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ORDINARY K3 SURFACES OVER A FINITE FIELD

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ABSTRACT. We give a description of the category of ordinary K3 surfaces over a finite field in terms of linear algebra data over $\mathbf{Z}$. This gives an analogue for K3 surfaces of Deligne's description of the category of ordinary abelian varieties over a finite field, and refines earlier work by N.O. Nygaard and J.-D. Yu.

Our main result is conditional on a conjecture on potential semi-stable reduction of K3 surfaces over p-adic fields. We give unconditional versions for K3 surfaces of large Picard rank and for K3 surfaces of small degree.

INTRODUCTION

Statement of the main results. A K3 surface X over a perfect field k of characteristic p is called *ordinary* if it satisfies the following equivalent conditions:

(i) the Hodge and Newton polygons of $\mathrm{H}^2_{\mathrm{crys}}(X/W(k))$ coincide,
(ii) the Frobenius endomorphism of $\mathrm{H}^2(X, \mathcal{O}_X)$ is a bijection,
(iii) the formal Brauer group of X (see [1]) has height 1.

If k is finite, then these are also equivalent with $|X(k)| \not\equiv 1 \bmod p$. Building on [1] and [7], Nygaard [14] has shown that such an ordinary X has a canonical lift X_{can} over the ring of Witt vectors $W(k)$.

Choose an embedding $\iota\colon W(\mathbf{F}_q) \to \mathbf{C}$. Then with every ordinary K3 surface over $\mathbf{F}_q$ we can associate a complex K3 surface $X^\iota_{\mathrm{can}} := X_{\mathrm{can}} \otimes_{W(\mathbf{F}_q)} \mathbf{C}$ and an integral lattice

$$M := \mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{Z}).$$

Using the Kuga-Satake construction, Nygaard [14] and Yu [21] have shown that there exists a (necessarily unique) endomorphism F of $M \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{p}]$ such that for every $\ell \neq p$ the canonical isomorphism

$$M \otimes \mathbf{Z}_\ell \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{\mathbf{F}}_q}, \mathbf{Z}_\ell)$$

matches F with the geometric Frobenius Frob on étale cohomology (see also § 3.1). We have:

(M1) the pairing $\langle -, - \rangle$ on M is unimodular, even, and of signature $(3, 19)$;
(M2) $\langle Fx, Fy \rangle = q^2 \langle x, y \rangle$ for every $x, y \in M$.

From Deligne's proof of the Weil conjectures for K3 surfaces [6] we also know

(M3) the endomorphism F of $M \otimes \mathbf{C}$ is semi-simple and all its eigenvalues have absolute value q.

Our first result is an integral p-adic property of the pair (M, F).

Theorem A. *The endomorphism F preserves the $\mathbf{Z}$-module M and satisfies*

(M4) *the $\mathbf{Z}_p[F]$-module $M \otimes \mathbf{Z}_p$ decomposes as $M^0 \oplus M^1 \oplus M^2$ with*
  (a) *$FM^s = q^s M^s$, for all s,*
  (b) *M^0, M^1 and M^2 are free $\mathbf{Z}_p$-modules of rank 1, 20, 1, respectively.*

For a $\mathbf{Z}$-lattice M equipped with an endomorphism F satisfying (M1)–(M4) we denote by $\mathrm{NS}(M,F)$ the group

$$\mathrm{NS}(M,F) := \{x \in M \mid F^d x = q^d x \text{ for some } d \geq 1\}.$$

Using the fact that all line bundles on an ordinary K3 surface extend uniquely to its canonical lift, one shows that there is a natural bijection $\operatorname{Pic} X_{\bar{\mathbf{F}}_q} \to \mathrm{NS}(M,F)$, and that the ample line bundles on $X_{\bar{\mathbf{F}}_q}$ span a real cone $\mathcal{K} \subset M \otimes \mathbf{R}$ satisfying

(M5) $\mathcal{K}$ is a connected component of

$$\left\{ x \in \mathrm{NS}(M,F) \otimes \mathbf{R} \;\middle|\; \langle x,x\rangle > 0,\ \langle x,\delta\rangle \neq 0 \text{ for all } \delta \in \mathrm{NS}(M,F) \text{ with } \delta^2 = -2 \right\}$$

satisfying $F\mathcal{K} = \mathcal{K}$.

See § 3.2 for more details.

Definition (see [11])**.** Let $\mathcal{O}_K$ be a complete discrete valuation ring with fraction field K. We say that a K3 surface X over K satisfies $(\star)$ if there exists a finite extension $K \subset L$ and algebraic space $\mathfrak{X}/\mathcal{O}_L$ satisfying

(i) $\mathfrak{X}_L \cong X_L$,
(ii) $\mathfrak{X}$ is regular,
(iii) the special fiber is a strict normal crossing divisor in $\mathfrak{X}$,
(iv) the relative dualizing sheaf of $\mathfrak{X}/\mathcal{O}_L$ is trivial.

This is a strong form of 'potential semi-stable reduction'. Over a complete dvr $\mathcal{O}_K$ of residue characteristic 0, it is known that all K3 surfaces satisfy $(\star)$. It is expected to hold in general, but currently only known under extra assumptions on X.

Our main result is the following description of the category of ordinary K3 surfaces over $\mathbf{F}_q$. It is an analogue for K3 surfaces of Deligne's theorem [5] on ordinary abelian varieties over a finite field.

Theorem B. *Fix an embedding* $\iota\colon W(\mathbf{F}_q) \to \mathbf{C}$. *Then the resulting functor* $X \mapsto (M,F,\mathcal{K})$ *is a fully faithful functor between the groupoids of*

(i) *ordinary K3 surfaces* X *over* $\mathbf{F}_q$, *and*
(ii) *triples* $(M,F,\mathcal{K})$ *consisting of*
  (a) *an integral lattice* M,
  (b) *an endomorphism* F *of the* $\mathbf{Z}$*-module* M, *and*
  (c) *a convex subset* $\mathcal{K} \subset M \otimes \mathbf{R}$,
  satisfying (M1)–(M5).

If every K3 surface over $\operatorname{Frac} W(\mathbf{F}_q)$ *satisfies* $(\star)$ *then the functor is essentially surjective.*

Fully faithfulness is essentially due to Nygaard [15] and Yu [21]. Our contribution is a description of the image of this functor.

Restricting to families for which $(\star)$ is known to hold, we also obtain unconditional equivalences of categories between ordinary K3 surfaces $X/\mathbf{F}_q$ satisfying one of the following additional conditions

(i) there is an ample $\mathcal{L} \in \operatorname{Pic} X_{\bar{\mathbf{F}}_q}$ with $\mathcal{L}^2 < p-4$,
(ii) $\operatorname{Pic} X_{\bar{\mathbf{F}}_q}$ contains a hyperbolic plane and $p \geq 5$,
(iii) X has geometric Picard rank ≥ 12 and $p \geq 5$,

and triples $(M,F,\mathcal{K})$ satisfying the analogous constraints. See Theorem 4.6 for the precise statement.

About the proofs. A crucial ingredient in the proof of Theorems A and B is the following criterion which distinguishes the canonical lift amongst all lifts using p-adic étale cohomology.

Theorem C. *Let $\mathcal{O}_K$ be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic 0. Let $\mathfrak{X}$ be a projective K3 surface over $\mathcal{O}_K$ and assume that $\mathfrak{X}_k$ is ordinary. Then the following are equivalent:*

(i) *$\mathfrak{X}$ is the base change from $W(k)$ to $\mathcal{O}_K$ of the canonical lift of $\mathfrak{X}_k$*
(ii) *$\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p) \cong H^0 \oplus H^1(-1) \oplus H^2(-2)$ with H^i unramified $\mathbf{Z}_p[\mathrm{Gal}_K]$-modules, free of rank 1, 20, 1 over $\mathbf{Z}_p$, respectively.*

Here the (-1) and (-2) in (ii) denote Tate twists. This theorem is an integral refinement of a theorem of Yu [21] which characterizes *quasi-canonical* lifts by the splitting of étale cohomology with $\mathbf{Q}_p$-coefficients.

The canonical lift of X is defined in terms of its enlarged formal Brauer group Ψ (a p-divisible group), and to prove Theorem C we need to compare the p-adic étale cohomology of the generic fiber of a K3 surface over $\mathcal{O}_K$ to the Tate module of its enlarged formal Brauer group. With $\mathbf{Q}_p$-coefficients, such comparison has been shown by Artin and Mazur [1]. We give a different argument leading to an integral version, see Theorem 2.1. Once Theorem C is established, Theorem A is an almost formal consequence.

Finally, we briefly sketch the argument for the proof of Theorem B. Fully faithfulness was shown by Nygaard [15] and Yu [21] (see § 3.3 for more details). The proof of essential surjectivity starts with the observation that the decomposition in (M4) induces (via the embedding $\iota\colon \mathbf{Z}_p \to \mathbf{C}$) a Hodge structure on M, for which $\mathrm{NS}(M, F)$ consists precisely of the Hodge classes. The Torelli theorem for K3 surfaces then shows that there is a canonical K3 surface $X/\mathbf{C}$ with $\mathrm{H}^2(X, \mathbf{Z}) = M$ and whose ample cone $\mathcal{K} \subset \mathrm{H}^2(X, \mathbf{R})$ coincides with $\mathcal{K} \subset M \otimes \mathbf{R}$. This K3 surface has complex multiplication, and hence can be defined over a number field. Using the strong version of the main theorem of CM for K3 surfaces of [19] we show that we can find a model of X over $K := \operatorname{Frac} W(\mathbf{F}_q) \subset \mathbf{C}$ such that

(i) the Gal_K-module $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_p) = M \otimes \mathbf{Z}_p$ decomposes as in Theorem C,
(ii) for $\ell \neq p$, the Gal_K-module $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_\ell) = M \otimes \mathbf{Z}_\ell$ is unramified, and Frobenius acts as F.

Assuming $(\star)$, it follows from Néron–Ogg–Shafarevich criterion of Liedtke and Matsumoto [11] that X has good reduction over an unramified extension L of K. Using Theorem C we show that X_L is the canonical lift of its reduction, and deduce from this that X has already a smooth projective model $\mathfrak{X}$ over $\mathcal{O}_K$. By construction, its reduction $\mathfrak{X}_k$ maps under our functor to the given triple $(M, F, \mathcal{K})$.

A question. We end this introduction with an essentially lattice-theoretical question to which we do not know the answer:

Question. Does there exist a triple $(M, F, \mathcal{K})$ satisfying (M1)–(M5) and the inequality $1 + \operatorname{tr} F + q^2 < 0$?

By (M3) such triple can only exist for small q. A positive answer to this question would imply that there exist K3 surfaces over p-adic fields that do not satisfy $(\star)$. Indeed, if $(M, F, \mathcal{K})$ came from a K3 surface $X/\mathbf{F}_q$ as in Theorem B we would have

$$|X(\mathbf{F}_q)| = \operatorname{tr}(\mathrm{Frob}, \mathrm{H}^\bullet(X_{\bar{\mathbf{F}}_q}, \mathbf{Q}_\ell)) = 1 + \operatorname{tr} F + q^2 < 0,$$

which is absurd.

1. p-divisible groups associated to K3 surfaces

Let Λ be a complete noetherian local ring with perfect residue field k of characteristic $p > 0$ and let $\mathfrak{X}$ be a K3 surface over $\operatorname{Spec}\Lambda$. We recall (and complement) some of the main results of Artin and Mazur [1] on the formal Brauer group and enlarged formal Brauer group of $\mathfrak{X}$.

1.1. **The formal Brauer group.** Let Art_Λ be the category of Artinian local Λ-algebras $(A, \mathfrak{m})$ with perfect residue field $A/\mathfrak{m}$. For an $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$ we denote by $\mathcal{U}_A$ the sheaf on $\mathfrak{X}_{\mathrm{\acute{e}t}}$ defined by the exact sequence

$$1 \to \mathcal{U}_A \to \mathcal{O}^\times_{\mathfrak{X}_A} \to \mathcal{O}^\times_{\mathfrak{X}_{A/\mathfrak{m}}} \to 1.$$

The *formal Brauer group* of $\mathfrak{X}$ is the functor

$$\hat{\mathrm{Br}}(\mathfrak{X})\colon\ \mathrm{Art}_\Lambda \to \mathrm{Ab},\ A \mapsto \mathrm{H}^2(\mathfrak{X}_{\mathrm{\acute{e}t}}, \mathcal{U}_A).$$

By [1] it is representable by a one-dimensional formal group, and if $\mathfrak{X}$ is not supersingular then $\hat{\mathrm{Br}}(\mathfrak{X})$ is a p-divisible group.

Lemma 1.1. $\mathrm{H}^\bullet(\mathfrak{X}_{\mathrm{\acute{e}t}}, \mathcal{U}_A) = \mathrm{H}^\bullet(\mathfrak{X}, \mathcal{U}_A)$ *and* $\mathrm{H}^1(\mathfrak{X}_{\mathrm{\acute{e}t}}, \mathcal{U}_A) = 0$.

Here $\mathrm{H}^\bullet(\mathfrak{X}, -)$ denotes Zariski cohomology.

Proof of Lemma 1.1. The sheaf $\mathcal{U}_A$ has a filtration whose graded pieces are

$$\mathfrak{m}^n/\mathfrak{m}^{n+1} \otimes_{A/\mathfrak{m}} \mathcal{O}_{\mathfrak{X}_{A/\mathfrak{m}}}$$

Since these are coherent, we have $\mathrm{H}^\bullet(\mathfrak{X}_{\mathrm{\acute{e}t}}, \mathcal{U}_A) = \mathrm{H}^\bullet(\mathfrak{X}, \mathcal{U}_A)$. Moreover, since $\mathrm{H}^1(\mathfrak{X}_{A/\mathfrak{m}}, \mathcal{O}_{\mathfrak{X}_{A/\mathfrak{m}}})$ vanishes, we conclude that $\mathrm{H}^1(\mathfrak{X}_{\mathrm{\acute{e}t}}, \mathcal{U}_A) = 0$. □

Lemma 1.2. *For every* $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$ *there is a natural exact sequence*

$$0 \longrightarrow \hat{\mathrm{Br}}(\mathfrak{X})[p^r](A) \longrightarrow \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_A, \mu_{p^r}) \longrightarrow \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r})$$

of abelian groups.

Here $\mathrm{H}^\bullet_{\mathrm{fl}}$ denotes cohomology in the fppf topology.

Proof of Lemma 1.2. Consider the complex $\mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A$ on $\mathfrak{X}_{\mathrm{\acute{e}t}}$ in degrees 0 and 1. We have a short exact sequence

$$1 \longrightarrow \mathrm{H}^1(\mathfrak{X}, \mathcal{U}_A) \otimes \mathbf{Z}/p^r\mathbf{Z} \longrightarrow \mathrm{H}^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A) \longrightarrow \mathrm{H}^2(\mathfrak{X}, \mathcal{U}_A)[p^r] \longrightarrow 1$$

and thanks to Lemma 1.1 we obtain canonical isomorphisms

$$\hat{\mathrm{Br}}(\mathfrak{X})[p^r](A) = \mathrm{H}^2(\mathfrak{X}, \mathcal{U}_A)[p^r] = \mathrm{H}^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A).$$

Now consider the short exact sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{U}_A & \longrightarrow & \mathcal{O}^\times_{\mathfrak{X}_A} & \longrightarrow & \mathcal{O}^\times_{\mathfrak{X}_{A/\mathfrak{m}}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle p^r} & & \downarrow{\scriptstyle p^r} & & \downarrow{\scriptstyle p^r} & & \\
1 & \longrightarrow & \mathcal{U}_A & \longrightarrow & \mathcal{O}^\times_{\mathfrak{X}_A} & \longrightarrow & \mathcal{O}^\times_{\mathfrak{X}_{A/\mathfrak{m}}} & \longrightarrow & 1
\end{array}$$

of length-two complexes concentrated in degrees 0 and 1. Using the Kummer sequence and the fact that $\mathbf{G}_m$ is smooth, we see that

$$\mathrm{H}^n_{\text{ét}}(\mathfrak{X}, \mathcal{O}^\times_{\mathfrak{X}_A} \xrightarrow{p^r} \mathcal{O}^\times_{\mathfrak{X}_A}) = \mathrm{H}^n_{\text{fl}}(\mathfrak{X}_A, \mathbf{G}_m \xrightarrow{p^r} \mathbf{G}_m) = \mathrm{H}^n_{\text{fl}}(\mathfrak{X}_A, \mu_{p^r})$$

and similarly for $\mathfrak{X}_{A/\mathfrak{m}}$. It follows that the above short exact sequence of complexes induces a long exact sequence of (hyper-)cohomology groups

$$\longrightarrow \mathrm{H}^1_{\text{fl}}(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r}) \longrightarrow \mathrm{H}^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A) \longrightarrow \mathrm{H}^2_{\text{fl}}(\mathfrak{X}_A, \mu_{p^r}) \longrightarrow \mathrm{H}^2_{\text{fl}}(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r}) \longrightarrow$$

Since $A/\mathfrak{m}$ is perfect and $\operatorname{Pic}\mathfrak{X}_{A/\mathfrak{m}}$ is torsion-free, we have $\mathrm{H}^1_{\text{fl}}(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r}) = 0$ and we conclude

$$\hat{\mathrm{Br}}(\mathfrak{X})[p^r](A) = \mathrm{H}^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A) = \ker\left[\mathrm{H}^2_{\text{fl}}(\mathfrak{X}_A, \mu_{p^r}) \to \mathrm{H}^2_{\text{fl}}(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r})\right],$$

which is what we had to show. □

1.2. **The enlarged formal Brauer group.** We now assume that $\mathfrak{X}_k$ is ordinary. Denote by μ_{p^∞} the sheaf $\operatorname{colim}_r \mu_{p^r}$ on the fppf site. The *enlarged formal Brauer group* of $\mathfrak{X}$ is the functor

$$\Psi(\mathfrak{X})\colon \operatorname{Art}_\Lambda \to \operatorname{Ab},\, A \mapsto \mathrm{H}^2_{\text{fl}}(\mathfrak{X}_A, \mu_{p^\infty}).$$

A priori this differs from the definition of Artin–Mazur [1, § IV.1] in two ways. First, Artin and Mazur restrict to A with algebraically closed residue fields, and then use Galois descent to extend their definition to non-closed perfect residue fields, and second, they restrict to those classes in $\mathrm{H}^2_{\text{fl}}(\mathfrak{X}_A, \mu_{p^\infty})$ that map to the p-divisible part of $\mathrm{H}^2_{\text{fl}}(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})$. Lemmas 1.4 and 1.5 below show that the above definition is equivalent to that of Artin–Mazur (under our standing condition that $\mathfrak{X}_k$ is an ordinary K3 surface). See also [14, Cor. 1.5].

The following lemma is well-known, and implicitly used in [1] and [14]. We include it for the sake of completeness.

Lemma 1.3. *For any quasi-compact quasi-separated scheme X there is a natural isomorphism*

$$\mathrm{H}^\bullet_{\text{fl}}(X, \mu_{p^\infty}) \xrightarrow{\sim} \operatorname{colim}_r \mathrm{H}^\bullet_{\text{fl}}(X, \mu_{p^r}).$$

Proof. This follows from [18, 0739], taking for $\mathcal{B}$ the class of quasi-compact and quasi-separated schemes, and for Cov the fpqc covers consisting of finitely many affine schemes. □

Lemma 1.4 ([14, Cor. 1.4])**.** *The group $\mathrm{H}^2_{\text{fl}}(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})$ is p-divisible.* □

Lemma 1.5. $\mathrm{H}^2_{\text{fl}}(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})^{\operatorname{Gal}(\bar{k}/k)} = \mathrm{H}^2_{\text{fl}}(\mathfrak{X}_k, \mu_{p^\infty})$.

Proof. Since the p-th power map $\bar{k}^\times \to \bar{k}^\times$ is a bijection, and since $\operatorname{Pic}\mathfrak{X}_{\bar{k}}$ is torsion-free, we have

$$\mathrm{H}^i_{\text{fl}}(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty}) = \operatorname{colim}_r \mathrm{H}^i_{\text{fl}}(\mathfrak{X}_{\bar{k}}, \mu_{p^r}) = 0$$

for $i \in \{0, 1\}$. It now follows from the Hochschild-Serre spectral sequence

$$E_2^{s,t} = \mathrm{H}^s\big(\operatorname{Gal}(\bar{k}/k),\, \mathrm{H}^t_{\text{fl}}(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})\big) \Rightarrow \mathrm{H}^{s+t}_{\text{fl}}(\mathfrak{X}_k, \mu_{p^\infty})$$

that $\mathrm{H}^2_{\text{fl}}(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})^{\operatorname{Gal}(\bar{k}/k)} = \mathrm{H}^2_{\text{fl}}(\mathfrak{X}_k, \mu_{p^\infty})$. □

The following theorem summarizes the properties of the enlarged formal Brauer group that we will use.

Theorem 1.6. *Let $\mathfrak{X}$ be a formal K3 surface over Λ with $\mathfrak{X}_k$ ordinary. Then the enlarged formal Brauer group $\Psi(\mathfrak{X})$ is representable by a p-divisible group over Λ. Its étale-local exact sequence*

$$0 \to \Psi^\circ(\mathfrak{X}) \to \Psi(\mathfrak{X}) \to \Psi^{\text{ét}}(\mathfrak{X}) \to 0$$

satisfies

(i) $\Psi^\circ(\mathfrak{X})$ *is a connected p-divisible group of height* 1*, with*

$$\Psi^\circ(\mathfrak{X})[p^r](A) = \ker\left(\mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_A, \mu_{p^r}) \to \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r})\right)$$

for all $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$*. It is canonically isomorphic to* $\hat{\mathrm{Br}}(\mathfrak{X})$*;*

(ii) $\Psi(\mathfrak{X})[p^r](A) = \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_A,\, \mu_{p^r})$ *for all* $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$*;*

(iii) $\Psi^{\text{ét}}(\mathfrak{X})[p^r](A) = \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_{A/\mathfrak{m}},\, \mu_{p^r})$ *for all* $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$*.*

Proof. The representability and (i) are shown in [1, Prop. IV.1.8].

To prove (ii), note that since $\mathrm{H}^1(\mathfrak{X}_A,\, \mathbf{G}_m) = \operatorname{Pic} \mathfrak{X}_A$ is torsion-free, we have a natural isomorphism

$$\mathrm{H}^1_{\mathrm{fl}}(\mathfrak{X}_A, \mu_{p^r}) = \mathrm{H}^0(\mathfrak{X}_A,\, \mathbf{G}_m) \otimes \mathbf{Z}/p^r\mathbf{Z}.$$

Taking the colimit over r we obtain a natural isomorphism

$$\mathrm{H}^1_{\mathrm{fl}}(\mathfrak{X}_A, \mu_{p^\infty}) = \mathrm{colim}_r\, \mathrm{H}^1_{\mathrm{fl}}(\mathfrak{X}_A, \mu_{p^r}) = \mathrm{H}^0(\mathfrak{X}_A,\, \mathbf{G}_m) \otimes (\mathbf{Q}_p/\mathbf{Z}_p),$$

and in particular we see that $\mathrm{H}^1_{\mathrm{fl}}(\mathfrak{X}_A,\, \mu_{p^\infty})$ is p-divisible. Now the long exact sequence associated to

$$1 \longrightarrow \mu_{p^r} \longrightarrow \mu_{p^\infty} \xrightarrow{p^r} \mu_{p^\infty} \longrightarrow 1$$

induces an isomorphism $\mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_A,\, \mu_{p^r}) \cong \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_A,\, \mu_{p^\infty})[p^r]$, which proves (ii).

A similar argument shows (iii) for A with $A/\mathfrak{m}$ algebraically closed, after which Lemma 1.5 implies the general case. □

1.3. **Canonical lifts.** Let X/k be an ordinary K3 surface. Since formal groups of height 1 are rigid, the p-divisible group $\Psi^\circ(X)$ over k extends uniquely to a p-divisible group $\Psi^\circ(X)_{\mathrm{can}}$ over Λ. Also the étale p-divisible group $\Psi^{\text{ét}}(X)$ over k extends uniquely to a p-divisible group $\Psi^{\text{ét}}(X)_{\mathrm{can}}$ over Λ.

To every lift $\mathfrak{X}/\Lambda$ of X/k we then have an associated short exact sequence of p-divisible groups

$$0 \longrightarrow \Psi^\circ(X)_{\mathrm{can}} \longrightarrow \Psi(\mathfrak{X}) \longrightarrow \Psi^{\text{ét}}(X)_{\mathrm{can}} \longrightarrow 0 \tag{1}$$

over Λ. In analogy with Serre-Tate theory, we have the following theorem.

Theorem 1.7 (Nygaard [14, Thm. 1.6])**.** *The map*

$$\left\{ \textit{formal lifts } \mathfrak{X}/\Lambda \textit{ of } X/k \right\} \to \mathrm{Ext}^1_\Lambda\left(\Psi^{\text{ét}}(X)_{\mathrm{can}},\, \Psi^\circ(X)_{\mathrm{can}}\right),\ \mathfrak{X} \mapsto \Psi(\mathfrak{X})$$

is a bijection. □

It follows that there exists a unique lift $\mathfrak{X}/\Lambda$ for which the sequence (1) splits. This $\mathfrak{X}$ is unique up to unique isomorphism, and is called the *canonical lift* of X. We denote it by X_{can}.

Proposition 1.8 ([14, Prop. 1.8])**.** $\operatorname{Pic} X_{\mathrm{can}} \to \operatorname{Pic} X$ *is a bijection.* □

Corollary 1.9 ([14, Prop. 1.8])**.** X_{can} *is algebraizable and projective.* □

2. p-ADIC ÉTALE COHOMOLOGY

Let $\mathcal{O}_K$ be a complete discrete valuation ring whose residue field k is perfect of characteristic p and whose fraction field K is of characteristic 0.

2.1. p-adic étale cohomology and the enlarged formal Brauer group.

Theorem 2.1. *Let $\mathfrak{X}$ be a projective K3 surface over $\mathcal{O}_K$. Assume that $\mathfrak{X}_k$ is ordinary. Then there is a natural injective map of* Gal_K*-modules*

$$T_p\Psi(\mathfrak{X})_{\bar{K}} \to \mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_p(1))$$

whose cokernel is a free $\mathbf{Z}_p$-module of rank 1.

Recall that if $\mathfrak{X}$ is ordinary, then $T_p\Psi(\mathfrak{X})_{\bar{K}}$ has rank 21. Up to possible torsion in the cokernel, Theorem 2.1 is shown in [1, § IV.2]. The proof of Artin and Mazur is based on Lefschetz pencils, reducing the problem on H^2 to a statement about H^1 and torsors. We give a proof working directly with the H^2 and their relation to Brauer groups to obtain the finer 'integral' statement above. This is made possible by the theorem of Gabber and de Jong [4] asserting that the Brauer group and the cohomological Brauer group of a quasi-projective scheme coincide.

Let $\mathfrak{X}$ be a formal K3 surface over $\mathcal{O}_K$. We denote by $\mathfrak{X}_n$ the truncation $\mathfrak{X}_{\mathcal{O}_K/\mathfrak{m}^n}$.

Lemma 2.2. *For all i the natural map*

$$\mathrm{H}^i_{\text{fl}}(\mathfrak{X}, \mu_{p^r}) \to \lim_n \mathrm{H}^i_{\text{fl}}(\mathfrak{X}_n, \mu_{p^r})$$

is an isomorphism.

Proof. As in the proof of Lemma 1.2, we have

$$\mathrm{H}^i_{\text{fl}}(\mathfrak{X}_n, \mu_{p^r}) = \mathrm{H}^i_{\text{ét}}(\mathfrak{X}, \mathcal{O}^\times_{\mathfrak{X}_n} \xrightarrow{p^r} \mathcal{O}^\times_{\mathfrak{X}_n})$$

and similarly

$$\mathrm{H}^i_{\text{fl}}(\mathfrak{X}, \mu_{p^r}) = \mathrm{H}^i_{\text{ét}}(\mathfrak{X}, \mathcal{O}^\times_{\mathfrak{X}} \xrightarrow{p^r} \mathcal{O}^\times_{\mathfrak{X}}).$$

Let $\mathcal{U}_n$ be the kernel of $\mathcal{O}^\times_{\mathfrak{X}_n} \to \mathcal{O}^\times_{\mathfrak{X}_1}$ and $\mathcal{U}$ the kernel of $\mathcal{O}^\times_{\mathfrak{X}} \to \mathcal{O}^\times_{\mathfrak{X}_1}$. Then by the usual dévissage arguments the lemma reduces to showing that

$$\mathrm{H}^i_{\text{ét}}(\mathfrak{X}, \mathcal{U}) \to \lim_n \mathrm{H}^i_{\text{ét}}(\mathfrak{X}, \mathcal{U}_n)$$

is an isomorphism for all i.

Since the maps $\mathcal{U}_{n+1} \to \mathcal{U}_n$ are surjective, we have $\mathrm{Rlim}_n \mathcal{U}_n = \mathcal{U}$. Since $\mathcal{U}$ has a filtration with graded pieces isomorphic to $\mathcal{O}_{\mathfrak{X}_1}$ it has cohomology concentrated in degrees 0 and 2. These two facts imply

$$\mathrm{R}\Gamma_{\text{ét}}(\mathfrak{X}, \mathrm{Rlim}_n \mathcal{U}_n) = \mathrm{H}^0_{\text{ét}}(\mathfrak{X}, \mathcal{U}) \oplus \mathrm{H}^2_{\text{ét}}(\mathfrak{X}, \mathcal{U})[-2]$$

in $\mathcal{D}(\mathrm{Ab})$. Similarly, we have

$$\mathrm{Rlim}_n \mathrm{R}\Gamma_{\text{ét}}(\mathfrak{X}, \mathcal{U}_n) = \lim_n \mathrm{H}^0_{\text{ét}}(\mathfrak{X}, \mathcal{U}_n) \oplus \lim_n \mathrm{H}^2_{\text{ét}}(\mathfrak{X}, \mathcal{U}_n)[-2]$$

in $\mathcal{D}(\mathrm{Ab})$. As $\mathrm{R}\Gamma_{\text{ét}}$ commutes with Rlim, the lemma follows. □

Corollary 2.3. *If $\mathfrak{X}_k$ is ordinary, then $\Psi(\mathfrak{X})(K)[p^r] = \mathrm{H}^2_{\text{fl}}(\mathfrak{X}, \mu_{p^r})$.*

Proof. Indeed, we have

$$\Psi(\mathfrak{X})(K)[p^r] = \lim_n \Psi(\mathfrak{X})[p^r](\mathcal{O}_K/\mathfrak{m}^n) = \lim_n \mathrm{H}^2_{\text{fl}}(\mathfrak{X}_n, \mu_{p^r}),$$

so the corollary follows from Lemma 2.2. □

Proposition 2.4. *If $\mathfrak{X}$ is a projective K3 surface over $\mathcal{O}_K$ then for all r the natural map $\mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}, \mu_{p^r}) \to \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_K, \mu_{p^r})$ is injective.*

Proof. The Kummer sequence gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\operatorname{Pic}\mathfrak{X}) \otimes \mathbf{Z}/p^r\mathbf{Z} & \longrightarrow & \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}, \mu_{p^r}) & \longrightarrow & (\mathrm{Br}'\,\mathfrak{X})[p^r] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\operatorname{Pic}\mathfrak{X}_K) \otimes \mathbf{Z}/p^r\mathbf{Z} & \longrightarrow & \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_K, \mu_{p^r}) & \longrightarrow & (\mathrm{Br}'\,\mathfrak{X}_K)[p^r] & \longrightarrow & 0 \end{array}$$

Since $\mathfrak{X}$ is projective, we have $\mathrm{Br} = \mathrm{Br}'$ for $\mathfrak{X}$ and $\mathfrak{X}_K$.

The left arrow in the diagram is an isomorphism since the special fiber $\mathfrak{X}_k$ is a principal divisor in $\mathfrak{X}$, so that $\operatorname{Pic}\mathfrak{X} \to \operatorname{Pic}\mathfrak{X}_K$ is an isomorphism. By [8, Cor 1.8] the natural maps of $\mathrm{Br}\,\mathfrak{X}$ and $\mathrm{Br}\,\mathfrak{X}_K$ to $\mathrm{Br}\,K(\mathfrak{X}_K)$ are injective, so that also the right arrow in the diagram is injective. We conclude that the middle map is injective. □

Proof of Theorem 2.1. The proof is now formal. By Corollary 2.3 and Proposition 2.4 we have for every r and every finite extension $K \subset L$ a canonical injection

$$\Psi(\mathfrak{X})[p^r](L) \to \mathrm{H}^2_{\mathrm{fl}}(\mathfrak{X}_L, \mu_{p^r}) = \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_L, \mathbf{Z}/p^r\mathbf{Z}(1)).$$

Taking the colimit over all L we obtain a Gal_K-equivariant injective map

$$\rho_r \colon \Psi(\mathfrak{X})[p^r](\bar{K}) \to \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}/p^r\mathbf{Z}(1)),$$

and taking the limit over r we obtain a Gal_K-equivariant injective map

$$\rho \colon T_p\Psi(\mathfrak{X})_{\bar{K}} \to \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)).$$

Denote the cokernel of ρ by Q. Tensoring ρ with $\mathbf{Z}/p\mathbf{Z}$ yields an exact sequence

$$0 \longrightarrow \mathrm{Tor}(Q, \mathbf{Z}/p\mathbf{Z}) \longrightarrow \Psi(\mathfrak{X})[p](\bar{K}) \xrightarrow{\rho\otimes\mathbf{Z}/p\mathbf{Z}} \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}/p\mathbf{Z}(1)) \longrightarrow Q\otimes\mathbf{Z}/p\mathbf{Z} \longrightarrow 0.$$

Since $\rho_1 = \rho \otimes \mathbf{Z}/p\mathbf{Z}$ is injective, we see that $\mathrm{Tor}(Q, \mathbf{Z}/p\mathbf{Z})$ vanishes and that Q is torsion-free. □

2.2. **Canonical lifts and p-adic étale cohomology.** In this section we prove Theorem C, characterizing the canonical lift in terms of p-adic étale cohomology.

Lemma 2.5. *Let U be a free $\mathbf{Z}_p$-module of rank 2 and $b \colon U \times U \to \mathbf{Z}_p$ a non-degenerate symmetric bilinear form. Let $L \subset U$ be a totally isotropic rank 1 submodule. If L is saturated in*

$$U^\vee := \{x \in \mathbf{Q}_p \otimes_{\mathbf{Z}_p} U \mid b(x, U) \subset \mathbf{Z}_p\},$$

then $U^\vee = U$.

Proof. Since L is saturated in $U^\vee$, it is also saturated in $U \subset U^\vee$ and we may choose a basis (e, f) for U with $L = \langle e \rangle$. Set $d := b(e, f)$. Since $b(e, e) = 0$, the determinant of b is $-d^2$. Since e/d lies in $U^\vee$, we must have that d is a unit and therefore $U^\vee = U$. □

Proof of Theorem C. Assume that (ii) holds. Then we have

$$\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) = H^0(1) \oplus H^1 \oplus H^2(-1)$$

with the H^i unramified. Since the Tate module of a p-divisible group is Hodge-Tate of weights 0 and -1, we have that $\mathrm{Hom}(T_p\Psi(\mathfrak{X})_{\bar{K}}, H^2(-1)) = 0$, and by Theorem 2.1 we see that $T_p\Psi(\mathfrak{X})_{\bar{K}} = H^0(1) \oplus H^1$. By Tate's theorem [20, Thm. 4] this implies that $\Psi(\mathfrak{X}) = \Psi^0(\mathfrak{X}) \oplus \Psi^{\text{ét}}(\mathfrak{X})$ with $T_p\Psi^0(\mathfrak{X})_{\bar{K}} = H^0(1)$ and $T_p\Psi^{\text{ét}}(\mathfrak{X})_{\bar{K}} = H^1$. It follows that $\mathfrak{X}$ is the base change of the canonical lift of $\mathfrak{X}_k$ to $\mathcal{O}_K$.

Conversely, assume that $\mathfrak{X}$ is the base change of the canonical lift of $\mathfrak{X}_k$ to $\mathcal{O}_K$. Let H^1 be the image of the direct summand $T_p\Psi^{\text{ét}}(\mathfrak{X})_{\bar{K}}$ under the embedding $T_p\Psi(\mathfrak{X})_{\bar{K}} \to \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ of Theorem 2.1. It is a primitive sub-module, and considering Hodge-Tate weights we see that the restriction of the bilinear form on H^2 to H^1 is non-degenerate. Let $U \subset \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ be its orthogonal complement. Then U is a rank 2 lattice over $\mathbf{Z}_p$. The inclusions of H^1 and U as mutual orthogonal complements inside the self-dual lattice $\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ induce an isomorphism

$$\alpha\colon U^\vee/U \xrightarrow{\sim} (H^1)^\vee/H^1$$

and an identification

$$\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) := \{(x,y) \in U^\vee \oplus (H^1)^\vee \mid \alpha(x) = y\}.$$

Consider the unramified Gal_K-module $H^0 := T_p\Psi^\circ(\mathfrak{X})_{\bar{K}}(-1)$. We have that $H^0(1)$ is a totally isotropic line in U. We claim that it is saturated in $U^\vee$. Indeed, if $x \in U^\vee$ satisfies $px \in H^0(1)$ then $(x, \alpha(x))$ defines a p-torsion element in the cokernel of $T_p\Psi(\mathfrak{X})_{\bar{K}} \to \mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$, which must be trivial by Theorem 2.1. By Lemma 2.5 we conclude that $U = U^\vee$ and that $\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) = U \oplus H^1$.

Now U is a unimodular $\mathbf{Z}_p$-lattice of rank 2 containing an isotropic line. Moreover, since the intersection pairing on $\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ is even, so is the lattice U. It follows that there is a unique isotropic line $H^2(-1) \subset U$ with $U = H^0(1) \oplus H^2(-1)$ and with H^0 and H^2 dual unramified representations. We find $\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) = H^0(1) \oplus H^1 \oplus H^2(-1)$, as claimed. □

Remark 2.6. Using the results on integral p-adic Hodge theory by Bhatt, Morrow, and Scholze [2] one can show that the splitting of $\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p)$ as in Theorem C implies an analogous splitting of the filtered crystal $\mathrm{H}^2_{\text{crys}}(\mathfrak{X}/W)$. If $p > 2$, then the splitting of $\mathrm{H}^2_{\text{crys}}(\mathfrak{X}/W)$ implies that $\mathfrak{X}$ is the canonical lift of $\mathfrak{X}_k$ (see [7] and [14, Lem. 1.11, Thm. 1.12]). For $p = 2$, however, the splitting of $\mathrm{H}^2_{\text{crys}}(\mathfrak{X}/W)$ is a weaker condition than the splitting of $\mathrm{H}^2_{\text{ét}}(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p)$, see also [7, 2.1.16.b].

3. The functor $X \mapsto (M, F, \mathcal{K})$

Let $\mathbf{F}_q$ be a finite field with $q = p^a$ elements. Let W be the ring of Witt vectors of $\mathbf{F}_q$, and K its fraction field. Fix an embedding $\iota\colon \bar{K} \to \mathbf{C}$. By § 1.3, every ordinary K3 surface X over $\mathbf{F}_q$ has a canonical lift X_{can} over W. We will denote by X^ι_{can} the complex K3 surface obtained by base changing X_{can} along $\iota\colon W \to \mathbf{C}$.

3.1. Construction of a pair (M, F).

Let X be an ordinary K3 surface over $\mathbf{F}_q$. The following theorem, due to Nygaard and Yu, says that the Frobenius on X can be lifted to an endomorphism of the Betti cohomology of X^ι_{can}, at least after inverting p. The proof relies on the Kuga–Satake construction.

Theorem 3.1 (Nygaard [14, § 3], Yu [21, Lemma 2.3])**.** *There is a unique endomorphism F of $\mathrm{H}^2(X^\iota_{\text{can}}, \mathbf{Z}[\frac{1}{p}])$ such that*

(i) *for every $\ell \neq p$ the map F corresponds under the comparison isomorphism*

$$\mathrm{H}^2(X^\iota_{\text{can}}, \mathbf{Z}[\tfrac{1}{p}]) \otimes \mathbf{Z}_\ell \xrightarrow{\sim} \mathrm{H}^2_{\text{ét}}(X_{\bar{\mathbf{F}}_q}, \mathbf{Z}_\ell)$$

to the geometric Frobenius Frob *on étale cohomology,*

(ii) *the map F corresponds under the comparison isomorphism*

$$\mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{Z}[\tfrac{1}{p}]) \otimes B_{\mathrm{crys}} \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{crys}}(X/W) \otimes_W B_{\mathrm{crys}}$$

to the endomorphism $\phi^a \otimes \mathrm{id}$, where ϕ denotes the crystalline Frobenius.

Moreover, F preserves the Hodge structure on $\mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{Q})$. □

For later use in the proofs of Theorems A and B, we record some well-known properties of Tate twists of unramified p-adic Galois representations.

Lemma 3.2. *Let V be an unramified Gal_K-representation over $\mathbf{Q}_p$, and let n be an integer. Let* Frob *be the geometric Frobenius endomorphism of V, relative to $\mathbf{F}_q$. Consider the Gal_K-module $V(-n) := V \otimes \mathbf{Q}_p(-n)$.*

(i) *The map*

$$K^\times \to \mathrm{GL}(V(-n)),\ x \mapsto \mathrm{Frob}^{v(x)} \otimes q^{nv(x)}\, \mathrm{Nm}_{K/\mathbf{Q}_p}(x)^{-n}$$

factors over the reciprocity map $K^\times \to \mathrm{Gal}_K^{\mathrm{ab}}$ and induces the action of Gal_K on $V(-n)$.

(ii) *$D_{\mathrm{crys}}(q^n\,\mathrm{Frob}) = \phi^a$ as endomorphisms of $D_{\mathrm{crys}}(V(-n))$.*

Proof. The first statement follows from Lubin–Tate theory, see for example [17, § 3.1, Theorem 2]. For the second, one uses that the functor D_{crys} commutes with Tate twists to reduce to the case $n = 0$. In this case, the statement follows from the observation that $\mathrm{Frob} \otimes 1 = 1 \otimes \phi^a$ as endomorphisms of

$$\left(V \otimes_{\mathbf{Z}_p} W(\bar{\mathbf{F}}_q)\right)^{\mathrm{Gal}_{\mathbf{F}_q}},$$

where the action of $\mathrm{Gal}_{\mathbf{F}_q}$ is the diagonal one. □

Theorem A is now an almost immediate consequence of Theorem C.

Proof of Theorem A. By its definition (in Theorem 3.1), the endomorphism F of $\mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{Q}) \otimes \mathbf{Q}_p = \mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_p)$ satisfies $D_{\mathrm{crys}}(F) = \phi^a$ on $\mathrm{H}^2_{\mathrm{crys}}(X/W)[\frac{1}{p}]$.

By Theorem C we have a decomposition $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_p) = H^0 \oplus H^1(-1) \oplus H^2(-2)$ with H^i unramified and by Lemma 3.2, also the endomorphism

$$F' := \mathrm{Frob}_{H^0} \oplus q\,\mathrm{Frob}_{H^1} \oplus q^2\,\mathrm{Frob}_{H^2}$$

of $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_p)$ satisfies $D_{\mathrm{crys}}(F') = \phi^a$. Since D_{crys} is fully faithful, we must have $F = F'$. But it then follows immediately that F preserves the $\mathbf{Z}_p$-lattice $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_p)$, and that $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_p)$ decomposes as described in the theorem. □

We thus have constructed from $X/\mathbf{F}_q$ an integral lattice $M := \mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{Z})$, equipped with an endomorphism F, satisfying (M1)–(M4). We end this paragraph by relating the p-adic decomposition in (M4) to the Hodge decomposition for X^ι_{can}.

Lemma 3.3. *Let (M, F) be a pair satisfying (M1)–(M4). Then complex conjugation on $M \otimes \mathbf{C}$ maps the subspace $M^s \otimes_{\mathbf{Z}_p,\iota} \mathbf{C}$ to $M^{2-s} \otimes_{\mathbf{Z}_p,\iota} \mathbf{C}$.*

In other words: the decomposition in (M4) induces under $\iota\colon \mathbf{Z}_p \to \mathbf{C}$ a $\mathbf{Z}$-Hodge structure on M.

Proof of Lemma 3.3. By (M4), the one-dimensional subspace $M^0 \otimes_{\mathbf{Z}_p,\iota} \mathbf{C}$ of $M \otimes \mathbf{C}$ is the unique eigenspace for the endomorphism F corresponding to an eigenvalue $u \in \mathbf{Q}_p \subset \mathbf{C}$ with $v_p(u) = 0$. By (M3) we have $u\bar{u} = q^2$, and hence also the eigenvalue $\bar{u} = q^2/u$ lies in $\mathbf{Q}_p \subset \mathbf{C}$. Since $v_p(\bar{u}) = v_p(q^2)$, we see that the corresponding eigenspace is $M^2 \otimes_{\mathbf{Z}_p,\iota} \mathbf{C}$. Similarly, complex conjugation maps M^2 to M^0. By (M2), the subspace M^1 is the orthogonal complement of $M^0 \oplus M^2$, and hence is preserved by complex conjugation. □

Proposition 3.4. *Let (M, F) be the pair associated to an ordinary K3 surface X over $\mathbf{F}_q$. Then we have $M^s \otimes_{\mathbf{Z}_p,\iota} \mathbf{C} = \mathrm{H}^{s,2-s}(X^\iota_{\mathrm{can}})$ as subspaces of $M \otimes \mathbf{C} = \mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{C})$.*

Proof. Indeed, under the 'Hodge–Tate' comparison isomorphism

$$\mathrm{H}^2_{\mathrm{dR}}(X_{\mathrm{can},K}/K) \otimes_K \mathbf{C}_p \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \cong M \otimes_{\mathbf{Z}_p} \mathbf{C}_p$$

the subspace $(\mathrm{Fil}^i \mathrm{H}^2_{\mathrm{dR}}(X_{\mathrm{can},K}/K)) \otimes_K \mathbf{C}_p$ is mapped to $\oplus_{s \geq i} M^s \otimes_{\mathbf{Z}_p} \mathbf{C}_p$. Extending ι to an embedding $\mathbf{C}_p \to \mathbf{C}$ we see that the Hodge filtration on $\mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{C})$ agrees with the filtration on $M \otimes \mathbf{C}$ induced by the decomposition on $M \otimes \mathbf{Z}_p$, and hence by Lemma 3.3 we have $M^s \otimes_{\mathbf{Z}_p} \mathbf{C} = \mathrm{H}^{s,2-s}(X^\iota_{\mathrm{can}})$. □

3.2. **Line bundles and ample cone.** Let X be an ordinary K3 surface over $\mathbf{F}_q$. Recall from § 1.3 that line bundles on X extend uniquely to X_{can}. We obtain isomorphisms

$$\operatorname{Pic} X \xrightarrow{\sim} \operatorname{Pic} X_{\mathrm{can}} \xrightarrow{\sim} \operatorname{Pic} X_{\mathrm{can},K}.$$

Let K^{nr} be the maximal unramified extension of K.

Proposition 3.5. *We have natural isomorphisms*

$$\operatorname{Pic} X_{\bar{\mathbf{F}}_q} \xrightarrow{\sim} \operatorname{Pic} X_{\mathrm{can},K^{\mathrm{nr}}} \xrightarrow{\sim} \operatorname{Pic} X_{\mathrm{can},\bar{K}},$$

and a class $\lambda \in \operatorname{Pic} X_{\bar{\mathbf{F}}_q}$ is ample if and only if its image in $\operatorname{Pic} X_{\mathrm{can},\bar{K}}$ is ample.

Proof. The first isomorphism follows from the fact that canonical lifts commute with finite unramified extensions. The second isomorphism follows from the triviality of the action of $\mathrm{Gal}(K/K^{\mathrm{nr}})$ on $\operatorname{Pic} X_{\mathrm{can},\bar{K}} \subset \mathrm{H}^2_{\mathrm{\acute{e}t}}(X_{\mathrm{can},\bar{K}}, \mathbf{Q}_\ell(1))$ and the vanishing of $\operatorname{Br} K^{\mathrm{nr}}$.

It remains to show that the isomorphism $\operatorname{Pic} X_{\bar{\mathbf{F}}_q} \xrightarrow{\sim} \operatorname{Pic} X_{\mathrm{can},\bar{K}}$ restricts to a bijection between the subsets of ample classes. Fix an ample line bundle H on X. Then by the structure theorem on the ample cone of a K3 surface over an algebraically closed field ([9, § 8.1]) we have that a line bundle L on $X_{\bar{\mathbf{F}}_q}$ is ample if and only if

(i) $L^2 > 0$
(ii) for every $D \in \operatorname{Pic} X_{\bar{\mathbf{F}}_q}$ with $D^2 = -2$ we have $L \cdot D \neq 0$ and $L \cdot D$ has the same sign as $H \cdot D$

and similarly for line bundles on $X_{\mathrm{can},\bar{K}}$. But the bijection $\operatorname{Pic} X_{\bar{\mathbf{F}}_q} \xrightarrow{\sim} \operatorname{Pic} X_{\mathrm{can},\bar{K}}$ is an isometry, and the canonical lift H_{can} of the ample line bundle H is itself ample, so we conclude that the bijection preserves ample classes. □

Proposition 3.6. *For every $d \geq 1$ the map*

$$\operatorname{Pic} X_{\mathbf{F}_{q^d}} \to \{\lambda \in \mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{Z}) \mid F^d\lambda = q^d\lambda\}$$

is an isomorphism.

Proof. Injectivity is clear, it suffices to show that the map is surjective. Without loss of generality we may assume that $d = 1$.

By Proposition 3.4 any $\lambda \in \mathrm{H}^2(X^\iota_{\mathrm{can}}, \mathbf{Z})$ satisfying $F\lambda = q\lambda$ is a Hodge class and by Theorem 3.1 we see that λ defines a Gal_K-invariant element of $\operatorname{Pic} X_{\mathrm{can},\bar{K}}$.

By the previous proposition, λ corresponds to a $\mathrm{Gal}_{\mathbf{F}_q}$-invariant class in $\operatorname{Pic} X_{\bar{\mathbf{F}}_q}$, which defines a line bundle $\mathcal{L}$ on X since the Brauer group of $\mathbf{F}_q$ vanishes. We conclude that the map is surjective as claimed. □

Proposition 3.7. *The real cone $\mathcal{K} \subset M \otimes_{\mathbf{Z}} R$ spanned by the classes of ample line bundles on* $\operatorname{Pic} X_{\bar{\mathbf{F}}_q}$ *satisfies (M5).*

Proof. This follows immediately from the Propositions 3.5 and 3.6, and the structure of the ample cone of a complex K3 surface. □

3.3. **Fully faithfulness.** In § 3.1 and § 3.2 we have constructed a functor $X \mapsto (M, F, \mathcal{K})$ from ordinary K3 surfaces over $\mathbf{F}_q$ to triples satisfying (M1)–(M5). We end this section by showing that this functor is fully faithful.

Proof of fully faithfulness in Theorem B. This is shown in [15] and [21, Theorem 3.3] for K3 surfaces equipped with an ample line bundle. The same argument works here, we repeat it for the convenience of the reader.

Faithfulness. Assume that $f, g\colon X_1 \to X_2$ are morphisms between ordinary K3 surfaces inducing the same maps $\mathrm{H}^2(X^\iota_{2,\mathrm{can}}, \mathbf{Z}) \to \mathrm{H}^2(X^\iota_{1,\mathrm{can}}, \mathbf{Z})$. Then $f^\iota_{\mathrm{can}} = g^\iota_{\mathrm{can}}$ as maps from $X^\iota_{1,\mathrm{can}}$ to $X^\iota_{2,\mathrm{can}}$ and therefore $f_{\mathrm{can}} = g_{\mathrm{can}}$ and $f = g$.

Fullness. Let X_1 and X_2 be ordinary K3 surfaces over $\mathbf{F}_q$. Let

$$\varphi\colon \mathrm{H}^2(X^\iota_{2,\mathrm{can}}, \mathbf{Z}) \to \mathrm{H}^2(X^\iota_{1,\mathrm{can}}, \mathbf{Z})$$

be an isometry commuting with F and respecting ample cones. By the description of the ample cones of X_1 and X_2, we may choose ample line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $X/\mathbf{F}_q$ such that φ maps $c_1(\mathcal{L}^\iota_{2,\mathrm{can}})$ to $c_1(\mathcal{L}^\iota_{1,\mathrm{can}})$.

By Proposition 3.4 the map φ respects the Hodge structures, and therefore the Torelli theorem shows that there is a unique isomorphism $f\colon X^\iota_{1,\mathrm{can}} \xrightarrow{\sim} X^\iota_{2,\mathrm{can}}$ with $f^* = \varphi$. Since $f^*F_2 = F_1 f^*$, and since the étale cohomology of the $X_{i,\mathrm{can},\bar{K}}$ is unramified, we have that

$$f^*\colon \mathrm{H}^2_{\text{ét}}(X_{2,\mathrm{can},\bar{K}}, \mathbf{Q}_\ell) \to \mathrm{H}^2_{\text{ét}}(X_{1,\mathrm{can},\bar{K}}, \mathbf{Q}_\ell)$$

is Gal_K-equivariant, and hence f descends to a morphism of polarized K3 surfaces over K. By Matsusaka–Mumford [12, Thm. 2] this extends to an isomorphism $f\colon X_{1,\mathrm{can}} \xrightarrow{\sim} X_{2,\mathrm{can}}$ and we conclude that φ comes from an isomorphism $f\colon X_1 \xrightarrow{\sim} X_2$ over $\mathbf{F}_q$. □

4. Essential surjectivity

4.1. **Models of K3 surfaces with complex multiplication.** We briefly recall a few facts about complex K3 surfaces with complex multiplication. We refer to [22] for proofs. Let $X/\mathbf{C}$ be a K3 surface. Its ($\mathbf{Q}$-)transcendental lattice V_X is defined as the orthogonal complement of $\mathrm{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ in $\mathrm{H}^2(X, \mathbf{Q}(1))$. The endomorphism algebra E of the $\mathbf{Q}$-Hodge structure V_X is a field, and we say that X has *complex multiplication* (by E) if V_X is one-dimensional as an E-vector space. In this case, E is necessarily a CM-field. Denote its complex conjugation by $\sigma\colon E \to E$. The Mumford-Tate group of the Hodge structure V_X is the algebraic torus $T/\mathbf{Q}$ defined by

$$T(A) = \{x \in (A \otimes E)^\times \mid x\sigma(x) = 1\}$$

for all $\mathbf{Q}$-algebras A. Note that T is an algebraic subgroup of $\mathrm{SO}(V_X)$.

If $X/\mathbf{C}$ is a K3 surface with CM by E, then it can be defined over a number field. In [19, Theorem 2] we classified the models $\mathcal{X}$ of X over finite extensions F of E in terms of their Galois representations on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{F}}, \hat{\mathbf{Z}})$. We will deduce from that result a version for models over local fields. In the statement we will need the composition

$$\mathrm{rec}\colon\ \mathrm{Gal}_E^{\mathrm{ab}} \cong \mathbf{A}_{E,f}^\times/E^\times \longrightarrow T(\mathbf{A}_f)/T(\mathbf{Q}),$$

where the isomorphism is given by global class field theory (note that E has no real places), and the second map is given by $z \mapsto \frac{z}{\sigma(z)}$.

Theorem 4.1. *Let $X/\mathbf{C}$ be a K3 surface with CM by E. Let K be a p-adic field containing E, and fix an embedding $\iota\colon K \to \mathbf{C}$ extending the embedding $E \to \mathbf{C}$ given by the action of E on $\mathrm{H}^{2,0}(X)$. Let*

$$\rho\colon\ \mathrm{Gal}_K \to \mathrm{O}(\mathrm{H}^2(X, \hat{\mathbf{Z}}(1)))$$

be a continuous homomorphism. Assume that for every $\sigma \in \mathrm{Gal}_K$ we have

- (i) *$\rho(\sigma)$ stabilizes $\mathrm{NS}(X) \subset \mathrm{H}^2(X, \hat{\mathbf{Z}}(1))$ and $\mathcal{K}_X \subset \mathrm{NS}(X) \otimes \mathbf{R}$,*
- (ii) *the restriction of $\rho(\sigma)$ to the transcendental lattice lands in the subgroup $T(\mathbf{A}_f) \subset \mathrm{O}(V_X \otimes \mathbf{A}_f)$ and its image in $T(\mathbf{A}_f)/T(\mathbf{Q})$ is $\mathrm{rec}(\sigma)$.*

Then there exists a model $\mathcal{X}/K$ of X so that the resulting action of Gal_K on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{K}}, \hat{\mathbf{Z}}(1)) = \mathrm{H}^2(X, \hat{\mathbf{Z}}(1))$ coincides with ρ.

We can reformulate the conditions on ρ as follows. Denote by $\Gamma \subset \mathrm{O}(\mathrm{H}^2(X, \hat{\mathbf{Z}}(1)))$ the subgroup consisting of those g satisfying

- (i) g stabilizes $\mathrm{NS}(X) \subset \mathrm{H}^2(X, \hat{\mathbf{Z}}(1))$ and $\mathcal{K}_X \subset \mathrm{NS}(X) \otimes \mathbf{R}$,
- (ii) the induced action on $V_X \otimes \mathbf{A}_f$ factors over $T(\mathbf{A}_f)$,

Then $\rho\colon\ \mathrm{Gal}_K \to \mathrm{O}(\mathrm{H}^2(X, \hat{\mathbf{Z}}(1)))$ satisfies the conditions in the theorem if and only if it factors over Γ, and makes the square

$$\begin{array}{ccc}
\mathrm{Gal}_K & \longrightarrow & \mathrm{Gal}_E \\
\Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\mathrm{rec}} \\
\Gamma & \xrightarrow{\ \delta\ } & T(\mathbf{A}_f)/T(\mathbf{Q})
\end{array}$$

commute. In particular, we can consider ρ as a lift of the map rec. This point of view will be useful in the proof of Theorem 4.1.

Lemma 4.2. *The map $\delta\colon \Gamma \to T(\mathbf{A}_f)/T(\mathbf{Q})$ is a continuous open homomorphism of profinite groups with finite kernel.*

Proof. Let $\Gamma_0 \subset \Gamma$ be the open subgroup of finite index consisting of those elements that act trivially on $\mathrm{NS}(X)$. The map $\Gamma_0 \to T(\mathbf{A}_f)$ is injective, and identifies Γ_0 with a compact open subgroup. Let $\mathcal{K}$ be the maximal compact open subgroup. Then it suffices to show that $\mathcal{K} \to T(\mathbf{A}_f)/T(\mathbf{Q})$ has finite kernel and cokernel. The kernel is $\{x \in \mathcal{O}_E^\times \mid x\sigma(x) = 1\}$, which is finite because E is a CM-field with complex conjugation σ. The finiteness of the cokernel $T(\mathbf{Q})\backslash T(\mathbf{A}_f)/\mathcal{K}$ is a property of arbitrary tori over $\mathbf{Q}$, see [16, Prop. 9 & Thm. 2]. $\square$

Lemma 4.3. *Let $\delta\colon G_0 \to G_1$ be an open continuous homomorphism of profinite groups with finite kernel. Let H be a closed subgroup of G_1, and $\rho\colon H \to G_0$ a* *continuous momorphism making the triangle*

$$\begin{tikzcd} H \arrow[d, "\rho"'] \arrow[dr, hook] & \\ G_0 \arrow[r, "\delta"] & G_1 \end{tikzcd}$$

commute. Then there exists an open subgroup $U \subset G_1$ containing H, and a continuous homomorphism $\rho'\colon U \to G_0$ making the square

$$\begin{tikzcd} H \arrow[r, hook] \arrow[d, "\rho"'] & U \arrow[d, hook] \arrow[dl, dashed, "\rho'"'] \\ G_0 \arrow[r, "\delta"] & G_1 \end{tikzcd}$$

commute.

Proof. Since the kernel of δ is finite, there exists an open subgroup $H_0 \subset G_0$ with $\ker\delta \cap H_0 = \{1\}$. The normalizer of H_0 has finite index in G_0, so the intersection $N_0 := \bigcap_{g\in G_0} gH_0g^{-1}$ is a normal open subgroup on which δ is injective. Denote by $N_1 \subset G_1$ its image, and by $s\colon N_1 \xrightarrow{\sim} N_0$ the inverse isomorphism. Note that $N_1 \subset G_1$ is open, and normalized by $H \subset G$.

Consider the continuous function

$$N_1 \cap H \to G_0,\ g \mapsto s(g)\rho(g)^{-1}.$$

It takes values in the finite subset $\ker\delta \subset G_0$, and maps 1 to 1. The collection of subgroups of the form $N_1 \cap H$ (for varying normal open $N_0 \subset G_0$) is a basis for the topology on H, so shrinking N_0 if necessary, we may without loss of generality assume that the above map is constant. We then have $s(g) = \rho(g)$ for all $g \in N_1 \cap H$.

The product $U := N_1 \cdot H$ is an open subgroup of G containing H, and by the above the map

$$\rho'\colon U \to G_0,\ gh \mapsto s(g)\rho(h) \qquad (g \in N_1,\ h \in H)$$

is a well-defined homomorphism satisfying the required properties. $\square$

Proof of Theorem 4.1. Let G_0 be the topological group defined by the cartesian square

$$\begin{tikzcd} G_0 \arrow[r, "\delta'"] \arrow[d] & \mathrm{Gal}_E \arrow[d, "\mathrm{rec}"] \\ \Gamma \arrow[r, "\delta"] & T(\mathbf{A}_f)/T(\mathbf{Q}) \end{tikzcd}$$

Note that δ' is open with finite kernel, and hence that G_0 is a profinite group. Now let ρ be as in the statement of the theorem. Then it induces a commutative triangle

$$\begin{tikzcd} \mathrm{Gal}_K \arrow[d, "\rho'"'] \arrow[dr, hook] & \\ G_0 \arrow[r, "\delta'"] & \mathrm{Gal}_E\,. \end{tikzcd}$$

By Lemma 4.3 there exists an intermediate field $E \subset F \subset K$ with F finite over E, and a continuous homomorphism $\rho''\colon \mathrm{Gal}_F \to G_0$ making the diagram

$$\begin{tikzcd} \mathrm{Gal}_K \arrow[r] \arrow[d, "\rho'"'] & \mathrm{Gal}_F \arrow[d, hook] \arrow[dl, "\rho''"'] \\ G_0 \arrow[r, "\delta'"] & \mathrm{Gal}_E\,. \end{tikzcd}$$

commute. Now [19, Theorem 2] guarantees the existence of a model $\mathcal{X}$ over F whose Galois action on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar F},\, \hat{\mathbf{Z}}(1)) = \mathrm{H}^2(X,\, \hat{\mathbf{Z}}(1))$ is given by the composition

$$\mathrm{Gal}_F \xrightarrow{\rho''} G_0 \longrightarrow \Gamma,$$

and hence the base change of $\mathcal{X}$ to K fulfils the requirements. □

4.2. **Criteria of good reduction.** Let $\mathcal{O}_K$ be a discrete valuation ring with fraction field K and perfect residue field k. In the introduction we defined a property $(\star)$ for K3 surfaces over K.

Theorem 4.4 (Liedtke–Matsumoto [11])**.** *Let X be a K3 surface over K satisfying $(\star)$. If for some ℓ different from the characteristic of k the action of Gal_K on $\mathrm{H}^2_{\text{ét}}(X_{\bar K},\, \mathbf{Z}_\ell)$ is unramified, then there exists a finite unramified extension $K \subset K'$ and a proper smooth algebraic space $\mathfrak{X}'$ over $\mathcal{O}_{K'}$ with $\mathfrak{X}_{K'} \cong X_{K'}$.* □

Analyzing the proof in the case where the specialization map on Picard groups is bijective, one obtains a stronger conclusion.

Proposition 4.5. *Let X be a K3 surface over K, let $K \subset K'$ be an unramified extension, and let $\mathfrak{X}'$ over $\mathcal{O}_{K'}$ be a proper smooth algebraic space with $\mathfrak{X}'_{K'} \cong X_{K'}$. If the reduction map $\mathrm{Pic}\, \mathfrak{X}'_{\bar K} \to \mathrm{Pic}\, \mathfrak{X}'_{\bar k}$ is bijective, then there exists a smooth projective $\mathfrak{X}$ over $\mathcal{O}_K$ with $\mathfrak{X}_K \cong X$.*

Proof. The map $\mathrm{Pic}\, \mathfrak{X}'_{\bar K} \xrightarrow{\sim} \mathrm{Pic}\, \mathfrak{X}'_{\bar k}$ identifies the (-2)-classes on generic and special fiber, and hence induces a bijection between the ample cones in $\mathrm{Pic}\, \mathfrak{X}'_{\bar K}$ and $\mathrm{Pic}\, \mathfrak{X}'_{\bar k}$ (see also the proof of Proposition 3.5). Now choose an ample line bundle $\mathcal{L}$ on X. It induces an ample line bundle $\mathcal{L}'$ on $X' := X_{K'}$, which extends to a relatively ample line bundle on $\mathfrak{X}'$. In particular, the canonical RDP model $P(X', \mathcal{L}')$ over $\mathcal{O}_{K'}$ of Liedtke and Matsumoto [11, Thm. 1.3] is non-singular. It follows from the construction of this model that $P(X', \mathcal{L}') = P(X, \mathcal{L}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ (see the end of section 6 in [3]), and hence $P(X, \mathcal{L})$ is a smooth projective model $\mathfrak{X}$ over $\mathcal{O}_K$. □

Alternatively, one can verify that under the hypothesis of Proposition 4.5 the group $\mathcal{W}^{\mathrm{nr}}_{X,\mathcal{L}}$ occurring in [3, Thm. 1.4] vanishes.

4.3. **Proof of Theorem B.** In § 3.3 we have established that the functor $X/\mathbf{F}_q \mapsto (M, F, \mathcal{K})$ is fully faithful. To finish the proof of Theorem B, it remains to show that the functor is essentially surjective, assuming $(\star)$ holds for K3 surfaces over p-adic fields.

Proof of essential surjectivity in Theorem B. Let $(M, F, \mathcal{K})$ be a triple satisfying (M1)–(M5). We will show that it lies in the essential image of our functor by constructing a suitable K3 surface over $\mathbf{F}_q$. We divide the construction in several steps.

Construction of a complex K3 surface. By Lemma 3.3 the decomposition

$$M_{\mathbf{C}} = M^{2,0} \oplus M^{1,1} \oplus M^{0,2}$$

with $M^{s,2-s} := M^s \otimes_{\mathbf{Z}_p,\iota} \mathbf{C}$ defines a $\mathbf{Z}$-Hodge structure on M. By the Torelli theorem for complex K3 surfaces, there is a projective K3 surface X and a Hodge isometry $f\colon \mathrm{H}^2(X, \mathbf{Z}) \xrightarrow{\sim} M$ mapping the ample cone of X to $\mathcal{K}$. The pair (X, f) is unique up to unique isomorphism.

X has complex multiplication. Let $V_X \subset \mathrm{H}^2(X, \mathbf{Q}(1))$ be the transcendental lattice. Note that F respects the decomposition $\mathrm{H}^2(X, \mathbf{Q}(1)) = \mathrm{NS}(X) \otimes \mathbf{Q} \oplus V_X$. Every $\mathbf{Q}$-linear endomorphism of V_X that commutes with F will respect the Hodge structure on V_X, and since the endomorphism algebra of the $\mathbf{Q}$-Hodge structure V_X is a field, we conclude that V_X is a cyclic $\mathbf{Q}[F]$-module, that $E := \operatorname{End} V_X$ is generated by F, and that $\dim_E V_X = 1$. In particular, X has complex multiplication by E, the field E is then a CM field, and if we denote the complex conjugation on E by σ, then the Mumford-Tate group T of V_X satisfies $T(\mathbf{Q}) = \{x \in E^\times \mid x\sigma(x) = 1\}$. Observe that $\sigma(F) = q^2/F$ on V_X, and hence that F/q defines an element of $T(\mathbf{Q})$.

The number field E has a unique place $v \mid p$ satisfying $v(F) > 0$. We have $E_v = \mathbf{Q}_p$. Let K be the fraction field of $W(\mathbf{F}_q)$, considered as a subfield of $\mathbf{C}$ via ι.

Descent to $K \subset \mathbf{C}$. For every $\ell \neq p$ consider the unramified Gal_K-representation

$$\rho_\ell\colon \mathrm{Gal}_K \to \mathrm{GL}(M \otimes \mathbf{Z}_\ell)$$

given by letting the geometric Frobenius Frob act as F. We also define a p-adic Gal_K-representation

$$\rho_p\colon \mathrm{Gal}_K \to \mathrm{GL}(M \otimes \mathbf{Z}_p) = \mathrm{GL}(\oplus_s M^s)$$

by declaring that the Tate twisted $\mathbf{Z}_p[\mathrm{Gal}_K]$-modules $M^s(s)$ are unramified with geometric Frobenius Frob acting as F/q^s. The ρ_ℓ and ρ_p assemble into an action of Gal_K on $M \otimes \hat{\mathbf{Z}}$. Denote by $M \otimes \hat{\mathbf{Z}}(1)$ its Tate twist. The resulting map

$$\rho\colon \mathrm{Gal}_K \to \mathrm{GL}(M \otimes \hat{\mathbf{Z}}(1)) = \mathrm{GL}(\mathrm{H}^2(X, \hat{\mathbf{Z}}(1)))$$

satisfies

(i) the image of ρ is contained in $O(\mathrm{H}^2(X, \hat{\mathbf{Z}}(1)))$,
(ii) the image of ρ preserves $\operatorname{Pic} X$ and the ample cone $\mathcal{K} \subset (\operatorname{Pic} X) \otimes \mathbf{R}$.

We claim that ρ also satisfies the reciprocity condition in Theorem 4.1.

Indeed, observe that the action of Gal_K on $M \otimes \hat{\mathbf{Z}}(1)$ is abelian. Let $x \in K^\times$. Using Lemma 3.2 we see that the action of the corresponding $\tau = \tau(x) \in \mathrm{Gal}_K^{\mathrm{ab}}$ on $M \otimes \hat{\mathbf{Z}}(1)$ satisfies

(i) τ acts on $M \otimes \mathbf{Z}_\ell(1)$ by $(F/q)^{v(x)}$ (for $\ell \neq p$),
(ii) τ acts on $M^0(1) \subset M \otimes \mathbf{Z}_p(1)$ by $(\mathrm{Nm}_{K/\mathbf{Q}_p} x)(F/q)^{v(x)}$,
(iii) τ acts on $M^1(1) \subset M \otimes \mathbf{Z}_p(1)$ by $(F/q)^{v(x)}$,
(iv) τ acts on $M^2(1) \subset M \otimes \mathbf{Z}_p(1)$ by $(\mathrm{Nm}_{K/\mathbf{Q}_p} x)^{-1}(F/q)^{v(x)}$.

(Note that by property (M4) these actions indeed preserve the $\mathbf{Z}_p$-lattices $M^s(1)$).

On the other hand, the decomposition of $M \otimes \mathbf{Z}_p$ induces a decomposition

$$V_X \otimes \mathbf{Q}_p = V_{-1} \oplus V_0 \oplus V_1$$

with $\dim V_{-1} = \dim V_1 = 1$. The group $E \otimes \mathbf{Q}_p$ acts on $V_X \otimes \mathbf{Q}_p$ on V_{-1} through the factor $E_v^\times \cong \mathbf{Q}_p^\times$, and on V_1 trough $E_{\sigma v}^\times \cong \mathbf{Q}_p^\times$. The inclusion of $E_v^\times \times E_{\sigma v}^\times \subset (E \otimes \mathbf{Q}_p)^\times$ defines a subgroup

$$T_{v,\sigma v} = \{(x_v, x_{\sigma v}) \in E_v^\times \times E_{\sigma v}^\times \mid x_v x_{\sigma v} = 1\} \subset T(\mathbf{Q}_p).$$

The compatibility between local and global class field theory and the definition of rec (see § 4.1) implies that the diagram

$$\begin{tikzcd} \mathrm{Gal}_K^{\mathrm{ab}} \arrow[rr] & & \mathrm{Gal}_E^{\mathrm{ab}} \arrow[d, "\mathrm{rec}"] \\ K^\times \arrow[u] \arrow[r] & T_{v,\sigma v} \arrow[r] & T(\mathbf{A}_f)/T(\mathbf{Q}) \end{tikzcd}$$

in which the map $K^\times \to T_{v,\sigma v}$ maps x to $(\mathrm{Nm}_{K/\mathbf{Q}_p}(x), \mathrm{Nm}_{K/\mathbf{Q}_p}(x)^{-1})$ commutes. We conclude that

$$K^\times \longrightarrow \mathrm{Gal}_K^{\mathrm{ab}} \longrightarrow \mathrm{Gal}_E^{\mathrm{ab}} \xrightarrow{\mathrm{rec}} T(\mathbf{A}_f)/T(\mathbf{Q})$$

maps an $x \in K^\times$ to the class of the element $\alpha = \alpha(x) \in T(\mathbf{A}_f)$ satisfying

(i) $\alpha_\ell = 1$ for all $\ell \neq p$,
(ii) α_p acts on $V_X \otimes \mathbf{Q}_p = V_{-1} \oplus V_0 \oplus V_1$ by $(\mathrm{Nm}_{K/\mathbf{Q}_p} x, 1, (\mathrm{Nm}_{K/\mathbf{Q}_p} x)^{-1})$.

Since F/q lies in $T(\mathbf{Q})$, we see that $\alpha(x)$ and $\tau(x)$ define the same element in $T(\mathbf{A}_f)/T(\mathbf{Q})$. This shows that ρ satisfies the requirements of Theorem 4.1, and we conclude that there is a model $\mathcal{X}/K$ of X whose Gal_K-action on $\mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{K}}, \hat{\mathbf{Z}}) = M \otimes \hat{\mathbf{Z}}$ is the prescribed one.

Extension to $\mathcal{O}_K$ and reduction to k. By construction, the action of Gal_K on $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}_\ell)$ is unramified. By Theorem 4.4, and since we are assuming $\mathcal{X}$ satisfies $(\star)$ there exists a finite unramified extension $K \subset K'$ so that $\mathcal{X}' := \mathcal{X}_{K'}$ has good reduction, and hence extends to a proper smooth $\mathfrak{X}'$ over $\mathcal{O}_{K'}$.

By Theorem C, this model $\mathfrak{X}'$ is the canonical lift of its reduction, and hence by Proposition 1.8 the map $\mathrm{Pic}\,\mathfrak{X}'_{\bar{K}} \to \mathrm{Pic}\,\mathfrak{X}_{\bar{\mathbf{F}}_q}$ is surjective. We conclude with Proposition 4.5 that X/K has a canonical smooth projective model $\mathfrak{X}/\mathcal{O}_K$. Again Theorem C guarantees that $\mathfrak{X}$ is the canonical lift of its reduction $\mathfrak{X}_k$, and we see that the functor of Theorem B maps $\mathfrak{X}_k$ to the given triple $(M, F, \mathcal{K})$. □

4.4. **Unconditional results.** As above, we fix an embedding $\iota\colon W(\mathbf{F}_q) \to \mathbf{C}$.

Theorem 4.6. *The functor $X \mapsto (M, F, \mathcal{K})$ restricts to an equivalence between the sub-groupoids consisting of:*

(i) *$X/\mathbf{F}_q$ for which there is an ample $\mathcal{L} \in \mathrm{Pic}\,X_{\bar{\mathbf{F}}_q}$ with $\mathcal{L}^2 < p - 4$,*
(ii) *$(M, F, \mathcal{K})$ for which there exists a $\lambda \in M \cap \mathcal{K}$ satisfying $\lambda^2 < p - 4$.*

Assuming $p \geq 5$ it also restricts to an equivalence between

(i) *$X/\mathbf{F}_q$ for which $\mathrm{Pic}\,X_{\bar{\mathbf{F}}_q}$ contains a hyperbolic lattice,*
(ii) *$(M, F, \mathcal{K})$ for which $\mathrm{NS}(M, F)$ contains a hyperbolic lattice,*

and between

(i) *$X/\mathbf{F}_q$ with $\mathrm{rk}\,\mathrm{Pic}\,X_{\bar{\mathbf{F}}_q} \geq 12$,*
(ii) *$(M, F, \mathcal{K})$ with $\mathrm{rk}\,\mathrm{NS}(M, F) \geq 12$.*

Proof. In view of Theorem B and Proposition 3.5, we only need to verify that any triple $(M, F, \mathcal{K})$ as in (ii) lies in the essential image of the functor $X \mapsto (M, F, \mathcal{K})$ on ordinary K3 surfaces. It suffices to show that the relevant X over $K = \operatorname{Frac} W(\mathbf{F}_q)$ occurring in the proof in § 4.3 satisfy $(\star)$.

By [13, Thm. 1.1] and [10, § 2] we know that any K3 surface over X with unramified $\mathrm{H}^2_{\text{ét}}(X_{\bar{K}}, \mathbf{Q}_\ell)$, and satisfying one of

(i) there is an ample $\mathcal{L} \in \operatorname{Pic} X_{\bar{K}}$ with $\mathcal{L}^2 < p - 4$,
(ii) $\operatorname{Pic} X_{\bar{K}}$ contains a hyperbolic plane and $p \geq 5$,
(iii) $\operatorname{Pic} X_{\bar{K}}$ has rank ≥ 12 and $p \geq 5$,

has potentially good reduction. In particular, any such K3 surface satisfies hypothesis $(\star)$. In all three cases the argument of § 4.3 goes through unconditionally. $\square$

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