristics whose residue field is k. There exists a family of smooth cubic hypersurfaces over R that lifts X. The lifting can be obtained by the universal parameter space U in the above. By the description locally U is the affine scheme of a finitely generated smooth R-algebra. It follows from the smooth lifting property that the map

$$\operatorname{Hom}_R(\operatorname{Spec} R, U) \to \operatorname{Hom}_R(\operatorname{Spec} k, U)$$

is surjective. Hence the k-point determined by X lifts to an R-point of U, which induces a family of smooth cubic hypersurfaces

$$\pi : \mathcal{X} \to \operatorname{Spec} R$$

Let $\psi : \mathcal{F} \to \operatorname{Spec} R$ be the family of Fano schemes of lines relatively to π. The morphism ψ is projective and smooth of relative dimension 4. Denoted by $\mathcal{P} \subset \mathcal{F} \times \mathcal{X}$ the relative incidence scheme. Consider the commutative diagram

$$\begin{tikzcd} & \mathcal{P} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{F} \arrow[dr, "\psi"'] & & \mathcal{X} \arrow[dl, "\pi"] \\ & \operatorname{Spec} R & \end{tikzcd}.$$

The prime number ℓ is invertible in R. Consider the relative cylinder homomorphism of étale sheaves

$$q_*p^* : R^6\psi_*\mathbb{Z}_\ell(3) \to R^4\pi_*\mathbb{Z}_\ell(2).$$

At the closed point of $\operatorname{Spec} R$ it equals the homomorphism (3.4.1).

The characteristic of R is zero. Hence we can embed its fraction field into the complex numbers. Let us denote by $\bar{\eta} : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} R$ the geometric point with respect to the embedding. By the proper and smooth base change theorem [49, Cor. 4.2], it suffices to prove the cylinder homomorphism at $\bar{\eta}$, i.e.

$$q_*p^*|_{\bar{\eta}} : H^6_{\text{et}}(F_{\mathbb{C}}, \mathbb{Z}_\ell(3)) \to H^4_{\text{et}}(X_{\mathbb{C}}, \mathbb{Z}_\ell(2))$$

is an isomorphism.

Note that X and F are smooth and proper over $\mathbb{C}$. Then it follows from the Comparison Theorem 2.9 that

$$H^6_{\text{et}}(F_{\mathbb{C}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^6_{\text{sing}}(F_{\text{an}}, \mathbb{Z}/\ell^n\mathbb{Z}),$$
$$H^4_{\text{et}}(X_{\mathbb{C}}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong H^4_{\text{sing}}(X_{\text{an}}, \mathbb{Z}/\ell^n\mathbb{Z}), \forall n.$$

By the functorial properties of the comparison theorem [26, §11], what remains to prove is the cylinder homomorphism with finite coefficients

$$\Psi_P : H^6(F_{an}, \mathbb{Z}/\ell^n\mathbb{Z}) \to H^4(X_{an}, \mathbb{Z}/\ell^n\mathbb{Z}), \forall n > 0. \tag{3.4.2}$$

are isomrophisms. Notice that X and F have no singular cohomology groups of odd degree. Then the isomorphism of (3.4.2) follows from the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot \ell^n} \mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z} \to 0$.

The homorphism q_*p^* (3.4.1) is defined by the universal family P as a correspondence. Hence q_*p^* is naturally compatible with the Galois actions.

When k is a number field, the fields $\bar{k} = \bar{\mathbb{Q}} \subset \mathbb{C}$ are separably closed. Then a corollary [49, Cor. 4.3] of the smooth and proper base change concludes that

$$H^{2i}_{et}(X_{\bar{k}}, \mathbb{Z}_\ell(i)) \cong H^{2i}_{et}(X_{\mathbb{C}}, \mathbb{Z}_\ell(i)), \forall i.$$

as well as F. Using the comparison theorem again, the assertions for the case of number fields follow from the same arguments above. □

Charles and Pirutka proved the integral Tate conjecture for codimension 2-cycles of cubic fourfolds.

Theorem 3.18. *[16, Theorem 1.1] Let k be a field finitely generated over its prime subfield with $\operatorname{char}(k) \neq 2, 3$, and let $X \subset \mathbb{P}^5_k$ be a smooth cubic fourfold. Let k_s be the separable closure of k, and let $\bar{X} = X \times_k k_s$ be the base change. Then for any prime number $\ell \neq \operatorname{char} k$, the cycle class map*

$$cl^2_{\bar{X},\mathbb{Z}_\ell} : \mathrm{CH}^2(\bar{X}) \otimes \mathbb{Z}_\ell \to \varinjlim_{k \subset k'} H^4_{\text{et}}(\bar{X}, \mathbb{Z}_\ell(2))^{G_{k'}}$$

is surjective, where k' runs over the intermediate fields of $k \subset k_s$ that finite over k.

Corollary 3.19. *Let X be a smooth cubic fourfold in $\mathbb{P}^5_k$ over a field k finitely generated over its prime subfield with $\operatorname{char}(k) \neq 2, 3$. Let $F := F(X)$ be the Fano variety of lines on X, and let $\bar{F} = F \times_k k_s$ be the base change. Then for any prime number $\ell \neq \operatorname{char} k$, the cycle class map*

$$cl^3_{\bar{F},\mathbb{Z}_\ell} : \mathrm{CH}^3(\bar{F}) \otimes \mathbb{Z}_\ell \to \varinjlim_{U} H^6_{\text{et}}(\bar{F}, \mathbb{Z}_\ell(3))^{G_{k'}} \tag{3.4.3}$$

is surjective, where k' runs over the intermediate fields of $k \subset k_s$ that finite over k. In the other words, the integral Tate conjecture for 1-cycles of F is true.

Proof. As pointed out in [16, p. 3], it suffices to prove the result for the finite extension of the prime subfield of k.

Let $\alpha \in H^6_{\text{et}}(\bar{F}, \mathbb{Z}_\ell(3))$ be any cohomology class fixed by some open subgroup $G_{k'}$. It follows from Theorem 3.18 that the class $q_*p^*\alpha \in H^4_{\text{et}}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^{G_{k'}}$ lifts to a 2-cycle $\Gamma \in \mathrm{CH}^2(\bar{X}) \otimes \mathbb{Z}_\ell$. By the surjectivity of the cylinder homomorphism, there is an 1-cycle $\gamma \in \mathrm{CH}_1(\bar{F}) \otimes \mathbb{Z}_\ell$ such that $q_*p^*\gamma = \Gamma$. By Lemma 3.17, the étale cylinder homomorphism $q_*p^* : H^6_{\text{et}}(\bar{F}, \mathbb{Z}_\ell(3))^{G_{k'}} \to H^4_{\text{et}}(\bar{X}, \mathbb{Z}_\ell(2))^{G_{k'}}$ is an isomorphism for all open subgroups $G_{k'}$. Thus α is algebraic represented by γ. □

Chapter 4

Motives of the Hilbert square of cubic hypersurface

4.1 Introduction

Let X be a smooth cubic hypersurface defined over a field k and $F(X)$ the Fano scheme of lines. In Section 2.4 we discussed the remarkable relation (2.4.5)

$$\mathbb{L}^2 \cdot [F(X)] + [X] \cdot [\mathbb{P}^n] = [X^{[2]}]$$

in the Grothendieck ring of varieties. This chapter aims to investigate the motivic nature of this relation. We notice that Laterveer [42] had proved the parallel relation for the Chow motives

$$\mathfrak{h}(F(X))(-2) \oplus \bigoplus_{i=0}^{\dim X} \mathfrak{h}(X)(-i) \simeq \mathfrak{h}(X^{[2]}). \tag{4.1.1}$$

Our main result is a formula of the decomposition of the diagonal (4.1.5) of the Hilbert squares of cubic hypersurfaces. To present our main statement we work in the settings of (3.2.1). The key ingredient is again the birational map (2.4.7)

$$\Phi : X^{[2]} \dashrightarrow P_X$$

that occurred repeatedly in the previous chapters. Let the graph closure $[\overline{\Gamma}_\Phi]$ be the degree 0 algebraic correspondence from $X^{[2]}$ to P_X. The transpose of $[\overline{\Gamma}_\Phi]$ is the graph closure $[\overline{\Gamma}_{\Phi^{-1}}]$ of the inverse map. For simplicity we write

$$\Phi_* := [\overline{\Gamma}_\Phi], \ \Phi^* := [\overline{\Gamma}_{\Phi^{-1}}].$$

In Lemma (4.10) we will prove that

$$\Phi_* \circ \Phi^* = \Delta_{P_X}. \tag{4.1.2}$$

By Lemma (2.5), the composition of correspondences $\Phi^* \circ \Phi_*$ is a projector on the Chow motive $X^{[2]}$. Meanwhile, the Chow motive $\mathfrak{h}(P_X)$ is isomorphic to, via the homomorphism Φ^*, the direct summand $(X^{[2]}, \Phi^* \circ \Phi_*, 0)$ of $\mathfrak{h}(X^{[2]})$. Therefore we have a direct sum decomposition

$$(X^{[2]}, \Delta_{X^{[2]}} - \Phi^* \circ \Phi_*, 0) \oplus \mathfrak{h}(P_X) \cong \mathfrak{h}(X^{[2]}). \tag{4.1.3}$$

Recall the picture (3.2.1). The relative symmetric product P_2 over the Fano scheme F is an algebraic correspondence of degree -2

$$[P_2] \in \mathrm{Corr}^{-2}(X^{[2]}, F).$$

We prove in Lemma 4.10 that

$$[P_2] \circ {}^{\top}[P_2] = \Delta_F. \tag{4.1.4}$$

The last strike is the following statement

Theorem 4.1. *(cf. Theorem 4.12) With the above notations, the diagonal class of the Hilbert square $X^{[2]}$ satisfies the formula*

$$\Delta_{X^{[2]}} = \Phi^* \circ \Phi_* + {}^{\top}[P_2] \circ [P_2] \tag{4.1.5}$$

in $\mathrm{CH}(X^{[2]} \times X^{[2]})$.

As a consequence, the Chow motive of the Fano scheme of lines $\mathfrak{h}(F)(-2)$ is isomorphic to the complement direct summand $(X^{[2]}, \Delta_{X^{[2]}} - \Phi^* \circ \Phi_*, 0)$ in (4.1.3). The expected motivic relation F and $X^{[2]}$ is explicitly given by the isomorphism

$${}^{\top}[P_2] + \Phi^* : \mathfrak{h}(F)(-2) \oplus \mathfrak{h}(P_X) \xrightarrow{\sim} \mathfrak{h}(X^{[2]}). \tag{4.1.6}$$

with the inverse map $([P_2], \Phi_*)$.

4.2 Chern classes of normal bundles

The key technique we use to prove the formulae (4.1.2), (4.1.4) and (4.1.5) is again the double blow-up formula A.7. To carry out our proof, it is crucial to know the Chern classes of associated normal bundles N_{P/P_X} and $N_{P_2/X^{[2]}}$.

On the one hand, the exceptional divisor $\mathcal{E}$ of the blowing up $\tau : \widetilde{X^{[2]}} \to X^{[2]}$ is isomorphic to $\mathbb{P}(N_{P_2/X^{[2]}})$. On the other hand, recall Proposition 3.3 that $\mathcal{E}$ is isomorphic to the fiber product $P_2 \times_F P$. Hence the normal sheaf $\mathcal{N}_{P_2/X^{[2]}}$ differs from the locally free sheaf $\pi_F^*(\mathscr{E}^\vee)$ by a line bundle in $\mathrm{Pic}(P_2)$. The aim of this section is to determine the normal sheaves $\mathcal{N}_{P_2/X^{[2]}}$ and $\mathcal{N}_{P/P_X}$ in terms of the universal quotient sheaf $\mathscr{E}$ and compute their Chern classes. By Proposition 4.5 and 4.8 the main conclusion is

1. the normal sheaf $\mathcal{N}_{P_2/X^{[2]}}$ is isomorphic to $\pi_F^*(\mathscr{E}^\vee \otimes \mathcal{O}_F(3)) \otimes \mathcal{O}_{S^2\mathscr{E}}(-1)$,
2. the normal sheaf $\mathcal{N}_{P/P_X}$ is isomorphic to $p^*(\mathrm{Sym}^2 \mathscr{E}^\vee \otimes \mathcal{O}_F(3)) \otimes \mathcal{O}_{\mathscr{E}}(-1)$.

Lemma 4.2. *[36, §V, Prop. 2.6] Let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the $\mathbb{P}^1$-bundle of a rank two locally free sheaf $\mathscr{E}$, and let $\mathcal{O}_{\mathscr{E}}(1)$ be the canonical line bundle on $\mathbb{P}(\mathscr{E})$. Suppose $\sigma : X \to \mathbb{P}(\mathscr{E})$ is a section determined by a surjective map*

$$\mathscr{E} \to \mathscr{L}$$

for some invertible sheaf $\mathscr{L}$ on X. Denote by D the image of the section σ and $\mathcal{N}$ the kernel of the map $\mathscr{E} \to \mathscr{L}$. Then we have

$$\pi_*(\mathcal{O}_{\mathscr{E}}(1) \otimes \mathcal{O}(-D)) \simeq \mathcal{N}$$

and

$$\mathcal{O}_{\mathscr{E}}(1) \otimes \mathcal{O}(-D) \simeq \pi^* \mathcal{N}.$$

Corollary 4.3. *Let $\mathscr{E}$ be locally free sheaf of rank two over a scheme S. Denote by $\mathcal{O}_E(1)$ the canonical line bundle on the projective bundle $\mathbb{P}(\mathscr{E})$. Let $\Delta_{\mathbb{P}(\mathscr{E})}$ be the relative diagonal of the $\mathbb{P}^1 \times \mathbb{P}^1$-bundle $\pi_S : \mathbb{P}(\mathscr{E}) \times_S \mathbb{P}(\mathscr{E}) \to S$. Then line bundle associated to the relative diagonal is*

$$\mathcal{O}(\Delta_{\mathbb{P}(\mathscr{E})}) = pr_1^*\mathcal{O}_{\mathscr{E}}(1) \otimes pr_2^*\mathcal{O}_{\mathscr{E}}(1) \otimes \pi_S^* \det(\mathscr{E})^\vee \tag{4.2.1}$$

Proof. Denote by $q : \mathbb{P}(\mathscr{E}) \to S$ the projection. The self product space $\mathbb{P}(\mathscr{E}) \times_S \mathbb{P}(\mathscr{E})$ can be regarded as the tautological projective bundle $\mathbb{P}(q^*\mathscr{E})$ over $\mathbb{P}(\mathscr{E})$. Then the relative diagonal embedding $\mathbb{P}(\mathscr{E}) \hookrightarrow \mathbb{P}(\mathscr{E}) \times_S \mathbb{P}(\mathscr{E})$ is exactly the tautological section

$$\begin{aligned} s : \mathbb{P}(\mathscr{E}) &\to \mathbb{P}(q^*\mathscr{E}) \\ ([l], s) &\mapsto (([l], s), [l]) \end{aligned}$$

that corresponds to the canonical surjection $q^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1)$. Let us identify the structure map $\mathbb{P}(q^*\mathscr{E}) \to \mathbb{P}(\mathscr{E})$ with the first projection $pr_1 : \mathbb{P}(\mathscr{E}) \times_S \mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E})$. Hence it follows from Lemma 4.2 that

$$\mathcal{O}_{q^*\mathscr{E}}(1) \otimes \mathcal{O}(-\Delta_{\mathbb{P}(\mathscr{E})}) \cong pr_1^*(\mathcal{N}),\ \mathcal{N} \text{ the kernel of } q^*\mathscr{E} \to \mathcal{O}_E(1)$$

It is easy to see that $\mathcal{N}$ is isomorphic to $q^* \det \mathscr{E} \otimes \mathcal{O}_{\mathscr{E}}(-1)$. Therefore we have

$$\begin{aligned} \mathcal{O}(\Delta_{\mathbb{P}(\mathscr{E})}) &= pr_1^*(q^* \det \mathscr{E}^\vee \otimes \mathcal{O}_{\mathscr{E}}(1)) \otimes \mathcal{O}_{q^*\mathscr{E}}(1) \\ &= \pi_S^* \det(\mathscr{E})^\vee \otimes pr_1^*\mathcal{O}_{\mathscr{E}}(1) \otimes pr_2^*\mathcal{O}_{\mathscr{E}}(1). \end{aligned}$$

□

Let $\mathbb{P}^1$ be the projective line. It is a classical result that the n-th symmetric power $(\mathbb{P}^1)^{(k)}$ is isomorphic to $\mathbb{P}^k$. Let us view $\mathbb{P}^1$ as $\mathbb{P}\Gamma(\mathcal{O}_{\mathbb{P}^1}(1))$. Then the quotient map by the symmetric group $\mathfrak{S}_k$

$$\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{k} \longrightarrow (\mathbb{P}^1)^{(k)}$$

can be rephrased as the product of k linear forms

$$\underbrace{\mathbb{P}\Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \times \cdots \times \mathbb{P}\Gamma(\mathcal{O}_{\mathbb{P}^1}(1))}_{k} \longrightarrow \mathbb{P}\Gamma(\mathcal{O}_{\mathbb{P}^1}(k)) \cong \mathbb{P}^k,$$

see [23, Proposition 10.5] for details. This alternative description has a relative counterpart for $\mathbb{P}^1$-bundles. Let $\pi_S : \mathbb{P}(\mathscr{E}) \to S$ be a $\mathbb{P}^1$-bundle over a scheme S. The relative k-th symmetric power of $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\pi_{S*}\mathcal{O}_{\mathscr{E}}(1))$ over S is isomorphic to $\mathbb{P}(\mathrm{Sym}^k \mathscr{E}) \cong \mathbb{P}(\pi_{S*}\mathcal{O}_{\mathscr{E}}(k))$. Then the realtive quotient map is given by

$$\pi : \underbrace{\mathbb{P}(\pi_{S*}\mathcal{O}_{\mathscr{E}}(1)) \times \cdots \times \mathbb{P}(\pi_{S*}\mathcal{O}_{\mathscr{E}}(1))}_{k} \to \mathbb{P}(\pi_{S*}\mathcal{O}_{\mathscr{E}}(k)). \tag{4.2.2}$$

Lemma 4.4. *Let $\mathcal{O}_{S^k\mathscr{E}}(1)$ be the canonical line bundle on $\mathbb{P}(\mathrm{Sym}^k \mathscr{E})$. Then we have*

$$\pi^*\mathcal{O}_{S^k\mathscr{E}}(1) = \underbrace{\mathcal{O}_{\mathscr{E}}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathscr{E}}(1)}_{k}$$

Proof. Let $\pi_S^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1)$ be the canonical surjection on $\mathbb{P}(\mathscr{E})$. By pulling back the surjections via projections $\mathbb{P}(\mathscr{E}) \times_S \cdots_S \times \mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E})$, we obtain the following $\mathfrak{S}_k$-equivariant surjective homomorphism

$$\pi_S^*\mathscr{E} \boxtimes \cdots \boxtimes \pi_S^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathscr{E}}(1).$$

By abuse of notation, we also use π_S to denote the projection $\mathbb{P}(\mathscr{E}) \times_S \cdots_S \times \mathbb{P}(\mathscr{E}) \to S$. Then there is $\pi_S^*\mathscr{E}^{\otimes k} = \pi_S^*\mathscr{E} \boxtimes \cdots \boxtimes \pi_S^*\mathscr{E}$. Since $\mathcal{O}_{\mathscr{E}}(1)$'s are line bundles the group $\mathfrak{S}_k$ acts trivially on $\mathcal{O}_{\mathscr{E}}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathscr{E}}(1)$. Therefore the above homomorphism factors through the surjection

$$\pi_S^* \operatorname{Sym}^k \mathscr{E} \twoheadrightarrow \mathcal{O}_{\mathscr{E}}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathscr{E}}(1)$$

which exactly gives arise to the quotient map π. It thus follows that $\pi^*\mathcal{O}_{S^k\mathscr{E}}(1) = \mathcal{O}_{\mathscr{E}}(1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathscr{E}}(1)$. □

Proposition 4.5. *Let $\mathcal{O}_{S^2\mathscr{E}}(1)$ be the canonical line bundle on the relative symmetric product $P_2 = \mathbb{P}(\operatorname{Sym}^2 \mathscr{E})$, and let $\mathcal{O}_F(1)$ be the Plücker polarization on F. Then the normal sheaf $\mathcal{N}_{P_2/X^{[2]}}$ is isomorphic to*

$$\pi_F^*(\mathscr{E}^\vee \otimes \mathcal{O}_F(3)) \otimes \mathcal{O}_{S^2\mathscr{E}}(-1). \tag{4.2.3}$$

Proof. The difference of $\mathcal{N}_{P_2/X^{[2]}}$ and $\pi_F^*(\mathscr{E}^\vee)$ is a line bundle $\mathscr{L}$. Since P_2 is a projective bundle over F the line bundle $\mathscr{L}$ is isomorphic to

$$\pi_F^*\mathcal{M} \otimes \mathcal{O}_{S^2\mathscr{E}}(m), \text{ for some } \mathcal{M} \in \operatorname{Pic}(F) \text{ and } m \in \mathbb{Z}$$

Let $\rho : \Xi \to X^{[2]}$ be the universal family of length two closed subschemes of X. The induced family over P_2 via the closed immersion $P_2 \hookrightarrow X^{[2]}$ is exactly $P \times_F P$. Moreover, the induced morphism of the family $\sigma : P \times_F P \to P_2$ is the quotient map (4.2.2) of the relative symmetric product. It gives the following cartesian diagram

$$\begin{tikzcd} P \times_F P \arrow[r, hook] \arrow[d, "\sigma"] & \Xi \arrow[d, "\rho"] \\ P_2 \arrow[r, hook] & X^{[2]}. \end{tikzcd}$$

Notice that the vertical maps are flat, and the horizontal closed embeddings are regular. Hence we have:

$$\sigma^*\mathcal{N}_{P_2/X^{[2]}} = \mathcal{N}_{P\times_F P/\Xi}.$$

With abuse of notation, we denote by π_F for both the structure maps $P_2 \to F$ and $P \times_F P \to F$. It follows from Lemma 4.4 that

$$\mathcal{N}_{P\times_F P/\Xi} = \pi_F^*(\mathscr{E}^\vee \otimes \mathcal{M}) \otimes pr_1^*\mathcal{O}_{\mathscr{E}}(m) \otimes pr_2^*\mathcal{O}_{\mathscr{E}}(m).$$

Hence the first Chern class of $\mathcal{N}_{P\times_F P/\Xi}$ is equal to

$$\pi_F^*(-c_1(\mathscr{E}) + 2c_1(\mathcal{M})) + 2(pr_1^* + pr_2^*)c_1(\mathcal{O}_{\mathscr{E}}(m))$$

From Lemma 4.7 we conclude that $m = -1$ and $c_1(\mathcal{M}) = c_1(\mathcal{O}_F(3))$. □

To calculate the Chern class of the normal $\mathcal{N}_{P\times_F P/\Xi}$, it is natural to consider the following exact sequence

$$0 \to \mathcal{T}_{P\times_F P} \to \mathcal{T}_\Xi|_{P\times_F P} \to \mathcal{N}_{P\times_F P/\Xi} \to 0 \tag{4.2.4}$$

with respect to the regular closed embedding $P \times_F P \hookrightarrow \Xi$. It is not hard to compute the Chern class of the tangent sheaf $\mathcal{T}_{P\times_F P}$. For the universal family Ξ, note that it is isomorphic to the blow-up $\widetilde{X \times X}$ of $X \times X$ along the diagonal such that the morphism

ρ is identified with the quotient map by the induced involution on $\widetilde{X \times X}$. Therefore it is natural to apply the following Proposition 4.6, which provides a standard method to compute Chern classes of blow-ups.

Here are the set-ups for Proposition 4.6. Let Y be a non-singular variety and W be a non-singular closed subvariety of codimension d. Consider the blow-up diagram

$$\begin{tikzcd} \tilde{W} \arrow[r, hook, "j"] \arrow[d, "g"] & \tilde{Y} \arrow[d, "f"] \\ W \arrow[r, hook] & Y. \end{tikzcd} \tag{4.2.5}$$

Let $N_{W/Y}$ be the normal bundle of W in Y. The exceptional divisor $\tilde{W}$ is isomorphic to the projective bundle $\mathbb{P}(N_{W/Y})$. Denote by $\mathcal{O}_{\tilde{W}}(-1)$ the tautological line bundle on $\mathbb{P}(N_{W/Y})$.

Proposition 4.6. *[27, §15.4] Use the above notations. Let Q be the universal quotient bundle*

$$0 \to \mathcal{O}_{\tilde{W}}(-1) \to g^* N_{W/Y} \to Q \to 0.$$

on the exceptional divisor $\tilde{W}$. Then there is a exact sequence

$$0 \to T_{\tilde{Y}} \to f^* T_Y \to j_* Q \to 0.$$

In particular, the low degree Chern classes of the coherent sheaf j_*Q can be expressed as follows:

$$\begin{aligned} c_1(j_*Q) &= (d-1)[\tilde{W}], \\ c_2(j_*Q) &= -j_*(\frac{d^2-d+2}{2} c_1(\mathcal{O}_{\tilde{W}}(1)) + g^* c_1(N_{W/Y})). \end{aligned}$$

Lemma 4.7. *The first Chern class of the normal sheaf $\mathcal{N}_{P\times_F P/\Xi}$ is*

$$c_1(\mathcal{N}_{P\times_F P/\Xi}) = -2(pr_1^* + pr_2^*) c_1(\mathcal{O}_{\mathscr{E}}(1)) + 5\pi_F^* c_1(\mathscr{E}).$$

Proof. By the exact sequence (4.2.4), the first Chern class $c_1(N_{P\times_F P/\Xi})$ is equal to $c_1(T_\Xi)|_{P\times_F P} - c_1(T_{P\times_F P})$. Consider the blow-up diagram

$$\begin{tikzcd} E_{\Delta,X} \arrow[r, hook, "j"] \arrow[d, "g"] & \Xi \arrow[d, "\rho"] \\ X \arrow[r, hook, "\Delta_X"] & X \times X. \end{tikzcd}$$

Let Q be the universal quotient bundle on $E_{\Delta,X}$. Using Proposition 4.6, we have

$$0 \to \mathcal{T}_\Xi \to \rho^* \mathcal{T}_{X\times X} \to j_*(Q) \to 0$$

with

$$c_1(\mathcal{T}_\Xi) = \rho^* c_1(X \times X) - (n-1)[E_{\Delta,X}].$$

On the one hand, for an n-dimensional hypersurface X of degree d, it is well-known that

$$c_1(X) = (n+2-d) \cdot H_X, H_X \text{ hyperplane section class.}$$

On the other hand, the intersection $E_{\Delta,X} \cap P \times_F P$ is the relative diagonal Δ_P in $P \times_F P$. Then we have

$$\begin{aligned} c_1(\mathcal{T}_\Xi)|_{P\times_F P} &= (n-1)(pr_1^* q^* H_X + pr_2^* q^* H_X) - (n-1)\cdot[\Delta_P] \\ &= (n-1)(pr_1^* c_1(\mathcal{O}_{\mathscr{E}}(1)) + pr_2^* c_1(\mathcal{O}_{\mathscr{E}}(1))) - (n-1)\cdot[\Delta_P] \\ &= (n-1)c_1(\mathscr{E}). \end{aligned}$$

The last equality is due to Corollary (4.3). □

The smooth morphism $\pi_F : P \times_F P \to F$ induces the exact sequence

$$0 \to \mathcal{T}_{P\times_F P/F} \to \mathcal{T}_{P\times_F P} \to \pi_F^* \mathcal{T}_F \to 0$$

The relative tangent bundle $\mathcal{T}_{P\times_F P/F}$ is isomorphic to $pr_1^*\mathcal{T}_{P/F} \oplus pr_2^*\mathcal{T}_{P/F}$, and $\mathcal{T}_{P/F} \cong p^* \det \mathscr{E}^\vee \otimes \mathcal{O}_{\mathscr{E}}(2)$. Then we have

$$\begin{aligned} c_1(\mathcal{T}_{P\times_F P}) &= c_1(\mathcal{T}_{P\times_F P/F}) + \pi_F^* c_1(\mathcal{T}_F) \\ &= (pr_1^* + pr_2^*)(p^* \det \mathscr{E}^\vee \otimes \mathcal{O}_{\mathscr{E}}(2)) + \pi_F^* c_1(F) \\ &= (pr_1^* + pr_2^*) c_1(\mathcal{O}_{\mathscr{E}}(2)) + (n-6)\pi_F^* c_1(\mathscr{E}). \end{aligned}$$

Finally, it concludes that

$$\begin{aligned} c_1(\mathcal{N}_{P\times_F P/\Xi}) &= c_1(\mathcal{T}_\Xi)|_{P\times_F P} - c_1((\mathcal{T}_{P\times_F P}) \\ &= -(pr_1^* + pr_2^*)(c_1(\mathcal{O}_{\mathscr{E}}(2))) + 5\pi_F^* c_1(\mathscr{E}). \end{aligned}$$

Proposition 4.8. *Let $p : P \to F$ be the universal family of lines over F with the canonical line bundle $\mathcal{O}_{\mathscr{E}}(1)$. Let $\mathcal{O}_F(1)$ be the Plücker polarization on F. Then the normal sheaf $\mathcal{N}_{P/P_X}$ is isomorphic to*

$$p^*(\mathrm{Sym}^2 \mathscr{E}^\vee \otimes \mathcal{O}_F(3)) \otimes \mathcal{O}_{\mathscr{E}}(-1).$$

Proof. Recall that the projective bundle $\mathbb{P}(T_{\mathbb{P}^{n+1}})$ over $\mathbb{P}^{n+1} := \mathbb{P}(V)$ can be regarded as the flag variety $\mathbf{Gr}(1,2,V)$ of lines and planes in the linear spaces V. So $P_X := \mathbb{P}(T_{\mathbb{P}^{n+1}}|_X)$ is a closed subvariety of $\mathbf{Gr}(1,2,V)$ From the natural closed embeddings

$$P \subset P_X \subset \mathbf{Gr}(1,2,V)$$

we obtain the short exact sequence of the normal sheaves

$$0 \to \mathcal{N}_{P/P_X} \to \mathcal{N}_{P/\mathbf{Gr}(1,2,V)} \to \mathcal{N}_{P_X/\mathbf{Gr}(1,2,V)}|_P \to 0. \tag{4.2.6}$$

The flag variety $\mathbf{Gr}(1,2,V)$ is a $\mathbb{P}^1$-bundle over $\mathbf{Gr}(2,V)$. The restriction the $\mathbb{P}^1$-bundle to the closed subscheme $F \subset \mathbf{Gr}(2,V)$ is exactly the universal family of lines $p : P \to F$. Hence we have

$$\mathcal{N}_{P/\mathbf{Gr}(1,2,V)} \cong p^*\mathcal{N}_{F/\mathbf{Gr}(2,V)} \cong p^* \mathrm{Sym}^3 \mathscr{E}.$$

Recall the last equality holds since F is the zeros of a regular section of the locally free sheaf $\mathrm{Sym}^3(\mathscr{E})$ on $\mathbf{Gr}(2,V)$, see Proposition 2.2. Let $\pi_X : P_X \to X$ be the structure morphism. It is easy to see that

$$\mathcal{N}_{P_X/\mathbf{Gr}(1,2,V)} \cong \pi_X^*\mathcal{N}_{X/\mathbb{P}^{n+1}} \cong \pi_X^*\mathcal{O}_X(3).$$

Therefore the exact sequence (4.2.6) is isomorphic to

$$0 \to \mathcal{N}_{P/P_X} \to p^* \mathrm{Sym}^3 \mathscr{E} \to q^*\mathcal{O}_X(3) \to 0. \tag{4.2.7}$$

Let K be the kernel of the canonical surjection $p^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1)$ on the universal family of lines P. By taking the symmetric power to the short exact sequence

$$0 \to K \to p^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1) \to 0,$$

we obtain the canonical right exact sequence

$$p^* \operatorname{Sym}^{j-1} \mathscr{E} \otimes K \to p^* \operatorname{Sym}^j \mathscr{E} \to \mathcal{O}_{\mathscr{E}}(j) \to 0,$$

which is also left exact since $\mathcal{O}_{\mathscr{E}}(1)$ is a line bundle. Recall that $\mathcal{O}_{\mathscr{E}}(1) = q^*\mathcal{O}_X(1)$. It follows that

$$\mathcal{N}_{P/P_X} \cong p^* \operatorname{Sym}^2 \mathscr{E} \otimes K. \tag{4.2.8}$$

It is easy to see that the kernel K is isomorphic to the line bundle $p^* \det \mathscr{E} \otimes \mathcal{O}_{\mathscr{E}}(-1)$. Hence $\mathcal{N}_{P/P_X}$ is isomorphic to $p^*(\operatorname{Sym}^2 \mathscr{E} \otimes \det \mathscr{E}) \otimes \mathcal{O}_{\mathscr{E}}(-1)$. In addition, the sheaf $\mathscr{E}$ is locally free of rank two. There is a natural non-degenerate pairing

$$\mathscr{E} \otimes \mathscr{E} \to \det \mathscr{E} =: \mathcal{O}_F(1)$$

that implies the isomorphism

$$\mathscr{E} \cong \mathscr{E}^\vee \otimes \mathcal{O}_F(1)$$

Then $\operatorname{Sym}^2 \mathscr{E} \cong \operatorname{Sym}^2 \mathscr{E}^\vee \otimes \mathcal{O}_F(2)$. As a consequence, we can show

$$\mathcal{N}_{P/P_X} \cong p^*(\operatorname{Sym}^2 \mathscr{E}^\vee \otimes \mathcal{O}_F(3)) \otimes \mathcal{O}_{\mathscr{E}}(-1). \tag{4.2.9}$$

□

Remark 9. Notice that the normal sheaf $\mathcal{N}_{P/P_X}$ is the sheaf of the sections on the normal bundle N_{P/P_X}. Then the exceptional divisor $\mathcal{E} := \mathbb{P}(N_{P/P_X})$ is defined to be $\operatorname{Proj}(\operatorname{Sym}^\bullet \mathcal{N}^\vee_{P/P_X})$. With respect to the cartesian diagram

$$\begin{tikzcd} E \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & P \arrow[d, "p"] \\ P_2 \arrow[r, "\pi_F"] & F, \end{tikzcd}$$

the normal sheaf $\mathcal{N}^\vee_{P/P_X}$ is a priori isomorphic to $\operatorname{Sym}^2 \mathscr{E}$ by twisting a line bundle. In order to track fiber product structure, we insist on the expression (4.2.9) rather than (4.2.8).

Corollary 4.9. *With the notations in the cartesian diagram* (3.2.2) *The canonical line bundle* $\mathcal{O}_{N_{P/P_X}}(1)$ *on the exceptional divisor* $\mathcal{E} \cong \mathbb{P}(N_{P/P_X})$ *is isomorphic to*

$$\pi_2^*\mathcal{O}_{S^2\mathscr{E}}(1) \otimes \pi_1^*\mathcal{O}_{\mathscr{E}}(1) \otimes \pi_F^*\mathcal{O}_F(-3). \tag{4.2.10}$$

Denote by $c_i = c_i(\mathscr{E})$ *and* $h = c_1(\mathcal{O}_{\mathscr{E}}(1))$. *The Chern classes of the normal bundle* N_{P/P_X} *is*

$$\begin{aligned} c_1(N_{P/P_X}) &= 6p^*c_1 - 3h \\ c_2(N_{P/P_X}) &= 11p^*c_1^2 + p^*c_2 - 9p^*c_1 \cdot h \\ c_3(N_{P/P_X}) &= 6p^*c_1^3 + 3p^*c_2 \cdot p^*c_1 - 6p^*c_1^2 \cdot h - 3p^*c_2 \cdot h. \end{aligned}$$

Proof. By the illustration of Remark 9, the line bundle $\mathcal{O}_{N_{P/P_X}}(1)$ is equal to the tautology $\mathcal{O}_{\mathcal{N}^\vee_{P/P_X}}(1)$. It follows from the result (4.2.9) that

$$\mathcal{N}^\vee_{P/P_X} \cong p^*(\operatorname{Sym}^2 \mathscr{E} \otimes \mathcal{O}_F(-3)) \otimes \mathcal{O}_{\mathscr{E}}(1).$$

Note that the canonical line bundle associates to the sheaf $p^* \operatorname{Sym}^2 \mathscr{E}$ is isomorphic to $\pi_2^*\mathcal{O}_{S^2\mathscr{E}}(1)$. Therefore we have

$$\mathcal{O}_{N_{P/P_X}}(1) = \pi_2^*\mathcal{O}_{S^2\mathscr{E}}(1) \otimes \pi_1^*\mathcal{O}_{\mathscr{E}}(1) \otimes \pi_F^*\mathcal{O}_F(-3)$$

For simplicity, we denote by N the normal bundle N_{P/P_X}. It follows from the result (4.2.8) that

$$\begin{aligned}
&c_1(N) = p^*c_1(\operatorname{Sym}^2 \mathscr{E}) + 3c_1(K)\\
&c_2(N) = p^*c_2(\operatorname{Sym}^2 \mathscr{E}) + 2p^*c_1(\operatorname{Sym}^2 \mathscr{E}) \cdot c_1(K) + 3c_1(K)^2\\
&c_3(N) = p^*c_3(\operatorname{Sym}^2 \mathscr{E}) + p^*c_2(\operatorname{Sym}^2 \mathscr{E}) \cdot c_1(K) + p^*c_1(\operatorname{Sym}^2 \mathscr{E}) \cdot c_1(K)^2 + c_1(K)^3.
\end{aligned}$$

It is easy to verify that $c_1(K) = c_1 - h$. Then we obtain

$$\begin{aligned}
&c_1(N) = 6p^*c_1 - 3h\\
&c_2(N) = 11p^*c_1^2 + p^*c_2 - 9p^*c_1 \cdot h\\
&c_3(N) = 6p^*c_1^3 + 3p^*c_2 \cdot p^*c_1 - 6p^*c_1^2 \cdot h - 3p^*c_2 \cdot h.
\end{aligned}$$

$\square$

Remark 10. The diagram (3.2.1) turns out to be a standard flip, a canonical operation in birational geometry, see [43] and [39, Section 3.2]. However, in the sense of standard flips, the vector bundles of the projective bundles $p : P \to F$ and $\pi_F : P_2 \to F$ should be simultaneously specified such that

$$\begin{aligned}
N_{P/P_X} &= \mathcal{O}_p(-1) \otimes p^*\mathcal{V}, \qquad (*)\\
N_{P_2/X^{[2]}} &= \mathcal{O}_{\pi_F}(-1) \otimes \pi_F^*\mathcal{V}'
\end{aligned}$$

where $\mathcal{V}$ (resp. $\mathcal{V}'$) is the vector bundle of P_2 (resp. P). Under this setting, the exceptional divisor $\mathcal{E}$ satisfies

$$\mathcal{O}_{\mathcal{E}}(\mathcal{E}) \cong \pi_1^*\mathcal{O}_p(-1) \otimes \pi_2^*\mathcal{O}_{\pi_F}(-1).$$

Let S be the tautological rank two vector bundles on F. Recall that the sheaf of sections of S is dual to the canonical quotient sheaf $\mathscr{E}$. By the canonical isomorphism $S \cong S^\vee \otimes \det S$, we can set

$$P = \mathbb{P}(S^\vee), \text{ and } P_2 = \mathbb{P}(\operatorname{Sym}^2 S^\vee).$$

We claim that such choice of notations satisfies the equalities $(*)$.

In fact, our computations of the normal bundles ((4.2.8) and (4.2.3)) show that

$$\begin{aligned}
N_{P/P_X} =& p^* \operatorname{Sym}^2 S^\vee \otimes \det(S^\vee) \otimes \mathcal{O}_S(-1),\\
N_{P_2/X^{[2]}} =& \pi_F^*(S \otimes \det S^{\vee\otimes 3}) \otimes \mathcal{O}_{\operatorname{Sym}^2 S}(-1).
\end{aligned}$$

By the isomorphism $S \cong S^\vee \otimes \det S$ we have

$$\mathcal{O}_S(-1) \cong \mathcal{O}_{S^\vee}(-1) \otimes p^* \det S,$$
$$\mathcal{O}_{\mathrm{Sym}^2 S}(-1) \cong \mathcal{O}_{\mathrm{Sym}^2 S^\vee}(-1) \otimes \pi_F^* \det S^{\otimes 2}.$$

As a result, we obtain

$$N_{P/P_X} = p^* \mathrm{Sym}^2 S^\vee \otimes \mathcal{O}_{S^\vee}(-1),$$
$$N_{P_2/X^{[2]}} = \pi_F^* S^\vee \otimes \mathcal{O}_{\mathrm{Sym}^2 S^\vee}(-1).$$

4.3 Decomposition of the diagonal

Lemma 4.10. *For the birational map $\Phi : X^{[2]} \dashrightarrow P_X$ in the diagram* (3.2.1)*, the graph correspondence $\Phi_* := [\overline{\Gamma}_\Phi]$(resp. $\Phi^* := [\overline{\Gamma}_{\Phi^{-1}}]$) in $\mathrm{Corr}^0(X^{[2]}, P_X)$(resp. $\mathrm{Corr}^0(P_X, X^{[2]})$) satisfies*

$$\Phi_* \circ \Phi^* = \Delta_{P_X}.$$

Proof. Recall that the birational map Φ can be resolved by blowing up τ followed by the morphism $\widetilde{\Phi}$, see diagram (3.2.1). Hence we have

$$\Phi_* = \widetilde{\Phi}_* \circ \tau^*, \Phi^* = \tau_* \circ \widetilde{\Phi}^*$$

By Lieberman's lemma 2.4, we have

$$\Phi_* \circ \Phi^* = \widetilde{\Phi}_* \circ \tau^* \circ \tau_* \circ \widetilde{\Phi}^* = (\widetilde{\Phi} \times \widetilde{\Phi})_* (\tau \times \tau)^* \Delta_{X^{[2]}}.$$

Recall that $\tau : \widetilde{X^{[2]}} \to X^{[2]}$ is the blow up of $X^{[2]}$ along the smooth center P_2 with the exceptional divisor $\mathcal{E}$. Let $j \times_{P_2} j : \mathcal{E} \times_{P_2} \mathcal{E} \hookrightarrow \widetilde{X^{[2]}} \times \widetilde{X^{[2]}}$ be the natural embedding. It follows from the double blow-up formula (A.7) that

$$\Phi_* \circ \Phi^* = (\widetilde{\Phi} \times \widetilde{\Phi})_* (\Delta_{\widetilde{X^{[2]}}} + j \times_{P_2} j_*(\alpha)),$$

where α is a cycle class of dimension $2n$ supported on $\mathcal{E} \times_{P_2} \mathcal{E}$. The dimension of $\mathcal{E} \times_{P_2} \mathcal{E}$ is $2n$, thus α is a multiple of the fundamental class $[\mathcal{E} \times_{P_2} \mathcal{E}]$. The morphism $\widetilde{\Phi} \times \widetilde{\Phi}$ maps $\mathcal{E} \times_{P_2} \mathcal{E}$ onto the subvariety $P \times_F P$ in $\widetilde{X^{[2]}} \times \widetilde{X^{[2]}}$. Notice that the dimension of $\dim P \times_F P$ is $2n-2$. Hence the cycle class of the image of $(j \times_{P_2} j)_*(\alpha)$ under $(\widetilde{\Phi} \times \widetilde{\Phi})_*$ is zero. We conclude that

$$\Phi_* \circ \Phi^* = (\widetilde{\Phi} \times \widetilde{\Phi})_* \Delta_{\widetilde{X^{[2]}}} = \Delta_{P_X}.$$

$\square$

Lemma 4.11. *Let $[P_2] \in \mathrm{Corr}^{-2}(X^{[2]}, F)$ be the algebraic correspondence of the relative symmetric product P_2. Then we have*

$$[P_2] \circ {}^\top[P_2] = \Delta_F$$

in $\mathrm{CH}_{2n-4}(F)$.

Proof. As an underlying incidence subvariety

$$\begin{tikzcd} P_2 \arrow[r, hook, "i_2"] \arrow[d, "\pi_F"] & X^{[2]} \\ F & \end{tikzcd},$$

the algebraic correspondence $[P_2]$ and its transpose ${}^\top[P_2]$ can be written as

$$[P_2] = \pi_{F*} \circ {i_2}^*, {}^\top[P_2] = i_{2*} \circ {\pi_F}^*.$$

By Lieberman's lemma (2.4), the composition $[P_2] \circ^t [P_2]$ is

$$\pi_{F*} \circ {i_2}^* \circ i_{2*} \circ \pi_F^* = (\pi_F \times \pi_F)_*(i_2 \times i_2)^*[\Delta_{X^{[2]}}].$$

Note that the codimension of P_2 in $X^{[2]}$ is two. By the refined Gysin homomorphism, we have

$$(i_2 \times i_2)^*[\Delta_{X^{[2]}}] = c_2(N_{P_2/X^{[2]}})$$

supported on the diagonal of $P_2 \times P_2$. It follows that

$$[P_2] \circ {}^\top[P_2] = \iota_{\Delta*}\pi_{F*}(c_2(N_{P_2/X^{[2]}})) = N \cdot [\Delta_F]$$

for some integer N. Let $\ell^{[2]}$ be any fibre of the projective bundle $P_2 \to F$. The integer N is equal to the intersection number

$$c_2(N_{P_2/X^{[2]}}) \cdot [\ell^{[2]}].$$

As a result of Proposition 4.5, there is

$$\mathcal{N}_{P_2/X^{[2]}}|_{\ell^{[2]}} \cong \pi_F^*(\mathscr{E}^\vee \otimes \mathcal{O}_F(3)) \otimes \mathcal{O}_{S^2\mathscr{E}}(-1)|_{\ell^{[2]}} \cong \mathcal{O}_{\ell^{[2]}}(-1)^{\oplus 2},$$

which implies

$$N = c_2(N_{P_2/X^{[2]}}) \cdot [\ell^{[2]}] = c_2(\mathcal{O}_{\ell^{[2]}}(-1)^{\oplus 2}) = 1.$$

□

Theorem 4.12. *Let $\Phi : X^{[2]} \dashrightarrow P_X$ be the birational map, and let P_2 be the relative symmetric product over the Fano scheme of lines F in the diagram* (3.2.1). *Then the diagonal class $\Delta_{X^{[2]}}$ can be decomposed as follows*

$$\Delta_{X^{[2]}} = \Phi^* \circ \Phi_* + {}^\top P_2 \circ P_2. \tag{4.3.1}$$

in $\mathrm{CH}_{2n}(X^{[2]} \times X^{[2]})$. *As a consequence, it implies the isomorphism of Chow motives*

$$\mathfrak{h}(F)(-2) \oplus \bigoplus_{i=0}^{n} \mathfrak{h}(X)(-i) \cong \mathfrak{h}(X^{[2]}) \tag{4.3.2}$$

Proof. The strategy of the proof is to compute the algebraic cycle $\Phi^* \circ \Phi_*$. Again it follows from the diagram (3.2.1) that

$$\Phi^* \circ \Phi_* = \tau_* \circ \widetilde{\Phi}^* \circ \widetilde{\Phi}_* \circ \tau^*.$$

By Lieberman's Lemma, we have

$$\Phi^* \circ \Phi_* = (\tau \times \tau)_* (\widetilde{\Phi} \times \widetilde{\Phi})^* \Delta_{P_X}.$$

Recall that the morphism $\widetilde{\Phi} : \widetilde{X^{[2]}} \to P_X$ is the blowing up on P_X along the smooth center P. Denote by $\xi = c_1(\mathcal{O}_{\mathcal{E}}(1))$ on the exceptional divisor $\mathcal{E} \cong \mathbb{P}(N_P/P_X)$, and ξ_i the pull-back of ξ to the i-th factor of $\mathcal{E} \times_P \mathcal{E}$. It follows from the double blow-up formula (A.7) that

$$\widetilde{\Phi}^* \circ \widetilde{\Phi}_* = \Delta_{\widetilde{X^{[2]}}} + (j \times_P j)_* \{ \frac{c(N_{P/P_X})}{(1-\xi_1)(1-\xi_2)} \}_{\dim P_X}.$$

where $j \times_P j : \mathcal{E} \times_P \mathcal{E} \hookrightarrow \widetilde{X^{[2]}} \times \widetilde{X^{[2]}}$ is the closed immersion and the term $\{\frac{c(N_{P/P_X})}{(1-\xi_1)(1-\xi_2)}\}_{\dim P_X}$ is a $2n$-dimensional cycle class supported on $\mathcal{E} \times_P \mathcal{E}$. Then we have

$$\Phi^* \circ \Phi_* = \Delta_{X^{[2]}} + (\tau \times \tau)_* (j \times_P j)_* \{ \frac{c(N_{P/P_X})}{(1-\xi_1)(1-\xi_2)} \}_{\dim P_X}. \qquad (4.3.3)$$

By the cartesian diagram (3.2.2), $\mathcal{E} \times_P \mathcal{E}$ is isomorphic to $P_2 \times_F P_2 \times_F P$. Then the product morphism $\tau \times \tau$ maps $\mathcal{E} \times_P \mathcal{E}$ onto $P_2 \times_F P_2$. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \times_P \mathcal{E} \arrow[r, hook, "j \times_P j"] \arrow[d, "\pi_2 \times_P \pi_2"'] & \widetilde{X^{[2]}} \times \widetilde{X^{[2]}} \arrow[d, "\tau \times \tau"] \\
P_2 \times_F P_2 \arrow[r, hook, "i_2 \times_F i_2"] & X^{[2]} \times X^{[2]}
\end{tikzcd}$$

Therefore the second term in (4.3.3) is equal to

$$(i_2 \times_F i_2)_* (\pi_2 \times_P \pi_2)_* \{ \frac{c(N_{P/P_X})}{(1-\xi_1)(1-\xi_2)} \}_{\dim P_X}$$

which supports on $P_2 \times_F P_2$. Note that the dimension of $\mathcal{E} \times_P \mathcal{E}$ is $2n+1$. It suffices to determine the cycle class of the image of the components

$$c_0(N) \cdot \xi_1, \; c_0(N) \cdot \xi_2, \; c_1(N)$$

in the bracket. Let $\eta = c_1(\mathcal{O}_{S^2\mathscr{E}}(1))$ on $P_2 = \mathbb{P}(\mathrm{Sym}^2 \mathscr{E})$, let $h = c_1(\mathcal{O}_{\mathscr{E}}(1))$ on $P = \mathbb{P}(\mathscr{E})$, and let $c_1 = c_1(\mathscr{E}) = c_1(\mathcal{O}_F(1))$ on F. It follows from (4.2.10) that

$$\xi = \pi_1^* h + \pi_2^* \eta - 3\pi_F^* c_1.$$

In addition, we have $c_1(N) = 6p^* c_1 - 3h$ by Corollary 4.9. Consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{E} \times_P \mathcal{E} \arrow[r, "\pi_P"] \arrow[d, "\pi_2 \times_P \pi_2"'] & P \arrow[d, "p"] \\
P_2 \times_F P_2 \arrow[r, "\pi_F"] & F.
\end{tikzcd}$$

Then the cycle classes that we need is

$$\begin{aligned}
(\pi_2 \times_P \pi_2)_* c_1(N) &= 6(\pi_2 \times_P \pi_2)_* \pi_p^* p^* c_1 - 3(\pi_2 \times_P \pi_2)_* \pi_p^* h \\
&= 6\pi_F^* p_* p^* c_1 - 3{\pi_F}^* p_* h \\
&= 0 - 3\pi_F^*[F] = -3[P_2 \times_F P_2],
\end{aligned}$$

and

$$\begin{aligned}(\pi_2 \times_P \pi_2)_*\xi_1 &= (\pi_2 \times_P \pi_2)_* \circ pr_1^*(\pi_1^* h + \pi_2^*\eta - 3\pi_F^* c_1)\\ &= pr_1^*(\pi_{2*}\pi_1^* h + \pi_{2*}\pi_2^*(\eta - 3\pi_F^* c_1))\\ &= pr_1^*[P_2] = [P_2 \times_F P_2].\end{aligned}$$

Symmetrically, we have $(\pi_2 \times_P \pi_2)_*\xi_2 = pr_2^*[P_2] = [P_2 \times_F P_2]$. In a conclusion, the term $(\pi_2 \times_p \pi_2)_*\{\frac{c(N_{P/P_X})}{(1-\xi_1)(1-\xi_2)}\}_{\dim P_X}$ is equal to

$$(\pi_2 \times_p \pi_2)_*(c_0(N) \cdot \xi_1 + c_0(N) \cdot \xi_2 + c_1(N)) = -[P_2 \times_F P_2].$$

It is straightforward to verify

$$[P_2 \times_F P_2] = {}^{\top}[P_2] \circ [P_2]$$

as algebraic cycles in $\mathrm{CH}_{2n}(X^{[2]} \times X^{[2]})$. As a consequence, we prove the formula of decomposition of the diagonal (4.1.5). $\square$

Integral cohomology of Fano scheme of lines

Let X be a smooth complex cubic n-fold. By Lefschetz hyperplane section theorem, there is an isomorphism of Hodge structures

$$H^*(X, \mathbb{Q}) = H^*(\mathbb{P}^{n+1}, \mathbb{Q}) \oplus H^n(X, \mathbb{Q})_{prim}.$$

One can apply the key relation (2.4.5) or the isomorphism (4.3.2) to compute the Hodge structure of $F(X)$ in terms of the Hodge structure

$$\mathcal{H}_X := H^n(X, \mathbb{Q})_{prim}(1).$$

Theorem 4.13. *[28, Thm. 6.1] Let X be a smooth complex cubic n-fold. The Hodge structure of the Fano variety of lines $F(X)$ has the following form*

$$H^*(F(X), \mathbb{Q}) \cong \mathrm{Sym}^2(\mathcal{H}_X) \oplus \bigoplus_{k=0}^{n-2} \mathcal{H}_X(-k) \oplus \bigoplus_{k=0}^{2n-4} \mathbb{Q}(-k)^{a_k}$$

where

$$a_k = \begin{cases} \lfloor \frac{k+2}{2} \rfloor, k < n-2;\\ \lfloor \frac{n-2}{2} \rfloor, k = n-2;\\ \lfloor \frac{2n-2-k}{2} \rfloor, k > n-2.\end{cases}$$

Example 6. Let X be a smooth cubic 4-fold. The primitive part $\mathcal{H}_X$ is a weight two Hodge structure of type $(0, 1, 20, 1, 0)$. Then the Hodge structure of the hyper-Kähler fourfold $F(X)$ can be represented as

$$\left[\begin{array}{c|c} H^8 & \mathbb{Q}(-4)\\ H^6 & \mathcal{H}_X(-2) \oplus \mathbb{Q}(-3)\\ H^4 & \mathrm{Sym}^2(\mathcal{H}_X) \oplus \mathcal{H}_X(-1) \oplus \mathbb{Q}(-2)\\ H^2 & \mathcal{H}_X \oplus \mathbb{Q}(-1)\\ H^0 & \mathbb{Q}\end{array}\right].$$

By the integral decomposition of the diagonal (4.3.1), the isomorphism of the Chow motives (4.3.2) implies the following consequences

Corollary 4.14. *There is an isomorphism of integral Hodge structures*

$$[P_2]^* + \Phi^* : H^*(F, \mathbb{Z})(-2) \oplus H^*(P_X, \mathbb{Z}) \cong H^*(X^{[2]}, \mathbb{Z}). \tag{4.3.4}$$

which preserves the intersection pairings. In particular, the integral cohomology $H^*(F, \mathbb{Z})$ is free of the rank determined by the formula in Theorem 4.13.

Proof. Notice that the isomorphism (4.3.2) and its inverse are given by integral algebraic correspondences. Hence it induces the isomorphism between the integral cohomology groups of the Chow motives. To show the isomorphism preserves the canonical intersection pairings on the two sides, we need to justify that for any given $\alpha, \beta \in H^*(F, \mathbb{Z})$ and $x, y \in H^*(P_X, \mathbb{Z})$ there is

$$(\alpha, \beta)_F + (x, y)_{P_X} = ([P_2]^*\alpha + \Phi^* x, [P_2]^*\beta + \Phi^* y)_{X^{[2]}}. \tag{4.3.5}$$

By the projection formula the right hand side is equal to

$$([P_2]_*[P_2]^*\alpha, \beta)_F + (\Phi_*[P_2]^*\alpha, y)_{P_X} + ([P_2]_*\Phi^* x, \beta)_F + (\Phi_*\Phi^* x, y)_{P_X}.$$

According to Lemma 4.10 and Lemma 4.11, there is

$$[P_2]_* \circ [P_2]^* = \mathrm{Id}_{H^*(F)}, \Phi_* \circ \Phi^* = \mathrm{Id}_{H^*(P_X)}.$$

Passing to the cohomology groups, the decomposition of the diagonal (4.3.1) yields the decomposition of the identity map

$$\mathrm{Id}_{X^{[2]}} = [P_2]^* \circ [P_2]_* + \Phi^* \circ \Phi_*,$$

which implies $[P_2]^* = [P_2]^* + \Phi^* \circ \Phi_* \circ [P_2]^*$. Hence the action $\Phi^* \circ \Phi_* \circ [P_2]^*$ is trivial. Compose with the action Φ_* we obtain $\Phi_* \circ [P_2]^* = 0$. The similar argument shows that $[P_2]_* \circ \Phi^* = 0$. Therefore the equation (4.3.5) follows.

The Lefschetz hyperplane theorem shows

$$\begin{cases} H^i(X, \mathbb{Z}) \cong H^i(\mathbb{P}^{n+1}, \mathbb{Z}), & i < n \\ H_i(X, \mathbb{Z}) \cong H_i(\mathbb{P}^{n+1}, \mathbb{Z}), & i < n. \end{cases}$$

The Poincaré duality asserts that $H^i(X, \mathbb{Z}) \cong \mathbb{Z}$ for $i \neq$. Moreover, it follows from the universal coefficients theorem

$$0 \to \mathrm{Ext}^1(H_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to \mathrm{Hom}(H_n(X, \mathbb{Z}), \mathbb{Z}) \to 0$$

that the middle cohomology $H^n(X, \mathbb{Z})$ is torsion-free. Totaro [66] showed that the integral cohomology of the Hilbert square of a complex manifold is torsion-free if the integral cohomology of the complex manifold is torsion-free. Therefore the isomorphism (4.3.4) implies the torsion-freeness of $H^*(F, \mathbb{Z})$.

Since $H^*(F, \mathbb{Z})$ is torsion-free the rank is equal to the dimension of the $\mathbb{Q}$-linear space $H^*(F, \mathbb{Q})$ that is characterized in Theorem 4.13. $\square$

Appendix A

Intersection theory

In several places of the thesis we use the Segre class and the refined Gysin homomorphism. They are fundamental subjects in Fulton's intersection theory [27]. More importantly, Lemma A.7 plays a central role in the proofs of several key statements. The main formula in the lemma relies on the notions of Segre class and refined Gysin homomorphisms. In the appendix we review some basic backgrounds for the sake of completeness.

A.1 Serge classes

Serge classes of vector bundles: Let E be a vector bundle of rank $e+1$ over a scheme X, and let $p : \mathbb{P}(E) \to X$ be the projective bundle over X. Denote by $\mathcal{O}_E(1)$ the canonical line bundle on $\mathbb{P}(E)$. The *total Serge class* of E, is a cycle class in $\mathrm{CH}^*(X)$ defined by the formula

$$s(E) = p_*(\sum_{i\geq 0} c_1(\mathcal{O}_E(1))^i).$$

The Serge classes and Chern classes of a vector bundle determine each other. In fact, the following canonical equation of Chern classes

$$\sum_{i=0}^{e} p^* c_i(E) \cdot c_1(\mathcal{O}_E(1))^{e-i} = 0$$

in sense of Grothendieck[34] implies that

$$s(E) \cap c(E) = 1.$$

Serge classes of cones: Let $\mathcal{S}^\bullet := \oplus_{i\geq 0}\mathcal{S}^i$ be a sheaf of graded $\mathcal{O}_X$-algebras of a scheme X. We assume that $\mathcal{S}^0$ is isomorphic to the structure sheaf $\mathcal{O}_X$ and $\mathcal{S}^\bullet$ is locally generated by $\mathcal{S}^1$. A cone C over X is the spectrum of some $\mathcal{S}^\bullet$. Let $P(C \oplus 1)$ be the space Proj($S^\bullet[z]$) with the augment graded $\mathcal{O}_X$-algebras $S^\bullet[z]$:

$$S^d[z] = S^d \oplus S^{d-1} \cdot z \oplus \ldots \oplus S^1 \cdot z^{d-1} \oplus S^0 \cdot z^d.$$

Let $\mathcal{O}_{C\oplus 1}(1)$ be the canonical line bundle on $P(C \oplus 1)$. The total Serge class of the cone C is a cycle class in $\mathrm{CH}^*(X)$ given by the formula

$$s(C) := p_*(\sum_{i\geq 0} c_1(\mathcal{O}_{C\oplus 1}(1))^i \cap [P(C \oplus 1)]).$$

Suppose X is a closed subscheme of a scheme Y. The normal cone C_XY to X in Y is defined by

$$C_XY := \mathrm{Spec}(\bigoplus_{n=0}^{\infty} \mathscr{I}^n/\mathscr{I}^{n+1})$$

where $\mathscr{I}$ is the ideal sheaf of X in Y. The *Serge class of a subscheme* X in Y, denote by $s(X,Y)$, is defined to be the Serge class of the cone C_XY. If the embedding of X into Y is regular, the normal cone C_XY is the normal bundle N_XY, and $s(X,Y) = c(N_XY)^{-1}$.

A.2 Refined Gysin homomorphism

Let $i : X \hookrightarrow Y$ be a closed regular embedding of codimension d and let N_XY be the normal bundle. Suppose V is a k-dimensional schme with a morphism $f : V \to Y$. Consider the fibre diagram

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "g"'] & V \arrow[d, "f"] \\ X \arrow[r, hook, "i"] & Y \end{tikzcd}$$

The normal cone C_WV is a purely k-dimensional closed subscheme of the vector bundle g^*N_XY. It thus defines a k-cycle class $[C_WV] \in \mathrm{CH}_k(g^*N_XY)$. The intersection product of X and V on Y,

$$X \cdot V(\text{ or } i^![V])$$

is defined to be a $(k-d)$-cycle class

$$s^*[C_WV] \in \mathrm{CH}_{k-d}(W)$$

where $s : W \to g^*N_XY$ is the zero section.

Lemma A.1. *The intersection product $X \cdot V$ can be characterized in terms of the Serge class of W in V*

$$X \cdot V = \{c(g^*N_XY) \cap s(W,V)\}_{k-d}.$$

By $\{\cdot\}_{k-d}$ we mean the $(k-d)$-dimensional component of the cycle class.

Proof. See [27, Proposition 6.1]. □

For any morphism $Y' \to Y$ with the fibre product $X' := X \times_Y Y'$, the *refined Gysin homomorphism* is a homomorphism of Chow groups

$$i^! : \mathrm{CH}_k(Y') \to \mathrm{CH}_{k-d}(X')$$

that assigns any k-cycle $\alpha = \sum_i n_i[V_i]$ with irreducible components V_i to the $(k-d)$-cycle $i^!(\alpha) = \sum_i n_i X \cdot V_i$. Notice that if the induced embedding $X' \hookrightarrow Y'$ is regular of codimension d, then the normal bundle $N_{X'}Y'$ is isomorphic to the $(N_XY)|_{X'}$, and we have $X \cdot V = X' \cdot V$ for any closed subvariety $V \subset Y'$. More functorial properties are given in the following proposition

Proposition A.2. *[27, Theorem 6.2,6.3] Consider a cartesian diagram*

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y. \end{array}$$

where i a regular embedding. The refined Gysin homomorphism satisfies the following.

1. *(Push-forward) If is p is proper, and $\alpha \in \mathrm{CH}_*(Y'')$, then*
$$i^!(p_*\alpha) = q_*(i^!\alpha).$$
2. *(Excess Intersection formula) Assume that i(resp. i') is a regular embedding of codimension d(resp. d'). Let N(resp. N') be the associated normal bundle of the regular embedding i(resp. i'), and let $E := g^*N/N'$ be the excess normal bundle of rank $e = d - d'$. Then for any $\alpha \in \mathrm{CH}_k(Y'')$,*
$$i^!(\alpha) = c_e(q^*E) \cap i'^!(\alpha).$$

A.3 Local complete intersections

Definition A.1. A morphism of schemes $f : X \to Y$ is called *local complete intersection* of codimension d if f factors through a regular closed embedding $i : X \hookrightarrow P$ of some codimension $e+d$, followed by a smooth morphism $p : P \to Y$ of relative dimension e.

The notion of local complete intersection is independent of the factorization, namely, if there are two different factorizations, one can find a third factorization dominating them. For instance, assume that $X \overset{i'}{\hookrightarrow} P' \overset{p'}{\to} Y$ is another factorization of f with codimension $e' + d$ of i and relative dimension e' of p'. By [1, VII 1.3], since the closed embedding i is regular the induced closed embedding $(i, i') : X \hookrightarrow P \times_Y P'$ is regular of codimension $e + e' + d$. The smooth morphism $(p, p') : P \times_Y P' \to Y$ is of relative dimension $e + e'$. Hence $P \times_Y P'$ is a factorization of f that dominates P and P'.

Definition A.2. For any local complete intersection morphism $f : X \to Y$ of codimension d, and any morphism $h : Y' \to Y$, consider the fibre diagram

$$\begin{array}{ccc} W & \xrightarrow{f'} & V \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & Y. \end{array}$$

The refined Gysin homormopshim

$$f^! : \mathrm{CH}_k(V) \to \mathrm{CH}_{k-d}(W)$$

is defined as follows: Factor f into $p \circ i$ as above, and consider the fibre diagram

$$\begin{array}{ccccc} W & \xrightarrow{i} & Q & \xrightarrow{p'} & V \\ \downarrow{\scriptstyle h'} & & \downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{i} & P & \xrightarrow{p} & Y. \end{array}$$

Then we define

$$f^!(\alpha) = i^!(p'^*(\alpha)).$$

for $\alpha \in \mathrm{CH}_k(Y')$.

Proposition A.3. *Let $f : X \to Y$ be any local complete intersection morphism.*

1. *The definition of $f^!$ is independent of factorizations of f.*
2. *If f is both flat and l.c.i, then $f^! = f^*$.*
3. *Suppose $j : V \hookrightarrow X$ is a regular embedding. Then the cycle class $f^!([V])$ can be identified with the intersection product $V \cdot X$.*

Proof. The proof of 1 and 2 can be found in [27, Proposition 6.6]. For the third assertion, suppose that $i \circ p$ is a factorization of f. Then we have

$$f^![V] = i^!(p^{-1}(V)) = X \cdot p^{-1}(V).$$

The induced embedding $p^{-1}(V) \hookrightarrow P$ is regular of codimension that is equal to the codimension of V in X. By Proposition A.2, there is

$$V \cdot X = p^{-1}(V) \cdot X.$$

Hence it suffices to prove the commutativity of intersection products, which is a direct consequence of [27, Theorem 6.4]. □

Proposition A.4 (commutativity of l.c.i morphisms)**.** *Suppose that $f : X \to Y$ is a l.c.i morphism of codimension d. Let $h : Y' \to Y$ be a proper morphism, and let $g : V \to Y$ be any morphism of schemes. Consider the induced cartesian cube*

$$\begin{tikzcd}
 & W' \arrow[rr] \arrow[dl, "h_W"'] \arrow[dd] & & V' \arrow[dl, "h_V"'] \arrow[dd] \\
W \arrow[rr] \arrow[dd] & & V \arrow[dd] & \\
 & X' \arrow[rr, "f'"] \arrow[dl] & & Y' \arrow[dl] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd}$$

Assume that the induced morphism $f' : X' \to Y'$ is also l.c.i of codimension d. Then for any algebraic cycle $\alpha \in \mathrm{CH}_k(V')$, we have

$$f^! h_{V*} \alpha = h_{W*} f'^! \alpha \tag{A.3.1}$$

Proof. By Definition A.1, the l.c.i morphism f admits a factorization $Y \overset{i}{\hookrightarrow} P \overset{p}{\to} X$ where i is a regular embedding of codimension $d + e$ and p is a smooth morphism of relative dimension e. Consider the fibre diagram

$$\begin{tikzcd}
X' \arrow[r, hook, "i'"] \arrow[d] & P' \arrow[r, "p'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, hook, "i"] & P \arrow[r, "p"] & Y.
\end{tikzcd}$$

We claim that the induced morphism i' and p' forms a factorization of f' as a l.c.i morphism.

It suffice to show the closed embedding $i' : Y' \hookrightarrow P'$ is regular of codimension $e+d$. By our assumption of f', the relative dimension $\dim X' - \dim Y'$ is equal to d. Then the codimension of Y' in P' is $e + d$ since p' is smooth of relative dimension e. Hence the closed embedding i' induced by the regular embedding i of codimension $e + d$ has the maximal codimension, which implies i' is regular too.

For the reason of notations let us consider the induced diagram of fibre products

$$\begin{tikzcd}
& W' \arrow[rr, hook] \arrow[dl, "h_W"'] \arrow[dd] & & Q' \arrow[rr, "q'"] \arrow[dl, "h_Q"'] \arrow[dd] & & V' \arrow[dl, "h_V"'] \arrow[dd] \\
W \arrow[rr, hook] \arrow[dd] & & Q \arrow[rr, "q"] \arrow[dd] & & V \arrow[dd] & \\
& X' \arrow[rr, hook, "i'"] \arrow[dl] & & P' \arrow[rr, "p'"] \arrow[dl] & & Y' \arrow[dl] \\
X \arrow[rr, hook, "i"] & & P \arrow[rr, "p"] & & Y &
\end{tikzcd}$$

For any algebraic cycle α on V', we have

$$\begin{aligned}
f^! h_{V*}(\alpha) &= i^! q^* h_{V*}(\alpha) \\
&= i^! h_{Q*} q'^*(\alpha) \\
&= h_{W*} i^! q'^*(\alpha) \text{ (Proposition A.2)}
\end{aligned}$$

Note that the codimensions of i and i' are the same. The excess normal bundle in Proposition A.2 (2) is empty. It follows that

$$i^! q'^*(\alpha) = i'^! q'^*(\alpha) = f'^!(\alpha)$$

Therefore we obtain the desired formula (A.3.1). □

Corollary A.5. *Let $f : X \to Y$ be a l.c.i morphism of codimension d. Let $h : Y' \to Y$ be a proper morphism that forms a fibre diagram*

$$\begin{tikzcd}
X' \arrow[r, "h'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "h"] & Y.
\end{tikzcd}$$

Assume that the induced morphism $f' : X' := X \times_Y Y' \to Y'$ is also l.c.i of codimension d. Then the push-forward and the refined Gysin homomorphism commute:

$$f^! h_*(\alpha) = h'_* f'^!(\alpha) \tag{A.3.2}$$

for $\alpha \in \mathrm{CH}(X')$. Very often we use the convention f^* instead of $f^!$ if it causes no ambiguity.

Proof. Just let V be Y itself in Proposition A.4, and the conclusion follows. □

A.3.1 Double blow-ups formula

Let $i : S \hookrightarrow X$ be a closed regular embedding. Let $\tilde{X} := \mathrm{Bl}_S X$ be the blowup of Y along S with the exceptional divisor E. It forms a fibre diagram

$$\begin{tikzcd}
E \arrow[r, hook, "j"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "\rho"'] \\
S \arrow[r, hook, "i"] & X
\end{tikzcd}$$

Note that the blowing-up ρ is not flat. Hence usual flat pull-back operator is invalid for ρ. The blowing up ρ of non-singular varieties is a local complete intersection morphism of codimension 0. Therefore we consider the refined Gysin homomorphism of morphisms of ρ. The following lemma characterizes the action of $\rho^!$ on subvarieties of X.

Lemma A.6 (Blow-up Formula)**.** *Let V be a k-dimensional subvariety of X, and let $\tilde{V} \subset \tilde{X}$ be the proper transform of V, i.e. the blow-up of V along $V \cap S$. Then we have*

$$\rho^![V] = [\tilde{V}] + j_*\{c(E) \cap g^*s(V \cap S, V)\}_k$$

in $\mathrm{CH}_k(\tilde{X})$. Here $\{\cdot\}_k$ indicates the k-dimensional component of the total cycle class in the bracket.

Proof. See [27, Thm. 6.7]. □

In the thesis, we significantly use the following lemma in many places.

Lemma A.7 (Double blow-ups formula)**.** *With the notations above. Assume that S and X are non-singular varieties and $\dim X = d$. Let ξ be the first Chern class $c_1(\mathcal{O}_E(1))$. Let $E \times_S E$ be the self-product of the exceptional divisor E over S with the natural embedding $j \times_S j : E \times_S E \hookrightarrow \tilde{X} \times \tilde{X}$. Then the formula*

$$(\rho \times \rho)^![\Delta_X] = [\Delta_{\tilde{X}}] + (j \times_S j)_*\{c(N) \cdot \sum_{i \geq 0} pr_1^*\xi^i \cdot \sum_{i \geq 0} pr_2^*\xi^i\}_d \tag{A.3.3}$$

holds in $\mathrm{CH}_d(\tilde{X} \times \tilde{X})$. Here $\{\cdot\}_d$ is the d-dimensional component of the total cycle class in the bracket, and $pr_i : E \times_S E \to E$ is the i-th projection.

Proof. Note that $\tilde{X} \times \tilde{X}$ can be regarded as two consecutive blow-ups. Hence one approach to prove the formula is applying Lemma A.6 twice. However, we prove it by a direct computation due to the remark of Proposition A.3 (3).

The morphism $\rho \times \rho$ is a local complete intersection, and the diagonal embedding of X is regular since X is non-singular. By Proposition A.3 (3), the cycle class $(\rho \times \rho)^!(\Delta_X)$ can be identified with the the intersection product

$$\Delta_X \cdot \tilde{X} \times \tilde{X}$$

along the diagonal embedding of X. It is easy to see

$$\Delta_X \cap (\tilde{X} \times \tilde{X}) = \Delta_{\tilde{X}} \cup E \times_S E.$$

The diagonal of $\tilde{X}$ has the correct codimension. Thus it forms one component of the intersection product. Another component of the intersection supports on $E \times_S E$. By Lemma A.1, it is equal to

$$\{c(N_{X/X \times X}) \cap s(E \times_S E, \tilde{X} \times \tilde{X})\}_d.$$

Since S and X are both non-singular, the embedding of $E \times_S E$ into $\tilde{X} \times \tilde{X}$ is regular. Hence the Segre class $s(E \times_S E, \tilde{X} \times \tilde{X})$ equals the inverse of the Chern class of the normal bundle of $E \times_S E$ in $\tilde{X} \times \tilde{X}$. Consider the exact sequence of the normal bundles

$$0 \to N_{E \times_S E/E \times E} \to N_{E \times_S E/\tilde{X} \times \tilde{X}} \to N_{E \times E/\tilde{X} \times \tilde{X}}|_{E \times_S E} \to 0.$$

On the one hand, the normal bundle $N_{E\times_S E/E\times E}$ is isomorphic to $\pi_S^* N_{S/S\times S}$. On the other hand, the $N_{E\times E/\tilde{X}\times\tilde{X}}$ is isomorphic to $\mathcal{O}_{\tilde{X}}(E)\boxtimes\mathcal{O}_{\tilde{X}}(E)$. Therefore, the cycle class $\{c(N_{X/X\times X})\cap s(E\times_S E, \tilde{X}\times\tilde{X})\}_d$ is equal to

$$\{\frac{c(N_{X/X\times X})}{c(N_{S/S\times S})\cdot c(\mathcal{O}_E(E)\boxtimes\mathcal{O}_E(E))}\}_d.$$

The normal bundle of the diagonal embeddings of X and S is isomorphic to each tangent bundle. By the canonical exact sequence

$$0\to T_S\to T_X\to N_S X\to 0$$

and the fact $\mathcal{O}_E(E)=\mathcal{O}_E(-1)$, we obtain

$$\begin{aligned}
\{\frac{c(N_{X/X\times X})}{c(N_{S/S\times S})\cdot c(\mathcal{O}_E(E)\boxtimes\mathcal{O}_E(E))}\}_d &= \{\frac{c(N_S X)}{c(\mathcal{O}_E(-1)\boxtimes\mathcal{O}_E(-1))}\}_d \\
&= \{\frac{c(N_S X)}{(1-pr_1^*\xi)\cdot(1-pr_2^*\xi)}\}_d \\
&= \{c(N_S X)\cdot\sum_{i\geq 0} pr_1^*\xi^i\cdot\sum_{i\geq 0} pr_2^*\xi^i\}_d
\end{aligned}$$

Therefore the equation (A.7) follows. □

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Summary

A cubic hypersurface is an algebraic variety defined by the set of zeros of a single polynomial equation of degree 3. The cubic hypersurface occupies a central place in classical to modern geometry. For instance, the one-dimensional cubic curves as elliptic curves, and the configuration of 27 lines on a cubic surface are geometric objects that have been studied tremendously since the nineteenth century.

In higher dimensions, the associated Fano variety of lines, which parametrizes the lines on a cubic hypersurface, is a common geometric object that appears in different branches, e.g. derived categories and enumerative geometry, to study the cubic hypersurfaces. In this dissertation, I study algebraic and geometric structures linking the cubic hypersurfaces and the associated Fano variety of lines in terms of algebraic cycles. The study consists of two parts.

The first part is the cylinder homomorphism. The family of lines over the Fano variety of a cubic hypersurface naturally defines the cylinder homomorphism. To any topological (algebraic) cycle on the cubic hypersurface this homormophism associates the cycle of lines parametrized by it. On the cohomology group, the cylinder homomorphism is fundamental to study other structures such as Hodge structures and intermediate Jacobians (by Clemens-Griffiths and Beauville-Donagi). Shimada illustrated more properties that are satified by the cylinder homomorphism. In particular, the map is always surjective. I prove that the cylinder homomorphism is universally surjective on the Chow groups, which generalizes the results for low dimensional cycles on the cubic hypersurface by Mingmin and René. The universally surjectivity means that after any field extension, the associated cylinder homomorphism remains surjective. Moreover, I applied the result to conclude the integral Hodge and Tate conjecture for one-cycles on the variety of lines.

The second part investigates certain relations between the Chow motives of cubic hypersurfaces and the Fano variety of lines. The general theory of motives is a program initiated by Grothendieck that aims to unify vast good cohomology theories of algebraic varieties. Using a group law-like structure on a smooth cubic hypersurface established by Galkin-Shinder, I prove a formula of decomposition of the diagonal of the Hilbert scheme of two points on the cubic hypersurface. The formula refines a result of Laterveer, which shows the Chow motive of the Fano variety of lines is essentially controlled by the Chow motive of the cubic hypersurface.

Samenvatting

Een kubisch hypervlak is een algebraïsche variëteit, gedefinieerd door de verzameling nulpunten van een derdegraads polynoom. Deze variëteiten staan centraal zowel in de klassieke als in de hedendaagse meetkunde. Bijvoorbeeld de elliptische krommen als een-dimensionale kubische krommen en de configuratie van 27 lijnen op een kubische oppervlak zijn meetkundige objecten die intensief bestudeerd worden sinds de negentiende eeuw.

Om in hogere dimensies de kubische hypervlakken te bestuderen wordt de geassocieerde Fano variëteit van lijnen gebruikt. De Fano variëteit parametriseert de lijnen op een kubisch hypervlak en komt voor in verschillende takken zoals afgeleide categorieën en enumeratieve meetkunde. In dit proefschrift bestudeer ik algebraïsche en meetkundige structuren die de kubische hypervlakken verbindt aan de Fano variëteiten van lijnen, in termen van algebraïsche cykels. Deze studie bestaat grofweg uit twee delen.

Het eerste deel gaat over het cylinder homomorfisme, wat op een natuurlijke manier gedefinieerd wordt aan de hand van de Fano variëteit van lijnen. Dit homomorfisme associeert aan een topologische (of algebraïsche) cykel, the cykel van lijnen die het parametriseert. Op de cohomologie groepen is het cylinder homomorfisme fundamenteel in het bestuderen van extra structuren als Hodge structuren en Jacobianen (gedaan door Clemens-Griffiths en Beauville-Donagi). Shimada beschreef meer eigenschappen van het cylinder homomorfisme. In het bijzonder is deze altijd surjectief. Ik laat zien dat het cylinder homomorfisme universeel surjectief is op Chow groepen, wat de resultaten van Mingmin en René over laag dimensionale cykels op kubische hypervlakken generaliseert. Universele surjectiviteit betekent dat het cylinder homomorfisme surjectief blijft na een willekeurige uitbreiding van het grondlichaam. Daarna pas ik dit resultaat toe om de integrale vermoeden van Hodge en Tate the bewijzen voor een-cykels op de variëteit van lijnen.

Het tweede deel onderzoekt bepaalde relaties tussen Chow motieven van kubische hypervlakken en de Fano variëteit van lijnen. De algemene theorie van motieven is een programma opgezet door Grothendieck wat zich richt op het samenbrengen van veel verschillende cohomologie theorieën voor algebraïsche variëteiten. Gebruikmakend van een groep-achtige structuur op een niet-singulier kubisch hypervlak, gedefinieerd door Galkin-Shinder, bewijs ik een formule voor de decompositie van de diagonaal van de Hilbert schema van twee punten op een kubisch hypervlak. Deze formule verfijnt een resultaat van Laterveer die bewijst dat de Chow motief van de Fano variëteit van lijnen werkelijk wordt bepaald door de Chow motief van het bijbehorende kubische hypervlak.