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Taelman, Lenny

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ORDINARY K3 SURFACES OVER A FINITE FIELD

LENNY TAE LMAN

ABSTRACT. We give a description of the category of ordinary K3 surfaces over a finite field in terms of linear algebra data over \mathbf{Z} . This gives an analogue for K3 surfaces of Deligne's description of the category of ordinary abelian varieties over a finite field, and refines earlier work by N.O. Nygaard and J.-D. Yu.

Our main result is conditional on a conjecture on potential semi-stable reduction of K3 surfaces over p -adic fields. We give unconditional versions for K3 surfaces of large Picard rank and for K3 surfaces of small degree.

INTRODUCTION

Statement of the main results. A K3 surface X over a perfect field k of characteristic p is called *ordinary* if it satisfies the following equivalent conditions:

- (i) the Hodge and Newton polygons of $H_{\text{crys}}^2(X/W(k))$ coincide,
- (ii) the Frobenius endomorphism of $H^2(X, \mathcal{O}_X)$ is a bijection,
- (iii) the formal Brauer group of X (see [1]) has height 1.

If k is finite, then these are also equivalent with $|X(k)| \not\equiv 1 \pmod{p}$. Building on [1] and [7], Nygaard [14] has shown that such an ordinary X has a canonical lift X_{can} over the ring of Witt vectors $W(k)$.

Choose an embedding $\iota: W(\mathbf{F}_q) \rightarrow \mathbf{C}$. Then with every ordinary K3 surface over \mathbf{F}_q we can associate a complex K3 surface $X_{\text{can}}^\iota := X_{\text{can}} \otimes_{W(\mathbf{F}_q)} \mathbf{C}$ and an integral lattice

$$M := H^2(X_{\text{can}}^\iota, \mathbf{Z}).$$

Using the Kuga-Satake construction, Nygaard [14] and Yu [21] have shown that there exists a (necessarily unique) endomorphism F of $M \otimes_{\mathbf{Z}} \mathbf{Z}[\frac{1}{p}]$ such that for every $\ell \neq p$ the canonical isomorphism

$$M \otimes_{\mathbf{Z}} \mathbf{Z}_\ell \xrightarrow{\sim} H_{\text{ét}}^2(X_{\mathbf{F}_q}, \mathbf{Z}_\ell)$$

matches F with the geometric Frobenius Frob on étale cohomology (see also § 3.1). We have:

- (M1) the pairing $\langle -, - \rangle$ on M is unimodular, even, and of signature $(3, 19)$;
- (M2) $\langle Fx, Fy \rangle = q^2 \langle x, y \rangle$ for every $x, y \in M$.

From Deligne's proof of the Weil conjectures for K3 surfaces [6] we also know

- (M3) the endomorphism F of $M \otimes \mathbf{C}$ is semi-simple and all its eigenvalues have absolute value q .

Our first result is an integral p -adic property of the pair (M, F) .

Theorem A. *The endomorphism F preserves the \mathbf{Z} -module M and satisfies*

- (M4) *the $\mathbf{Z}_p[F]$ -module $M \otimes \mathbf{Z}_p$ decomposes as $M^0 \oplus M^1 \oplus M^2$ with*
 - (a) $FM^s = q^s M^s$, for all s ,
 - (b) M^0, M^1 and M^2 are free \mathbf{Z}_p -modules of rank 1, 20, 1, respectively.

For a \mathbf{Z} -lattice M equipped with an endomorphism F satisfying (M1)–(M4) we denote by $\mathrm{NS}(M, F)$ the group

$$\mathrm{NS}(M, F) := \{x \in M \mid F^d x = q^d x \text{ for some } d \geq 1\}.$$

Using the fact that all line bundles on an ordinary K3 surface extend uniquely to its canonical lift, one shows that there is a natural bijection $\mathrm{Pic} X_{\mathbf{F}_q} \rightarrow \mathrm{NS}(M, F)$, and that the ample line bundles on $X_{\mathbf{F}_q}$ span a real cone $\mathcal{K} \subset M \otimes \mathbf{R}$ satisfying

(M5) \mathcal{K} is a connected component of

$$\left\{ x \in \mathrm{NS}(M, F) \otimes \mathbf{R} \mid \langle x, x \rangle > 0, \langle x, \delta \rangle \neq 0 \text{ for all } \delta \in \mathrm{NS}(M, F) \text{ with } \delta^2 = -2 \right\}$$

satisfying $F\mathcal{K} = \mathcal{K}$.

See § 3.2 for more details.

Definition (see [11]). Let \mathcal{O}_K be a complete discrete valuation ring with fraction field K . We say that a K3 surface X over K satisfies (\star) if there exists a finite extension $L \subset K$ and algebraic space $\mathfrak{X}/\mathcal{O}_L$ satisfying

- (i) $\mathfrak{X}_L \cong X_L$,
- (ii) \mathfrak{X} is regular,
- (iii) the special fiber is a strict normal crossing divisor in \mathfrak{X} ,
- (iv) the relative dualizing sheaf of $\mathfrak{X}/\mathcal{O}_L$ is trivial.

This is a strong form of ‘potential semi-stable reduction’. Over a complete dvr \mathcal{O}_K of residue characteristic 0, it is known that all K3 surfaces satisfy (\star) . It is expected to hold in general, but currently only known under extra assumptions on X .

Our main result is the following description of the category of ordinary K3 surfaces over \mathbf{F}_q . It is an analogue for K3 surfaces of Deligne’s theorem [5] on ordinary abelian varieties over a finite field.

Theorem B. *Fix an embedding $\iota: W(\mathbf{F}_q) \rightarrow \mathbf{C}$. Then the resulting functor $X \mapsto (M, F, \mathcal{K})$ is a fully faithful functor between the groupoids of*

- (i) ordinary K3 surfaces X over \mathbf{F}_q , and
- (ii) triples (M, F, \mathcal{K}) consisting of
 - (a) an integral lattice M ,
 - (b) an endomorphism F of the \mathbf{Z} -module M , and
 - (c) a convex subset $\mathcal{K} \subset M \otimes \mathbf{R}$,
satisfying (M1)–(M5).

If every K3 surface over $\mathrm{Frac} W(\mathbf{F}_q)$ satisfies (\star) then the functor is essentially surjective.

Fully faithfulness is essentially due to Nygaard [15] and Yu [21]. Our contribution is a description of the image of this functor.

Restricting to families for which (\star) is known to hold, we also obtain unconditional equivalences of categories between ordinary K3 surfaces X/\mathbf{F}_q satisfying one of the following additional conditions

- (i) there is an ample $\mathcal{L} \in \mathrm{Pic} X_{\mathbf{F}_q}$ with $\mathcal{L}^2 < p - 4$,
- (ii) $\mathrm{Pic} X_{\mathbf{F}_q}$ contains a hyperbolic plane and $p \geq 5$,
- (iii) X has geometric Picard rank ≥ 12 and $p \geq 5$,

and triples (M, F, \mathcal{K}) satisfying the analogous constraints. See Theorem 4.6 for the precise statement.

About the proofs. A crucial ingredient in the proof of Theorems A and B is the following criterion which distinguishes the canonical lift amongst all lifts using p -adic étale cohomology.

Theorem C. *Let \mathcal{O}_K be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic 0. Let \mathfrak{X} be a projective K3 surface over \mathcal{O}_K and assume that \mathfrak{X}_k is ordinary. Then the following are equivalent:*

- (i) \mathfrak{X} is the base change from $W(k)$ to \mathcal{O}_K of the canonical lift of \mathfrak{X}_k
- (ii) $H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p) \cong H^0 \oplus H^1(-1) \oplus H^2(-2)$ with H^i unramified $\mathbf{Z}_p[\text{Gal}_K]$ -modules, free of rank 1, 20, 1 over \mathbf{Z}_p , respectively.

Here the (-1) and (-2) in (ii) denote Tate twists. This theorem is an integral refinement of a theorem of Yu [21] which characterizes *quasi-canonical* lifts by the splitting of étale cohomology with \mathbf{Q}_p -coefficients.

The canonical lift of X is defined in terms of its enlarged formal Brauer group Ψ (a p -divisible group), and to prove Theorem C we need to compare the p -adic étale cohomology of the generic fiber of a K3 surface over \mathcal{O}_K to the Tate module of its enlarged formal Brauer group. With \mathbf{Q}_p -coefficients, such comparison has been shown by Artin and Mazur [1]. We give a different argument leading to an integral version, see Theorem 2.1. Once Theorem C is established, Theorem A is an almost formal consequence.

Finally, we briefly sketch the argument for the proof of Theorem B. Fully faithfulness was shown by Nygaard [15] and Yu [21] (see § 3.3 for more details). The proof of essential surjectivity starts with the observation that the decomposition in (M4) induces (via the embedding $\iota: \mathbf{Z}_p \rightarrow \mathbf{C}$) a Hodge structure on M , for which $\text{NS}(M, F)$ consists precisely of the Hodge classes. The Torelli theorem for K3 surfaces then shows that there is a canonical K3 surface X/\mathbf{C} with $H^2(X, \mathbf{Z}) = M$ and whose ample cone $\mathcal{K} \subset H^2(X, \mathbf{R})$ coincides with $\mathcal{K} \subset M \otimes \mathbf{R}$. This K3 surface has complex multiplication, and hence can be defined over a number field. Using the strong version of the main theorem of CM for K3 surfaces of [19] we show that we can find a model of X over $K := \text{Frac } W(\mathbf{F}_q) \subset \mathbf{C}$ such that

- (i) the Gal_K -module $H_{\text{ét}}^2(X_{\bar{K}}, \mathbf{Z}_p) = M \otimes \mathbf{Z}_p$ decomposes as in Theorem C,
- (ii) for $\ell \neq p$, the Gal_K -module $H_{\text{ét}}^2(X_{\bar{K}}, \mathbf{Z}_\ell) = M \otimes \mathbf{Z}_\ell$ is unramified, and Frobenius acts as F .

Assuming (\star) , it follows from Néron–Ogg–Shafarevich criterion of Liedtke and Matsumoto [11] that X has good reduction over an unramified extension L of K . Using Theorem C we show that X_L is the canonical lift of its reduction, and deduce from this that X has already a smooth projective model \mathfrak{X} over \mathcal{O}_K . By construction, its reduction \mathfrak{X}_k maps under our functor to the given triple (M, F, \mathcal{K}) .

A question. We end this introduction with an essentially lattice-theoretical question to which we do not know the answer:

Question. Does there exist a triple (M, F, \mathcal{K}) satisfying (M1)–(M5) and the inequality $1 + \text{tr } F + q^2 < 0$?

By (M3) such triple can only exist for small q . A positive answer to this question would imply that there exist K3 surfaces over p -adic fields that do not satisfy (\star) . Indeed, if (M, F, \mathcal{K}) came from a K3 surface X/\mathbf{F}_q as in Theorem B we would have

$$|X(\mathbf{F}_q)| = \text{tr}(\text{Frob}, H^\bullet(X_{\bar{\mathbf{F}}_q}, \mathbf{Q}_\ell)) = 1 + \text{tr } F + q^2 < 0,$$

which is absurd.

1. p -DIVISIBLE GROUPS ASSOCIATED TO K3 SURFACES

Let Λ be a complete noetherian local ring with perfect residue field k of characteristic $p > 0$ and let \mathfrak{X} be a K3 surface over $\text{Spec } \Lambda$. We recall (and complement) some of the main results of Artin and Mazur [1] on the formal Brauer group and enlarged formal Brauer group of \mathfrak{X} .

1.1. The formal Brauer group. Let Art_Λ be the category of Artinian local Λ -algebras (A, \mathfrak{m}) with perfect residue field A/\mathfrak{m} . For an $(A, \mathfrak{m}) \in \text{Art}_\Lambda$ we denote by \mathcal{U}_A the sheaf on $\mathfrak{X}_{\text{ét}}$ defined by the exact sequence

$$1 \rightarrow \mathcal{U}_A \rightarrow \mathcal{O}_{\mathfrak{X}_A}^\times \rightarrow \mathcal{O}_{\mathfrak{X}_{A/\mathfrak{m}}}^\times \rightarrow 1.$$

The *formal Brauer group* of \mathfrak{X} is the functor

$$\hat{\text{Br}}(\mathfrak{X}): \text{Art}_\Lambda \rightarrow \text{Ab}, A \mapsto \text{H}^2(\mathfrak{X}_{\text{ét}}, \mathcal{U}_A).$$

By [1] it is representable by a one-dimensional formal group, and if \mathfrak{X} is not supersingular then $\hat{\text{Br}}(\mathfrak{X})$ is a p -divisible group.

Lemma 1.1. $\text{H}^\bullet(\mathfrak{X}_{\text{ét}}, \mathcal{U}_A) = \text{H}^\bullet(\mathfrak{X}, \mathcal{U}_A)$ and $\text{H}^1(\mathfrak{X}_{\text{ét}}, \mathcal{U}_A) = 0$.

Here $\text{H}^\bullet(\mathfrak{X}, -)$ denotes Zariski cohomology.

Proof of Lemma 1.1. The sheaf \mathcal{U}_A has a filtration whose graded pieces are

$$\mathfrak{m}^n / \mathfrak{m}^{n+1} \otimes_{A/\mathfrak{m}} \mathcal{O}_{\mathfrak{X}_{A/\mathfrak{m}}}$$

Since these are coherent, we have $\text{H}^\bullet(\mathfrak{X}_{\text{ét}}, \mathcal{U}_A) = \text{H}^\bullet(\mathfrak{X}, \mathcal{U}_A)$. Moreover, since $\text{H}^1(\mathfrak{X}_{A/\mathfrak{m}}, \mathcal{O}_{\mathfrak{X}_{A/\mathfrak{m}}})$ vanishes, we conclude that $\text{H}^1(\mathfrak{X}_{\text{ét}}, \mathcal{U}_A) = 0$. \square

Lemma 1.2. For every $(A, \mathfrak{m}) \in \text{Art}_\Lambda$ there is a natural exact sequence

$$0 \longrightarrow \hat{\text{Br}}(\mathfrak{X})[p^r](A) \longrightarrow \text{H}_{\mathbb{H}}^2(\mathfrak{X}_A, \mu_{p^r}) \longrightarrow \text{H}_{\mathbb{H}}^2(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r})$$

of abelian groups.

Here $\text{H}_{\mathbb{H}}^\bullet$ denotes cohomology in the fppf topology.

Proof of Lemma 1.2. Consider the complex $\mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A$ on $\mathfrak{X}_{\text{ét}}$ in degrees 0 and 1. We have a short exact sequence

$$1 \longrightarrow \text{H}^1(\mathfrak{X}, \mathcal{U}_A) \otimes \mathbf{Z}/p^r\mathbf{Z} \longrightarrow \text{H}^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A) \longrightarrow \text{H}^2(\mathfrak{X}, \mathcal{U}_A)[p^r] \longrightarrow 1$$

and thanks to Lemma 1.1 we obtain canonical isomorphisms

$$\hat{\text{Br}}(\mathfrak{X})[p^r](A) = \text{H}^2(\mathfrak{X}, \mathcal{U}_A)[p^r] = \text{H}^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A).$$

Now consider the short exact sequence

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{U}_A & \longrightarrow & \mathcal{O}_{\mathfrak{X}_A}^\times & \longrightarrow & \mathcal{O}_{\mathfrak{X}_{A/\mathfrak{m}}}^\times \longrightarrow 1 \\ & & \downarrow p^r & & \downarrow p^r & & \downarrow p^r \\ 1 & \longrightarrow & \mathcal{U}_A & \longrightarrow & \mathcal{O}_{\mathfrak{X}_A}^\times & \longrightarrow & \mathcal{O}_{\mathfrak{X}_{A/\mathfrak{m}}}^\times \longrightarrow 1 \end{array}$$

of length-two complexes concentrated in degrees 0 and 1. Using the Kummer sequence and the fact that \mathbf{G}_m is smooth, we see that

$$H_{\acute{e}t}^n(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_A}^\times \xrightarrow{p^r} \mathcal{O}_{\mathfrak{X}_A}^\times) = H_{\acute{e}t}^n(\mathfrak{X}_A, \mathbf{G}_m \xrightarrow{p^r} \mathbf{G}_m) = H_{\acute{e}t}^n(\mathfrak{X}_A, \mu_{p^r})$$

and similarly for $\mathfrak{X}_{A/\mathfrak{m}}$. It follows that the above short exact sequence of complexes induces a long exact sequence of (hyper-)cohomology groups

$$\longrightarrow H_{\acute{e}t}^1(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r}) \longrightarrow H^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A) \longrightarrow H_{\acute{e}t}^2(\mathfrak{X}_A, \mu_{p^r}) \longrightarrow H_{\acute{e}t}^2(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r}) \longrightarrow$$

Since A/\mathfrak{m} is perfect and $\text{Pic } \mathfrak{X}_{A/\mathfrak{m}}$ is torsion-free, we have $H_{\acute{e}t}^1(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r}) = 0$ and we conclude

$$\hat{\text{Br}}(\mathfrak{X})[p^r](A) = H^2(\mathfrak{X}, \mathcal{U}_A \xrightarrow{p^r} \mathcal{U}_A) = \ker [H_{\acute{e}t}^2(\mathfrak{X}_A, \mu_{p^r}) \rightarrow H_{\acute{e}t}^2(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r})],$$

which is what we had to show. \square

1.2. The enlarged formal Brauer group. We now assume that \mathfrak{X}_k is ordinary. Denote by μ_{p^∞} the sheaf $\text{colim}_r \mu_{p^r}$ on the fppf site. The *enlarged formal Brauer group* of \mathfrak{X} is the functor

$$\Psi(\mathfrak{X}): \text{Art}_\Lambda \rightarrow \text{Ab}, A \mapsto H_{\acute{e}t}^2(\mathfrak{X}_A, \mu_{p^\infty}).$$

A priori this differs from the definition of Artin–Mazur [1, § IV.1] in two ways. First, Artin and Mazur restrict to A with algebraically closed residue fields, and then use Galois descent to extend their definition to non-closed perfect residue fields, and second, they restrict to those classes in $H_{\acute{e}t}^2(\mathfrak{X}_A, \mu_{p^\infty})$ that map to the p -divisible part of $H_{\acute{e}t}^2(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})$. Lemmas 1.4 and 1.5 below show that the above definition is equivalent to that of Artin–Mazur (under our standing condition that \mathfrak{X}_k is an ordinary K3 surface). See also [14, Cor. 1.5].

The following lemma is well-known, and implicitly used in [1] and [14]. We include it for the sake of completeness.

Lemma 1.3. *For any quasi-compact quasi-separated scheme X there is a natural isomorphism*

$$H_{\acute{e}t}^\bullet(X, \mu_{p^\infty}) \xrightarrow{\sim} \text{colim}_r H_{\acute{e}t}^\bullet(X, \mu_{p^r}).$$

Proof. This follows from [18, 0739], taking for \mathcal{B} the class of quasi-compact and quasi-separated schemes, and for Cov the fpqc covers consisting of finitely many affine schemes. \square

Lemma 1.4 ([14, Cor. 1.4]). *The group $H_{\acute{e}t}^2(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})$ is p -divisible.* \square

Lemma 1.5. $H_{\acute{e}t}^2(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})^{\text{Gal}(\bar{k}/k)} = H_{\acute{e}t}^2(\mathfrak{X}_k, \mu_{p^\infty})$.

Proof. Since the p -th power map $\bar{k}^\times \rightarrow \bar{k}^\times$ is a bijection, and since $\text{Pic } \mathfrak{X}_{\bar{k}}$ is torsion-free, we have

$$H_{\acute{e}t}^i(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty}) = \text{colim}_r H_{\acute{e}t}^i(\mathfrak{X}_{\bar{k}}, \mu_{p^r}) = 0$$

for $i \in \{0, 1\}$. It now follows from the Hochschild-Serre spectral sequence

$$E_2^{s,t} = H^s(\text{Gal}(\bar{k}/k), H_{\acute{e}t}^t(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})) \Rightarrow H_{\acute{e}t}^{s+t}(\mathfrak{X}_k, \mu_{p^\infty})$$

that $H_{\acute{e}t}^2(\mathfrak{X}_{\bar{k}}, \mu_{p^\infty})^{\text{Gal}(\bar{k}/k)} = H_{\acute{e}t}^2(\mathfrak{X}_k, \mu_{p^\infty})$. \square

The following theorem summarizes the properties of the enlarged formal Brauer group that we will use.

Theorem 1.6. *Let \mathfrak{X} be a formal K3 surface over Λ with \mathfrak{X}_k ordinary. Then the enlarged formal Brauer group $\Psi(\mathfrak{X})$ is representable by a p -divisible group over Λ . Its étale-local exact sequence*

$$0 \rightarrow \Psi^\circ(\mathfrak{X}) \rightarrow \Psi(\mathfrak{X}) \rightarrow \Psi^{\text{ét}}(\mathfrak{X}) \rightarrow 0$$

satisfies

- (i) $\Psi^\circ(\mathfrak{X})$ is a connected p -divisible group of height 1, with

$$\Psi^\circ(\mathfrak{X})[p^r](A) = \ker(\mathrm{H}_{\mathfrak{h}}^2(\mathfrak{X}_A, \mu_{p^r}) \rightarrow \mathrm{H}_{\mathfrak{h}}^2(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r}))$$

for all $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$. It is canonically isomorphic to $\widehat{\mathrm{Br}}(\mathfrak{X})$;

- (ii) $\Psi(\mathfrak{X})[p^r](A) = \mathrm{H}_{\mathfrak{h}}^2(\mathfrak{X}_A, \mu_{p^r})$ for all $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$;
 (iii) $\Psi^{\text{ét}}(\mathfrak{X})[p^r](A) = \mathrm{H}_{\mathfrak{h}}^2(\mathfrak{X}_{A/\mathfrak{m}}, \mu_{p^r})$ for all $(A, \mathfrak{m}) \in \mathrm{Art}_\Lambda$.

Proof. The representability and (i) are shown in [1, Prop. IV.1.8].

To prove (ii), note that since $\mathrm{H}^1(\mathfrak{X}_A, \mathbf{G}_m) = \mathrm{Pic} \mathfrak{X}_A$ is torsion-free, we have a natural isomorphism

$$\mathrm{H}_{\mathfrak{h}}^1(\mathfrak{X}_A, \mu_{p^r}) = \mathrm{H}^0(\mathfrak{X}_A, \mathbf{G}_m) \otimes \mathbf{Z}/p^r \mathbf{Z}.$$

Taking the colimit over r we obtain a natural isomorphism

$$\mathrm{H}_{\mathfrak{h}}^1(\mathfrak{X}_A, \mu_{p^\infty}) = \mathrm{colim}_r \mathrm{H}_{\mathfrak{h}}^1(\mathfrak{X}_A, \mu_{p^r}) = \mathrm{H}^0(\mathfrak{X}_A, \mathbf{G}_m) \otimes (\mathbf{Q}_p/\mathbf{Z}_p),$$

and in particular we see that $\mathrm{H}_{\mathfrak{h}}^1(\mathfrak{X}_A, \mu_{p^\infty})$ is p -divisible. Now the long exact sequence associated to

$$1 \longrightarrow \mu_{p^r} \longrightarrow \mu_{p^\infty} \xrightarrow{p^r} \mu_{p^\infty} \longrightarrow 1$$

induces an isomorphism $\mathrm{H}_{\mathfrak{h}}^2(\mathfrak{X}_A, \mu_{p^r}) \cong \mathrm{H}_{\mathfrak{h}}^2(\mathfrak{X}_A, \mu_{p^\infty})[p^r]$, which proves (ii).

A similar argument shows (iii) for A with A/\mathfrak{m} algebraically closed, after which Lemma 1.5 implies the general case. \square

1.3. Canonical lifts. Let X/k be an ordinary K3 surface. Since formal groups of height 1 are rigid, the p -divisible group $\Psi^\circ(X)$ over k extends uniquely to a p -divisible group $\Psi^\circ(X)_{\mathrm{can}}$ over Λ . Also the étale p -divisible group $\Psi^{\text{ét}}(X)$ over k extends uniquely to a p -divisible group $\Psi^{\text{ét}}(X)_{\mathrm{can}}$ over Λ .

To every lift \mathfrak{X}/Λ of X/k we then have an associated short exact sequence of p -divisible groups

$$(1) \quad 0 \longrightarrow \Psi^\circ(X)_{\mathrm{can}} \longrightarrow \Psi(\mathfrak{X}) \longrightarrow \Psi^{\text{ét}}(X)_{\mathrm{can}} \longrightarrow 0$$

over Λ . In analogy with Serre-Tate theory, we have the following theorem.

Theorem 1.7 (Nygaard [14, Thm. 1.6]). *The map*

$$\{ \text{formal lifts } \mathfrak{X}/\Lambda \text{ of } X/k \} \rightarrow \mathrm{Ext}_\Lambda^1(\Psi^{\text{ét}}(X)_{\mathrm{can}}, \Psi^\circ(X)_{\mathrm{can}}), \mathfrak{X} \mapsto \Psi(\mathfrak{X})$$

is a bijection. \square

It follows that there exists a unique lift \mathfrak{X}/Λ for which the sequence (1) splits. This \mathfrak{X} is unique up to unique isomorphism, and is called the *canonical lift* of X . We denote it by X_{can} .

Proposition 1.8 ([14, Prop. 1.8]). *$\mathrm{Pic} X_{\mathrm{can}} \rightarrow \mathrm{Pic} X$ is a bijection.* \square

Corollary 1.9 ([14, Prop. 1.8]). *X_{can} is algebraizable and projective.* \square

2. p -ADIC ÉTALE COHOMOLOGY

Let \mathcal{O}_K be a complete discrete valuation ring whose residue field k is perfect of characteristic p and whose fraction field K is of characteristic 0.

2.1. p -adic étale cohomology and the enlarged formal Brauer group.

Theorem 2.1. *Let \mathfrak{X} be a projective K3 surface over \mathcal{O}_K . Assume that \mathfrak{X}_k is ordinary. Then there is a natural injective map of Gal_K -modules*

$$T_p \Psi(\mathfrak{X})_{\bar{K}} \rightarrow H_{\text{ét}}^2(X_{\bar{K}}, \mathbf{Z}_p(1))$$

whose cokernel is a free \mathbf{Z}_p -module of rank 1.

Recall that if \mathfrak{X} is ordinary, then $T_p \Psi(\mathfrak{X})_{\bar{K}}$ has rank 21. Up to possible torsion in the cokernel, Theorem 2.1 is shown in [1, § IV.2]. The proof of Artin and Mazur is based on Lefschetz pencils, reducing the problem on H^2 to a statement about H^1 and torsors. We give a proof working directly with the H^2 and their relation to Brauer groups to obtain the finer ‘integral’ statement above. This is made possible by the theorem of Gabber and de Jong [4] asserting that the Brauer group and the cohomological Brauer group of a quasi-projective scheme coincide.

Let \mathfrak{X} be a formal K3 surface over \mathcal{O}_K . We denote by \mathfrak{X}_n the truncation $\mathfrak{X}_{\mathcal{O}_K/\mathfrak{m}^n}$.

Lemma 2.2. *For all i the natural map*

$$H_{\text{fl}}^i(\mathfrak{X}, \mu_{p^r}) \rightarrow \lim_n H_{\text{fl}}^i(\mathfrak{X}_n, \mu_{p^r})$$

is an isomorphism.

Proof. As in the proof of Lemma 1.2, we have

$$H_{\text{fl}}^i(\mathfrak{X}_n, \mu_{p^r}) = H_{\text{ét}}^i(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}_n}^{\times} \xrightarrow{p^r} \mathcal{O}_{\mathfrak{X}_n}^{\times})$$

and similarly

$$H_{\text{fl}}^i(\mathfrak{X}, \mu_{p^r}) = H_{\text{ét}}^i(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\times} \xrightarrow{p^r} \mathcal{O}_{\mathfrak{X}}^{\times}).$$

Let \mathcal{U}_n be the kernel of $\mathcal{O}_{\mathfrak{X}_n}^{\times} \rightarrow \mathcal{O}_{\mathfrak{X}_1}^{\times}$ and \mathcal{U} the kernel of $\mathcal{O}_{\mathfrak{X}}^{\times} \rightarrow \mathcal{O}_{\mathfrak{X}_1}^{\times}$. Then by the usual dévissage arguments the lemma reduces to showing that

$$H_{\text{ét}}^i(\mathfrak{X}, \mathcal{U}) \rightarrow \lim_n H_{\text{ét}}^i(\mathfrak{X}, \mathcal{U}_n)$$

is an isomorphism for all i .

Since the maps $\mathcal{U}_{n+1} \rightarrow \mathcal{U}_n$ are surjective, we have $\text{Rlim}_n \mathcal{U}_n = \mathcal{U}$. Since \mathcal{U} has a filtration with graded pieces isomorphic to $\mathcal{O}_{\mathfrak{X}_1}$ it has cohomology concentrated in degrees 0 and 2. These two facts imply

$$\text{R}\Gamma_{\text{ét}}(\mathfrak{X}, \text{Rlim}_n \mathcal{U}_n) = H_{\text{ét}}^0(\mathfrak{X}, \mathcal{U}) \oplus H_{\text{ét}}^2(\mathfrak{X}, \mathcal{U})[-2]$$

in $\mathcal{D}(\text{Ab})$. Similarly, we have

$$\text{Rlim}_n \text{R}\Gamma_{\text{ét}}(\mathfrak{X}, \mathcal{U}_n) = \lim_n H_{\text{ét}}^0(\mathfrak{X}, \mathcal{U}_n) \oplus \lim_n H_{\text{ét}}^2(\mathfrak{X}, \mathcal{U}_n)[-2]$$

in $\mathcal{D}(\text{Ab})$. As $\text{R}\Gamma_{\text{ét}}$ commutes with Rlim , the lemma follows. \square

Corollary 2.3. *If \mathfrak{X}_k is ordinary, then $\Psi(\mathfrak{X})(K)[p^r] = H_{\text{fl}}^2(\mathfrak{X}, \mu_{p^r})$.*

Proof. Indeed, we have

$$\Psi(\mathfrak{X})(K)[p^r] = \lim_n \Psi(\mathfrak{X})[p^r](\mathcal{O}_K/\mathfrak{m}^n) = \lim_n H_{\text{fl}}^2(\mathfrak{X}_n, \mu_{p^r}),$$

so the corollary follows from Lemma 2.2. \square

Proposition 2.4. *If \mathfrak{X} is a projective K3 surface over \mathcal{O}_K then for all r the natural map $H_{\mathfrak{h}}^2(\mathfrak{X}, \mu_{p^r}) \rightarrow H_{\mathfrak{h}}^2(\mathfrak{X}_K, \mu_{p^r})$ is injective.*

Proof. The Kummer sequence gives a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{Pic } \mathfrak{X}) \otimes \mathbf{Z}/p^r \mathbf{Z} & \longrightarrow & H_{\mathfrak{h}}^2(\mathfrak{X}, \mu_{p^r}) & \longrightarrow & (\text{Br}' \mathfrak{X})[p^r] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (\text{Pic } \mathfrak{X}_K) \otimes \mathbf{Z}/p^r \mathbf{Z} & \longrightarrow & H_{\mathfrak{h}}^2(\mathfrak{X}_K, \mu_{p^r}) & \longrightarrow & (\text{Br}' \mathfrak{X}_K)[p^r] \longrightarrow 0 \end{array}$$

Since \mathfrak{X} is projective, we have $\text{Br} = \text{Br}'$ for \mathfrak{X} and \mathfrak{X}_K .

The left arrow in the diagram is an isomorphism since the special fiber \mathfrak{X}_k is a principal divisor in \mathfrak{X} , so that $\text{Pic } \mathfrak{X} \rightarrow \text{Pic } \mathfrak{X}_K$ is an isomorphism. By [8, Cor 1.8] the natural maps of $\text{Br } \mathfrak{X}$ and $\text{Br } \mathfrak{X}_K$ to $\text{Br } K(\mathfrak{X}_K)$ are injective, so that also the right arrow in the diagram is injective. We conclude that the middle map is injective. \square

Proof of Theorem 2.1. The proof is now formal. By Corollary 2.3 and Proposition 2.4 we have for every r and every finite extension $K \subset L$ a canonical injection

$$\Psi(\mathfrak{X})[p^r](L) \rightarrow H_{\mathfrak{h}}^2(\mathfrak{X}_L, \mu_{p^r}) = H_{\text{ét}}^2(\mathfrak{X}_L, \mathbf{Z}/p^r \mathbf{Z}(1)).$$

Taking the colimit over all L we obtain a Gal_K -equivariant injective map

$$\rho_r: \Psi(\mathfrak{X})[p^r](\bar{K}) \rightarrow H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}/p^r \mathbf{Z}(1)),$$

and taking the limit over r we obtain a Gal_K -equivariant injective map

$$\rho: T_p \Psi(\mathfrak{X})_{\bar{K}} \rightarrow H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)).$$

Denote the cokernel of ρ by Q . Tensoring ρ with $\mathbf{Z}/p \mathbf{Z}$ yields an exact sequence

$$0 \rightarrow \text{Tor}(Q, \mathbf{Z}/p \mathbf{Z}) \rightarrow \Psi(\mathfrak{X})[p](\bar{K}) \xrightarrow{\rho \otimes \mathbf{Z}/p \mathbf{Z}} H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}/p \mathbf{Z}(1)) \rightarrow Q \otimes \mathbf{Z}/p \mathbf{Z} \rightarrow 0.$$

Since $\rho_1 = \rho \otimes \mathbf{Z}/p \mathbf{Z}$ is injective, we see that $\text{Tor}(Q, \mathbf{Z}/p \mathbf{Z})$ vanishes and that Q is torsion-free. \square

2.2. Canonical lifts and p -adic étale cohomology. In this section we prove Theorem C, characterizing the canonical lift in terms of p -adic étale cohomology.

Lemma 2.5. *Let U be a free \mathbf{Z}_p -module of rank 2 and $b: U \times U \rightarrow \mathbf{Z}_p$ a non-degenerate symmetric bilinear form. Let $L \subset U$ be a totally isotropic rank 1 submodule. If L is saturated in*

$$U^\vee := \{x \in \mathbf{Q}_p \otimes_{\mathbf{Z}_p} U \mid b(x, U) \subset \mathbf{Z}_p\},$$

then $U^\vee = U$.

Proof. Since L is saturated in U^\vee , it is also saturated in $U \subset U^\vee$ and we may choose a basis (e, f) for U with $L = \langle e \rangle$. Set $d := b(e, f)$. Since $b(e, e) = 0$, the determinant of b is $-d^2$. Since e/d lies in U^\vee , we must have that d is a unit and therefore $U^\vee = U$. \square

Proof of Theorem C. Assume that (ii) holds. Then we have

$$H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) = H^0(1) \oplus H^1 \oplus H^2(-1)$$

with the H^i unramified. Since the Tate module of a p -divisible group is Hodge-Tate of weights 0 and -1 , we have that $\text{Hom}(T_p \Psi(\mathfrak{X})_{\bar{K}}, H^2(-1)) = 0$, and by Theorem 2.1 we see that $T_p \Psi(\mathfrak{X})_{\bar{K}} = H^0(1) \oplus H^1$. By Tate's theorem [20, Thm. 4] this implies

that $\Psi(\mathfrak{X}) = \Psi^0(\mathfrak{X}) \oplus \Psi^{\text{ét}}(\mathfrak{X})$ with $T_p\Psi^0(\mathfrak{X})_{\bar{K}} = H^0(1)$ and $T_p\Psi^{\text{ét}}(\mathfrak{X})_{\bar{K}} = H^1$. It follows that \mathfrak{X} is the base change of the canonical lift of \mathfrak{X}_k to \mathcal{O}_K .

Conversely, assume that \mathfrak{X} is the base change of the canonical lift of \mathfrak{X}_k to \mathcal{O}_K . Let H^1 be the image of the direct summand $T_p\Psi^{\text{ét}}(\mathfrak{X})_{\bar{K}}$ under the embedding $T_p\Psi(\mathfrak{X})_{\bar{K}} \rightarrow H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ of Theorem 2.1. It is a primitive sub-module, and considering Hodge-Tate weights we see that the restriction of the bilinear form on H^2 to H^1 is non-degenerate. Let $U \subset H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ be its orthogonal complement. Then U is a rank 2 lattice over \mathbf{Z}_p . The inclusions of H^1 and U as mutual orthogonal complements inside the self-dual lattice $H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ induce an isomorphism

$$\alpha: U^\vee/U \xrightarrow{\sim} (H^1)^\vee/H^1$$

and an identification

$$H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) := \{(x, y) \in U^\vee \oplus (H^1)^\vee \mid \alpha(x) = y\}.$$

Consider the unramified Gal_K -module $H^0 := T_p\Psi^0(\mathfrak{X})_{\bar{K}}(-1)$. We have that $H^0(1)$ is a totally isotropic line in U . We claim that it is saturated in U^\vee . Indeed, if $x \in U^\vee$ satisfies $px \in H^0(1)$ then $(x, \alpha(x))$ defines a p -torsion element in the cokernel of $T_p\Psi(\mathfrak{X})_{\bar{K}} \rightarrow H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$, which must be trivial by Theorem 2.1. By Lemma 2.5 we conclude that $U = U^\vee$ and that $H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) = U \oplus H^1$.

Now U is a unimodular \mathbf{Z}_p -lattice of rank 2 containing an isotropic line. Moreover, since the intersection pairing on $H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1))$ is even, so is the lattice U . It follows that there is a unique isotropic line $H^2(-1) \subset U$ with $U = H^0(1) \oplus H^2(-1)$ and with H^0 and H^2 dual unramified representations. We find $H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p(1)) = H^0(1) \oplus H^1 \oplus H^2(-1)$, as claimed. \square

Remark 2.6. Using the results on integral p -adic Hodge theory by Bhatt, Morrow, and Scholze [2] one can show that the splitting of $H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p)$ as in Theorem C implies an analogous splitting of the filtered crystal $H_{\text{crys}}^2(\mathfrak{X}/W)$. If $p > 2$, then the splitting of $H_{\text{crys}}^2(\mathfrak{X}/W)$ implies that \mathfrak{X} is the canonical lift of \mathfrak{X}_k (see [7] and [14, Lem. 1.11, Thm. 1.12]). For $p = 2$, however, the splitting of $H_{\text{crys}}^2(\mathfrak{X}/W)$ is a weaker condition than the splitting of $H_{\text{ét}}^2(\mathfrak{X}_{\bar{K}}, \mathbf{Z}_p)$, see also [7, 2.1.16.b].

3. THE FUNCTOR $X \mapsto (M, F, \mathcal{K})$

Let \mathbf{F}_q be a finite field with $q = p^a$ elements. Let W be the ring of Witt vectors of \mathbf{F}_q , and K its fraction field. Fix an embedding $\iota: \bar{K} \rightarrow \mathbf{C}$. By § 1.3, every ordinary K3 surface X over \mathbf{F}_q has a canonical lift X_{can} over W . We will denote by X_{can}^ι the complex K3 surface obtained by base changing X_{can} along $\iota: W \rightarrow \mathbf{C}$.

3.1. Construction of a pair (M, F) . Let X be an ordinary K3 surface over \mathbf{F}_q . The following theorem, due to Nygaard and Yu, says that the Frobenius on X can be lifted to an endomorphism of the Betti cohomology of X_{can}^ι , at least after inverting p . The proof relies on the Kuga–Satake construction.

Theorem 3.1 (Nygaard [14, § 3], Yu [21, Lemma 2.3]). *There is a unique endomorphism F of $H^2(X_{\text{can}}^\iota, \mathbf{Z}[\frac{1}{p}])$ such that*

- (i) *for every $\ell \neq p$ the map F corresponds under the comparison isomorphism*

$$H^2(X_{\text{can}}^\iota, \mathbf{Z}[\frac{1}{p}]) \otimes \mathbf{Z}_\ell \xrightarrow{\sim} H_{\text{ét}}^2(X_{\bar{\mathbf{F}}_q}, \mathbf{Z}_\ell)$$

to the geometric Frobenius Frob on étale cohomology,

(ii) the map F corresponds under the comparison isomorphism

$$\mathrm{H}^2(X_{\mathrm{can}}^\iota, \mathbf{Z}[\frac{1}{p}]) \otimes B_{\mathrm{crys}} \xrightarrow{\sim} \mathrm{H}_{\mathrm{crys}}^2(X/W) \otimes_W B_{\mathrm{crys}}$$

to the endomorphism $\phi^a \otimes \mathrm{id}$, where ϕ denotes the crystalline Frobenius.

Moreover, F preserves the Hodge structure on $\mathrm{H}^2(X_{\mathrm{can}}^\iota, \mathbf{Q})$. \square

For later use in the proofs of Theorems A and B, we record some well-known properties of Tate twists of unramified p -adic Galois representations.

Lemma 3.2. *Let V be an unramified Gal_K -representation over \mathbf{Q}_p , and let n be an integer. Let Frob be the geometric Frobenius endomorphism of V , relative to \mathbf{F}_q . Consider the Gal_K -module $V(-n) := V \otimes \mathbf{Q}_p(-n)$.*

(i) The map

$$K^\times \rightarrow \mathrm{GL}(V(-n)), x \mapsto \mathrm{Frob}^{v(x)} \otimes q^{nv(x)} \mathrm{Nm}_{K/\mathbf{Q}_p}(x)^{-n}$$

factors over the reciprocity map $K^\times \rightarrow \mathrm{Gal}_K^{\mathrm{ab}}$ and induces the action of Gal_K on $V(-n)$.

(ii) $D_{\mathrm{crys}}(q^n \mathrm{Frob}) = \phi^a$ as endomorphisms of $D_{\mathrm{crys}}(V(-n))$.

Proof. The first statement follows from Lubin–Tate theory, see for example [17, § 3.1, Theorem 2]. For the second, one uses that the functor D_{crys} commutes with Tate twists to reduce to the case $n = 0$. In this case, the statement follows from the observation that $\mathrm{Frob} \otimes 1 = 1 \otimes \phi^a$ as endomorphisms of

$$(V \otimes_{\mathbf{Z}_p} W(\bar{\mathbf{F}}_q))^{\mathrm{Gal}_{\mathbf{F}_q}},$$

where the action of $\mathrm{Gal}_{\mathbf{F}_q}$ is the diagonal one. \square

Theorem A is now an almost immediate consequence of Theorem C.

Proof of Theorem A. By its definition (in Theorem 3.1), the endomorphism F of $\mathrm{H}^2(X_{\mathrm{can}}^\iota, \mathbf{Q}) \otimes \mathbf{Q}_p = \mathrm{H}_{\mathrm{ét}}^2(X_{\bar{K}}, \mathbf{Q}_p)$ satisfies $D_{\mathrm{crys}}(F) = \phi^a$ on $\mathrm{H}_{\mathrm{crys}}^2(X/W)[\frac{1}{p}]$.

By Theorem C we have a decomposition $\mathrm{H}_{\mathrm{ét}}^2(X_{\bar{K}}, \mathbf{Z}_p) = H^0 \oplus H^1(-1) \oplus H^2(-2)$ with H^i unramified and by Lemma 3.2, also the endomorphism

$$F' := \mathrm{Frob}_{H^0} \oplus q \mathrm{Frob}_{H^1} \oplus q^2 \mathrm{Frob}_{H^2}$$

of $\mathrm{H}_{\mathrm{ét}}^2(X_{\bar{K}}, \mathbf{Q}_p)$ satisfies $D_{\mathrm{crys}}(F') = \phi^a$. Since D_{crys} is fully faithful, we must have $F = F'$. But it then follows immediately that F preserves the \mathbf{Z}_p -lattice $\mathrm{H}_{\mathrm{ét}}^2(X_{\bar{K}}, \mathbf{Z}_p)$, and that $\mathrm{H}_{\mathrm{ét}}^2(X_{\bar{K}}, \mathbf{Z}_p)$ decomposes as described in the theorem. \square

We thus have constructed from X/\mathbf{F}_q an integral lattice $M := \mathrm{H}^2(X_{\mathrm{can}}^\iota, \mathbf{Z})$, equipped with an endomorphism F , satisfying (M1)–(M4). We end this paragraph by relating the p -adic decomposition in (M4) to the Hodge decomposition for X_{can}^ι .

Lemma 3.3. *Let (M, F) be a pair satisfying (M1)–(M4). Then complex conjugation on $M \otimes \mathbf{C}$ maps the subspace $M^s \otimes_{\mathbf{Z}_p, \iota} \mathbf{C}$ to $M^{2-s} \otimes_{\mathbf{Z}_p, \iota} \mathbf{C}$.*

In other words: the decomposition in (M4) induces under $\iota: \mathbf{Z}_p \rightarrow \mathbf{C}$ a \mathbf{Z} -Hodge structure on M .

Proof of Lemma 3.3. By (M4), the one-dimensional subspace $M^0 \otimes_{\mathbf{Z}_p, \iota} \mathbf{C}$ of $M \otimes \mathbf{C}$ is the unique eigenspace for the endomorphism F corresponding to an eigenvalue $u \in \mathbf{Q}_p \subset \mathbf{C}$ with $v_p(u) = 0$. By (M3) we have $u\bar{u} = q^2$, and hence also the eigenvalue $\bar{u} = q^2/u$ lies in $\mathbf{Q}_p \subset \mathbf{C}$. Since $v_p(\bar{u}) = v_p(q^2)$, we see that the

corresponding eigenspace is $M^2 \otimes_{\mathbf{Z}_p, \iota} \mathbf{C}$. Similarly, complex conjugation maps M^2 to M^0 . By (M2), the subspace M^1 is the orthogonal complement of $M^0 \oplus M^2$, and hence is preserved by complex conjugation. \square

Proposition 3.4. *Let (M, F) be the pair associated to an ordinary K3 surface X over \mathbf{F}_q . Then we have $M^s \otimes_{\mathbf{Z}_p, \iota} \mathbf{C} = \mathbf{H}^{s, 2-s}(X_{\text{can}}^\iota)$ as subspaces of $M \otimes \mathbf{C} = \mathbf{H}^2(X_{\text{can}}^\iota, \mathbf{C})$.*

Proof. Indeed, under the ‘Hodge–Tate’ comparison isomorphism

$$\mathbf{H}_{\text{dR}}^2(X_{\text{can}, K}/K) \otimes_K \mathbf{C}_p \xrightarrow{\sim} \mathbf{H}_{\text{ét}}^2(X_{\bar{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \cong M \otimes_{\mathbf{Z}_p} \mathbf{C}_p$$

the subspace $(\text{Fil}^i \mathbf{H}_{\text{dR}}^2(X_{\text{can}, K}/K)) \otimes_K \mathbf{C}_p$ is mapped to $\bigoplus_{s \geq i} M^s \otimes_{\mathbf{Z}_p} \mathbf{C}_p$. Extending ι to an embedding $\mathbf{C}_p \rightarrow \mathbf{C}$ we see that the Hodge filtration on $\mathbf{H}^2(X_{\text{can}}^\iota, \mathbf{C})$ agrees with the filtration on $M \otimes \mathbf{C}$ induced by the decomposition on $M \otimes_{\mathbf{Z}_p} \mathbf{C}$, and hence by Lemma 3.3 we have $M^s \otimes_{\mathbf{Z}_p} \mathbf{C} = \mathbf{H}^{s, 2-s}(X_{\text{can}}^\iota)$. \square

3.2. Line bundles and ample cone. Let X be an ordinary K3 surface over \mathbf{F}_q . Recall from § 1.3 that line bundles on X extend uniquely to X_{can} . We obtain isomorphisms

$$\text{Pic } X \xrightarrow{\sim} \text{Pic } X_{\text{can}} \xrightarrow{\sim} \text{Pic } X_{\text{can}, K}.$$

Let K^{nr} be the maximal unramified extension of K .

Proposition 3.5. *We have natural isomorphisms*

$$\text{Pic } X_{\bar{\mathbf{F}}_q} \xrightarrow{\sim} \text{Pic } X_{\text{can}, K^{\text{nr}}} \xrightarrow{\sim} \text{Pic } X_{\text{can}, \bar{K}},$$

and a class $\lambda \in \text{Pic } X_{\bar{\mathbf{F}}_q}$ is ample if and only if its image in $\text{Pic } X_{\text{can}, \bar{K}}$ is ample.

Proof. The first isomorphism follows from the fact that canonical lifts commute with finite unramified extensions. The second isomorphism follows from the triviality of the action of $\text{Gal}(K/K^{\text{nr}})$ on $\text{Pic } X_{\text{can}, \bar{K}} \subset \mathbf{H}_{\text{ét}}^2(X_{\text{can}, \bar{K}}, \mathbf{Q}_\ell(1))$ and the vanishing of $\text{Br } K^{\text{nr}}$.

It remains to show that the isomorphism $\text{Pic } X_{\bar{\mathbf{F}}_q} \xrightarrow{\sim} \text{Pic } X_{\text{can}, \bar{K}}$ restricts to a bijection between the subsets of ample classes. Fix an ample line bundle H on X . Then by the structure theorem on the ample cone of a K3 surface over an algebraically closed field ([9, § 8.1]) we have that a line bundle L on $X_{\bar{\mathbf{F}}_q}$ is ample if and only if

- (i) $L^2 > 0$
- (ii) for every $D \in \text{Pic } X_{\bar{\mathbf{F}}_q}$ with $D^2 = -2$ we have $L \cdot D \neq 0$ and $L \cdot D$ has the same sign as $H \cdot D$

and similarly for line bundles on $X_{\text{can}, \bar{K}}$. But the bijection $\text{Pic } X_{\bar{\mathbf{F}}_q} \xrightarrow{\sim} \text{Pic } X_{\text{can}, \bar{K}}$ is an isometry, and the canonical lift H_{can} of the ample line bundle H is itself ample, so we conclude that the bijection preserves ample classes. \square

Proposition 3.6. *For every $d \geq 1$ the map*

$$\text{Pic } X_{\mathbf{F}_{q^d}} \rightarrow \{\lambda \in \mathbf{H}^2(X_{\text{can}}^\iota, \mathbf{Z}) \mid F^d \lambda = q^d \lambda\}$$

is an isomorphism.

Proof. Injectivity is clear, it suffices to show that the map is surjective. Without loss of generality we may assume that $d = 1$.

By Proposition 3.4 any $\lambda \in \mathbf{H}^2(X_{\text{can}}^\iota, \mathbf{Z})$ satisfying $F\lambda = q\lambda$ is a Hodge class and by Theorem 3.1 we see that λ defines a Gal_K -invariant element of $\text{Pic } X_{\text{can}, \bar{K}}$.

By the previous proposition, λ corresponds to a $\text{Gal}_{\mathbf{F}_q}$ -invariant class in $\text{Pic } X_{\mathbf{F}_q}$, which defines a line bundle \mathcal{L} on X since the Brauer group of \mathbf{F}_q vanishes. We conclude that the map is surjective as claimed. \square

Proposition 3.7. *The real cone $\mathcal{K} \subset M \otimes_{\mathbf{Z}} R$ spanned by the classes of ample line bundles on $\text{Pic } X_{\mathbf{F}_q}$ satisfies (M5).*

Proof. This follows immediately from the Propositions 3.5 and 3.6, and the structure of the ample cone of a complex K3 surface. \square

3.3. Fully faithfulness. In § 3.1 and § 3.2 we have constructed a functor $X \mapsto (M, F, \mathcal{K})$ from ordinary K3 surfaces over \mathbf{F}_q to triples satisfying (M1)–(M5). We end this section by showing that this functor is fully faithful.

Proof of fully faithfulness in Theorem B. This is shown in [15] and [21, Theorem 3.3] for K3 surfaces equipped with an ample line bundle. The same argument works here, we repeat it for the convenience of the reader.

Faithfulness. Assume that $f, g: X_1 \rightarrow X_2$ are morphisms between ordinary K3 surfaces inducing the same maps $\text{H}^2(X_{2,\text{can}}^t, \mathbf{Z}) \rightarrow \text{H}^2(X_{1,\text{can}}^t, \mathbf{Z})$. Then $f_{\text{can}}^t = g_{\text{can}}^t$ as maps from $X_{1,\text{can}}^t$ to $X_{2,\text{can}}^t$ and therefore $f_{\text{can}} = g_{\text{can}}$ and $f = g$.

Fullness. Let X_1 and X_2 be ordinary K3 surfaces over \mathbf{F}_q . Let

$$\varphi: \text{H}^2(X_{2,\text{can}}^t, \mathbf{Z}) \rightarrow \text{H}^2(X_{1,\text{can}}^t, \mathbf{Z})$$

be an isometry commuting with F and respecting ample cones. By the description of the ample cones of X_1 and X_2 , we may choose ample line bundles \mathcal{L}_1 and \mathcal{L}_2 on X/\mathbf{F}_q such that φ maps $c_1(\mathcal{L}_{2,\text{can}}^t)$ to $c_1(\mathcal{L}_{1,\text{can}}^t)$.

By Proposition 3.4 the map φ respects the Hodge structures, and therefore the Torelli theorem shows that there is a unique isomorphism $f: X_{1,\text{can}}^t \xrightarrow{\sim} X_{2,\text{can}}^t$ with $f^* = \varphi$. Since $f^*F_2 = F_1f^*$, and since the étale cohomology of the $X_{i,\text{can},\bar{K}}$ is unramified, we have that

$$f^*: \text{H}_{\text{ét}}^2(X_{2,\text{can},\bar{K}}, \mathbf{Q}_\ell) \rightarrow \text{H}_{\text{ét}}^2(X_{1,\text{can},\bar{K}}, \mathbf{Q}_\ell)$$

is Gal_K -equivariant, and hence f descends to a morphism of polarized K3 surfaces over K . By Matsusaka–Mumford [12, Thm. 2] this extends to an isomorphism $f: X_{1,\text{can}} \xrightarrow{\sim} X_{2,\text{can}}$ and we conclude that φ comes from an isomorphism $f: X_1 \xrightarrow{\sim} X_2$ over \mathbf{F}_q . \square

4. ESSENTIAL SURJECTIVITY

4.1. Models of K3 surfaces with complex multiplication. We briefly recall a few facts about complex K3 surfaces with complex multiplication. We refer to [22] for proofs. Let X/\mathbf{C} be a K3 surface. Its (\mathbf{Q} -)transcendental lattice V_X is defined as the orthogonal complement of $\text{NS}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ in $\text{H}^2(X, \mathbf{Q}(1))$. The endomorphism algebra E of the \mathbf{Q} -Hodge structure V_X is a field, and we say that X has *complex multiplication* (by E) if V_X is one-dimensional as an E -vector space. In this case, E is necessarily a CM-field. Denote its complex conjugation by $\sigma: E \rightarrow E$. The Mumford-Tate group of the Hodge structure V_X is the algebraic torus T/\mathbf{Q} defined by

$$T(A) = \{x \in (A \otimes E)^\times \mid x\sigma(x) = 1\}$$

for all \mathbf{Q} -algebras A . Note that T is an algebraic subgroup of $\text{SO}(V_X)$.

If X/\mathbf{C} is a K3 surface with CM by E , then it can be defined over a number field. In [19, Theorem 2] we classified the models \mathcal{X} of X over finite extensions F of E in terms of their Galois representations on $H_{\text{ét}}^2(\mathcal{X}_{\bar{F}}, \hat{\mathbf{Z}})$. We will deduce from that result a version for models over local fields. In the statement we will need the composition

$$\text{rec}: \text{Gal}_E^{\text{ab}} \cong \mathbf{A}_{E,f}^\times / E^\times \longrightarrow T(\mathbf{A}_f)/T(\mathbf{Q}),$$

where the isomorphism is given by global class field theory (note that E has no real places), and the second map is given by $z \mapsto \frac{z}{\sigma(z)}$.

Theorem 4.1. *Let X/\mathbf{C} be a K3 surface with CM by E . Let K be a p -adic field containing E , and fix an embedding $\iota: K \rightarrow \mathbf{C}$ extending the embedding $E \rightarrow \mathbf{C}$ given by the action of E on $H^{2,0}(X)$. Let*

$$\rho: \text{Gal}_K \rightarrow \text{O}(H^2(X, \hat{\mathbf{Z}}(1)))$$

be a continuous homomorphism. Assume that for every $\sigma \in \text{Gal}_K$ we have

- (i) $\rho(\sigma)$ stabilizes $\text{NS}(X) \subset H^2(X, \hat{\mathbf{Z}}(1))$ and $\mathcal{K}_X \subset \text{NS}(X) \otimes \mathbf{R}$,
- (ii) the restriction of $\rho(\sigma)$ to the transcendental lattice lands in the subgroup $T(\mathbf{A}_f) \subset \text{O}(V_X \otimes \mathbf{A}_f)$ and its image in $T(\mathbf{A}_f)/T(\mathbf{Q})$ is $\text{rec}(\sigma)$.

Then there exists a model \mathcal{X}/K of X so that the resulting action of Gal_K on $H_{\text{ét}}^2(\mathcal{X}_{\bar{K}}, \hat{\mathbf{Z}}(1)) = H^2(X, \hat{\mathbf{Z}}(1))$ coincides with ρ .

We can reformulate the conditions on ρ as follows. Denote by $\Gamma \subset \text{O}(H^2(X, \hat{\mathbf{Z}}(1)))$ the subgroup consisting of those g satisfying

- (i) g stabilizes $\text{NS}(X) \subset H^2(X, \hat{\mathbf{Z}}(1))$ and $\mathcal{K}_X \subset \text{NS}(X) \otimes \mathbf{R}$,
- (ii) the induced action on $V_X \otimes \mathbf{A}_f$ factors over $T(\mathbf{A}_f)$,

Then $\rho: \text{Gal}_K \rightarrow \text{O}(H^2(X, \hat{\mathbf{Z}}(1)))$ satisfies the conditions in the theorem if and only if it factors over Γ , and makes the square

$$\begin{array}{ccc} \text{Gal}_K & \longrightarrow & \text{Gal}_E \\ \downarrow \rho & & \downarrow \text{rec} \\ \Gamma & \xrightarrow{\delta} & T(\mathbf{A}_f)/T(\mathbf{Q}) \end{array}$$

commute. In particular, we can consider ρ as a lift of the map rec . This point of view will be useful in the proof of Theorem 4.1.

Lemma 4.2. *The map $\delta: \Gamma \rightarrow T(\mathbf{A}_f)/T(\mathbf{Q})$ is a continuous open homomorphism of profinite groups with finite kernel.*

Proof. Let $\Gamma_0 \subset \Gamma$ be the open subgroup of finite index consisting of those elements that act trivially on $\text{NS}(X)$. The map $\Gamma_0 \rightarrow T(\mathbf{A}_f)$ is injective, and identifies Γ_0 with a compact open subgroup. Let \mathcal{K} be the maximal compact open subgroup. Then it suffices to show that $\mathcal{K} \rightarrow T(\mathbf{A}_f)/T(\mathbf{Q})$ has finite kernel and cokernel. The kernel is $\{x \in \mathcal{O}_E^\times \mid x\sigma(x) = 1\}$, which is finite because E is a CM-field with complex conjugation σ . The finiteness of the cokernel $T(\mathbf{Q}) \backslash T(\mathbf{A}_f)/\mathcal{K}$ is a property of arbitrary tori over \mathbf{Q} , see [16, Prop. 9 & Thm. 2]. \square

Lemma 4.3. *Let $\delta: G_0 \rightarrow G_1$ be an open continuous homomorphism of profinite groups with finite kernel. Let H be a closed subgroup of G_1 , and $\rho: H \rightarrow G_0$ a*

continuous morphism making the triangle

$$\begin{array}{ccc} H & & \\ \rho \downarrow & \searrow & \\ G_0 & \xrightarrow{\delta} & G_1 \end{array}$$

commute. Then there exists an open subgroup $U \subset G_1$ containing H , and a continuous homomorphism $\rho': U \rightarrow G_0$ making the square

$$\begin{array}{ccc} H & \hookrightarrow & U \\ \rho \downarrow & \swarrow \rho' & \downarrow \\ G_0 & \xrightarrow{\delta} & G_1 \end{array}$$

commute.

Proof. Since the kernel of δ is finite, there exists an open subgroup $H_0 \subset G_0$ with $\ker \delta \cap H_0 = \{1\}$. The normalizer of H_0 has finite index in G_0 , so the intersection $N_0 := \bigcap_{g \in G_0} gH_0g^{-1}$ is a normal open subgroup on which δ is injective. Denote by $N_1 \subset G_1$ its image, and by $s: N_1 \xrightarrow{\sim} N_0$ the inverse isomorphism. Note that $N_1 \subset G_1$ is open, and normalized by $H \subset G$.

Consider the continuous function

$$N_1 \cap H \rightarrow G_0, g \mapsto s(g)\rho(g)^{-1}.$$

It takes values in the finite subset $\ker \delta \subset G_0$, and maps 1 to 1. The collection of subgroups of the form $N_1 \cap H$ (for varying normal open $N_0 \subset G_0$) is a basis for the topology on H , so shrinking N_0 if necessary, we may without loss of generality assume that the above map is constant. We then have $s(g) = \rho(g)$ for all $g \in N_1 \cap H$.

The product $U := N_1 \cdot H$ is an open subgroup of G containing H , and by the above the map

$$\rho': U \rightarrow G_0, gh \mapsto s(g)\rho(h) \quad (g \in N_1, h \in H)$$

is a well-defined homomorphism satisfying the required properties. \square

Proof of Theorem 4.1. Let G_0 be the topological group defined by the cartesian square

$$\begin{array}{ccc} G_0 & \xrightarrow{\delta'} & \text{Gal}_E \\ \downarrow & & \downarrow \text{rec} \\ \Gamma & \xrightarrow{\delta} & T(\mathbf{A}_f)/T(\mathbf{Q}) \end{array}$$

Note that δ' is open with finite kernel, and hence that G_0 is a profinite group. Now let ρ be as in the statement of the theorem. Then it induces a commutative triangle

$$\begin{array}{ccc} \text{Gal}_K & & \\ \rho' \downarrow & \searrow & \\ G_0 & \xrightarrow{\delta'} & \text{Gal}_E. \end{array}$$

By Lemma 4.3 there exists an intermediate field $E \subset F \subset K$ with F finite over E , and a continuous homomorphism $\rho'' : \text{Gal}_F \rightarrow G_0$ making the diagram

$$\begin{array}{ccc} \text{Gal}_K & \longrightarrow & \text{Gal}_F \\ \rho' \downarrow & \nearrow \rho'' & \downarrow \\ G_0 & \xrightarrow{\delta'} & \text{Gal}_E. \end{array}$$

commute. Now [19, Theorem 2] guarantees the existence of a model \mathcal{X} over F whose Galois action on $H_{\text{ét}}^2(\mathcal{X}_{\bar{F}}, \hat{\mathbf{Z}}(1)) = H^2(X, \hat{\mathbf{Z}}(1))$ is given by the composition

$$\text{Gal}_F \xrightarrow{\rho''} G_0 \longrightarrow \Gamma,$$

and hence the base change of \mathcal{X} to K fulfils the requirements. \square

4.2. Criteria of good reduction. Let \mathcal{O}_K be a discrete valuation ring with fraction field K and perfect residue field k . In the introduction we defined a property (\star) for K3 surfaces over K .

Theorem 4.4 (Liedtke–Matsumoto [11]). *Let X be a K3 surface over K satisfying (\star) . If for some ℓ different from the characteristic of k the action of Gal_K on $H_{\text{ét}}^2(X_{\bar{K}}, \mathbf{Z}_{\ell})$ is unramified, then there exists a finite unramified extension $K \subset K'$ and a proper smooth algebraic space \mathfrak{X}' over $\mathcal{O}_{K'}$ with $\mathfrak{X}_{K'} \cong X_{K'}$.* \square

Analyzing the proof in the case where the specialization map on Picard groups is bijective, one obtains a stronger conclusion.

Proposition 4.5. *Let X be a K3 surface over K , let $K \subset K'$ be an unramified extension, and let \mathfrak{X}' over $\mathcal{O}_{K'}$ be a proper smooth algebraic space with $\mathfrak{X}'_{K'} \cong X_{K'}$. If the reduction map $\text{Pic } \mathfrak{X}'_{\bar{K}} \rightarrow \text{Pic } \mathfrak{X}'_{\bar{k}}$ is bijective, then there exists a smooth projective \mathfrak{X} over \mathcal{O}_K with $\mathfrak{X}_K \cong X$.*

Proof. The map $\text{Pic } \mathfrak{X}'_{\bar{K}} \xrightarrow{\sim} \text{Pic } \mathfrak{X}'_{\bar{k}}$ identifies the (-2) -classes on generic and special fiber, and hence induces a bijection between the ample cones in $\text{Pic } \mathfrak{X}'_{\bar{K}}$ and $\text{Pic } \mathfrak{X}'_{\bar{k}}$ (see also the proof of Proposition 3.5). Now choose an ample line bundle \mathcal{L} on X . It induces an ample line bundle \mathcal{L}' on $X' := X_{K'}$, which extends to a relatively ample line bundle on \mathfrak{X}' . In particular, the canonical RDP model $P(X', \mathcal{L}')$ over $\mathcal{O}_{K'}$ of Liedtke and Matsumoto [11, Thm. 1.3] is non-singular. It follows from the construction of this model that $P(X', \mathcal{L}') = P(X, \mathcal{L}) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ (see the end of section 6 in [3]), and hence $P(X, \mathcal{L})$ is a smooth projective model \mathfrak{X} over \mathcal{O}_K . \square

Alternatively, one can verify that under the hypothesis of Proposition 4.5 the group $\mathcal{W}_{X, \mathcal{L}}^{\text{nr}}$ occurring in [3, Thm. 1.4] vanishes.

4.3. Proof of Theorem B. In § 3.3 we have established that the functor $X/\mathbf{F}_q \mapsto (M, F, \mathcal{K})$ is fully faithful. To finish the proof of Theorem B, it remains to show that the functor is essentially surjective, assuming (\star) holds for K3 surfaces over p -adic fields.

Proof of essential surjectivity in Theorem B. Let (M, F, \mathcal{K}) be a triple satisfying (M1)–(M5). We will show that it lies in the essential image of our functor by constructing a suitable K3 surface over \mathbf{F}_q . We divide the construction in several steps.

Construction of a complex K3 surface. By Lemma 3.3 the decomposition

$$M_{\mathbf{C}} = M^{2,0} \oplus M^{1,1} \oplus M^{0,2}$$

with $M^{s,2-s} := M^s \otimes_{\mathbf{Z}_p, \iota} \mathbf{C}$ defines a \mathbf{Z} -Hodge structure on M . By the Torelli theorem for complex K3 surfaces, there is a projective K3 surface X and a Hodge isometry $f: \mathbf{H}^2(X, \mathbf{Z}) \xrightarrow{\sim} M$ mapping the ample cone of X to \mathcal{K} . The pair (X, f) is unique up to unique isomorphism.

X has complex multiplication. Let $V_X \subset \mathbf{H}^2(X, \mathbf{Q}(1))$ be the transcendental lattice. Note that F respects the decomposition $\mathbf{H}^2(X, \mathbf{Q}(1)) = \text{NS}(X) \otimes \mathbf{Q} \oplus V_X$. Every \mathbf{Q} -linear endomorphism of V_X that commutes with F will respect the Hodge structure on V_X , and since the endomorphism algebra of the \mathbf{Q} -Hodge structure V_X is a field, we conclude that V_X is a cyclic $\mathbf{Q}[F]$ -module, that $E := \text{End } V_X$ is generated by F , and that $\dim_E V_X = 1$. In particular, X has complex multiplication by E , the field E is then a CM field, and if we denote the complex conjugation on E by σ , then the Mumford-Tate group T of V_X satisfies $T(\mathbf{Q}) = \{x \in E^\times \mid x\sigma(x) = 1\}$. Observe that $\sigma(F) = q^2/F$ on V_X , and hence that F/q defines an element of $T(\mathbf{Q})$.

The number field E has a unique place $v \mid p$ satisfying $v(F) > 0$. We have $E_v = \mathbf{Q}_p$. Let K be the fraction field of $W(\mathbf{F}_q)$, considered as a subfield of \mathbf{C} via ι .

Descent to $K \subset \mathbf{C}$. For every $\ell \neq p$ consider the unramified Gal_K -representation

$$\rho_\ell: \text{Gal}_K \rightarrow \text{GL}(M \otimes \mathbf{Z}_\ell)$$

given by letting the geometric Frobenius Frob act as F . We also define a p -adic Gal_K -representation

$$\rho_p: \text{Gal}_K \rightarrow \text{GL}(M \otimes \mathbf{Z}_p) = \text{GL}(\oplus_s M^s)$$

by declaring that the Tate twisted $\mathbf{Z}_p[\text{Gal}_K]$ -modules $M^s(s)$ are unramified with geometric Frobenius Frob acting as F/q^s . The ρ_ℓ and ρ_p assemble into an action of Gal_K on $M \otimes \hat{\mathbf{Z}}$. Denote by $M \otimes \hat{\mathbf{Z}}(1)$ its Tate twist. The resulting map

$$\rho: \text{Gal}_K \rightarrow \text{GL}(M \otimes \hat{\mathbf{Z}}(1)) = \text{GL}(\mathbf{H}^2(X, \hat{\mathbf{Z}}(1)))$$

satisfies

- (i) the image of ρ is contained in $O(\mathbf{H}^2(X, \hat{\mathbf{Z}}(1)))$,
- (ii) the image of ρ preserves $\text{Pic } X$ and the ample cone $\mathcal{K} \subset (\text{Pic } X) \otimes \mathbf{R}$.

We claim that ρ also satisfies the reciprocity condition in Theorem 4.1.

Indeed, observe that the action of Gal_K on $M \otimes \hat{\mathbf{Z}}(1)$ is abelian. Let $x \in K^\times$. Using Lemma 3.2 we see that the action of the corresponding $\tau = \tau(x) \in \text{Gal}_K^{\text{ab}}$ on $M \otimes \hat{\mathbf{Z}}(1)$ satisfies

- (i) τ acts on $M \otimes \mathbf{Z}_\ell(1)$ by $(F/q)^{v(x)}$ (for $\ell \neq p$),
- (ii) τ acts on $M^0(1) \subset M \otimes \mathbf{Z}_p(1)$ by $(\text{Nm}_{K/\mathbf{Q}_p} x)(F/q)^{v(x)}$,
- (iii) τ acts on $M^1(1) \subset M \otimes \mathbf{Z}_p(1)$ by $(F/q)^{v(x)}$,
- (iv) τ acts on $M^2(1) \subset M \otimes \mathbf{Z}_p(1)$ by $(\text{Nm}_{K/\mathbf{Q}_p} x)^{-1}(F/q)^{v(x)}$.

(Note that by property (M4) these actions indeed preserve the \mathbf{Z}_p -lattices $M^s(1)$).

On the other hand, the decomposition of $M \otimes \mathbf{Z}_p$ induces a decomposition

$$V_X \otimes \mathbf{Q}_p = V_{-1} \oplus V_0 \oplus V_1$$

with $\dim V_{-1} = \dim V_1 = 1$. The group $E \otimes \mathbf{Q}_p$ acts on $V_X \otimes \mathbf{Q}_p$ on V_{-1} through the factor $E_v^\times \cong \mathbf{Q}_p^\times$, and on V_1 through $E_{\sigma v}^\times \cong \mathbf{Q}_p^\times$. The inclusion of $E_v^\times \times E_{\sigma v}^\times \subset (E \otimes \mathbf{Q}_p)^\times$ defines a subgroup

$$T_{v,\sigma v} = \{(x_v, x_{\sigma v}) \in E_v^\times \times E_{\sigma v}^\times \mid x_v x_{\sigma v} = 1\} \subset T(\mathbf{Q}_p).$$

The compatibility between local and global class field theory and the definition of rec (see § 4.1) implies that the diagram

$$\begin{array}{ccc} \text{Gal}_K^{\text{ab}} & \longrightarrow & \text{Gal}_E^{\text{ab}} \\ \uparrow & & \downarrow \text{rec} \\ K^\times & \longrightarrow & T(\mathbf{A}_f)/T(\mathbf{Q}) \end{array}$$

in which the map $K^\times \rightarrow T_{v,\sigma v}$ maps x to $(\text{Nm}_{K/\mathbf{Q}_p}(x), \text{Nm}_{K/\mathbf{Q}_p}(x)^{-1})$ commutes. We conclude that

$$K^\times \longrightarrow \text{Gal}_K^{\text{ab}} \longrightarrow \text{Gal}_E^{\text{ab}} \xrightarrow{\text{rec}} T(\mathbf{A}_f)/T(\mathbf{Q})$$

maps an $x \in K^\times$ to the class of the element $\alpha = \alpha(x) \in T(\mathbf{A}_f)$ satisfying

- (i) $\alpha_\ell = 1$ for all $\ell \neq p$,
- (ii) α_p acts on $V_X \otimes \mathbf{Q}_p = V_{-1} \oplus V_0 \oplus V_1$ by $(\text{Nm}_{K/\mathbf{Q}_p} x, 1, (\text{Nm}_{K/\mathbf{Q}_p} x)^{-1})$.

Since F/q lies in $T(\mathbf{Q})$, we see that $\alpha(x)$ and $\tau(x)$ define the same element in $T(\mathbf{A}_f)/T(\mathbf{Q})$. This shows that ρ satisfies the requirements of Theorem 4.1, and we conclude that there is a model \mathcal{X}/K of X whose Gal_K -action on $\text{H}_{\text{ét}}^2(\mathcal{X}_{\bar{K}}, \hat{\mathbf{Z}}) = M \otimes \hat{\mathbf{Z}}$ is the prescribed one.

Extension to \mathcal{O}_K and reduction to k . By construction, the action of Gal_K on $\text{H}_{\text{ét}}^2(X_{\bar{K}}, \mathbf{Z}_\ell)$ is unramified. By Theorem 4.4, and since we are assuming \mathcal{X} satisfies (\star) there exists a finite unramified extension $K \subset K'$ so that $\mathcal{X}' := \mathcal{X}_{K'}$ has good reduction, and hence extends to a proper smooth \mathfrak{X}' over $\mathcal{O}_{K'}$.

By Theorem C, this model \mathfrak{X}' is the canonical lift of its reduction, and hence by Proposition 1.8 the map $\text{Pic } \mathfrak{X}'_{\bar{K}} \rightarrow \text{Pic } \mathfrak{X}'_{\bar{\mathbf{F}}_q}$ is surjective. We conclude with Proposition 4.5 that X/K has a canonical smooth projective model $\mathfrak{X}/\mathcal{O}_K$. Again Theorem C guarantees that \mathfrak{X} is the canonical lift of its reduction \mathfrak{X}_k , and we see that the functor of Theorem B maps \mathfrak{X}_k to the given triple (M, F, \mathcal{K}) . \square

4.4. Unconditional results. As above, we fix an embedding $\iota: W(\mathbf{F}_q) \rightarrow \mathbf{C}$.

Theorem 4.6. *The functor $X \mapsto (M, F, \mathcal{K})$ restricts to an equivalence between the sub-groupoids consisting of:*

- (i) X/\mathbf{F}_q for which there is an ample $\mathcal{L} \in \text{Pic } X_{\bar{\mathbf{F}}_q}$ with $\mathcal{L}^2 < p - 4$,
- (ii) (M, F, \mathcal{K}) for which there exists a $\lambda \in M \cap \mathcal{K}$ satisfying $\lambda^2 < p - 4$.

Assuming $p \geq 5$ it also restricts to an equivalence between

- (i) X/\mathbf{F}_q for which $\text{Pic } X_{\bar{\mathbf{F}}_q}$ contains a hyperbolic lattice,
- (ii) (M, F, \mathcal{K}) for which $\text{NS}(M, F)$ contains a hyperbolic lattice,

and between

- (i) X/\mathbf{F}_q with $\text{rk Pic } X_{\bar{\mathbf{F}}_q} \geq 12$,
- (ii) (M, F, \mathcal{K}) with $\text{rk NS}(M, F) \geq 12$.

Proof. In view of Theorem B and Proposition 3.5, we only need to verify that any triple (M, F, \mathcal{K}) as in (ii) lies in the essential image of the functor $X \mapsto (M, F, \mathcal{K})$ on ordinary K3 surfaces. It suffices to show that the relevant X over $K = \text{Frac } W(\mathbf{F}_q)$ occurring in the proof in § 4.3 satisfy (\star) .

By [13, Thm. 1.1] and [10, § 2] we know that any K3 surface over X with unramified $H_{\text{ét}}^2(X_{\bar{K}}, \mathbf{Q}_\ell)$, and satisfying one of

- (i) there is an ample $\mathcal{L} \in \text{Pic } X_{\bar{K}}$ with $\mathcal{L}^2 < p - 4$,
- (ii) $\text{Pic } X_{\bar{K}}$ contains a hyperbolic plane and $p \geq 5$,
- (iii) $\text{Pic } X_{\bar{K}}$ has rank ≥ 12 and $p \geq 5$,

has potentially good reduction. In particular, any such K3 surface satisfies hypothesis (\star) . In all three cases the argument of § 4.3 goes through unconditionally. \square

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