A Quantum Multiparty Packing Lemma and the Relay Channel

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A Quantum Multiparty Packing Lemma and the Relay Channel

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Abstract—Optimally encoding classical information in a quantum system is one of the oldest and most fundamental challenges of quantum information theory. Holevo’s bound places a hard upper limit on such encodings, while the Holevo-Schumacher-Westmoreland (HSW) theorem addresses the question of how many classical messages can be “packed” into a given quantum system. In this article, we use Sen’s recent quantum joint typicality results to prove a one-shot multiparty quantum packing lemma generalizing the HSW theorem. The lemma is designed to be easily applicable in many network communication scenarios. As an illustration, we use it to straightforwardly obtain quantum generalizations of well-known classical coding schemes for the relay channel: multihop, coherent multihop, decode-forward, and partial decode-forward. We provide both finite blocklength and asymptotic results, the latter matching existing classical formulas.

Given the key role of the classical packing lemma in network asymptotic results, the latter matching existing classical formulas. For the case when \( U = \emptyset \) and when \( Y^n \sim p_Y^{(n)} \), the quantum generalization of the packing lemma is known: the Holevo-Schumacher-Westmoreland (HSW) theorem [5], [6]. This can be proven using a conditional typicality lemma for a classical-quantum state with one classical and one quantum system. However, until recently no such typicality lemma was known for the case of multiple encoding systems, and so a quantum version of Lemma 1 was lacking. Furthermore, while in classical Shannon theory Lemma 1 can be used repeatedly in scenarios where the message is encoded into multiple random variables, this approach fails in the quantum case due to measurement disturbance, specifically the influence of one decoding on subsequent decodings. Hence, while it is sufficient to solve the full multiparty packing problem in the classical case with just two encoding systems and repeated measurements, a general multiparty packing lemma with \( k \in \mathbb{N} \) encoding systems is required in the quantum case. The bottleneck is again the lack of a general quantum joint typicality lemma with multiple systems. However, we can obtain partial results in the quantum case for some network scenarios, as we will describe below.

In this paper we use the quantum joint typicality lemma, established recently by Sen [7] to prove a quantum one-shot multiparty packing lemma for \( k \) classical encoding systems. We then demonstrate the wide applicability of the lemma by using it to generalize classical network information theory protocols to the quantum case. The lemma allows us to construct and prove the correctness of these simple generalizations and, we believe, should help to open the field of classical network information theory to direct quantum generalization.

One feature of the lemma is that it leads naturally to demonstrations of the achievability of rate regions without having to resort to time-sharing, a desirable property known as simultaneous decoding. Simultaneous decoding is

\[ R < I(X; Y | U) - \delta(\varepsilon) \]

if \( R < I(X; Y | U) - \delta(\varepsilon) \), where \( \mathcal{T}_0^{(n)} \) is the set of \( \varepsilon \)-typical strings of length \( n \) with respect to \( p_{UXY} \).

The packing lemma provides a unified approach to many, if not most, of the achievability results in Shannon theory. Despite its broad utility, it is a simple consequence of the union bound and the standard joint typicality lemma with the three variables \( U, X, Y \). The usual channel coding theorem directly follows from taking \( U = \emptyset \) and when \( Y^n \sim p_Y^{(n)} \).

Note that a simultaneous smoothing result for the max-relative entropy is still missing, which would be necessary e.g. for a “multiparty covering lemma.” To prove such a result is a major open problem in the field.

1See, e.g., [4]. Our formulation is slightly paraphrased and uses a notation that is more suitable for the following.

2Sen modestly calls his result a lemma, but the highly ingenious proof more justifies calling it a theorem.

3Note that a simultaneous smoothing result for the max-relative entropy is still missing, which would be necessary e.g. for a “multiparty covering lemma.” To prove such a result is a major open problem in the field.
often necessary in network information theory to obtain one-
shot rates for the full achievable rate region. This region is
often a convex closure of the union of different regions, where
convex combinations of rates are usually achieved through time-
sharing. This is not possible in a one-shot setting. Furthermore,
different receivers could have different effective rate regions and
therefore require incompatible time-sharing strategies. Indeed,
this is a frequent source of incomplete or incorrect results
even in classical information theory [8]. A general construction
leading to simultaneous decoding in the quantum setting has
therefore been sought for many years [8], [9], [10], [11], [12],
[13], [14]. Sen’s quantum joint typicality lemma achieves this
goal, as does our packing lemma, which can be viewed as a
user-friendly interface for Sen’s lemma.

Recall that network information theory is the study of
communication with multiple parties and is a generalization
of the conventional single-sender single-receiver two-party
scenario, commonly known as point-to-point communication.
Common network scenarios include having multiple senders
encoding different messages, as in the case of the multiple
access channel [15], multiple receivers decoding the messages,
as in the broadcast channel [16], or a combination of both, as
for the interference channel [17]. However, the above examples
are all instances of what is called single hop communication,
where the message directly travels from a sender to a receiver.
In multihop communication, there is one or even multiple
intermediate nodes where the message is decoded or partially
decoded before being transmitted to the final receiver. Examples
of such communication scenarios include the relay channel [18],
which we focus on in this paper, and more generally, graphical
multi-cast networks [19], [20].

Research in quantum joint typicality has generally been
driven by the need to establish quantum generalizations of
results in classical network information theory. Examples
include the quantum multiple access channel [11], [21], the
quantum broadcast channel [22], [23], and the quantum
interference channel [12]. Indeed, some partial results on joint
typicality had been established or conjectured in order to
prove achievability bounds for various network information
processing tasks [10], [24], [25]. Subsequent work made some
headway on the abstract problem of joint typicality for quantum
states, but not enough to affect coding theorems [26], [27] prior
to Sen’s breakthrough [7].

The quantum relay channel was studied previously in [28],
where the authors constructed a partial decode-forward protocol.
Here we develop finite blocklength results for the relay channel
in addition to reproducing the earlier conclusions and avoiding
a resolvable issue with error accumulation from successive
measurements in their partial decode-forward bound. (We
construct a joint decoder which obtains all the messages
from the multiple rounds of communication simultaneously.)
Our analysis makes extensive use of our quantum multiparty
packing lemma. Once the coding strategy is specified, a direct
application of the packing lemma in the asymptotic limit gives
a list of inequalities which describe the rate region, which
we then simplify using entropy inequalities to the usual rate
region of the partial decode-forward lower bound. There has
also been related work in [29], which considered concatenated
channels, a special case of the more general relay channel
model. As noted in [28], work on quantum relay channels may
have applications to designing quantum repeaters [30]. Note
that Sen has already used his joint typicality lemma to prove
achievability results for the quantum multiple access, broadcast,
and interference channels [7], [31], but here we give a general
packing lemma which can be used as a black box for quantum
network information applications. The relay channel serves as
a demonstration of this.

Our paper is structured as follows. In Section II, we
establish notation and discuss some preliminaries. In Section III,
we describe the setting and state the quantum multiparty
packing lemma. The statement very much resembles a one-
shot, multiparty generalization of Lemma 1, but, to reiterate,
while the multiparty generalization is trivial in the classical
case, it requires the power of a full joint typicality lemma
in the quantum case. In Section IV we describe the classical-
quantum (c-q) relay channel and systematically describe coding
schemes that generalize known schemes for the classical
relay channel: multihop, coherent multihop, decode-forward,
and partial decode-forward [32]. In addition to the one-shot
bounds, we show that the asymptotic bounds are obtained
by taking the limit of large blocklength, thereby obtaining
quantum generalizations of known capacity lower bounds
for the classical case. In Section V we prove the quantum
multiparty packing lemma via Sen’s quantum joint typicality
lemma [7]. For convenience, we restate a special case of the
Sen’s joint typicality lemma and suppress some of the details.
In Section VI we give a conclusion.

II. Preliminaries

We first establish some notation and recall some basic results.

Classical and quantum systems: A classical system $X$ is
identified with an alphabet $\mathcal{X}$ and a Hilbert space of dimension $|\mathcal{X}|$, while a quantum system $B$ is given by a Hilbert space of dimension $d_B$. Classical states are modeled by diagonal density operators such as $\rho_X = \sum_{x \in \mathcal{X}} p_X(x) |x\rangle\langle x|_X$, where $p_X$ is a probability distributions, quantum states are described by density operator $\rho_A$ etc, and classical-quantum states are described by density operators of the form

$$\rho_{XB} = \sum_{x \in \mathcal{X}} p_X(x) |x\rangle\langle x|_X \otimes \rho_B(x).$$

Probability bound: Denote by $E_1$, $E_2$ two events. We use the following inequality repeatedly in the paper:

$$\Pr(E_1) = \Pr(E_1|E_2) \Pr(E_2) + \Pr(E_1|\overline{E_2}) \Pr(\overline{E_2}) \leq \Pr(E_2) + \Pr(E_1|\overline{E_2}),$$

where we use $\overline{E_2}$ to denote the complement of $E_2$ and used the fact that $\Pr(E_2), \Pr(E_1|\overline{E_2}) \leq 1$.

Hypothesis-testing relative entropy: The hypothesis-testing relative entropy [33] is defined as

$$D^H_{\epsilon}(\rho||\sigma) = \max_{0\leq \Pi \leq I} \frac{-\log \text{tr}(\Pi \sigma)}{\text{tr}(\Pi \rho) \geq 1-\epsilon}.$$  

$^4$As always in information theory, log here is base 2.
For $n$ copies of states $\rho$ and $\sigma$, [34], [35], [36] establishes the following inequalities:

$$D(\rho\|\sigma) - \frac{F_1(\epsilon)}{\sqrt{n}} \leq \frac{1}{n} D_H^c(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq D(\rho\|\sigma) + \frac{F_2(\epsilon)}{\sqrt{n}},$$

(3)

where $F_1(\epsilon), F_2(\epsilon) \geq 0$ are given by $F_1(\epsilon) \equiv 4\sqrt{2} \log \frac{1}{\epsilon} - \log \eta$, $F_2(\epsilon) \equiv 4\sqrt{2} \log \frac{1}{\epsilon n} - \log \eta$, with $\eta \equiv 1 + tr \rho^{3/2}\sigma^{-1/2} + tr \rho^{1/2}\sigma^{1/2}$ and $D(\rho\|\sigma) = tr(\rho \log \rho) - tr(\rho \log \sigma)$ being the quantum relative entropy. In the limit of large $n$, we obtain the quantum Stein’s lemma [37], [38]:

$$\lim_{n \to \infty} \frac{1}{n} D_H^c(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma).$$

(4)

**Conditional density operators:** Let a classical system $X$ consist of subsystems $X_v$, for $v$ in some index set $V$, with alphabet $X = \bigotimes_{v \in V} X_v$, where $\otimes$ denotes the Cartesian product of sets. Consider a classical-quantum state $\rho_{XB}$ as in Eq. (1) and a subset $S \subseteq V$. We can write

$$\rho_{XB} = \sum_{x_S} p_{X_S|x}(x_S|x_S)\langle x_S|x_S \rangle_X \otimes \rho_{X_SB},$$

(5)

where $S \equiv V \setminus S$ and

$$\rho_{X_SB}^{(x_S)} = \sum_{x_S} p_{X_S|x}(x_S|x_S)\langle x_S|x_S \rangle_S \otimes \rho_{X_SB}.$$
random coding scheme via an algorithm we give below which takes the multiplex Bayesian network as input and generates random codewords $x(m)$ with components $x_v(m)$ for $v \in V$. However, different components of a codeword may only depend on particular message sets, as in the case of the MAC where we only generate $x_1(m_1)$ for all $m_1 \in M_1$. We model this situation by an index set $J$ which index the different message sets $M_j$ for $j \in J$, and a function $\text{ind}: V \rightarrow \mathcal{P}(J)$, where $\text{ind}(v) \subseteq J$ corresponds to the subset of indices (hence the name) that index the message sets the codeword component $X_v$ depends on. Now, for our random codebook construction to be well-defined, we require that given $v \in V$,
\[
\text{ind}(v') \subseteq \text{ind}(v) \text{ for every } v' \in \text{pa}(v),
\]
where for $v \in V$, \[
\text{pa}(v) = \{ v' \in V \mid (v', v) \in E \}
\]
denote the set of parents of $v$. This is a natural requirement: in our algorithm we generate the codewords in an iterative manner following the edges in the DAG, and so we should require that all codeword components also depend on the message sets that the components upstream depend on. That is, codeword components “inherit” the indices of their parents.

We call the tuple $B = (G, X, M, \text{ind})$, where $M \equiv \times_{j \in J} M_j$, a multiplex Bayesian network. We can visualize a multiplex Bayesian network by adjoining to the DAG $G$ additional vertices $M_j$, one for each $j \in J$, and edges that connect each $X_v$ to every $M_j$ such that $j \in \text{ind}(v)$. Again, as an example consider the MAC multiplex Bayesian network with three random variables in Fig. 1. Note that in the figure the random variable $U$ is not connected with any message set. We define our algorithm to treat $U$ as if it is connected with a singleton.

![Figure 1](Image)

Figure 1. An example of a multiplex Bayesian network with vertices $U$, $X_1$, and $X_2$ and message sets $M_1$ and $M_2$. This network can be used to generate a random code for the two-sender c-q MAC, where we generate $u$ according to $p_U$, then $x_1(m_1)$ for $m_1 \in M_1$ according to $p_{X_1|U}(\cdot|u)$ and $x_2(m_2)$ according to $p_{X_2|U}(\cdot|u)$.

We next give the algorithm that generates the random codebook. Given a multiplex Bayesian network $B = (G, X, M, \text{ind})$, we would like to generate a random codebook
\[
\{x_v(m)\}_{v \in V, m \in M},
\]
where $x_v$ is a random variable with alphabet $X_v$. The vertices represent random codeword components and the graph $G$ describes the dependencies between different components. Moreover, each component $x_v(m)$ only depends on $m_j \in M_j$ for which $j \in \text{ind}(v)$. That is, $x_v(m)$ and $x_v(m')$ are equal as random variables provided $m_j = m'_j$ for every $j \in \text{ind}(v)$.

We now give the algorithm for generating the random codebook. Since $G$ is a DAG, it has a topological ordering, that is, a total ordering on $V$ such that for every $(v', v) \in E$, $v'$ precedes $v$ in the ordering. We also pick an arbitrary total ordering on $J$ and on $M_j$ for every $j \in J$. This then induces a lexicographical ordering on Cartesian products of $M_j$, which we denote by $M_j := \times_{j \in J} M_j$ for any $J' \subseteq J$.

We define $M_\emptyset = \{ \emptyset \}$ as a singleton so that we can identify $M_j \times M_{j'} = M_{j \cup j'}$ for any two disjoint subsets $J', J'' \subseteq J$. Note that these total orderings determine the order in which we perform the for loops below, but do not impact the joint distribution of the codewords. We can now define the following algorithm:

**Algorithm 1:** Codebook generation from multiplex Bayesian network

for $v \in V$ do

for $m_v \in M_{\text{ind}(v)}$ do

generate $x_v(m_v)$ according to $p_{X_v|\text{pa}(v)}(\cdot|x_{\text{pa}(v)}(m_{\text{pa}(v)})))$

for $m_v \in M_{\text{ind}(v)}$ do

$x_v(m_v, m_v) = x_v(m_v)$

end for

end for

end for

Here, $\text{ind}(v) \equiv J \setminus \text{ind}(v)$, $m_{\text{pa}(v)}$ is the restriction of $m_v$ to $M_{\text{ind}(\text{pa}(v))}$ (this makes sense by Eq. (7)), $X_{\text{pa}(v)} \equiv (X_{v'})_{v' \in \text{pa}(v)}$ and similarly for $x_{\text{pa}(v)}(m_{\text{pa}(v)})$, and the pair $(m_v, m_v)$ is interpreted as an element of $M$ with the appropriate components. The topological ordering on $V$ ensures that $x_{\text{pa}(v)}(m_{\text{pa}(v)})$ is generated before $x_v(m_v)$, so this algorithm can be run. We thus obtain a random codebook as in Eq. (8).

We make a few observations.

1) By construction, for all $m \in M$ and $\xi \in \mathcal{X}$,

\[
\Pr(x(m) = \xi) = \Pr(x(\xi) = \prod_{v \in V} p_{X_v|\text{pa}(v)}(\xi_v|x_{\text{pa}(v)})),
\]

That is, $x(m)$ is a Bayesian network with respect to $G$ and equal in distribution to $X$.

2) By construction, given $v \in V$ and $m_v \in M_{\text{ind}(v)}$, all $x_v(m_v, m_v)$ for $m_v \in M_{\text{ind}(v)}$ are equal as random variables.

3) Generalizing observation 1, the joint distribution of all codewords can be split into factors in a simple manner. Specifically, given $\xi(m) \in \mathcal{X}$ for every $m \in M$, we have

\[
\Pr(x(m) = \xi(m) \text{ for all } m \in M) = \prod_{v \in V} \prod_{m_v \in M_{\text{ind}(v)}} p_{X_v|\text{pa}(v)}(\xi_v(m_v)|x_{\text{pa}(v)}(m_{\text{pa}(v)}))
\]

provided $\xi_v(m) = \xi_v(m')$ for all $m, m'$ with $m_v = m_v'$. Otherwise, the joint probability is zero.
To allow for the variety of coding schemes encountered in network information theory, we introduce a few additional elements. For instance, we would like the freedom to construct multiple different quantum decoders for the same random codebook. This is a very natural requirement when there are multiple receivers involved or when a particular receiver has to make multiple measurements in an interactive communication scenario (and therefore cannot simply make a single joint measurement). To realize this, let $H$ be the induced subgraph of $G$ for some $V_H \subseteq V$ where for all $v \in V_H$, $pa(v) \subseteq V_H$. We call $H$ an ancestral subgraph. Then, we can naturally define $X_H$ to be the set of random variables corresponding to $V_H$, $J_H = \bigcup_{v \in V_H} \text{ind}(v) \subseteq J$, $M_H = \times_{j \in J_H} M_j$, and $C_H = \{x_H(m_H)\}_{m_H \in M_H}$.

Finally, the classical variables are encoded into a quantum system via a family of quantum states $\{\rho_B^{(x_H)}\}_{x_H \in X_H}$, where $B$ is some quantum system.

The next element we introduce allows for receivers to decode a number of message sets using a guess for the other message sets. This is naturally motivated by iterative decoding schemes in which a receiver makes multiple measurements where latter measurements take into account results from previous measurements. Again, this is mainly relevant in interactive scenarios. To realize this, let $D \subseteq J_H$ be a subset of indices which index the message sets to be decoded. This means we have a guess for the remaining message sets indexed by $D = J_H \setminus D$.

We can now state our quantum multiparty packing lemma:

**Lemma 2** (One-shot quantum multiparty packing lemma). Let $B = (G, X, M, \text{ind})$ be a multiplex Bayesian network and run Algorithm 1 to obtain a random codebook $C = \{x(m)\}_{m \in M}$. Let $H \subseteq G$ be an ancestral subgraph. $\{\rho_B^{(x_H)}\}_{x_H \in X_H}$ a family of quantum states, $D \subseteq J_H$, and $\varepsilon \in (0, 1)$. Then there exists a POVM $\{Q_B^{(m_D, m_{\overline{D}})}\}_{m_D \in M_D}$ for each $m_D \in M_D$ such that, for all $(m_D, m_{\overline{D}}) \in M_H$,

$$\mathbb{E}_{C_H} \left[ \text{tr} \left[ (I - Q_B^{(m_D, m_{\overline{D}})}(x_H(m_D, m_{\overline{D}}))) \rho_B^{(x_H)} \right] \right] \leq f(|V_H|, \varepsilon) + 4 \sum_{\emptyset \neq T \subseteq D} 2^{\left( \sum_{\{j \in T\}} r_j \right) - D_H(\rho_{x_H B}(x_H x_{m_{\overline{D}}}))} 2^{-D_H(\rho_{x_H B}(x_H x_{m_{\overline{D}}}))}, \quad (9)$$

Here, $\mathbb{E}_{C_H}$ denotes the expectation over the random codebook $C_H = \{x_H(m_H)\}_{m_H \in M_H}$, $R_t \equiv \log |M_t|$, $S_T = \{v \in V_H \mid \text{ind}(v) \cap T = \emptyset\}$, and

$$\rho_{x_H B} \equiv \sum_{x_H \in X_H} p_{x_H}(x_H) |x_H\rangle \langle x_H|_{X_H} \otimes \rho_B^{(x_H)}.$$  

Furthermore, $f(k, \varepsilon)$ is a universal function (independent of our setup) that tends to zero as $\varepsilon \to 0$.

Remark. The bound in Eq. (9) can also be written as

$$\mathbb{E}_{C_H} \left[ \text{tr} \left[ (I - Q_B^{(m_D, m_{\overline{D}})}(x_H(m_D, m_{\overline{D}}))) \rho_B^{(x_H)} \right] \right] \leq f(|V_H|, \varepsilon) + 4 \sum_{m_D' \neq m_D} 2^{-D_H(\rho_{x_H B}(x_H x_{m_{\overline{D}}}))}, \quad (10)$$

where

$$S = \{v \in V_H \mid \exists j \in D \cap \text{ind}(v) \text{ such that } (m_D)_j \neq (m_D')_j\}.$$

In words, $S$ is the set of random codewords that depend on a part of the message that differs between $m_D$ and $m_D'$. This is similar to decoding error bounds obtained with conventional methods, such as the Hayashi-Nagaoka lemma [40]. We obtain Eq. (9) from Eq. (10) by parametrizing the different $m_D'$ with respect to the indices that differ from $m_D$.

Remark. Note Eq. (9) assumes that the decoder’s guess of $m_{\overline{D}}$ is correct. That is, they choose the POVM $\{Q_B^{(m_D, m_{\overline{D}})}\}_{m_D \in M_D}$, where $m_{\overline{D}}$ is exactly the $m_{\overline{D}}$ in the encoded state $\rho_B^{(x_H(m_D, m_{\overline{D}}))}$. If the decoder’s guess is incorrect, then this bound does hold in general. In applications, $m_{\overline{D}}$ typically corresponds to message estimates of previous rounds, which we will assume to be correct by invoking a classical union bound.

That is, we bound the total probability of error by summing the probabilities of error of a decoding assuming that all previous decodings were correct. Note that the decodings must be performed on disjoint quantum systems for this argument to hold.

The following is the explicit form of $f(k, \varepsilon)$ for $k \in \mathbb{N}$ from [7] and our proof of the packing lemma in Section V:

$$f(k, \varepsilon) = (1 + 6 \times 2^{k+1} + 4 \times 2^{3k+5+k^2+2k}) \varepsilon^{1/3}. \quad (11)$$

For simplicity, we can make some coarse approximations to obtain an upper bound:

$$f(k, \varepsilon) \leq 2^{27k} \varepsilon^{1/3}. \quad (12)$$

Using Lemma 2 and Eq. (6), we can naturally obtain the asymptotic version where we simply take $n \in \mathbb{N}$ copies of the codebook and take the limit of large $n$. By the quantum Stein’s lemma Eq. (4), the error in Eq. (9) will vanish if the rates of encoding are bounded by conditional mutual information quantities. We present this as a self-contained statement.

**Lemma 3** (Asymptotic quantum multiparty packing lemma). Let $B = (G, X, M, \text{ind})$ be a multiplex Bayesian network. Run Algorithm 1 $n$ times to obtain a random codebook $C^n = \{x^n(m)\}_{m \in M}$. Let $H \subseteq G$ be an ancestral subgraph, $\{p_B^{(x_H)}\}_{x_H \in X_H}$ a family of quantum states, and $D \subseteq J_H$. Then there exists a POVM $\{Q_B^n(m_D, m_{\overline{D}})\}_{m_D \in M_D}$ for each $m_D \in M_D$ such that, for all $(m_D, m_{\overline{D}}) \in M_H$,

$$\lim_{n \to \infty} \mathbb{E}_{C^n_H} \left[ \text{tr} \left[ (I - Q_B^n(m_D, m_{\overline{D}})) \otimes \rho_B^{(x_H(m_D, m_{\overline{D}}))} \right] \right] = 0,$$
provided that\footnote{Note that $R_t$ is defined differently here. This is due to the difference in the definition of “rate” for one-shot and asymptotic settings.}

$$\sum_{t \in T} R_t < I(X_{ST};B|X_{ST}^r)_{p} - \delta(n) \quad \text{for all } \emptyset \neq T \subseteq D.$$  

Above, $\mathbb{E}_{C^n}$ is the expectation over the random codebook $C^n_H = \{x^n_H(m_H)\}_{m_H \in M_H}$, $R_t \equiv \frac{1}{n} \log |M_t|$, $\delta(n)$ is some function that tends to 0 as $n \rightarrow \infty$.

$$S_T \equiv \{v \in V_H \mid \text{ind}(v) \cap T \neq \emptyset\}, \quad \overline{S_T} \equiv V_H \setminus S_T,$$

and

$$\rho_{X_{ST}} = \sum_{x_H \in V_H} p_{X_{H|H}}(x_H) |x_H\rangle\langle x_H| \otimes \rho_B^{(x_H)}.$$  

\textbf{Example.} To clarify the definitions and illustrate the applications of Lemma 2 and Lemma 3, we use them to code over the two-sender c-q MAC. Consider the multiplex Bayesian network given in Fig. 1. We apply Algorithm 1 to obtain a random codebook $\{u, x_1(m_1), x_2(m_2)\}_{m_1 \in M_1, m_2 \in M_2}$. We then simply let each sender transmit their message via the corresponding codeword. Now, choosing $H = G$ and $D = J = \{1, 2\}$, by Lemma 3 we obtain a POVM $\{Q_B^{(m_1, m_2)}\}_{m_1 \in M_1, m_2 \in M_2}$. The mapping from $T \subseteq D$ to $S_T \subseteq V = \{U, X_1, X_2\}$ is given in Table I.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$T$ & $S_T$ \\
\hline
\{1\} & $\{X_1\}$ \\
\{2\} & $\{X_2\}$ \\
\{1, 2\} & $\{X_1, X_2\}$ \\
\hline
\end{tabular}
\caption{$S_T \subseteq V$ for various $\emptyset \neq T \subseteq D$.}
\end{table}

Thus, letting the receiver use this POVM achieves the rate region

$$R_1 < I(X_1;B|X_2U)_{p},$$

$$R_2 < I(X_2;B|X_1U)_{p},$$

$$R_1 + R_2 < I(X_1X_2;B|U)_{p},$$

where

$$\rho_{U_{X_{1}X_{2}B}} = \sum_{u, x_1, x_2} p_U(u)p_{X_1|U}(x_1|u)p_{X_2|U}(x_2|u)$$

$$|u, x_1, x_2\rangle\langle u, x_1, x_2|U_{X_{1}X_{2}} \otimes \rho_B^{(x_1, x_2)}$$

and $\rho_B^{(x_1, x_2)}$ is the output of the MAC with input $(x_1, x_2)$. Hence, with our quantum multiparty packing lemma we readily achieve the capacity found in [11].

We can also get one-shot results for the MAC. Let $R_1, R_2, \varepsilon \in \mathbb{R}_{\geq 0}$ such that

$$R_1 \leq D_H^c(\rho_{U_{X_{1}X_{2}B}}||\rho_B^{(X_1, X_2)}) - 2 - \log \frac{1}{\varepsilon}$$

$$R_2 \leq D_H^c(\rho_{U_{X_{1}X_{2}B}}||\rho_B^{(X_1, X_2)}) - 2 - \log \frac{1}{\varepsilon}$$

Then, applying Lemma 2, the probability of error in decoding is at most

$$P_e \leq f(\varepsilon) + 4 \left(2^{-2-\log \frac{1}{\varepsilon}}\right) \times 3$$

$$\leq 2^{2^2 \varepsilon} \varepsilon^{1/3} + 3 \varepsilon \leq (2^{2^2} + 3)\varepsilon^{1/3},$$

where we used the coarse approximation in Eq. (12) and that $\varepsilon \in (0, 1)$. Using that $X_1$ and $X_2$ are independent conditional on $U$, we obtain up to constants Theorem 2 of [7].

We expect that Lemma 2 and Lemma 3 can be used in a variety of scenarios to directly generalize results from classical network information theory, which often hinge on Lemma 1, to the quantum case. In fact, it is not too difficult to see that an i.i.d. variant\footnote{This is because we assume i.i.d. codewords in Lemma 3, which is sufficient for, e.g., relay, multiple access [7], and broadcast channels [31].} of Lemma 1 can be derived from Lemma 3. More precisely, let $(U, X, Y) \sim p_{UXY}$ be a triple of random variables as in the former. Consider a DAG $G$ consisting of two vertices, corresponding to random variables $U$ and $X$ with joint distribution $p_{UX}$, and an edge going from the former to the latter. We set $J = \{\emptyset\}$, $\text{ind}(X) = \{\emptyset\}$, and $M_1 = M$ as the message set. A visualization of this simple multiplex Bayesian network $(G, (U, X), M, \text{ind})$ is given in Fig. 2.

![Figure 2. The multiplex Bayesian network](image)

By running Algorithm 1 $n$ times, we obtain codewords which we can identify as $\tilde{U}^n$ and $\tilde{X}^n(m)$. Conditioned on $\tilde{U}^n$, it is clear that for each $m \in M$, $\tilde{X}^n(m) \sim \otimes_{i=1}^n p_{X|U=\tilde{U}_i}$. Next, choose the subgraph to be all of $G$, set of quantum states to be the classical states

$$\left\{\rho_{X_{U}^{(u,x)}} \equiv \sum_{y \in Y} p_{Y|U}(y|u, x) |y\rangle\langle y| \right\}_{u \in U, x \in X},$$

and decoding subset $D = \{\emptyset\}$, corresponding to $M$. We see that if we consider the entire system consisting of $\tilde{U}^n$, $\tilde{X}^n(m)$ and $\otimes_{i=1}^n \rho_{Y_i}^{(U_i, X_i(m_i))}$ for $m' \neq m$, it is clear that $\tilde{X}^n(m)$ is conditionally independent of $\tilde{Y}^n$ given $\tilde{U}^n$ due to the conditional independence of $\tilde{X}^n(m)$ and $\tilde{X}^n(m')$ given $\tilde{U}^n$. By Lemma 3, we obtain a POVM $\{Q_{Y_i}^{(m_i)}\}_{m_i \in M}$ such that, for all $m \in M$,

$$\lim_{n \rightarrow \infty} \mathbb{E}_{C^n} \left[ \text{tr} \left( (I - Q_{Y_i}^{(m_i)}) \otimes_{i=1}^n \rho_{Y_i}^{(U_i, X_i(m_i))} \right) \right] = 0.$$
provided $R < I(X;Y|U) - \delta(n)$, which is analogous to Lemma 1 if we “identify” the POVM measurement with the typicality test.

In Section V we prove Lemma 2 using Sen’s quantum joint typicality lemma with $|V|$ classical systems and a single quantum system. We then prove Lemma 3. In the proof of our packing lemma, we actually prove a more general, albeit more abstract, statement.

IV. APPLICATION TO THE CLASSICAL-QUANTUM RELAY CHANNEL

To illustrate the wide applicability of Lemma 2 and demonstrate how to use it, we prove a series of achievability results for the classical-quantum relay channel. The first three results make use of the packing lemma in situations where the number of random variables involved in the decoding is at most two ($|V_H| \leq 2$). This situation can be dealt with using existing techniques [28]. The final partial decode-forward lower bound, however, applies the packing lemma with $|V_H|$ unbounded with increasing blocklength, thus requiring its full strength. These lower bounds are well-known for classical relay channels [4], and our packing lemma allows us to straightforwardly generalize them to the quantum and even finite blocklength case. We can then invoke Lemma 3 to obtain lower bounds on the capacity, which match exactly those of the classical setting with the quantum generalization of mutual information. Note that the partial decode-forward asymptotic bound for the classical-quantum relay channel was first established in [28].

First we give some definitions. A classical-quantum relay channel [28], [29] is a classical-quantum channel $N_{X_1, X_2 \rightarrow B_2 B_3}$, $\rho_{B_2 B_3}$, is a family of quantum states which defines the classical-quantum relay channel.

Now, before looking at specific coding schemes, we first give a general upper bound, a direct generalization of the cutset bound for the classical relay channel:

1) A message set $M$ with cardinality $2^nR$.
2) An encoding $x_i^2(m) \in X_i^n$ for each $m \in M$.
3) A relay encoding and decoding $R_{j-1}(B_{2j-1}^{j-1} \rightarrow (X_{2j}(B_{2j})_j)$ for $j \in [n]$. Here, $(B_{2j})_j$ is isomorphic to $B_2$ and $(X_{2j})_j$ isomorphic to $X_2$ while $(B_{2j})_j$ is some arbitrary quantum system. The relay starts with some trivial (dimension 1) quantum system $(B_2)_0(B_{2j})_0$.
4) A receiver decoding POVM $\{Q_{B_3}^{(m)}\}_{m \in M}$.

On round $j$, the sender transmits $(x_1)_j(m)$ while the relay applies $R_{j}(B_{2j-1}^{j-1} \rightarrow (X_{2j}(B_{2j})_j)$ to their $(B_{2j})_j(B_{2j})_{j-1}$ system and transmits the $(X_{2j})_j$ state while keeping the $(B_{2j})_j$ system. After the completion of $n$ rounds, the receiver applies the decoding POVM $\{Q_{B_3}^{(m)}\}_{m \in M}$ on their received systems $\rho_{B_3}^{(m)}$ to obtain their estimate for the message. See Fig. 4 for a visualization of a protocol with $n = 3$ rounds. The average probability of error of a general protocol is given by

$$p_e = \frac{1}{|M|} \sum_{m \in M} tr \left( (I - Q_{B_3}^{(m)}) \rho_{B_3}^{(m)} \right).$$

In the protocols we give below, we use random codebooks. We can derandomize in the usual way to conform to the above definition of a code. Furthermore, in our protocols the relay only leaves behind a classical system during intermediate stages of the protocol. Since our relay channels are classical-quantum, it is not clear if higher rates can be achieved by letting the relay leave behind a quantum system after every round. We leave the possibility of leaving behind a quantum system in our definition to allow for the most general protocols.

Given $R \in \mathbb{R}_{\geq 0}$, $n \in \mathbb{N}$, $\delta \in [0, 1]$, we say that a triple $(R, n, \delta)$ is achievable for a relay channel if there exists a $(n, 2^nR, \delta)$ code such that

$$R' \geq R \quad \text{and} \quad p_e \leq \delta.$$
Proposition 4 (Cutset Bound). Given a classical-quantum relay channel $N_{X_1 X_2 \rightarrow B_2 B_3}$, its capacity is bounded from above by
\[
C(N_{X_1 X_2 \rightarrow B_2 B_3}) \leq \max_{p_{X_1 X_2}} \min \{ I(X_1; B_3), I(X_1; B_2 B_3 | X_2) \}.
\]

Proof. See Section A.

For some special relay channels, Proposition 4 along with some of the lower bounds proven below will be sufficient to determine the capacity.

A. Multihop Scheme

The multihop lower bound is obtained by a simple two-step process where the sender transmits the message to the relay and the relay then transmits it to the receiver. That is, the relay simply “relays” the message. The protocol we give below is exactly analogous to the classical case [4], right down to the structure of the codebook. In other words, with our packing lemma, the classical protocol can be directly generalized to the quantum case. The only difference is that the channel outputs a quantum state and the decoding uses a POVM measurement.

Consider a relay channel $N_{X_1 X_2 \rightarrow B_2 B_3} : X_1 \times X_2 \rightarrow H_{B_2} \otimes H_{B_3}$, $(x_1, x_2) \mapsto \rho_{B_2 B_3}^{(x_1 x_2)}$.

Let $R \geq 0$, $b \in \mathbb{N}$, $\varepsilon \in (0, 1)$, where $b$ is number of blocks. Again, $R$ is the log of the size of the message set and $b$ the number of relay uses, while $\varepsilon$ is the small parameter input to Lemma 2. We show that we can achieve the triple $(\frac{R}{b}, b, \varepsilon)$ for some $\delta$ a function of $R, b, \varepsilon$. Let $p_{X_1}, p_{X_2}$ be probability distributions over $X_1, X_2$, respectively. Throughout, we use
\[
\rho_{X_1 X_2 B_2 B_3} = \sum_{x_1, x_2} p_{X_1}(x_1) p_{X_2}(x_2) |x_1 x_2\rangle \langle x_1 x_2|_{X_1 X_2} \otimes \rho_{B_2 B_3}^{(x_1 x_2)}.
\]

We also define $\rho_{B_3} \equiv \sum_{x_1} p_{X_1}(x_1) \rho_{B_3}^{(x_1 x_2)}$ to be the reduced state on $B_3$ induced by tracing out $X_1 B_2$ and fixing $X_2$.

Code: Throughout, $j \in [b]$. Let $G$ be a graph with $2b$ vertices corresponding to independent random variables $(X_1)_j \sim p_{X_1}, (X_2)_j \sim p_{X_2}$. Since all the random variables are independent, there are no edges. Furthermore, let $M_0, M_j$ be index sets, where $|M_0| = 1$ and $|M_j| = 2^j$. That is, our index set for the different message sets should be $J = [0 : b]$. The $M_j$ are the sets from which the messages for each round is taken. We use a singleton $M_0$ to make the effect of the first and the last blocks more explicit. Finally, the function ind maps $(X_1)_j$ to $\{j\}$ and $(X_2)_j$ to $\{j - 1\}$. Then, letting $X \equiv X_1^{X_2} X_3^{X_3}$ and $M \equiv \bigcup_{j=0}^{b} M_j$, $B \equiv (G, X, M, \text{ind})$ is a multiplex Bayesian network. See Fig. 5 for a visualization when $b = 3$. Now, run Algorithm 1 with $B$ as the argument. This returns a random codebook
\[
C = \bigcup_{j=1}^{b} \{(x_1)_j (m_j)_j (x_2)_j (m_j - 1)_j\}_{m_j \in M_j, m_{j-1} \in M_{j-1}},
\]
where we restricted to the message indices the codewords are dependent on via ind. For decoding we apply Lemma 2 with this codebook and use the assortment of POVMs that are given for different ancestral subgraphs and other parameters.

Encoding: On the $j$th transmission, the sender transmits a message $m_j \in M_j$ via $(x_1)_j (m_j)_j \in C$.

Relay encoding: Set $\tilde{m}_0$ to be the sole element of $M_0$. On the $j$th transmission, the relay sends their estimate $\tilde{m}_{j-1}$ via $(x_2)_j (\tilde{m}_{j-1}) \in C$. Note that this is the relay’s estimate of the message $m_{j-1}$ transmitted by the sender on the $(j - 1)$th transmission.

Relay decoding: Consider the $j$th transmission. We invoke Lemma 2 with the ancestral subgraph containing the two vertices $(X_1)_j$ and $(X_2)_j$, the set of quantum states $\{\rho_{B_3}^{(x_1 x_2)}\}_{x_1 \in X_1, x_2 \in X_2}$, decoding subset $\{j\} \subseteq \{j - 1, j\}$, and small parameter $\varepsilon \in (0, 1)$. The relay picks the POVM corresponding to the message estimate for the previous round $\tilde{m}_{j-1}$, which is denoted by $\{Q_{B_3}^{(m'_{j-1})}\}_{m'_{j-1} \in M_j}$. The relay applies this on their received state to obtain a measurement result $\tilde{m}_{j}$. Note that this is the relay’s estimate for message $m_j$.

Decoding: On the $j$th transmission, we again invoke Lemma 2 and let the receiver use the POVM corresponding to the ancestral subgraph containing just the vertex $(X_2)_j$, the set of quantum states $\{\rho_{B_3}^{(x_2)}\}_{x_2 \in X_2}$, decoding subset $\{j - 1\} \subseteq \{j - 1, j\}$, and small parameter $\varepsilon$. Note that we don’t have a message guess here since the decoding subset is not a proper subset. In this case we suppress the conditioning for conciseness. We denote the POVM by $\{Q_{B_3}^{(m'_{j-1})}\}_{m'_{j-1} \in M_{j-1}}$, and the receiver applies this on their received state to obtain a measurement result $\tilde{m}_{j-1}$. Note that this is the receiver’s estimate of the $(j - 1)$th message. $\tilde{m}_0$ is trivially be the sole element of $M_0$.

Error analysis: Set $m_0$ to be the sole element of $M_0$. Fix $\tilde{m} \equiv (m_0, \ldots, m_{b-1})$. Note that $m_b$ is never decoded by the receiver since it is the message sent in the last block and thus, we can ignore it without loss of generality. Let $\tilde{m} \equiv (m_0, \ldots, m_{b-1})$, $\tilde{m}_j \equiv (m_0, \ldots, m_{b-1})$ denote the aggregation of the message estimates of the relay and receiver.
respectively. The probability of error averaged over the random codebook $C$ is given by
\[
p_e(C) = E_C[p(\hat{m} \neq m)],
\]
where $p$ here denotes the probability for a fixed codebook. Now, by Eq. (2),
\[
p_e(C) \leq E_C[p(\hat{m} \neq m)] + E_C[p(\hat{m} \neq m | m = m)]. \quad (14)
\]
We consider the first term corresponding to the relay decoding. By the union bound,
\[
E_C[p(\hat{m} \neq m)] \leq \sum_{j=1}^{b-1} E_C[p(\hat{m}_j \neq m_j | m_{j-1} = m_{j-1})].
\]
By the definition of $\hat{m}_{n0}$, the first term is zero. Now, we can apply Eq. (9) to bound each summand in the second term as follows: \cite{11}
\[
E_C[p(\hat{m}_j \neq m_j | m_{j-1} = m_{j-1})] = E_C[\text{tr}((I - Q_j)_{B_2}^{(m_{j-1})})^{(x_1, y_j, (m_j, m_{j-1}))}] = E_{C_j}(x_j, y_j)[\text{tr}((I - Q_j)_{B_2}^{(m_{j-1})})^{(x_j, y_j, (m_j, m_{j-1}))}] \leq f(\varepsilon, \rho) + \sum_{T} 2^{-D_{H}(\rho^{(X_1, X_2)}, \rho^{(X_1, B_2)}_j)}.
\]
where $C_j(x_j, y_j)$ is the corresponding subset of the codebook $C$, and we used $S_{\varepsilon, \rho}(x_j, y_j) = \{ (X_1, X_2) \}$. We dropped the index $j$ in the last equality since $(X_1)_j (X_2)_j \sim p_{X_1} \times p_{X_2}$. Hence, overall,
\[
E_C[p(\hat{m} \neq m)] \leq b \left[ f(\varepsilon, \rho) + 4 \times 2^{-D_{H}(\rho^{X_1, X_2, B_2})} \right].
\]
We now consider the second term in Eq. (14), corresponding to the receiver decoding. By the union bound,
\[
E_C[p(\hat{m} \neq m | \hat{m} = m)] \leq E_C[p(\hat{m}_{n0} \neq m_0 | \hat{m} = m)] + \sum_{j=1}^{b-1} E_C[p(\hat{m}_j \neq m_j | \hat{m} = m)].
\]
Again by definition, the first term vanishes. Now, the receiver on the $(j+1)^{th}$ transmission obtains the state $\rho_{B_3}^{(x_j, y_j, (m_j, m_{j-1}))}$. Averaging over $(X_1)_j (X_2)_j (m_j, m_{j-1})$, this becomes $\rho_{B_3}^{(x_j, y_j, (m_j, m_{j-1}))}$. Hence, the summands in second term are also bounded via Eq. (9): \cite{12}
\[
E_C[p(\hat{m}_j \neq m_j | \hat{m} = m)] = E_C[\text{tr}((I - Q_j)_{B_3}^{(x_j, y_j, (m_j, m_{j-1}))})] = E_{C_j}(x_j, y_j)[\text{tr}((I - Q_j)_{B_3}^{(x_j, y_j, (m_j, m_{j-1}))})] \leq f(\varepsilon, \rho) + \sum_{T} 2^{-D_{H}(\rho^{X_1, X_2, B_2})}.
\]
where we used $S_{\varepsilon, \rho}(x_j, y_j) = \{ (X_1, X_2)_j \}$ and again dropped indices in the last inequality. Hence, overall
\[
E_C[p(\hat{m} \neq m | \hat{m} = m)] \leq b \left[ f(\varepsilon, \rho) + 4 \times 2^{-D_{H}(\rho^{X_1, X_2, B_2})} \right].
\]
Note that since $X_1, X_2$ are independent, $\rho_{B_3}^{(X_1, X_2, B_2)} = \rho_{X_1} \otimes \rho_{X_2}$. We have therefore established the following:

**Proposition 5 (Multihop).** Given $R \in \mathbb{R}_{\geq 0}, \varepsilon \in (0, 1), b \in \mathbb{N}$, the triple $(\frac{k-1}{b} R, b, \delta)$ is achievable for the classical-quantum relay channel, where \cite{13}
\[
\delta = b \left[ f(\varepsilon, \rho) + f(2, \rho) + 4 \times 2^{-D_{H}(\rho^{X_1, X_2, B_2})} \right].
\]
In the asymptotic limit we use the channel $n/b$ times in each of the $b$ blocks. The protocol is analogous to one-shot protocol, except the relay channel has a tensor product form $N_{X_1 X_2 B_2, X_1 X_2 B_3}^{(n/b)}$ characterized by a family of quantum states $\rho^{(x_1, y_1, (m_1))}_{X_1 X_2 B_2}$ (for finite $b$ and large $n$). The codebook is $C^{(n/b)}$ and for finite $b$ and large $n$ we invoke Lemma 3 (instead of Lemma 2) to construct POVM’s for the relay and the receiver such that the decoding error vanishes if the rate satisfies $R < \text{min} \{ I(X_1; B_2 | X_2), I(X_2; B_3 | X_1, B_2) \}$, thereby obtaining the quantum equivalent of the classical multihop bound for sufficiently large $n, b$.

\[
C \geq \max_{P_{X_1}, P_{X_2}} \min \{ I(X_1; B_2 | X_2), I(X_2; B_3 | X_1, B_2) \}. \quad (15)
\]

**B. Coherent Multihop Scheme**

In the multihop scheme, we obtained a rate optimized over product distributions, specifically Eq. (15). For the coherent multihop scheme we obtain the same rate except optimized over all possible two-variable distributions $p_{X_1, X_2}$ by conditioning codewords on each other.

Again, let $R \geq 0$ be our rate, $\varepsilon \in (0, 1)$, and total blocklength $b \in \mathbb{N}$. We show that we can achieve the triple $(\frac{k-1}{b} R, b, \delta)$

\footnote{Note that we need $R, b, \delta$ to be sufficiently small so that $\delta \in [0, 1]$. Otherwise, a block Markov scheme can be employed to obtain a meaningful error bound.}

\footnote{Note that our rate is $\frac{k-1}{b} R$. To achieve rate $R$ we need $\frac{k-1}{b} \to 1$, and so we take the large $n$ limit followed by the large $b$ limit.}
We also again define where we restricted to the message indices the codewords (Relay encoding): With an analysis essentially identical to that of the multihop protocol we arrive at the following.

For decoding we apply Lemma 2 to random variables \( x \) and \( X \) as the argument to obtain a random function \( g \). Furthermore, let the index set \( X \) be our rate, and total number of blocks \( b \in \mathbb{N} \). The classical-quantum state \( \rho_{X_1 X_2 B_3} \) is identical to that of the coherent multipath scenario.

**Code:** Let \( G \) be a graph with \( 2b \) vertices corresponding to random variables \( (X_1)_j (X_2)_j \sim p_{X_1 X_2} \), independent of other pairs, with edges from \( (X_1)_j \) to \( (X_2)_j \). Furthermore, let \( M_0, M_j \) be index sets, where \( |M_0| = 1 \) and \( |M_j| = 2^R \).

Finally, the function \( \text{ind} \) maps \( (X_1)_j \) to \( \{j\} \) and \( (X_2)_j \) to \( \{j-1\} \). Then, letting \( X = X_1^b X_2^b \) and \( M = X_j \cap M_0 \), it is easy to see that \( B \equiv (G, X, M, \text{ind}) \) is a multiplex Bayesian network. See Fig. 6 for a visualization when \( b = 3 \). Now, run Algorithm 1 with \( B \) as the argument to obtain a random codebook \( C \) given by

\[
\rho_{X_1 X_2 B_3} = \sum_{x_1, x_2} p_{X_1 X_2} (x_1, x_2) \langle x_1 x_2 | X_1 X_2 \rangle \otimes \rho_{B_3}.
\]

We also again define \( \rho_{B_3} = \sum_{x_1} p_{X_1} (x_1) \rho_{B_3} \otimes \rho_{B_3} \) to be the reduced state on \( B_3 \) by tracing out \( X_1 B_2 \) and fixing \( X_2 \). Our coding scheme is similar to that of the multipath.

**Proposition 6 (Coherent Multihop).** Given \( R \in \mathbb{R}_{\geq 0} \), \( \varepsilon \in (0, 1) \), \( b \in \mathbb{N} \), the triple \( (\frac{1}{6} R, b, \delta) \) is achievable for the classical-quantum relay channel, where

\[
\delta = \beta [f(1, \varepsilon) + f(2, \varepsilon) + 4 \times 2^{R - D'_R (\rho_{X_1 X_2 B_3}, \rho_{X_1 X_2 B_3})} + 4 \times 2^{R - D'_R (\rho_{X_1 X_2 B_3}, \rho_{X_1 X_2 B_3})}].
\]

Asymptotically, this vanishes if

\[
R < \min \left\{ I (X_1 | B_2) / \rho, I (X_2 | B_3) / \rho \right\},
\]

thereby obtaining the quantum equivalent of the coherent multipath bound for sufficiently large \( b \):

\[
C \geq \max \left\{ I (X_1 | B_2), I (X_2 | B_3) \right\}.
\]

**C. Decode-Forward Scheme**

In the decode-forward protocol we make an incremental improvement on the coherent multipath protocol by letting the receiver’s decoding also involve \( X_1 \).

Again, let \( R \geq 0 \) be our rate, \( \varepsilon \in (0, 1) \), and total number of blocks \( b \in \mathbb{N} \). The classical-quantum state \( \rho_{X_1 X_2 B_3} \) is identical to that of the coherent multipath scenario.

**Code:** The codebook is generated in the same way as in the coherent multipath protocol save with the index set \( M_b \) having cardinality 1 to take into account boundary effects for the backward decoding protocol\(^{17}\) we implement.

**Encoding:** Set \( m_0 \) to be the sole element of \( M_0 \). On the \( j \)-th transmission, the sender transmits the message \( m_j \in M_j \) via \( (x_1)_j (m_{j-1}, m_{j}) \in C \). Note that there is only one message \( m_b \in M_b \) they can choose on the \( b \)-th round.

**Relay encoding:** Same as that of coherent multipath.

**Relay decoding:** Same as that of coherent multipath.

However, note that on \( b \)-th round, since \( |M_0| = 1 \), the decoding is trivial and the estimate \( \hat{m}_b \) is the sole element of \( M_b \).

**Decoding:** The receiver waits until all \( b \) transmissions are finished. Then, they implement a backward decoding protocol, that is, starting with the last system they obtain. Set \( \hat{m}_b \) to be the sole element of \( M_b \). On the \( j \)-th system they use the POVM corresponding to the ancestral subgraph containing vertices \( (X_1)_j \) and \( (X_2)_j \), the set of quantum states \( \{\rho_{B_3}^{(x_j z_j)}\} \), decoding subset \( \{j - 1\} \subseteq \{j - 1, j\} \), and small parameter \( \varepsilon \). We denote the POVM by \( \{Q_{B_3}^{(m_j - 1, m_j)}\} \), where we use the estimate \( \hat{m}_j \), and the obtained measurement result \( \hat{m}_{j-1} \). Note that trivially \( \hat{m}_0 \) is the sole element of \( M_0 \).

**Error analysis:** Fix some \( \hat{m} = (m_0, \ldots, m_b) \in M \). Let \( \hat{m} = (\hat{m}_0, \ldots, \hat{m}_b) \). Assume that the aggregation of the messages estimates of the relay and receiver,

\(^{16}\)Note, however, that the POVM the relay uses from Lemma 2 is not be the same as that of the multipath case since the multiplex Bayesian networks are not the same.

\(^{17}\)In [4] multiple decoding protocols are given. We here give the quantum generalization of the backward decoding protocol.
respectively. Then, the probability of error averaged over \( C \) is given by
\[
\bar{p}_c(C) = \mathbb{E}_C \left[ p(\hat{m} \neq m) \right].
\]

Again, by the bound in Eq. (2),
\[
\bar{p}_c(C) \leq \mathbb{E}_C \left[ p(\hat{m} \neq m) \right] + \mathbb{E}_C \left[ p(\hat{m} \neq m | m = \tilde{m}) \right].
\]

The bound on the first term is identical to that of the coherent multithop protocol and is given by
\[
\mathbb{E}_C \left[ p(\hat{m} \neq m) \right] \leq b \left[ f(2, \varepsilon) + 4 \times 2^{R - D'_{\mu}(p_{X_1, X_2} b_3 \| p_{X_1, X_2} b_3)} \right].
\]

For the second term, we first apply the union bound:
\[
\mathbb{E}_C \left[ p(\hat{m} \neq m | m = \tilde{m}) \right] \leq \mathbb{E}_C \left[ \sum_{j=1}^{b+1} p(\hat{m}_j \neq m_j | m_j + 1 \wedge \hat{m} = \tilde{m}) \right],
\]

where we take into account that the terms corresponding to 0 and \( b \) vanish by definition. Each of the summands can be upper bounded via Lemma 2:
\[
\mathbb{E}_C \left[ p(\hat{m}_j \neq m_j | m_j + 1 = m_j + 1 \wedge \hat{m} = \tilde{m}) \right] = \mathbb{E}_C \left[ \text{tr} \left( (I - Q_{B_3}^{m_j}) \rho B_{3}^{m_j} \right) \right] = \mathbb{E}_C^{(X_1)_{j+1}(X_2)_{j+1}} \left[ \text{tr} \left( (I - Q_{B_3}^{m_j}) \rho B_{3}^{m_j} \right) \right] \leq f(2, \varepsilon) + 4 \sum_{T=\{j\}} 2^{R - D'_{\mu}(p_{X_1, X_2} b_3 \| p_{X_1, X_2} b_3)} \leq f(2, \varepsilon) + 4 \times 2^{R - D'_{\mu}(p_{X_1, X_2} b_3 \| p_{X_1, X_2} b_3)}.
\]

D. Partial Decode-Forward Scheme

We now derive the partial decode-forward lower bound. This requires the full power of Lemma 2 as the receiver decodes all the messages simultaneously by performing a joint measurement on all \( b \) blocks. Intuitively, the partial decode-forward builds on the decode-forward by letting the relay only decode and pass on a part, what we call \( P \), of the overall message.

We split the message into two parts \( P \) and \( Q \) with respective rates \( R_P, R_Q \) \( \geq 0 \). Let \( \varepsilon \in (0, 1) \) and \( b \in \mathbb{N} \) be the total blocklength. Choose some distribution \( p_{X_1, X_2} \) but also a random variable \( U \) correlated with \( X_1, X_2 \) so that the overall distribution is \( p_{UX_1, X_2} \). The classical-quantum state of interest is
\[
\rho_{UX_1, X_2} L B_3 \equiv \sum_{u,x_1,x_2} p_{UX_1, X_2}^{(ux)} \langle ux_1 x_2 | U X_1 X_2 \otimes \rho_{B_3}^{(x_1 x_2)} \rangle.
\]

Note that \( \rho_{B_3}^{(x_1 x_2)} \) does not depend on \( u \), but sometimes we will write \( \rho_{B_3}^{(ux)} \) to keep notation explicit. However, if we trace over \( X_1 \), we induce a \( u \) dependence via the correlation between \( U \) and \( X_1, X_2 \):
\[
\rho_{UX_2} L B_3 \equiv \sum_{u,x_2} p_{UX_2} \langle ux_2 | U X_2 \otimes \rho_{B_3}^{(ux)} \rangle.
\]

This state will be important for the relay decoding.

Code: Let \( G \) be a graph with \( 3b \) vertices corresponding to random variables \( (U)_j, (X_1)_j, (X_2)_j \sim p_{UX_1, X_2} \). The graph has edges going from \( (X_2)_j \) to \( (U)_j \) and \( (U)_j \) to \( (X_1)_j \) for all \( j \) and no edges going across blocks with different \( j \)'s. Furthermore, let \( P_0, P_2 \) and \( Q_2 \) be index sets, so that \( J = [0 : b] \cup [b] \), where \( |P_0| = |P_2| = |Q_2| = 1 \), \( |P_2| = 2^{R_P} \) and \( |Q_2| = 2^{R_Q} \) otherwise. Finally, the function \( \text{ind} \) maps \( (X_1)_j \) to \( \{P_j, Q_j, P_{j-1}\} \), \( (U)_j \) to \( \{P_j, P_{j-1}\} \), and \( (X_2)_j \) to \( \{P_{j-1}\} \). Then, letting \( X \equiv U^b X_1^b X_2^b \), \( M_p = \chi_{j=0}^b P_j \), \( M_q = \chi_{j=1}^b Q_j \) and \( M = M_p \times M_q \), it is easy to see that \( B \equiv (G, X, M, \text{ind}) \) is a multiplex Bayesian network. See Fig. 7 for a visualization when \( b = 3 \). Now, run Algorithm 1 with \( B \) as the argument. This returns a random codebook
\[
C = \bigcup_{j=1}^b \{(x_1)_j (p_{j-1}, p_j, q_j), (u)_j (p_{j-1}, p_j), (x_2)_j (p_{j-1}) \}_{p_j \in P_j, p_{j-1} \in P_{j-1}, q_j \in Q_j},
\]

where we restricted to the message indices the codewords are dependent on via \( \text{ind} \). For decoding we apply Lemma 2 with this codebook and use the assortment of POVMs that are given for different ancestral subgraphs and other parameters.

\footnote{For convenience we denote the elements of \( J \) by the index sets they correspond to.}
Encoding: Set \( p_0 \) to be the sole element of \( P_0 \). On the \( j \)th transmission, the sender transmits the two-part message \( (p_j, q_j) \in P_j \times Q_j \) via \((x_1)_{(p_j-1)}(p_j, q_j) \). Note that on the \( j \)th transmission the sender has to send a fixed message \((p_0, q_0)\) being the sole element of \( P_0 \times Q_0 \).

Relay encoding: Let \( \tilde{p}_0 \) to be the sole element of \( P_0 \). On the \( j \)th transmission, the relay sends \( \tilde{p}_{j-1} \) via \((x_2)_{(j-1)}(\tilde{p}_{j-1}) \) from codebook \( C \). Note that this is the relay’s estimate of the message sent by the sender on the \((j-1)\)th transmission.

Relay decoding: The relay tries to recover the \( p \)-part of the sender’s message using the same technique as in the previous protocols. On the \( j \)th transmission the relay uses the POVM corresponding to the ancestral subgraph containing the two vertices \((U)_{j}\) and \((X)_{j}\), the set of quantum states \( \{\rho_{p}=\rho_{p}\}_{p=p_{j}} \), decoding subset \( \{p_{j}\} \subseteq \{p_{j-1}, p_{j}\} \), and small parameter \( \varepsilon \). The POVM is denoted by \( \{\rho_{p_{j}}\}_{p_{j} \in p_{j}} \), where we use the estimate \( \tilde{p}_{j-1} \), and the relay applies this on their received state to obtain a measurement result \( \tilde{p}_j \). Note that \( \tilde{p}_0 \) is trivially the sole element of \( P_0 \).

Decoding: The decoder waits until all \( b \) transmissions are completed. The receiver uses the POVM corresponding to the ancestral subgraph the entire graph \( G \), the set of quantum states \( \{\otimes_{j=1}^{b} \rho_{p_{k}}\}_{p_{k}=\rho_{p_{k}}} \), where the \((u)_{j} \) dependence here is trivial, decoding set \( \times_{j=1}^{b-1} P_j \times \times_{j=1}^{b} Q_j \), and small parameter \( \varepsilon \). We denote the POVM by \( \{\rho_{p_{b}}\}_{p_{b} \in p_{b}} \), to their received state on \( B_{b}^{2} \) to obtain their estimate of the entire string of messages, which we call \( \tilde{m}_{p} \equiv (\tilde{p}_0, \ldots, \tilde{p}_b) \), \( \tilde{m}_{q} \equiv (\tilde{q}_1, \ldots, \tilde{q}_b) \), where \( p_0, \tilde{p}_0, \tilde{q}_b \) are set to be the sole elements of the respective index sets.

Error analysis: We fix the strings of messages \( m_p = (p_0, \ldots, p_b) \) and \( m_q = (q_1, \ldots, q_b) \). By the bound in Eq. (2),

\[
P_{e}(C) \equiv E_{C}[p(\tilde{m}_p m_q \neq m_p m_q)] 
\leq E_{C}[p(\tilde{m}_p \neq m_p)] + E_{C}[p(m_q \neq m_q | \tilde{m}_p = m_p)].
\]

We can bound the first term just as we did for the other protocols. First, use the union bound.

\[
E_{C}[p(\tilde{m}_p \neq m_p)] \leq \sum_{j=1}^{b-1} E_{C}[p(\tilde{p}_j \neq p_j | \tilde{p}_{j-1} = p_{j-1})].
\]

By Lemma 2 we can bound each summand as follows:

\[
E_{C}[p(\tilde{p}_j \neq p_j | \tilde{p}_{j-1} = p_{j-1})] = E_{C}[\text{tr}(I - Q(\rho_{p_{j}})) \rho_{B_{2}}(\{(x_1)_{(p_j-1)}(p_j, q_j) \})]
= E_{C}(U_{j}(X), \{x_2\}) \text{tr}(I - Q(\rho_{p_{j}})) \rho_{B_{2}}(\{(x_1)_{(p_j-1)}(p_j, q_j) \})
\leq f(2, \varepsilon) + 4 \sum_{p_{j}} \rho^{2p_{j} - D_{H}(\langle p_{j} \rangle_{(x_2)})} \rho^{((x_2_{(p_j-1)})_{(x_2_{(p_j-1)})})}
\leq f(2, \varepsilon) + 4 \times 2^{R_{c} - D_{H}(\rho_{U_{j}X_{2}\rho_{B_{2}}})} \rho^{((U_{j}, B_{2})_{(x_2)})},
\]

where we used \( S(\langle p_{j} \rangle_{(x_2)}) \equiv \{(U_{j})_{(x_2)}\} \). We dropped the index \( j \) in the last equality since \((U_{j}(X))_{(x_2)} \sim \rho_{U_{j}X_{2}B_{3}} \). Hence, overall,

\[
E_{C}[p(\tilde{m}_p \neq m_p)] 
\leq b \left[ f(2, \varepsilon) + 4 \times 2^{R_{c} - D_{H}(\rho_{U_{j}X_{2}\rho_{B_{2}}})} \right] + f(3b, \varepsilon) + 4 \sum_{j_{p}, j_{q} \leq \{b-1\}; j_{p} + j_{q} > 0} j_{p} R_{p} + j_{q} R_{q} - D_{H}(\rho_{U_{j}X_{2}\rho_{B_{2}}}) \rho^{((X_{2})_{(x_2)}^{U_{j}(X_{2})})}
\leq 2(1) \left[ f(2, \varepsilon) + 4 \times 2^{R_{c} - D_{H}(\rho_{U_{j}X_{2}\rho_{B_{2}}})} \right] + f(3b, \varepsilon) + 4 \sum_{j_{p}, j_{q} \leq \{b-1\}; j_{p} + j_{q} > 0} j_{p} R_{p} + j_{q} R_{q} - D_{H}(\rho_{U_{j}X_{2}\rho_{B_{2}}}) \rho^{((X_{2})_{(x_2)}^{U_{j}(X_{2})})}
\leq 2(1) \left[ f(2, \varepsilon) + 4 \times 2^{R_{c} - D_{H}(\rho_{U_{j}X_{2}\rho_{B_{2}}})} \right] + f(3b, \varepsilon) + 4 \sum_{j_{p}, j_{q} \leq \{b-1\}; j_{p} + j_{q} > 0} j_{p} R_{p} + j_{q} R_{q} - D_{H}(\rho_{U_{j}X_{2}\rho_{B_{2}}}) \rho^{((X_{2})_{(x_2)}^{U_{j}(X_{2})})}.
\]

In the asymptotic limit, the error vanishes provided

\[
R_{p} < I(U; B_{2}|X_{2}) \quad (18)
\]

and, for all \( J_{p}, J_{q} \leq \{b-1\} \),

\[
J_{p} R_{p} + J_{q} R_{q} < I(X_{2}^{U_{j}} X_{2}^{U_{j}}) \rho^{((U_{j}, B_{2})_{(x_2)})} \rho^{((X_{2})_{(x_2)}^{U_{j}(X_{2})})}.
\]

Note \( J_{p}, J_{q} \leq \{b-1\} \) and \( J_{p} \leq \{2 : b\} \). However, we use the convention that all complementary sets are with respect to largest containing set \( b \). We can simplify Eq. (19) via a general lemma:

\[\text{The } b \text{th messages and estimates match, but in general the } b \text{th } x_{1}, x_{2}, u \text{ depend also on the } (b - 1)\text{th messages and estimates.} \]
Lemma 9. Let $\rho_{B_1 \ldots B_m}$ be $m$-partite quantum state. We consider the state $\rho_{B_1 \ldots B_m}^n$ for some $n \in \mathbb{N}$. Now, let $B, B', C$ be disjoint subsystems of $(B_1 \ldots B_m)^{\otimes n}$ and such that $B, B'$ are supported on disjoint tensor factors. Then,

$$I(B; B'(C)) = 0.$$  

Proof. We prove this by the definition of the conditional mutual information and the fact that $\rho_{B_1 \ldots B_m}^n$ is a tensor product state:

$$I(B; B'(C)) = S(BC) + S(B'C) - S(B'B'C) - S(C)$$

$$= S(BC_B) + S(C_{\overline{B}}) + S(B'C_{\overline{B}}) + S(C_{\overline{B}'})$$

$$- S(BC_B) - S(B'C_{\overline{B}'}) - S(C_{\overline{B}'})$$

$$= 0.$$  

where $C_B$ is the subsystem of $C$ supported on the tensor factors that support $B$ and $C_{\overline{B}}$ is the rest of $C$. □

Using Lemma 9 this and the chain rule, for any conditional mutual information quantity we can remove conditioning systems which are supported on tensor factors disjoint from those that support the non-conditioning systems. This is key in the following analyses. For instance, in Eq. (19), $J$ and $J' \cup J'' \cup J'' = J$ are supported on disjoint tensor factors, and so we can remove the conditioning on the $X_J$ system:

$$I(X_J X_{J'} X_{J''} U_J \mid b; B'_p | X_{J'} X_{J''} U_J)$$

$$= I(X_J X_{J'} X_{J''} U_J \mid b; B'_p | X_{J'} X_{J''} U_J)$$

$$+ I(X_J X_{J'} U_J \mid b; X_{J'} X_{J''} U_J)$$

$$- I(X_J X_{J'} U_J \mid b; X_{J'} X_{J''} U_J)$$

$$= I(X_J X_{J'} U_J \mid b; B'_p | X_{J'} X_{J''} U_J).$$

Thus, Eq. (19) reduces to

$$j_p R_p + j_q R_q < I(X_J X_{J'} U_J \mid b; B'_p | X_{J'} X_{J''} U_J).$$

We claim that the set of pairs $(R_p, R_q)$ that satisfy these bounds gives the classical partial decode-forward lower bound with quantum mutual information quantities in the limit of large $b$.\textsuperscript{21} In particular, we show:

\textsuperscript{21}This will also cause $b \frac{R_p + R_q}{b} \to 1$ so that the rate we achieve really is $R_p + R_q$.  

---

**Figure 8.** $R_p - R_q$ rate region. Gray region is $S$, defined by the blue (solid) lines that correspond to Eq. (21) and Eq. (22). The red (dashed) line corresponds to Eq. (20) for a fixed $j_p, j_q \subseteq [b - 1]$.  

**Proposition 10.** Let

$$S(b) = \{(R_p, R_q) \in \mathbb{R}^2_{\geq 0} \mid \forall j_p, j_q \subseteq [b - 1]$$

$$\text{such that } j_p + j_q > 0,$$

$$j_p R_p + j_q R_q < I(X_J X_{J'} U_J \mid b; B'_p | X_{J'} X_{J''} U_J)\}$$

and

$$S = \{(R_p, R_q) \in \mathbb{R}^2_{\geq 0} \mid R_p + R_q < I(X_J X_{J'} U_J \mid b; B'_p | X_{J'} X_{J''} U_J)\}$$

where $\rho_{X_J X_{J'} U_J}$ is given by Eq. (16). Then, $\lim_{b \to \infty} S(b)$ exists and is equal to $S$.

Note that the bounds that define $S$ do not match the bounds given for instance in [4] since we do not first decode $P$ and thereby $Q$, but instead jointly decode to obtain all messages simultaneously. However, in the end we still obtain the same lower bound on the capacity.

Proof. For reference, we list the bounds:

$$j_p R_p + j_q R_q < I(X_J X_{J'} U_J \mid b; B'_p | X_{J'} X_{J''} U_J)\}$$

(20)
and

\[ R_q < I(X_1; B_3 | U X_2)_{ρ_{UX}, x_2 b_3} \tag{21} \]

\[ R_p + R_q < I(X_1 X_2; B_3)_{p_{UX}, x_2 b_3} \tag{22} \]

We first claim \( \limsup_{b \to \infty} S(b) \subseteq S \). Consider \( J_p, J_q = [b - 1] \), in which case Eq. (20) becomes

\[(b - 1)(R_p + R_q) < I(X_1^b (X_2^b U)^b ; B_3^b | (X_2)_1), \]

which, using Lemma 9, can be manipulated into

\[ R_p + R_q < \frac{b}{b - 1} I(X_1 X_2 U; B_3) - \frac{1}{b - 1} I(X_2; B_3) = I(X_1 X_2; B_3) + \frac{1}{b - 1} I(X_1; B_3 | X_2). \]

In the limit of large \( b \), this becomes Eq. (22). To obtain Eq. (21), take \( j_p = 0 \). Then, Eq. (20) by Lemma 9

\[ J_q R_q < I(X_1^j ; B_3^j | X_2^j U^j) = J_q I(X_1; B_3 | X_2 U). \]

Now, since \( j_p = 0 \), \( J_q \) cannot be zero, so this is equivalent to

\[ R_q < I(X_1; B_3 | X_2 U). \]

The claim thus follows.

We next claim \( S(b) \supseteq S \) for all \( b \) and \( b \liminf_{b \to \infty} S(b) \supseteq S \). We only need to consider when \( j_p > 0 \) since otherwise we obtain Eq. (21) as shown above, which holds for all \( b \). Now, interpret each of the inequalities above as a linear bound on an \( R_p, R_q \) diagram (see Fig. 8). We show that none of the lines corresponding to Eq. (20) cuts into \( S \). First, fixing \( J_p, J_q \subseteq [b - 1] \), we find the \( R_p \) intercept of said line

\[ \frac{1}{J_p} I(X_1^j X_2^j U^j ; B_3^j | X_2^j U^j) \]

\[ = \frac{1}{J_p} \left( I(X_1^j X_2^j U^j ; B_3^j | X_2^j U^j) + \cdots \right) \]

\[ \geq \frac{1}{J_p} I(X_1^j X_2^j U^j ; B_3^j) \]

\[ = I(X_1 X_2 U; B_3) = I(X_1 X_2; B_3), \]

where \( \cdots \) stands for some conditional mutual information quantity and therefore is non-negative. Thus, the \( R_p \) intercept is at least as large as that of Eq. (22), as shown in Fig. 8. This determines one of the points of the line.

We now find another point. We observe that \( I(X_1; B_3 | X_2 U) \leq I(X_1 X_2 U; B_3) \) so the line associated with Eq. (21) intersects that of Eq. (22) in \( \mathbb{R}^2_0 \). Hence, it is sufficient to show the bound on \( R_p \) when \( R_q = I(X_1; B_3 | X_2 U) \) in Eq. (20) is weaker than \( I(X_1 X_2 U; B_3) - I(X_1; B_3 | X_2 U) = I(X_2 U; B_3) \). To see this, we substitute \( R_q = I(X_1; B_3 | X_2 U) \) into Eq. (20):

\[ J_p R_p + J_q I(X_1; B_3 | X_2 U) \]

\[ \leq I(X_1^j X_2^j U^j ; B_3^j | X_2^j U^j) \]

\[ = \frac{1}{J_p} I(X_1^j X_2^j U^j ; B_3^j | X_2^j U^j) \]

\[ + I(X_1^j X_2^j U^j ; B_3^j | X_2^j U^j) \]

\[ = I(X_1^j; B_3^j | X_2^j U^j) + I(X_2^j U^j; B_3^j | X_2^j U^j) + \cdots \]

\[ = J_q I(X_1; B_3 | X_2 U) + J_p I(X_2 U; B_3) + \cdots. \]

This establishes our claim and completes the proof. \( \square \)

Therefore, combining the bounds Eqs. (18), (21) and (22), the overall rate \( R_p + R_q \) of the entire protocol is achievable if

\[ R_p + R_q < \min \{ I(X_1; B_3 | X_2 U)_ρ + I(U; B_2 | X_2)_ρ, \]

\[ I(X_1 X_2; B_3)_ρ \}. \]

This is sufficient since if it holds we can choose \( R_p, R_q \) to satisfy the bounds. It is also necessary since if it is violated, then one of the bounds has to be violated. We can optimize over \( p_{UX}, x_2 \), so we obtain the partial decode-forward lower bound:

\[ C \geq \max \left\{ \min \{ I(X_1; B_3 | X_2 U)_ρ + I(U; B_2 | X_2)_ρ, \right. \]

\[ \left. I(X_1 X_2; B_3)_ρ \} \right\}. \tag{23} \]

Remark. This coding scheme is optimal in the case when \( X_1, x_2 \rightarrow B_2 \) is semideterministic, namely \( B_2 \) is classical and \( ρ_{B_2} = \rho_{x_2 | x_2} \) is pure for all \( x_1, x_2 \). This is because in this case the partial decode-forward lower bound Eq. (23) with \( U = B_2 \) as random variables matches the cutset upper bound Eq. (13). This is possible because of the purity condition, which essentially means \( B_2 \) is a deterministic function of \( X_1, X_2 \). The semideterministic classical relay channel was defined and analyzed in [41].

V. PROOF OF THE QUANTUM MULTIPARTY PACKING LEMMA

In this section we prove Lemma 2 via Sen’s joint typicality lemma [7]. We then use Lemma 2 to prove the asymptotic version, Lemma 3. We shall state a special case of the joint typicality lemma, the \( t = 1 \) intersection case in the notation of [7], as a theorem. For the sake of conciseness, we suppress some of the detailed expressions.

We first give some definitions. A subpartition \( \mathcal{L} \) of some set \( S \) is a collection of nonempty, pairwise disjoint subsets of \( S \). We define \( \bigcup(\mathcal{L}) \) to be their union, that is, \( \bigcup(\mathcal{L}) \equiv \bigcup_{L \in \mathcal{L}} L \). Note that \( \bigcup(\mathcal{L}) \subseteq S \). We say a subpartition \( \mathcal{L} \) of \( S \) covers \( T \subseteq S \) if \( T \subseteq \bigcup(\mathcal{L}) \).

Theorem 11 (One-shot Quantum Joint Typicality Lemma [7]). Let

\[ \rho_{X A} = \sum_x p_x(x) |x⟩⟨x|_X \otimes ρ_A^{(s)} \]

be a classical-quantum state where \( A = A_1 \ldots A_N \) and \( X = X_1 \ldots X_M \). Let \( ε \in (0, 1) \) and let \( Y = Y_1 \ldots Y_{N + M} \) consist of \( N + M \) identical copies of some classical system, with total dimension \( d_Y \). Then there exist quantum systems \( A_k \) and isometries \( J_k : A_k \rightarrow A_k \) for \( k \in [N] \), as well as a cqc-state of the form

\[ \hat{ρ}_{X A Y} = \frac{1}{d_Y} \sum_{x, y} p_x(x) |x⟩⟨x|_X \otimes ρ_A^{(s)} \otimes |y⟩⟨y|_Y, \]

and a cqc-POVM \( \hat{Π}_{X A Y} \) such that, with \( \hat{J} \equiv \bigoplus_{k \in [N]} J_k \),

1) \[ \left\| \hat{ρ}_{X A Y} - (I_X \otimes \hat{J}) \rho_{X A} (I_X \otimes \hat{J})^† \right\|_1 \leq f(N, M, ε), \]

where \( γ_Y = \frac{1}{d_Y} \sum_y |y⟩⟨y|_Y \) denotes the maximally mixed state on \( Y \).
We denote \( \hat{k} \) for each pseudosubpartition of \( \mathcal{L} \) that satisfies \( \hat{k} = \epsilon / (0, 1) \) a small parameter. Now, let \( C = \{ x(i) \} \) for each \( k \in \mathcal{L} \) be a family of random variables such that for every \( i \in \mathcal{I} \), \( x(i) \sim p_X \), and there exists a map \( \Psi : \mathcal{I} \times \mathcal{I} \rightarrow \mathcal{P}([k]) \) such that for every \( i, i' \in \mathcal{I} \), letting \( T = \Psi(i, i') \),

1) \( x_T(i) = x_T(i') \) as random variables
2) \( x_T(i), x_T(i') \) are independent conditioned on \( x_T(i) = x_T(i') \),

where \( \hat{T} = [k] \setminus T \). Then, for each \( i_1 \in \mathcal{I}_1 \) there exists a POVM \( \{ Q^{(x_{i_1}(x))}_i \}_{x \in \mathcal{I}_2} \) dependent on the random variables in \( C \) such that for all \( i = (i_1, i_2) \in \mathcal{I} \),

\[
\mathbb{E}_C \left[ \| I - Q^{(x_{i_1}(x))}_i \|_2 \right] \leq 4 \sum_{i_1, i_2} 2^{-D_H(\rho_{X_{i_1}} \parallel (p^{(x_{i_1}(x))}))},
\]

where \( \mathbb{E}_C \) is the expectation over the random variables in \( C \), \( \hat{S} = \Psi([i_1], [i_1, i_2]) \), and

\[
\rho_{X_{i_1}} = \sum_x p_X(x) |x\rangle \langle x| \otimes \rho_B^{(x)}.
\]

Furthermore, \( f(k, \epsilon) \) is a universal function such that \( \lim_{\epsilon \to 0} f(k, \epsilon) = 0 \).

Before we prove Lemma 12, we first show that Lemma 2 follows from Lemma 12.

Proof of Lemma 2. Fix subgraph \( H, \{ \rho_B^{(x_{i_1})} \}_{x \in \mathcal{X}_H} \) with \( \mathcal{H} \). Let \( \epsilon \in (0, 1) \). We invoke Lemma 12 with the ensemble \( \{ p_{X_{i_1}}, \rho_B^{(x_{i_1})} \} \) for each \( \mathcal{I}_1 = \mathcal{I} \) and \( \mathcal{I}_2 = \mathcal{I}_1 \), the same \( \epsilon \), and the family of random variables \( C = C_H \). We thus identify \( \mathcal{I} = \mathcal{H} = \mathcal{D} \times \mathcal{D} \). We also define an arbitrary ordering on \( \mathcal{X}_H \) such that we can identify it with \( [k] \).

We check that \( C_H \) satisfies the required properties using the observations we made regarding Algorithm 1. First, for every \( m_H \in \mathcal{H} \), \( x_H(m_H) \sim p_X \) by observation 1 on p. 4.

Next, we claim the map

\[
\Psi(m_H, m_H') \equiv \{ v \in \mathcal{V}_H \mid \exists j \in \text{ind}(v) \text{ such that } (m_H)_j \neq (m_H')_j \}
\]

satisfies the required conditions. Let \( m_H, m_H' \in \mathcal{H} \) and \( T = \Psi(m_H, m_H') \). By definition, given \( v \in \mathcal{T} \), for all \( j \in \text{ind}(v) \), \( (m_H)_j = (m_H')_j \). Hence, \( m_H' \sim \text{ind}(v) = m_H' \sim \text{ind}(v) \), so by observation 2 on p. 4, \( x_v(m_H) = x_v(m_H') \) as random

\[22\]Note that the bound does not depend on the specific choice of the map.
variables. Thus, \( x_T(m_H) = x_T(m'_H) \) as random variables, so we have established condition 1.

We now prove the conditional independence statement in condition 2 is satisfied. For \( \xi_T \in X_T \), observation 1 shows that

\[
\Pr(x_T(m_H) = \xi_T) = \prod_{v \in T} p_{X_v|X_{pa(v)}}(\xi_v|\xi_{pa(v)}),
\]

where we used that \( pa(T) \subseteq T \) as a consequence of Eq. (7).

Next, observation 3 implies that the joint distribution of \( x_T(m_H), x_T(m'_H) \), and \( x_T(m_H) \) is given as follows. For \( \xi, \xi' \in X \) such that \( \xi_T = \xi_T' \),

\[
\Pr(x_T(m_H) = \xi_T, x_T(m'_H) = \xi_T') = \Pr(x_T(m_H) = \xi, x_T(m'_H) = \xi')
\]

\[
= \left( \prod_{v \in T} p_{X_v|X_{pa(v)}}(\xi_v|\xi_{pa(v)}) \right) \left( \prod_{v \in T} p_{X_v|X_{pa(v)}}(\xi'_v|\xi_{pa(v)}) \right).
\]

Hence, \( x_T(m_H) \) and \( x_T(m'_H) \) are independent conditional on \( x_T(m_H) \). Lemma 2 in the form given in Eq. (10) then directly follows from applying Lemma 12.

Next, we prove that Lemma 3 follows from Lemma 2.

**Proof of Lemma 3.** This follows from Lemma 2 by replacing \( X \) with \( n \in \mathbb{N} \) i.i.d. copies of itself, \( X^n \). Then, associating each \( v \in V \) with \( X^n_v \), \((G, X^n, M, \text{ind})\) is a multiplex Bayesian network.

We now apply Algorithm 1 with \((G, X^n, M, \text{ind})\) as input. This is equivalent to applying it with \((G, X, M, \text{ind})\) \( n \) times. Then, applying Lemma 2 with inputs \( H = \{ \bigotimes_{i=1}^n \rho_B^{(x_i)} \}, X^n \in X^n, D \in \mathcal{D}(\epsilon(n) \in (0,1)), \) we obtain a POVM \( \{Q_B^{(x_i)}(m_H)\}_{m_H \in \mathcal{M}_D} \) for each \( m_H \in M_D \) such that for \((m_D, m_H) \in M_H \),

\[
\mathbb{E}_{C_{H}} \left[ \left| \left( I - Q_B^{(m_H)} \right) \rho_{B^{n}} \right| \right] \leq f((|V_H|, \epsilon(n))
\]

\[
+ 4 \sum_{\forall \theta \subseteq D} 2^{(\sum_{t \in T} R_t) - D^{(n)}(\rho_{B^{n}})} \rho_{B^{n}} \left( \left| \left( \bigotimes_{i=1}^n \rho_B^{(x_i)} \right) \right| \right).
\]

Consider now

\[
\rho_{X^n_B} = \sum_{x_H} \rho_{X_H}^{n} \left| x_H \right| \left| x_H \right| X^n_H \otimes \bigotimes_{i=1}^n \rho_B^{(x_i)}
\]

and

\[
\rho_{X^n_B}^{(x_H)} = \sum_{x_H} \rho_{X_H}^{n} \left| x_H \right| \left| x_H \right| X^n_H \otimes \rho_B^{(x_H)}.
\]

It is not difficult to see that

\[
\rho_{X^n_B} = \left( \sum_{x_H} \rho_{X_H}^{n} \left| x_H \right| \left| x_H \right| X_H \otimes \rho_B^{(x_H)} \right)^{\otimes n}
\]

which conveniently justifies this slight abuse of notation. Furthermore, considering

\[
\rho_{X^n_B}^{(x_H)} = \sum_{x_H} \rho_{X_H}^{n} \left| x_H \right| \left| x_H \right| X_H \otimes \rho_B^{(x_H)}
\]

\[
\rho_B^{(x_H)} = \bigotimes_{i=1}^n \rho_B^{(x_i)}
\]

we likewise conclude

\[
\rho_{X_B^n} \left( \left| \left( \bigotimes_{i=1}^n \rho_B^{(x_i)} \right) \right| \right) \rightarrow 0
\]

when the rate inequalities are satisfied. Given Eq. (3), one possibility is \( \epsilon(n) = 1/n \) and \( \delta(n) = n^{-1/4} \). This concludes the proof.

Finally, we prove Lemma 12. Note that Theorem 11 gives a pair \( \tilde{\rho}, \tilde{\Pi} \) that satisfy joint typicality properties but live in a larger Hilbert space. In order to prove Lemma 2, which claims the existence of a POVM on the original Hilbert space, we need to construct the corresponding POVM in the larger Hilbert space and then appropriately invert the isometry. There is also an extra classical system \( Y \) associated with the \( X \) systems, which we can interpret as an additional random codebook. We use a conventional derandomization argument to eliminate it from the statement. The extra \( Y \)'s associated with the \( B \) systems we simply trace over.

**Proof of Lemma 12.** We invoke Theorem 11 with inputs the \( \rho_{X_B}, \epsilon, \) and a classical system \( YZ \). Here \( X \equiv X_1 \ldots X_k, Y \equiv Y_1 \ldots Y_k \) and \( Z \) is a classical system associated with \( B \), to obtain a quantum state \( \tilde{\rho}_{XBYZ} \) and POVM \( \tilde{\Pi}_{XBYZ} \) which we can expand as follows:

\[
\tilde{\rho}_{XBYZ} = \bigoplus_{x,y} \rho_{X}(x) |x⟩⟨x| \otimes \frac{1}{dy} |y⟩⟨y| \otimes \tilde{\rho}_{BZ}^{(x,y)}
\]

\[
\tilde{\Pi}_{XBYZ} = \bigoplus_{x,y} |x⟩⟨x| \otimes |y⟩⟨y| \otimes \tilde{\Pi}_{BZ}^{(x,y)}
\]

Now, for every \( x_j \in X \), draw \( y_j(x_j) \) uniformly at random from \( Y_j \), and consider the random vectors \( y(x) := (y_1(x_1), \ldots, y_k(x_k)) \). We use these random vectors and the codebook \( C = \{(x_i)\}_{i \in \mathcal{I}} \) to define a codebook \( C' = \{(y(i))\}_{i \in \mathcal{I}} \), where we set \( y(i) := y(x(i)) \). We also define the joint codebook \( C'' = \{(x(i), y(i))\}_{i \in \mathcal{I}} \). Then, for every \( i, i' \in \mathcal{I}, \) letting \( T = \Psi(i, i') \), the following holds:

1) \( x_T(i) y_T(i) = x_T(i') y_T(i') \) as random variables,

2) \( x_T(i) y_T(i) \) and \( x_T(i') y_T(i') \) are independent conditioned on \( x_T(i) y_T(i) = x_T(i') y_T(i') \),

with probabilities

\[
p_{X_T Y_T}(x_T, y_T) = p_{X_T}(x_T) \cdot p_{Y_T}(y_T) = \frac{1}{dy} p_{X_T}(x_T)
\]
We then define the square-root measurement \( \rho^{(i)}_{BZ} \equiv \rho^{(x(i),y(i))}_{BZ} \) and \( \hat{\Pi}^{(i)}_{BZ} \equiv \hat{\Pi}^{(x(i),y(i))}_{BZ} \).

We then define the square-root measurement
\[
\hat{Q}^{(i_1,i_2)}_{BZ} = \left( \sum_{\hat{\nu} \in \mathcal{I}_2} \hat{\Pi}^{(i_1,\hat{\nu})}_{BZ} \right)^{-1/2} \hat{\Pi}^{(i_1,i_2)}_{BZ} \left( \sum_{\hat{\nu} \in \mathcal{I}_2} \hat{\Pi}^{(i_1,\hat{\nu})}_{BZ} \right)^{-1/2}
\]
and “invert” the isometry \( \hat{J} \) to obtain the following family of POVM’s on the Hilbert space:
\[
\hat{Q}^{(i_1,i_2)}_{BZ} = \sum_{\hat{\nu} \in \mathcal{I}_2} \hat{\Pi}^{(i_1,\hat{\nu})}_{BZ} \hat{J}_{B \rightarrow \hat{B}}.
\]

Note that we have a POVM for each value of \( i_1 \) and these POVM’s are dependent on our random encoding \( x(i) \) and random choice of \( y(i) \).

Now, fixing \( i = (i_1,i_2) \in \mathcal{I} \), we compute the probability of error averaged over the random choice of \( x(i) \) and \( y(i) \), denoting this by \( \mathbb{E} \equiv \mathbb{E}_{C'} \):

\[
\mathbb{E} \text{tr} \left[ (I - Q^{(i)}_B) \rho^{(i)}_B \right] \\
= 1 - \mathbb{E} \text{tr} \left[ Q^{(i)}_B \rho^{(i)}_B \right] \\
= 1 - \mathbb{E} \text{tr} \left[ \hat{Q}^{(i)}_{BZ} \left( \hat{J}_{B \rightarrow \hat{B}} \rho^{(i)}_{B} \hat{J}_{B \rightarrow \hat{B}} \otimes \tau_{Z} \right) \right] \\
\leq 1 - \mathbb{E} \text{tr} \left[ \hat{Q}^{(i)}_{BZ} \rho^{(i)}_{BZ} \right] + \mathbb{E} \left\| \hat{J}_{B \rightarrow \hat{B}} \rho^{(i)}_{B} \hat{J}_{B \rightarrow \hat{B}} \otimes \tau_{Z} - \rho^{(i)}_{BZ} \right\|_1 \\
= 1 - \mathbb{E} \text{tr} \left[ \hat{Q}^{(i)}_{BZ} \rho^{(i)}_{BZ} \right] + \mathbb{E} \left\| \hat{J}_{B \rightarrow \hat{B}} \rho^{(i)}_{B} \hat{J}_{B \rightarrow \hat{B}} \otimes \tau_{Y \rightarrow Z} - \rho^{(i)}_{X \rightarrow Y \rightarrow Z} \right\|_1 \\
\leq 1 - \mathbb{E} \text{tr} \left[ \hat{Q}^{(i)}_{BZ} \rho^{(i)}_{BZ} \right] + f(1,k,\varepsilon) \\
\leq 2 \left( 1 - \mathbb{E} \text{tr} \left[ \hat{\Pi}^{(i_1,\hat{\nu})}_{BZ} \rho^{(i_1,\hat{\nu})}_{BZ} \right] \right) + 4 \sum_{i_2 \neq i_2} \mathbb{E} \text{tr} \left[ \hat{\Pi}^{(i_1,i_2)}_{BZ} \rho^{(i_1,i_2)}_{BZ} \right] + f(1,k,\varepsilon) \\
\leq 4 \sum_{i_2 \neq i_2} \mathbb{E} \text{tr} \left[ \hat{\Pi}^{(i_1,i_2)}_{BZ} \rho^{(i_1,i_2)}_{BZ} \right] + f(1,k,\varepsilon) + 2g(1,k,\varepsilon).
\]

where in the last three inequalities we used Theorem 11 and the Hayashi-Nagaoka lemma [40, 42].

We consider the first term. Let \( S = \Psi((i_1,i_2), (i_1,i_2)) \). Note that by our conditions on the random codebook, the codewords are equal as random variables on \( S \) and hence we obtain Eq. (24). In the first two equalities we use the notation \( X \equiv x(i_1,i_2), X' \equiv x(i_1,i_2)' \) and similarly for \( Y, Y' \). In the fourth equality \( \rho^{(x,y)}_{BZ} \) is the marginal of the conditional density operator \( \rho^{(x,y)}_{X \rightarrow Y \rightarrow Z} \). In the last inequality we use Theorem 11 and choose the dimensions of \( Y, Z \) to be sufficiently large so that \( h(1,k,d_B, d_Y d_Z) \leq \varepsilon^{1/3} \).

Finally, we can invoke the usual derandomization argument to remove the dependency of our POVM on the choice of \( y(i) \). That is, we know that
\[
\mathbb{E} \text{tr} \left[ (I - Q^{(i)}_B) \rho^{(i)}_B \right] \\
\leq \varepsilon^{1/3} + f(1,k,\varepsilon) + 2g(1,k,\varepsilon).
\]

VI. CONCLUSIONS

The packing lemma is a cornerstone of classical network information theory, used as a black box in the analyses of all kinds of network communication protocols. At its core, the packing lemma follows from properties of the set of jointly typical sequences for multiple random variables. In this letter, we provide an analogous statement in the quantum setting that we believe can serve a similar purpose for quantum network information theory. We illustrate this by using it as a black box to prove achievability results for the classical-quantum relay channel. Our result is based on a joint typicality lemma recently proved by Sen [7]. This result, at a high level, provides a single POVM which achieves the hypothesis testing bound for all possible divisions of a multiparty state into a tensor product of its marginals. This result allows for the construction of finite blocklength protocols for quantum multiple access, relay, broadcast, and interference channels [31].

Two alternative formulations of joint typicality were proposed in [10] and [26]. In the first work, the author conjectured the existence of the jointly typical state that is close to an i.i.d. multiparty state but with marginals whose purities satisfy certain bounds. This notion of typicality was then used in the analysis of multiparty state merging and assisted entanglement distillation protocols. In the second work, the authors provided a similar statement for the one-shot case. Specifically, for a given multiparty state, they conjectured the existence of a state that is close to the initial state but has a min-entropy bounded by the smoothed min-entropy of the initial state for all marginals. In a follow up paper we will try to understand the relationship between these various notions of quantum joint typicality and whether Sen’s results can be extended to prove the other notions or to realize the applications they are designed for.

Also, as noted in the corresponding section, our protocol for the partial decode-forward bound is not a straightforward generalization of the classical protocol in [4]. Our algorithm involves a joint measurement of all the transmitted blocks instead of performing a backward decoding followed by a
Another idea is to study networks of relay channels, where the relays are operating in series or in parallel. Some preliminary work has been done in [29], and the most general notion of this in the classical literature is a multicast network [4]. Lastly, relay channels with feedback would also be interesting to investigate.

There are still several interesting questions that remain open regarding quantum relay channels. The most obvious one is proving converses for the given achievability lower bounds. There are known converses for special classical relay channels, and it would be interesting to extend them to the quantum case as we did for semideterministic relay channels. Another, albeit less trivial, direction is to prove a quantum equivalent of the compress-forward lower bound [4]. We might need to analyze this in the entanglement-assisted case since it is only then that a single-letter quantum rate-distortion theorem is known [43].

Another idea is to study networks of relay channels, where the relays are operating in series or in parallel. Some preliminary work was done in [29], and the most general notion of this in the classical literature is a multicast network [4]. Lastly, relay channels with feedback would also be interesting to investigate.

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We give a proof of Proposition 4, essentially identical to that of [4]:

Proof. Consider an $(n, 2^R)$ code for $N_{X_1, X_2 \rightarrow B_3 B_3}$. Suppose we have a uniform distribution over the message set $M$, and denote the final classical system obtained by Bob from the POVM measurement by $\hat{M}$. By the classical Fano’s inequality, $nR = H(M) = I(M; \hat{M}) + H(\hat{M}; M) \leq I(M; \hat{M}) + n\delta(n)$, where $\delta(n)$ satisfies $\lim_{n \to \infty} \delta(n) = 0$ if the decoding error is to vanish in asymptotic limit.

We denote by $(X_1)_j, (X_2)_j, (B_2)_j, (B_3)_j$ the respective classical and quantum systems induced by our protocol. We argue

$$I(M; \hat{M}) \leq I(M; B_3^1) = \sum_{j=1}^n I(M; (B_3)_j | B_3^{j-1})$$

$$\leq \sum_{j=1}^n I(M; B_3^j; (B_3)_j)$$

$$\leq \sum_{j=1}^n I((X_1)_j, (X_2)_j, MB_3^j; (B_3)_j)$$

$$= \sum_{j=1}^n I((X_1)_j | (X_2)_j; (B_3)_j).$$

The last step follows from the i.i.d. nature of the $n$ channel uses and the channel is classical-quantum. More explicitly, we can write out the overall state as the protocol progresses, and since the input to the channel on each round is classical, it is not difficult to see that given $(X_1)_j, (X_2)_j, (B_2)_j, (B_3)_j$ is in tensor product with the other systems. This would not hold if the channel takes quantum inputs, for which we would expect an upper bound that involves regularization. Now, similarly,

$$I(M; \hat{M}) \leq I(M; B_3^1)$$

$$\leq I(M; B_3^1, n B_3^1)$$

$$= \sum_{j=1}^n I(M; (B_2)_j, (B_3)_j | B_2^{j-1} B_3^{j-1})$$

$$\leq \sum_{j=1}^n I(M; (B_2)_j, (B_3)_j | B_2^{j-1} B_3^{j-1} (X_2)_j)$$

$$\leq \sum_{j=1}^n I(M; B_2^{j-1} B_3^{j-1}; (B_2)_j, (B_3)_j (X_2)_j)$$

$$\leq \sum_{j=1}^n I((X_1)_j; B_2^{j-1} B_3^{j-1}; (B_2)_j, (B_3)_j (X_2)_j).$$

We consider the following scenarios:

1. **Scenario 1:**
   $$I((X_1)_j; B_2^{j-1} B_3^{j-1}; (B_2)_j, (B_3)_j (X_2)_j).$$

2. **Scenario 2:**
   $$I(M; B_3^1)$$

3. **Scenario 3:**
   $$I(M; B_3^1, n B_3^1)$$

Similarly, we consider the following cases:

1. **Case 1:**
   $$I(M; B_3^1)$$

2. **Case 2:**
   $$I(M; B_3^1, n B_3^1)$$

The final step follows from the i.i.d. nature of the $n$ channel uses and the channel is classical-quantum. More explicitly, we can write out the overall state as the protocol progresses, and since the input to the channel on each round is classical, it is not difficult to see that given $(X_1)_j, (X_2)_j, (B_2)_j, (B_3)_j$ is in tensor product with the other systems. This would not hold if the channel takes quantum inputs, for which we would expect an upper bound that involves regularization. Now, similarly,
where the second equality follows since given $B_2^{j-1}$, one can obtain $(X_2)_j$ by a series of $R$ operations (Note that $(B_2)_0(B_2)_0$ is a trivial system and thus independent of the code.).

Define the state

$$\sigma_{QX_1X_2B_2B_3} = \frac{1}{n} \sum_{q=1}^{n} |q\rangle \langle q|_Q \otimes \sigma_{X_1X_2B_2B_3}^{(q)}$$

where $\sigma^{(q)}$ is the classical-quantum state on the $q^{th}$ round of the protocol, that is, the state on the system $(X_1)_q(X_2)_q(B_2)_q(B_3)_q$. Now, $I(B_2B_3; Q|X_1X_2)_\sigma = 0$, so

$$\sum_{j=1}^{n} I((X_1)_j(X_2)_j; (B_3)_j) = bI(X_1X_2; B_3|Q)_\sigma$$

$$\leq nI(X_1X_2Q; B_3)_\sigma$$

$$= nI(X_1X_2; B_3)_\sigma$$

and similarly

$$\sum_{j=1}^{n} I((X_1)_j; (B_2)_j(B_3)_j(X_2)_j) = nI(X_1; B_2B_3|X_2Q)_\sigma$$

$$\leq nI(X_1Q; B_2B_3|X_2)_\sigma$$

$$= nI(X_1; B_2B_3|X_2)_\sigma.$$ 

Hence,

$$R \leq \min \{I(X_1X_2; B_3)_\sigma, I(X_1; B_2B_3|X_2)_\sigma\} + \delta(n).$$

Now, $\sigma_{X_1X_2B_2B_3}$ is simply a uniform average of all the classical-quantum states from each round of the protocol, it is also a possible classical-quantum state induced by $N_{X_1X_2B_2B_3}$ acting on some classical input distribution $p_{X_1X_2}$. In particular, $R$ is therefore upper bounded by the input distribution which maximizes the quantity on the right-hand side:

$$R \leq \max_{p_{X_1X_2}} \min \{I(X_1X_2; B_3), I(X_1; B_2B_3|X_2)\} + \delta(n).$$

Taking the $n \to \infty$ limit completes the proof. □