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DOI

[10.1016/0165-1684\(94\)90058-2](https://doi.org/10.1016/0165-1684(94)90058-2)

Publication date

1994

Document Version

Final published version

Published in

Signal Processing

[Link to publication](#)

Citation for published version (APA):

Dorst, L., & Van den Boomgaard, R. (1994). Morphological signal processing and the slope transform. *Signal Processing*, 38(1), 79-98. [https://doi.org/10.1016/0165-1684\(94\)90058-2](https://doi.org/10.1016/0165-1684(94)90058-2)

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ELSEVIER

Signal Processing 38 (1994) 79–98

**SIGNAL
PROCESSING**

Morphological signal processing and the slope transform

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Received 13 September 1993

Abstract

This paper presents the operation of *tangential dilation*, which describes the touching of differentiable surfaces. It generalizes the classical dilation, but is invertible. It is shown that line segments are eigenfunctions of this dilation, and are parallel transported, and that curvature is additive. We then present the slope transform which provides tangential morphology with the analytical power which the Fourier transform lends to linear signal processing, in particular: dilation becomes addition (just as under a Fourier transform, convolution becomes multiplication). We give a discrete slope transform suited for implementation, and discuss the relationships to the Legendre transform, the Young–Fenchel conjugate, and the $\mathcal{A}$ -transform. We exhibit a logarithmic correspondence of this tangential morphology to linear systems theory, and touch on the consequences for morphological data analysis of a scanning tunnelling microscope.

Zusammenfassung

Dieser Artikel stellt die Operationen der *tangentialen Dilatation* vor, die die Berührung differenzierbarer Oberflächen beschreiben. Sie verallgemeinert die klassische Dilatation, ist aber umkehrbar. Es wird gezeigt, daß Liniensegmente Eigenfunktionen dieser Dilatation sind und parallel verschoben werden, und daß die Krümmung additiv ist. Wir präsentieren dann die Steigungstransformation, die eine tangentielle Morphologie mit der analytischen Leistung liefert, wie sie die Fouriertransformation für die lineare Signalverarbeitung darstellt, speziell wird Dilatation zu Addition (genauso wie unter der Fouriertransformation Faltung zu Multiplikation wird). Wir präsentieren eine diskrete Steigungstransformation, die für eine Implementierung geeignet ist und besprechen die Beziehungen zur Legendre-Transformation, zur Young–Fenchel Konjugierten und der $\mathcal{A}$ -Transformation. Wir stellen eine logarithmische Übereinstimmung dieser tangentialen Morphologie mit der linearen Systemtheorie dar und berühren die Konsequenzen für die morphologische Datenanalyse eines abtastenden Tunnelmikroskops.

Résumé

Cet article présente l'opération de *dilatation tangentielle*, qui décrit le contact de surfaces dérivables. Elle généralise la dilatation classique mais est inversible. On démontre que les droites sont des fonctions propres de cette dilatation, qu'elles sont simplement déplacées parallèlement à elle même par cette dilatation et que la courbature est additive. On présente ensuite la transformation de pente qui fournit à la morphologie tangentielle un outil analytique similaire à la transformation de Fourier en traitement linéaire du signal, en particulier: la dilatation devient une addition (de façon analogue à la convolution qui devient une multiplication après transformation de Fourier). On donne une version

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numérique de la transformation adaptée à une implémentation et ses relations avec les transformées de Legendre, de Young–Fenchel et $\mathcal{A}$ sont discutées. Une correspondance logarithmique entre la morphologie tangentielle et la théorie des systèmes linéaires est mise en évidence. Finalement, l'analyse morphologique de données de microscope à effet tunnel est considérée.

Key words: Tangential dilation; Slope transform; Morphology

1. Introduction

In the applications and the theory of linear signal processing, the Fourier transform plays a central role. It provides a dual description in the frequency domain, in which the involved basic operation of spatial convolution becomes a multiplication. This then inspires a host of techniques for filter design, deconvolution, noise suppression, etc. It is in fact what makes the linear theory into a theory of *linear signal processing*.

Mathematical morphology is an alternative way of combining signals. Where linear convolution is a good approximation to, for instance, the blurring in optical systems, morphological dilation is a good approximation to the contact-sensing process of a scanning tunnelling microscope [7]. Thus, mathematical morphology and linear systems theory are two different tools to describe different aspects of nature. For the linear phenomena, an analytical theory is well established. It now becomes necessary to develop a theory of *morphological signal processing* to analyze the morphological phenomena. The classical algebraic, set-based and topological approaches to morphology do not provide, by themselves, this – a more quantitative theory is needed to obtain an analytical power comparable to that of linear systems theory.

In this paper, we present the *slope transform*. It is to morphology what the Fourier transform is to linear filtering. It provides a dual description of morphological dilation in a *slope domain*, in which spatial dilation becomes addition. The slope representation provides renewed inspiration for morphological filter design, morphological sampling theorems, and the like. Some advanced theoretical results in morphology can be derived and understood more easily via their dual formulation.

The reader should be prepared for a surprise, however. At first it may have been unexpected that

complex numbers should play such a central role in the description of the convolution of *real* signals, but they do: the Fourier transform is complex-valued. Apparently, in linear signal processing, one gets the most symmetric theory by embedding the real signals as real-valued complex signals. In morphology, something similar happens: the slope transform of general signals is set-valued, and therefore the most symmetric description is achieved by embedding *real-valued* functions into *set-valued* 'functions'. We believe that this is a consequence of the fact that the most natural description of morphology is actually as *the theory of the contact of surfaces*, for which the parametrized descriptions of differential geometry are appropriate. In that description, single-valuedness pertains, but if one chooses a functional description, set-valuedness results. Nevertheless, in this paper we follow the functional description, to connect to both standard morphological notation and to linear filtering.

Because of this embedding, we have to generalize dilations slightly. Indeed, Ghosh [3] first pointed out that a dilation can be split into a highly symmetrical operation followed by a projection operation. We introduce the *tangential dilation*. It is a purely *local* operation on surfaces (or functions), differing from the classical dilation in disregarding the zeroth-order and second-order demands for the actual non-overlapping touching of umbral sets. This symmetrical operation coincides with the classical dilation on convex functions, but is invertible. The slope transform is the canonical description of this tangential set-valued dilation, just as the Fourier transform is the canonical description of complex-valued convolution. Thus, for tangential dilations it is most easy to develop a signal processing theory, and we begin to do so in this paper.

The need for a morphological signal processing theory has been clearly felt in recent literature, and there are numerous recent papers that point

towards the slope transform. Ghosh [3] was among the first to recognize the central importance of *slope* for the description of mathematical morphology (his slope diagrams contain the ‘phase’ information of the slope transform). Van den Boomgaard proved that dilation is a parallel transport of slopes, and formulated the differential equations of mathematical morphology purely in terms of the slope [14]. Maragos’ $\mathcal{A}$ -transform [9] (which appeared simultaneous with the slope transform [2]) will turn out to be a special case of the slope transform, for convex functions. Mattioli [10] gave differential equations for morphological operations in terms of the Young–Fenchel conjugate, which is also a special case of the slope transform. Physicists working on the scanning tunnelling microscope STM [7] use the classic Legendre transform to restore signals obtained by contact probing – as we will show, the Legendre transform is yet another special case of the slope transform.

The paper is organized as follows. We first introduce the tangential dilation in Section 2, with examples to show how it extends classical dilation. Then we give the dual description of arbitrary functions by their slopes, which is the slope transform (Section 3), and prove some relevant properties. We also derive a discrete slope transform suited for implementation. Section 4 shows how these results point towards a theory of morphological signal processing similar to linear signal processing, and indicate the potential application on the scanning tunnelling microscope. As usual, the paper ends with some conclusions. Appendix A specifies the connection to the other known transforms mentioned above.

2. Making morphology differentiable

2.1. The slope theorem

The definition of the dilation of two real-valued functions is

$$(f \oplus g)(x) = \sup_u [f(u) + g(x - u)]. \quad (1)$$

Consider this dilation locally, for two concave¹ functions f and g , at abscissa $\bar{x}$. Let the corresponding supremum be reached at the abscissa $\bar{u}$, so that $(f \oplus g)(\bar{x}) = f(\bar{u}) + g(\bar{x} - \bar{u})$. It is common use to interpret the dilation as the contact of the probe g^* with f , ‘from above’ (see Fig. 1(a)), where g^* is called the *transpose* of g , and defined as

$$g^*(x) = -g(-x). \quad (2)$$

In that view, $\bar{u}$ is the point of contact of f and g^* corresponding to the outcome $f \oplus g$ at abscissa $\bar{x}$. The correctness of this construction follows easily: g^* centered at the point $(\bar{x}, (f \oplus g)(\bar{x})) = (\bar{x}, f(\bar{u}) + g(\bar{x} - \bar{u}))$ is $-g(-(x - \bar{x})) + f(\bar{u}) + g(\bar{x} - \bar{u}) = f(\bar{u}) + g(\bar{x} - \bar{u}) - g(\bar{x} - x)$, which indeed equals $f(\bar{u})$ for $x = \bar{u}$. The value of $\bar{u}$ follows from the following demands:

- The contact is *touching*, so the derivative of the argument of ‘sup’ with respect to u equals 0;

$$\nabla f(\bar{u}) - \nabla g(\bar{x} - \bar{u}) = 0. \quad (3)$$

- The contact is *non-intersecting*: at $\bar{x}$, there is no $\bar{u}$ such that $f(\bar{u}) + g(\bar{x} - \bar{u}) > f(\bar{u}) + g(\bar{x} - \bar{u})$. Locally, this gives a demand on the second-order derivatives of f and g . At longer range from the contact point, it gives a demand on the function values, or ‘zeroth-order’ derivatives.

Thus, for concave functions the dilation satisfies $\nabla f(\bar{u}) = \nabla g(\bar{x} - \bar{u})$. This applies at all $\bar{x}$, leading to the function $(f \oplus g)$ for all x .

Note that the derivative of $(f \oplus g)(x)$ in Eq. (1) with respect to x at $\bar{x}$ equals $\nabla g(\bar{x} - \bar{u})$. We have proved the *slope theorem*.²

Slope theorem. Under a dilation of two concave functions f and g , an infinitesimal line segment tangent to f at $\bar{u}$ is parallel transported from $(\bar{u}, f(\bar{u}))$ to

¹There is a difference in usage of the term ‘convex’ for functions in function theory and mathematical morphology. In function theory, $f: x \mapsto x^2$ is convex, $g: x \mapsto -x^2$ is concave. In morphology, g is called convex since its lower umbra is a convex set. Other than a sign difference, the definitions describe the same objects.

²In the morphological literature, the slope theorem is first found in [14]. However, it is also present as an observation in a physics paper on scanning tunnelling microscopes [12], under the name ‘parallel transport’.

Fig. 1. Dilation and the slope theorem.

$(\bar{x}, f(\bar{u}) + g(\bar{x} - \bar{u}))$, where $\bar{x}$ is given implicitly by $\nabla f(\bar{u}) = \nabla g(\bar{x} - \bar{u})$. As a consequence,

$$\nabla(f \oplus g)(\bar{x}) = \nabla f(\bar{u}) = \nabla g(\bar{x} - \bar{u}). \quad (4)$$

Thus, for concave functions, dilation is a *local* operation, in the sense that the outcome $f \oplus g$ at $\bar{x}$ is only dependent on the local properties of the input function f (namely at $\bar{u}$) and the structuring function g (at $\bar{x} - \bar{u}$, which is where the slope equals $\nabla f(\bar{u})$). Thus, in this *forward formulation* (starting from f and g and indicating the result) we have a local property; in a *backward formulation* (starting from the result and looking for the points that caused it) the slope theorem is less immediate. Therefore, we retain the forward view in this paper.

For non-convex functions, this simple result becomes more involved: not all points generated by the procedure of the slope theorem may be found in

the output, since other points of the input may generate points at the same x with a higher function value – and the zeroth order demand retains the latter only. (Also, there might be more than one location where the slope of g equals $\nabla f(\bar{u})$.)

2.2. Tangential dilation

We now define a new operation, the *tangential dilation*, denoted ' $\check{\oplus}$ '. The recipe for its construction is to apply the slope theorem everywhere, for all points in the domain of f . Pictorially, this is indicated in Figs. 1(a) and (b).

To compute $f \check{\oplus} g$, start from f at a point $\bar{u}$. Determine $\nabla f(\bar{u})$, the slope of f at $\bar{u}$. Find the point $\bar{v}$ such that $\nabla g(\bar{v})$ equals $\nabla f(\bar{u})$. Then the tangential dilation result at $\bar{x} = \bar{u} + \bar{v}$ equals $f(\bar{u}) + g(\bar{v})$. We constrain the definition in this paper to *strictly convex or concave structuring functions* g . We will include an occasional non-strictly convex or concave structuring function as the limit of such a function. (We believe these restrictions will be waived eventually; for now, the lack of some proper formalizations prevents us from doing so.) The input function f will not be restricted: it may contain both convex and concave parts.

The slope theorem states that the slope of $f \check{\oplus} g$ is parallel transported: the input slope $\nabla f(\bar{u})$ equals the output slope $\nabla(f \check{\oplus} g)(\bar{x})$. We have indicated a line connecting input and output point in Fig. 1(a). For concave f , the construction of $f \check{\oplus} g$ is thus the same as for $f \oplus g$.

Fig. 1(b) shows the construction for non-concave f : there may be a 'crossing-over' of the results for two flanks of f . Here tangential dilation and classical dilation differ. Classical dilation retains only the supremum of the result, a single real number: the point of crossing over corresponds to a multiple contact of f and g^* , and the crossed-over part corresponds to a contact with one part of f , but an intersection with another. Tangential dilation notes tangent contact only, and retains all curves. It may therefore have more than one outcome for a given abscissa.

By its construction, the tangential dilation result can be described as a parametrized curve (parametrized by $\bar{u}$) or as a set-valued function of x .

Although we believe that the parametrized description is most natural for the description of these differentiable surfaces, we choose the functional description in this paper. We do this to show most clearly the relationship of the results to classical morphology and to linear signal processing.

The central operator for such a functional notation is the ‘stat’. In words, $\text{stat}_u f(u)$ is the set of all stationary values of $f(u)$ with respect to u . In formula,

$$\text{stat}_u f(u) = \{f(\bar{u}) \mid \nabla f(\bar{u}) = 0\}. \quad (5)$$

There may be more than one such stationary value if f is not convex, or no such value if f is strictly monotonic. Thus, $\text{stat}_u f(u)$ is set-valued, with the empty set included as legitimate outcome.

The equation involving the derivative has a subtlety for non-smooth functions: at kinks in such functions, the derivative may be interval-valued.³ The equation $\nabla f(\bar{u}) = 0$ then holds at those points $\bar{u}$ for which this interval contains 0. The notion of an interval-valued derivative is well-developed in convex analysis, where it is called a *subdifferential* (see [6]). We prefer to view such functions as limits of smooth functions with well-defined derivatives – as such, they seem to present no essential problems. Both interpretations appear to be consistent.

With this ‘stat’ notation, the tangential dilation is defined as

$$(f \check{\oplus} g)(x) = \text{stat}_u [f(u) + g(x - u)]. \quad (6)$$

Note that indeed the slope theorem holds: by the definition of ‘stat’, we have for $\bar{u}$, $\nabla f(\bar{u}) - \nabla g(x - \bar{u}) = 0$, and by differentiation we obtain $\nabla(f \check{\oplus} g)(x) = \nabla g(x - \bar{u})$. Thus, Eq. (6) is indeed the tangential dilation formula.

The relationship between $\oplus$ and $\check{\oplus}$ is simply

$$\sup(f \check{\oplus} g)(x) = (f \oplus g)(x). \quad (7)$$

Proof. This is most easily seen in the interpretation of $f \check{\oplus} g$ as a probe g^* hitting a surface f . Construct this hit as follows. At position $\bar{x}$, lower the function

g^* from above (so form $g^*(x - \bar{x}) + b$ with b decreasing from ∞) until it hits the original function f . Let the point of contact be $(\bar{u}, f(\bar{u}))$, at b^* (we have seen above that $b^* = f(\bar{u}) + g(\bar{x} - \bar{u})$). At that point, $\nabla(g^*(\bar{u} - \bar{x}) + b^*) = \nabla g^*(\bar{u} - \bar{x}) = \nabla g(\bar{x} - \bar{u})$ must be contained in $\nabla f(\bar{u})$ (with equality if f is differentiable at $\bar{u}$). Therefore, the slope theorem holds, and $(f \oplus g)(x)$ must be contained in $(f \check{\oplus} g)(x)$ (which contains all points for which the slope theorem holds). By construction, b^* is the supremal vertical offset such that the hit occurs, and the result follows. $\square$

Note that there are in fact two differences between the classical dilation of Eq. (1) and the tangential dilation: the local nature of the extremum (supremum or infimum) is ignored (no demand on the second derivative) and the stationary value need not be global (no demand on the zeroth-order derivative). Taking the ‘sup’ restores both demands, and makes a tangential dilation result into a result for classical morphology – if the sup exists!

2.3 Tangential erosion is tangential dilation

Tangential dilations with strictly convex (concave) structuring functions are invertible:

$$(f \check{\oplus} g) \check{\oplus} g^* = f. \quad (8)$$

Proof. $(F \check{\oplus} g^*)(x) = \text{stat}_u [F(u) - g(u - x)] = F(\bar{u}) - g(\bar{u} - x)$ with $\nabla F(\bar{u}) = \nabla g(\bar{u} - x)$. Now let $F(x) = f \check{\oplus} g(x) = f(\bar{u}) + g(x - \bar{u})$, such that $\nabla f(\bar{u}) = \nabla g(x - \bar{u}) = \nabla F(x)$. Then we obtain $(F \check{\oplus} g^*)(x) = f(\bar{u}) + g(\bar{u} - \bar{u}) - g(\bar{u} - x)$, with $\nabla g(\bar{u} - \bar{u}) = \nabla g(\bar{u} - x)$. For strictly convex (concave) g , this condition implies $x = \bar{u}$, and we have $((f \check{\oplus} g) \check{\oplus} g^*)(x) = f(x)$, for all x . $\square$

In tangential morphology we permit the transpose g^* of g as a legitimate structuring function, so there is no need to define a separate operator for this inversion. However, to correspond to classical morphology, we might define a *tangential erosion* by

$$\begin{aligned} (f \check{\ominus} g)(x) \\ = (f \check{\oplus} g^*)(x) = \text{stat}_u [f(u) - g(u - x)]. \end{aligned} \quad (9)$$

³If the function has a kink at a , the value of the directional derivative in direction n is $\nabla_n f(a) = [\lim_{x \uparrow a} \nabla_n f(x), \lim_{x \downarrow a} \nabla_n f(x)]$.

In classical morphology on functions, the definition of erosion is [5]

$$(f \ominus g)(x) = \inf_h [f(x+h) - g(h)] \\ = \inf_u [f(u) - g(u-x)], \quad (10)$$

which is obviously contained in Eq. (9). However, we re-emphasize that tangential morphology does not view the erosion as an operator different from dilation.

2.4. Examples of tangential dilation

We give some examples of tangential dilation.

(1) *Parabola dilated with parabola:* Let $f: x \mapsto -x^2/2a_1$ and $g: x \mapsto -x^2/2a_2$. Then

$$(f \oplus g)(x) = \frac{x^2}{2(a_1 + a_2)}. \quad (11)$$

Proof. $\text{stat}_u[-u^2/(2a_1) - (x-u)^2/(2a_2)] = -\bar{u}^2/(2a_1) - (x-\bar{u})^2/(2a_2)$ with $\bar{u}/a_1 = (x-\bar{u})/a_2$, so $\bar{u} = a_1 x/(a_1 + a_2)$ and the result follows. $\square$

Thus, for this case of concave functions, the result coincides with the classical dilation result.

(2) *Planar function dilated by parabola yields parabola:* Let $f: x \mapsto \langle \omega, x \rangle + b$, with $\langle \cdot, \cdot \rangle$ denoting the standard inner product. Thus, f denotes a planar function with ‘slope’ ω . Let $g: x \mapsto -x^2/2a$. Then

$$(f \oplus g)(x) = \langle \omega, x \rangle + b + \frac{1}{2}a\omega^2. \quad (12)$$

Proof. $\text{stat}_u[\langle \omega, u \rangle + b - (x-u)^2/(2a)] = \langle \omega, \bar{u} \rangle + b - (x-\bar{u})^2/(2a)$ with $(x-\bar{u})/a = -\nabla \langle \omega, \bar{u} \rangle = -\omega$, and the result follows. $\square$

Note that the dilation result is the original function f , vertically shifted over $a\omega^2/2$. We will come back to this!

(3) *Parabola dilated by half circles:* Let $f: x \mapsto x^2/2$ and $g_a: x \mapsto \sqrt{a^2 - x^2}$. The dilation of these functions is most easily expressed in parametrization:

$$(x, (f \oplus g_a)(x)) = \left(u - \frac{au}{\sqrt{1+u^2}}, \frac{u^2}{2} + \frac{a}{\sqrt{1+u^2}} \right). \quad (13)$$

Proof. Put f and g_a in parametric form: $(u, u^2/2)$ and $(t, \sqrt{a^2 - t^2})$, with $t \in [-a, a]$. According to the slope theorem, for a dilation result at $x = t + u$ we need to determine the relationship between u and t such that: $f'(u) = g'_a(t)$, so $u = -t\sqrt{a^2 - t^2}$. It follows that $t = -au/\sqrt{1+u^2}$, valid for $t \in (-a, a)$. The slope theorem then gives Eq. (13) for the dilation result. $\square$

The dilation result is shown in Fig. 2 (for $a = 0.5, 1, 1.5, 2, 2.5$). Note that the dilation of these two innocent functions leads to cusped, non-functional objects (at $a = 1$, a *swallowtail catastrophe* occurs at the point $(0, 1)$; see [11]). This is a consequence of the consistent application of the slope theorem. Therefore, the cusps are not kinks: the curves that result are still differentiable. Also these cusped curves can be tangentially dilated: the dilation by a circle of size 2.5, for instance, can be achieved by tangential dilation of the result at $a = 2.0$ by a circle with radius 0.5. For consistency, we thus need to include those cusped objects as permissible ‘functions’ on which tangential morphology can be performed.

(4) *Sinusoid dilated by parabola:* This example shows that a function with both concave and convex sections can be handled in a straightforward way by tangential dilation. Let $f: x \mapsto \sin x$ and $g: x \mapsto -x^2/(2a)$. The dilation of these functions is most easily expressed in parametrization:

$$(x, (f \oplus g)(x)) \\ = (u - a \cos u, \sin u - a/2 \cos^2 u). \quad (14)$$

Proof. Put f and g in parametric form: $(u, \sin u)$ and $(t, -t^2/(2a))$. According to the slope theorem, we need to determine the relationship between u and t such that $u = x - t$ and $f'(u) = g'(t)$, so $\cos u = -t/a$. It follows that $t = -a \cos u$. The slope theorem then gives Eq. (14) for the dilation result. $\square$

Fig. 2. Dilation of a parabola with half-circles of increasing radius.

The dilation result was already shown in Fig. 1(b) (for $a = 2$).

(5) *Parabola dilated by cone:* This examples involves an interval-valued derivative. Let $f: x \mapsto x^2$ and $g: x \mapsto |x|$:

$$(f \oplus g)(x) = \begin{cases} x - 1/4 & \text{if } x \geq 1/2, \\ x^2 & \text{if } x \in [-1/2, 1/2], \\ -x - 1/4 & \text{if } x \leq -1/2. \end{cases} \quad (15)$$

Proof. We have $\text{stat}_u[u^2 + |x - u|] = \bar{u}^2 + |x - \bar{u}|$ with, for $x - \bar{u} > 0$, $2\bar{u} - 1 = 0$ and, for $(x - \bar{u}) < 0$, $2\bar{u} + 1 = 0$. The interval-valued differentiation is needed for $x - \bar{u} = 0$. At that point the function $|x|$ changes slope from -1 to 1 , and the derivative at that point equals the interval $[-1, 1]$. We thus get the following for the derivative: $2\bar{u} + [-1, 1] = 0$, together with $x = \bar{u}$. The result is therefore x^2 with $x \in [-1/2, 1/2]$. $\square$

2.5. How morphology changes curvature

We can use the tangential dilation to establish a result on second derivatives and radius of curvature. We prove the n -dimensional result on the Hessians $\mathcal{H}$ (matrix of second derivatives) of the functions involved:

$$\mathcal{H}_{f \oplus g}^{-1}(x) = \mathcal{H}_f^{-1}(\bar{u}) + \mathcal{H}_g^{-1}(x - \bar{u}). \quad (16)$$

Proof. By repeated application of the slope theorem. Let us introduce a convenient notation. Superscript i denotes the i th component, subscript i denotes differentiation to x^i . Denote $f \oplus g$ by h . The slope theorem reads in this notation:

$$h_i(x) = f_i(\bar{u}) = g_i(x - \bar{u}). \quad (17)$$

Note that $\bar{u}$ depends on x . Differentiating the slope theorem, we obtain

$$h_{ij}(x) = f_{ik}(\bar{u})\bar{u}_j^k = g_{ik}(x - \bar{u})(x_j^k - \bar{u}_j^k). \quad (18)$$

The second inequality can be rewritten as

$$\begin{aligned} (f_{ik}(\bar{u}) + g_{ik}(x - \bar{u}))\bar{u}_j^k \\ = g_{ik}(x - \bar{u})x_j^k = g_{ij}(x - \bar{u}). \end{aligned} \quad (19)$$

This is a matrix equation for the components of the Hessian matrices. Let $\mathcal{H}_f(x)$ denote the Hessian matrix of f at x ; then $g_{ij}(x - \bar{u}) = [\mathcal{H}_g(x - \bar{u})]_{ij}$, etc. We obtain

$$(\mathcal{H}_f(\bar{u}) + \mathcal{H}_g(x - \bar{u}))J = \mathcal{H}_g(x - \bar{u}), \quad (20)$$

where we introduced the Jacobian matrix $J: [J]_{kj} = u_j^k$. This gives

$$J = (\mathcal{H}_f(\bar{u}) + \mathcal{H}_g(x - \bar{u}))^{-1} \mathcal{H}_g(x - \bar{u}). \quad (21)$$

The first inequality in Eq. (18) gives

$$\mathcal{H}_h(x) = \mathcal{H}_f(\bar{u})J. \quad (22)$$

Therefore, we have

$$\mathcal{H}_h(x) = \mathcal{H}_f(\bar{u})(\mathcal{H}_f(\bar{u}) + \mathcal{H}_g(x - \bar{u}))^{-1} \mathcal{H}_g(x - \bar{u}), \quad (23)$$

which implies

$$\begin{aligned} \mathcal{H}_h(x)^{-1} \\ = \mathcal{H}_g(x - \bar{u})^{-1}(\mathcal{H}_f(\bar{u}) + \mathcal{H}_g(x - \bar{u}))\mathcal{H}_f(\bar{u})^{-1} \end{aligned}$$

$$= \mathcal{H}_g(x - \bar{u})^{-1} + \mathcal{H}_f(\bar{u})^{-1}. \quad \square \quad (24)$$

This result (but for classical dilation of convex functions) may be found (unproven) in an article by Keller on data analysis for a scanning tunnelling microscope [7]! In Section 4.2 we will discuss this morphological machine in more detail.

In one dimension we thus have for the second derivatives:

$$\frac{1}{(f \oplus g)''(x)} = \frac{1}{f''(\bar{u})} + \frac{1}{g''(x - \bar{u})}. \quad (25)$$

For a concave paraboloid g , $g''(x - \bar{u})$ is a negative constant, and therefore dilation of a concave function f by g brings its second derivative closer to 0: the function ‘flattens out’, locally.

For two dimensions the bounding of the Laplacian in a classical opening (closing) by a paraboloid was derived in [4]. The connection between Eq. (16) and that result is through the projection operator Eq. (7), as we hope to show in a future paper.

With the derivation of the second-order properties, we have obtained an interesting sequence of equations for dilations. Let us consider the one-dimensional case, and change notation for clarity: let the abscissae of the contact points be x , $x_f (= \bar{u})$ and $x_g (= x - \bar{u})$. We have

$$x = x_f + x_g, \quad (26)$$

$$(f \oplus g)(x) = f(x_f) + g(x_g), \quad (27)$$

$$(f \oplus g)'(x) = f'(x_f) = g'(x_g), \quad (28)$$

$$\frac{1}{(f \oplus g)''(x)} = \frac{1}{f''(x_f)} + \frac{1}{g''(x_g)}. \quad (29)$$

If we consider the functions as local descriptions of the contours of objects, in some coordinate system, we are interested in results that are independent of the choice of that coordinate system. The slope theorem is such a result, if cast in the form of ‘parallel normal transport’ [14]. A result involving second-order properties is the addition of radii of curvature. Let $R_f(x)$ be the radius of curvature of a function f at x . Then, in the above notation,

$$R_{f \oplus g}(x) = R_f(x_f) + R_g(x_g). \quad (30)$$

Proof. Standard geometry gives $R_f(x) = (1 + f'(x)^2)^{3/2} / f''(x)$. Thus, $R_f(x_f) + R_g(x_g) = (1 + f'(x_f)^2)^{3/2} / f''(x_f) + (1 + g'(x_g)^2)^{3/2} / g''(x_g) = (1 + (f \oplus g)'(x))^3 / ((1 + f'(x_f)^2)^{3/2} f''(x_f) + (1 + g'(x_g)^2)^{3/2} g''(x_g)) = 1 + (f \oplus g)'(x)^3 / ((f \oplus g)''(x)) = R_{f \oplus g}(x)$. Since the equation holds in one coordinate system, and is coordinate-independent, it holds in all. $\square$

This result is again found in the paper by Keller [7]. One of the consequences of Eq. (30) for morphology is that an inflection point (infinite radius of curvature) in f or g leads to an inflection point in $f \oplus g$: *inflection points are preserved*.

3. The slope transform

3.1. Morphological eigenfunctions

Since a tangential dilation does not change slopes locally, but just translates the point carrying that slope, it follows that it translates a function with a constant slope as a whole. Thus, such functions are *morphological eigenfunctions*: they may change their location (which we will describe like amplitude and phase), but not their shape. Obviously, such functions are the *planar functions* $e_\omega: x \mapsto \langle \omega, x \rangle$.

Planar functions $e_\omega: x \mapsto \langle \omega, x \rangle$ are eigenfunctions of tangential morphology. We have

$$(e_\omega \oplus g)(x) = e_\omega(x) + \mathcal{S}[g](\omega), \quad (31)$$

where $\mathcal{S}[g]$ is a function only dependent on g , which we may consider as an additive ‘phase factor’.

Proof. $(e_\omega \oplus g)(x) = \text{stat}_u [\langle \omega, u \rangle + g(x - u)] = \langle \omega, \bar{u} \rangle + g(x - \bar{u})$ with $\omega = \nabla g(x - \bar{u})$, so $\bar{u} = x - (\nabla g)^{-1}(\omega)$. The inverse exists because of the demand of strict convexity (concavity) on g . Therefore, $(e_\omega \oplus g)(x) = \langle \omega, x \rangle + [g((\nabla g)^{-1}(\omega)) - \langle \omega, (\nabla g)^{-1}(\omega) \rangle] = \langle \omega, x \rangle + \mathcal{S}[g](\omega)$. $\square$

Each eigenfunction e_ω thus gets translated by an amount that only depends on the structuring function g (and of course on ω). This amount is characteristic for g . If we can now invertibly decompose arbitrary functions into the planar eigenfunctions,

then a dilation by g can be described as the composition of the shifted eigenfunctions. Further, if we also decompose g into eigenfunctions, then the description of dilation involves the eigenfunctions only. The idea leads immediately to the slope transform.

3.2. Caustic decomposition

Fig. 3 gives a geometric construction on which a dual representation of functions by slopes can be based. Consider first Fig. 3(a), where f is a concave function. At a point $(\bar{x}, f(\bar{x}))$, draw the tangent plane. The slope of this tangent plane is $\nabla f(\bar{x})$; denote it by $\bar{\omega}$. With ∇f invertible, we thus have $\bar{x} = (\nabla f)^{-1}(\bar{\omega})$. Now note the intercept of the tangent plane with the functional axis. This real

Fig. 3. The definition of the slope transform for (a) concave and (b) non-concave functions.

Fig. 4. Some enveloping tangent lines of $\sin x$.

number equals $f(\bar{x}) - \langle \bar{\omega}, \bar{x} \rangle$. We call this the *slope transform* of f at $\bar{\omega}$, denoted by $\mathcal{S}[f](\bar{\omega})$.

Using the ‘stat’ operator, we may write

$$\mathcal{S}[f](\omega) = \text{stat}_x[f(x) - \langle \omega, x \rangle], \quad (32)$$

since this implies that $\omega = \nabla \langle \omega, x \rangle = \nabla f(x)$. This definition, motivated for the concave case, actually also serves well for the case where a given slope ω is assumed at more than one value x (see Fig. 3(b)), and even when f is set-valued. Therefore, we take it as the definition of the slope transform $\mathcal{S}[f]$ for arbitrary set-valued functions f .

Given the intercepts as a function of the slope, the original function can be reconstructed. Geometrically, this amounts to considering the function as the *caustic* of its tangent planar functions (see Fig. 4). The corresponding formula reconstructs f by determining the stationary points on the tangent planes when they vary with ω . This *inverse slope transform* is given by

$$f(x) = \text{stat}_\omega[\mathcal{S}[f](\omega) + \langle \omega, x \rangle]. \quad (33)$$

Proof. Let the argument of Eq. (32) be stationary for $\bar{x}$, so $\nabla f(\bar{x}) = \omega$ and $\mathcal{S}[f](\omega) = f(\bar{x}) - \langle \omega, \bar{x} \rangle$.

original	transform
$\frac{1}{2a}x^2$	$-\frac{1}{2}a\omega^2$
$\pm\sqrt{1-x^2}$	$\pm\sqrt{1+\omega^2}$
$\frac{1}{3}x^3$	$\mp\frac{2}{3}\omega^{3/2}$
$-\frac{1}{x}$	$\mp 2\sqrt{\omega}$
$\log x$	$-1 - \log \omega$
$\Delta(x-a)+b$	$-a\omega+b$
$ax+b$	$\Delta(\omega-a)+b$
$ x $	$I(\omega)$

Fig. 5. Slope transform of common functions.

Then $\text{stat}_\omega[\mathcal{S}[f](\omega) + \langle \omega, x \rangle] = \text{stat}_\omega[f(\bar{x}) + \langle \omega, x - \bar{x} \rangle]$. Let the stationary value be reached for $\bar{\omega}$, so $0 = \langle \nabla f(\bar{x}), \nabla_\omega \bar{x}(\bar{\omega}) \rangle + (x - \bar{x}) - \langle \omega, \nabla_\omega \bar{x}(\bar{\omega}) \rangle = x - \bar{x}$ (remember that $\bar{x}$ is an implicit function of ω through $\nabla f(\bar{x}) = \omega$). It follows that $\bar{\omega}$ exists iff $x = \bar{x}$, and then $\text{stat}_\omega[\mathcal{S}[f](\omega) + \langle \omega, x \rangle] = f(x)$. $\square$

Note the duality between the two formulas (32) and (33). The argument in the proof shows that

$$\nabla f(\bar{x}) = \bar{\omega} \Leftrightarrow \nabla \mathcal{S}[f](\bar{\omega}) = -\bar{x}. \quad (34)$$

Note that $\mathcal{S}[g]$ in Eq. (31) is indeed the slope transform of g (as its notation suggests): when ∇g is invertible, $\mathcal{S}[g](\omega) = g((\nabla g)^{-1}(\omega)) - \langle \omega, (\nabla g)^{-1}(\omega) \rangle$, in agreement with the expression in the proof of Eq. (31).

The slope transform extends some known transforms: the Legendre transformation, the Young–Fenchel conjugate and the $\mathcal{A}$ -transform. The relationship to these transforms is given in Appendix A.

3.3. Slope transforms of some common functions

Fig. 5 gives the slope transform of some standard functions. As in the text, x represents a spatial vector, and ω a slope vector. Note that the slope transform of a concave function is a convex function. Note also that the slope transform of a non-convex function (such as $\frac{1}{3}x^3$) is generally

set-valued. We derive a few of the entries. The notation $\xleftrightarrow{\mathcal{S}}$ will denote transition to the dual domain.

(1) *Paraboloid becomes paraboloid*: Let $A = A^T$ and $\det A \neq 0$. Then

$$\frac{1}{2} x^T A x \xleftrightarrow{\mathcal{S}} -\frac{1}{2} \omega^T A^{-1} \omega. \quad (35)$$

Proof. $\text{stat}_x [x^T A x / 2 - \langle \omega, x \rangle] = [\bar{x}^T A \bar{x} / 2 - \langle \omega, \bar{x} \rangle]$ with $\nabla(x^T A x / 2)(\bar{x}) = A \bar{x} = \omega$, so $\bar{x} = A^{-1} \omega$ and the result follows. $\square$

The one-dimensional version is

$$\frac{1}{2a} x^2 \xleftrightarrow{\mathcal{S}} -\frac{1}{2} a \omega^2. \quad (36)$$

Note that this gives an alternative computation for Eq. (12): Eq. (31) shows that the dilation of a line by a function adds the slope transform of that function (see proof of Eq. (31)), which in case of the parabola $-x^2/(2a)$ equals $a\omega^2/2$.

(2) *Half circle becomes half hyperbola*:

$$\pm \sqrt{1 - x^2} \xleftrightarrow{\mathcal{S}} \pm \sqrt{1 + \omega^2}. \quad (37)$$

Proof. In $F(\omega) = \text{stat}_x [\pm \sqrt{1 - x^2} - \omega x]$, we have $\mp \bar{x} / \sqrt{1 - \bar{x}^2} = \omega$, so $\bar{x} = \mp \omega / \sqrt{1 + \omega^2}$, and the result follows. $\square$

We introduce some special functions. The first is a *morphological Delta function*:

$$\Delta(x) = \begin{cases} 0 & \text{if } x = 0, \\ -\infty & \text{if } x \neq 0. \end{cases} \quad (38)$$

It is the functional representation of a point at $(0, 0)$.

We also need an *interval function* $I(x)$:

$$I(x) = \begin{cases} 0 & \text{if } \|x\| \leq 1, \\ -\infty & \text{if } \|x\| > 1. \end{cases} \quad (39)$$

This is the morphological counterpart of the block function in linear filtering. In the linear theory, *multiplication* by a block function results in effective

restriction of a function to an interval – in morphology, *addition* of the interval function $I(\cdot)$ results in such a restriction.

These special functions can be transformed:

(1) *Planar function to point*: The slope transform of a planar eigenfunctions $x \mapsto \langle a, x \rangle + b$ is a point (a, b) in slope space, which as a function is just the morphological Delta function $\Delta(\omega - a) + b$:

$$\langle a, x \rangle + b \xleftrightarrow{\mathcal{S}} \Delta(\omega - a) + b. \quad (40)$$

Proof. $\text{stat}_x [\langle a, x \rangle + b - \langle \omega, x \rangle] = \langle a - \omega, \bar{x} \rangle + b$ with $\bar{x}$ defined by $a - \omega = 0$. Therefore, only for $a = \omega$ does ‘stat’ have a non-empty value, equal to b , and the result follows. $\square$

(2) *Cone to interval*:

$$|x| \xleftrightarrow{\mathcal{S}} I(\omega). \quad (41)$$

Proof. We derive the result by taking limits of functions. We refer to [6] for proofs on the permissibility of taking limits, in the analogous case of convex functions. Now, $|x| = \lim_{a \rightarrow 0} a \sqrt{1 + (x/a)^2} \xleftrightarrow{\mathcal{S}} \lim_{a \rightarrow 0} a \sqrt{1 - \omega^2} = I(\omega)$.

Using interval-valued gradients, one has an alternative derivation of the result: $\text{stat}_x [|x| - \langle \omega, x \rangle] = |\bar{x}| - \langle \omega, \bar{x} \rangle$ with $\nabla |x|(\bar{x}) = \omega$. So for $\bar{x} > 0$, $\omega = 1$ and for $\bar{x} < 0$, $\omega = -1$. At $\bar{x} = 0$ the gradient is interval-valued and equal to $[-1, 1]$. The result follows. $\square$

3.4. Properties of the slope transform

Important properties of the slope transform are summarized in Fig. 6. Most are easily proved, for instance the property

$$a f(x) \xleftrightarrow{\mathcal{S}} a \mathcal{S}[f] \left(\frac{\omega}{a} \right). \quad (42)$$

So, multiplication in the spatial domain becomes umbral scaling in the slope domain.

Proof. $\text{stat}_x [a f(x) - \langle \omega, x \rangle] = a \text{stat}_x [f(x) - \langle \omega/a, x \rangle] = a \mathcal{S}[f](\omega/a)$. $\square$

original	transform
$f(x)$	$\mathcal{S}[f](\omega) = \text{stat}_x[f(x) - \langle \omega, x \rangle]$
$\mathcal{S}^{-1}[F](x) = \text{stat}_\omega[F(\omega) + \langle \omega, x \rangle]$	$F(\omega)$
$a + f(x)$	$a + \mathcal{S}[f](\omega)$
$f(x - a)$	$-\langle \omega, a \rangle + \mathcal{S}[f](\omega)$
$ax + f(x)$	$\mathcal{S}[f](\omega - a)$
$f(ax)$	$\mathcal{S}[f](\frac{\omega}{a})$
$af(x)$	$a\mathcal{S}[f](\frac{\omega}{a})$
$af(\frac{x}{a})$	$a\mathcal{S}[f](\omega)$
$f^*(x) \equiv -f(-x)$	$-\mathcal{S}[f](\omega)$
$f^{-1}(x)$	$-\omega\mathcal{S}^{-1}[f](-1/\omega)$
$F(x)$	$f(-\omega)$
$(f \oplus g)(x)$	$\mathcal{S}[f](\omega) + \mathcal{S}[g](\omega)$
$f(x) + g(x)$	$(\mathcal{S}[f] \oplus \mathcal{S}[g])(\omega)$

Fig. 6. Properties of the slope transform.

A property which saves work is the *duality property*:

$$\text{if } f(x) \xleftrightarrow{\mathcal{S}} F(\omega) \text{ then } F(x) \xleftrightarrow{\mathcal{S}} f(-\omega). \quad (43)$$

Proof. $\mathcal{S}[F](\omega) = \text{stat}_x[F(x) - \langle \omega, x \rangle] = \text{stat}_x[F(x) + \langle -\omega, x \rangle] = \mathcal{S}^{-1}[F](-\omega) = f(-\omega)$. $\square$

So, once we have the slope transform of a function f in the spatial domain, we can easily obtain the inverse slope transform of the same function in the slope domain. Example: $|x| \xleftrightarrow{\mathcal{S}} I(\omega)$, so $I(x) \xleftrightarrow{\mathcal{S}} |-\omega| = |\omega|$.

The property for the slope transform that makes it like the Fourier transform of tangential morphology is: *dilation becomes addition*.

$$\mathcal{S}[f \oplus g](\omega) = \mathcal{S}[f](\omega) + \mathcal{S}[g](\omega). \quad (44)$$

We will refer to this as the *dilation theorem*.

Proof.

$$\begin{aligned} \mathcal{S}[f](\omega) &= \text{stat}_x[(f \oplus g)(x) - \langle \omega, x \rangle] \\ &= \text{stat}_x[f(x) - \langle \omega, x \rangle] + \text{stat}_x[g(x) - \langle \omega, x \rangle] \\ &= \mathcal{S}[f](\omega) + \mathcal{S}[g](\omega). \quad \square \end{aligned}$$

$$\begin{aligned} &= \{(f \oplus g)(\bar{x}) - \langle \omega, \bar{x} \rangle | \omega \in \nabla(f \oplus g)(\bar{x})\} \\ &= \{\text{stat}_u[f(u) + g(\bar{x} - u)] - \langle \omega, \bar{x} \rangle | \\ &\quad \omega \in \nabla(f \oplus g)(\bar{x})\} \\ &= \{f(\bar{u}) + g(\bar{x} - \bar{u}) - \langle \omega, \bar{x} \rangle | \omega \in \nabla(f \oplus g)(\bar{x}) \\ &\quad \text{and } \nabla f(\bar{u}) = \nabla g(\bar{x} - \bar{u})\} \\ &= [f(\bar{u}) + g(\bar{x} - \bar{u}) - \langle \omega, \bar{x} \rangle | \omega \in \nabla f(\bar{u}) \\ &\quad = \nabla g(\bar{x} - \bar{u})\} \text{ (slope theorem)} \\ &= \{f(\bar{u}) + g(\bar{v}) - \langle \omega, \bar{u} - \bar{v} \rangle | \omega \in \nabla f(\bar{u}) \\ &\quad = \nabla g(\bar{v})\} \\ &= \{[f(\bar{u}) - \langle \omega, \bar{u} \rangle] + [g(\bar{v}) - \langle \omega, \bar{v} \rangle] | \\ &\quad \omega \in \nabla f(\bar{u}) = \nabla g(\bar{v})\} \\ &= \{f(\bar{u}) - \langle \omega, \bar{u} \rangle | \omega \in \nabla f(\bar{u}) = \nabla g(\bar{v})\} \\ &\quad + \{g(\bar{v}) - \langle \omega, \bar{v} \rangle | \omega \in \nabla g(\bar{v}) = \nabla f(\bar{u})\} \\ &= \{f(\bar{u}) - \langle \omega, \bar{u} \rangle | \omega \in \nabla f(\bar{u})\} \\ &\quad + \{g(\bar{v}) - \langle \omega, \bar{v} \rangle | \omega \in \nabla g(\bar{v})\} \\ &= \text{stat}_u[f(u) - \langle \omega, u \rangle] + \text{stat}_v[g(v) - \langle \omega, v \rangle] \\ &= \mathcal{S}[f](\omega) + \mathcal{S}[g](\omega). \quad \square \end{aligned}$$

Note that the $+$ in Eq. (44) is an *addition of the one-dimensional sets* $\mathcal{S}[f](\omega)$ and $\mathcal{S}[g](\omega)$ at fixed ω . It is therefore in fact a Minkowski sum, commonly denoted by $\oplus$. Since g is strictly convex (concave), $\mathcal{S}[g](\omega)$ is a one-element set, the Minkowski sum becomes trivial. We have chosen to denote it by $+$ to prevent confusion with the n -dimensional dilation operation in the spatial domain.

3.5. Tangential dilation in the slope domain

(1) *Dilation of parabola by parabola:* We computed this result in Eq. (11), using the definition of the tangential dilation. Using the dilation theorem and the slope transform of a parabola, the result is immediate. In a somewhat sloppy but self-explanatory notation,

$$\left(-\frac{x^2}{2a_1}\right) \oplus \left(-\frac{x^2}{2a_2}\right) \xleftrightarrow{\mathcal{S}} \left(a_1 \frac{\omega^2}{2}\right) + \left(a_2 \frac{\omega^2}{2}\right) \\ = (a_1 + a_2) \frac{\omega^2}{2} \xleftrightarrow{\mathcal{S}} -\frac{x^2}{2(a_1 + a_2)}. \quad (45)$$

(2) *Parabola dilated by half-circle:* We have performed the dilation of $x^2/2$ by $\sqrt{a^2 - x^2}$ in Eq. (13). Fig. 7(a) shows the operation in the slope domain, for $a = 2$. The hyperbola is the slope transform of the half-circle, according to Eq. (37); addition of the parabola (which is the slope transform of the spatial parabola by Eq. (36)) gives the dotted shape. Inverse slope transformation of the figure gives Fig. 7(b). Compare this to Fig. 2, for $a = 2$.

(3) *Sinusoid dilated by parabola:* We have performed the dilation of $\sin x$ by $-x^2/4$ in Eq. (14). Fig. 8(a) shows the operation in the slope domain. The star-like shape is the set-valued slope transform of the sinusoid; addition of the parabola (which is the slope transform of the spatial parabola) gives the dotted star. Inverse slope transformation of the figure gives Fig. 8(b), which may be compared to Fig. 1(b).

(4) *Dilation by a cone:* When we dilate a function by a cone, we limit the slopes of the result:

$$f(x) \oplus |x| \xleftrightarrow{\mathcal{S}} \mathcal{S}[f](\omega) + I(\omega) = \mathcal{S}[f](\omega)$$

$$\text{such that } \omega \in [-1, 1]. \quad (46)$$

Proof. Direct application of the dilation theorem and the dual of Eq. (41). $\square$

We have seen an example of this in Eq. (15). The slope transform gives the perspective that slopes higher than 1 and lower than -1 are ‘suppressed’, or ‘filtered out’. This results in a Lipschitz function.

(5) *Dilation by a flat structuring function:* Dilation by a flat structuring function leads to a spreading out of the local extrema, and a shift of the function at non-extremal points. Let us consider this in one dimension:

$$(f \oplus I)(x) \xleftrightarrow{\mathcal{S}} \mathcal{S}[f](\omega) + |\omega|. \quad (47)$$

Proof. Dilation theorem and dual of Eq. (41). $\square$

Thus, for positive ω , there is an addition of ω , which according to Fig. 6 corresponds to a shift of the function to the left, and similarly we have a shift to the right for negative ω . At $\omega = 0$, the gradient of the slope transform acquires an additive factor of $\nabla|\omega| = [-1, 1]$, i.e. it becomes interval-valued. According to Eq. (34), $\nabla\mathcal{S}[f](\omega)$ equals $-x$, so we obtain the same value $f(x)$ at an interval of abscissae: the extremum spreads out. (A more detailed analysis would show that local maxima spread out a positive amount, local minima a negative amount.)

(6) *Dilation by a family of umbral scaled functions:* Suppose we have a family of functions g^t , obtained from some strictly convex (concave) function g by umbral scaling parametrized by t :

$$g^t(x) = t g\left(\frac{x}{t}\right). \quad (48)$$

Tangential dilation of a function f by members of this family forms an additive group in t :

$$(f \oplus g^{t_1}) \oplus g^{t_2} = f \oplus g^{t_1+t_2}. \quad (49)$$

This is obvious in the slope domain. With $g^t \xleftrightarrow{\mathcal{S}} t\mathcal{S}[g]$ (by the dual of Eq. (42)) we obtain $((f \oplus g^{t_1}) \oplus g^{t_2}) \xleftrightarrow{\mathcal{S}} (\mathcal{S}[f] + t_1\mathcal{S}[g]) + t_2\mathcal{S}[g]$

Fig. 7. The tangential dilation of a parabola f by a half-circle g (a) in the slope domain and (b) in the spatial domain.

Fig. 8. The tangential dilation of a sinusoid f by a parabola g (a) in the slope domain and (b) in the spatial domain.

$=\mathcal{S}[f] + (t_1 + t_2)\mathcal{S}[g] \xrightarrow{\mathcal{L}} f \oplus g^{(t_1+t_2)}$. In classical dilation, umbral scaling leads to an additive semi-group in t [13].

3.6. The discrete slope transform

Figs. 7 and 8 were generated using a discrete version of the slope transform, in one dimension. It is

based on the parametric representation of curves. In the parametric representation in one dimension the slope transform is

$$(x(t), y(t)) \xleftrightarrow{\mathcal{S}} \left(\frac{\dot{y}}{\dot{x}}(t), \frac{\dot{x}y - x\dot{y}}{\dot{x}}(t) \right) \equiv (X(t), Y(t)). \quad (50)$$

Proof. $(x(t), y(t)) = (x(t), f(x(t))) \xleftrightarrow{\mathcal{S}} \omega, \mathcal{S}[f](\omega)$ with $\omega = f'(x) = y'(t)/x'(t) = (\dot{y}/\dot{x})(t)$. Here $\dot{y} = (\partial/\partial t)y$ and $\dot{x} = (\partial/\partial t)x$. We obtain $\mathcal{S}[f](\omega) = f(x) - \omega x = (\dot{x}y - x\dot{y})/\dot{x}$. $\square$

This formula shows clearly that the slope transform is a *local* transformation: each oriented point in the spatial domain determines a point in the slope domain.

The *inverse parametrized slope transform* is

$$\left(-\frac{\dot{Y}}{\dot{X}}(t), \frac{\dot{X}Y - X\dot{Y}}{\dot{X}}(t) \right) \xleftrightarrow{\mathcal{S}^{-1}} (X(t), Y(t)). \quad (51)$$

Proof. $X = \dot{y}/\dot{x}$, so $\dot{X} = (\ddot{y}\dot{x} - \dot{y}\ddot{x})/(\dot{x})^2$, and $Y = (y\dot{x} - x\dot{y})/\dot{x}^2$, so $\dot{Y} = -x(\ddot{y}\dot{x} - \dot{y}\ddot{x})/(\dot{x})^2$. Therefore, $x = -\dot{Y}/\dot{X}$. The expression for y follows by straightforward substitution. $\square$

If we have a polygonal arc given as a sequence of points (x_i, y_i) , with $i \in [0, n]$, we can make the slope transform by making a discrete version of Eq. (50). We have to choose some sensible approximation for the derivatives with respect to i . Let us take the simplest:

$$\dot{x}(i) \rightarrow x_i - x_{i-1}, \quad i \in [1, n]. \quad (52)$$

The *discrete parametrized slope transform* then reads

$$(x_i, y_i) \xleftrightarrow{\mathcal{S}} \left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}}, \frac{x_i y_{i-1} - x_{i-1} y_i}{x_i - x_{i-1}} \right) \equiv (X_i, Y_i), \quad (53)$$

and the *inverse discrete parametrized slope transform* is

$$\left(-\frac{Y_{i+1} - Y_i}{X_{i+1} - X_i}, \frac{X_{i+1} Y_i - X_i Y_{i+1}}{X_{i+1} - X_i} \right) \xleftrightarrow{\mathcal{S}^{-1}} (X_i, Y_i). \quad (54)$$

Proof. Follows by straightforward calculation. (It should be noted that the inverse is an exact algebraic inverse. It can be obtained from the inverse parametrized slope transform by approximating $\dot{X}(i)$ by $\dot{X}(i) \rightarrow X_{i+1} - X_i$, for $i \in [0, n-1]$. This compensates exactly for the asymmetry introduced by the definition of the discrete derivative in Eq. (52). $\square$

We can now make an implementation of the slope transform as transformations on discrete sequences of points. The addition of the slope transforms for strictly convex (concave) structuring functions is straightforward since it involves a trivial Minkowski addition of their slope transforms (see the remark after Eq. (44)). The addition of such sequences at fixed abscissae requires interpolation, since the sequences do not necessarily contain points with the same abscissa in the slope domain, even though they may have been sampled equidistantly in the spatial domain. This interpolation presents no problem – the figures were generated with a linear interpolation.

4. Towards morphological signal processing

We believe that the slope transform adds an analytical power to mathematical morphology which helps bring it on a par with linear signal processing. We motivate this view in this section, and enquire what more is needed to create a morphological systems theory.

4.1. The analogy with linear systems theory

We have listed some results from tangential morphology and from linear signal processing in Fig. 9. There appears to be an almost logarithmic correspondence between the two in their basic properties, which an appropriately abstract description might explain. For now, one can only be pleasantly surprised at this structure.

The existence of this correspondence implies that many techniques from linear theory may be applied to the development of a theory of morphological signal processing. It appears that we do not need to

	tangential morphology	linear systems theory
elementary operation	$(f \oplus g)(x) = \text{stat}_u [f(x) + g(x - u)]$	$(f * g)(x) = \int f(u)g(x - u)dx$
eigenfunctions	$\langle \omega, x \rangle$	$e^{-i\langle \omega, x \rangle}$
canonical transform	$S[f](\omega) = \text{stat}_x [f(x) - \langle \omega, x \rangle]$	$\mathcal{F}[f](\omega) = \int f(x)e^{-i\langle \omega, x \rangle} dx$
translation $f(x - a)$	$S[f](\omega) - \langle \omega, a \rangle$	$\mathcal{F}[f](\omega)e^{-i\langle \omega, a \rangle}$
stretching $f(ax)$	$S[f](\omega/a)$	$\mathcal{F}[f](\omega/a)/ a $
scaling $af(x)$	$aS[f](\omega/a)$	$a\mathcal{F}[f](\omega)$
central theorem	$S[f \oplus g] = S[f] + S[g]$	$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$
band-limitation	dilation with $ x $	convolution with $\frac{\sin x}{x}$

Fig. 9. Comparison of two signal processing theories.

develop a new way of thinking about morphological signals – we can just apply the methods from linear systems to the morphological basic operations and formulas. Therefore, one may expect rapid progress in the development of the theory of morphological signals and systems. Impressive illustrations of the effectiveness of applying the usual methods from signals analysis to morphology have recently appeared [7, 9]. We expect many more to follow.

A second train of development can be the establishment of *morphological scale spaces*, analogous to the linear scale spaces [8]. In that context, it is interesting to note that the result (35) shows that the quadratic functions $x^T Ax$ are eigenfunctions of the slope transform, just as the Gaussian functions $\exp(x^T Ax)$ are eigenfunctions of the Fourier transform. This explains the analogy between Gaussian functions as convolution kernels and quadratic structuring functions as dilation kernels in the development of scale spaces [15].

4.2. System identification: STM and AFM

In machines called the scanning tunnelling microscope (STM) and in the atomic force microscope (AFM), an unknown atomic probe g is run over an unknown atomic surface f , and its height measured h as a function of the scanning location. The probe is servoed to maintain a constant current (STM) or force (AFM) from probe to surface. To a first approximation, this has the effect that the height function h is the dilation $f \oplus g$ of the surface

by the probe function (see e.g. [12], where the dilation is called convolution).

The problem now is to reconstruct the atomic surface from the height data h . This is straightforward once the shape of the probe is known. However, it is not known, and it needs to be estimated from the input/output analysis of the system. Physicists have made some progress in this using the Legendre transform [7] – in fact, they have derived some theorems that properly belong to morphology, such as the theorems on second derivatives of Eqs. (16), (25) and (30)! Here we sketch this solution in terms of the slope transform.

Let the atomic surface along the i th scan be given by the function f_i , and the probe by g . The output of the scan is the function $f_i \oplus g$. Let us assume that over some interval this equals $f_i \oplus g$ (no double contacts), so that all is well-behaved: differentiable and invertible. Then in the slope domain, the output equals $\mathcal{S}[f_i] + \mathcal{S}[g]$. The average of this over a large number of scans is $1/n \sum_{i=1}^n \mathcal{S}[f_i] + \mathcal{S}[g]$. If the surface is sufficiently uncorrelated and sufficiently rich in slopes, the average $1/n \sum_{i=1}^n \mathcal{S}[f_i]$ will tend to 0 over the domain of $\mathcal{S}[g]$. Under those assumptions, one can thus retrieve $\mathcal{S}[g]$ and hence, by the inverse slope transform, g . Using this, each of the outputs of the scans can be eroded by g to yield the original f_i .

There are a number of assumptions in this outline which are definitely not satisfied in reality, and to specialists in linear systems theory they should look familiar. They are the logarithmic counterpart of the assumptions that underly the principle of deconvolution. In the linear theory, various

filtering methods have been designed to put this idea to work, and their morphological counterpart needs to be developed. Thus, we may expect the notions of correlation, white noise, etc., so familiar in linear system identification, to crop up in morphology. For STM, such results will aid in the manufacture of artificial surfaces with certain correlation properties which permit the analysis to within a certain accuracy.

At this point, we cannot say much more than this. However, note that this new issue in morphology, *de-dilation of an unknown structuring function*, appears to be tractable, and that the linear theory can be the direct inspiration to develop appropriate techniques, with the slope transform playing the role of Fourier transform. The structural similarity of both theories (Fig. 9) can thus be made to pay off immediately.

5. Conclusions

We have developed an analytical theory of basic aspects in mathematical morphology. We first introduced a weaker version of the dilation operation, *tangential dilation*. Application of the tangential dilation leads to set-valued functions (or self-intersecting surfaces) with a nicely tractable *differentiable* structure.

The slope theorem shows that a key property of dilation is that it leads to *parallel transport* of (infinitesimal) planar segments. This implies that *planar functions are the eigenfunctions of morphology*. We then based a dual representation of functions on planar functions: arbitrary functions can be written as the caustic of planes; see Fig. 4. We characterize these planes by their intercept, as a function of their slope. This dual representation of the function is the *slope transform*. For common signals, it is set-valued.

In the dual representation in the slope domain, *dilation becomes addition*. This makes the slope transform like the ‘Fourier transform of morphology’ in that it converts an involved elementary operation in the spatial domain into a simple elementary operation in the transformed domain. It makes one wonder whether a ‘fast slope transform’ might exist.

The slope transform is invertible for a wide class of functions – but exactly what that class is still needs to be established. It does appear to include all functions that would be interesting to morphological signal processing. The slope transform needs to be formalized properly. Two ways seem promising: one is the analogy to *convex analysis* (see e.g. [6]), of which the applicability to classical morphology has already been demonstrated by Mattioli [10]; the other approach is *differential geometry*, which has the natural means of dealing with self-intersecting differentiable surfaces. We plan to investigate the latter.

There exist transforms that are structurally very similar to the slope transform, but which have a more limited applicability: the Legendre transform, the Young–Fenchel conjugate and the $\mathcal{A}$ -transform. These have recently surfaced in morphological literature [14, 9, 10, 7]. We show in Appendix A that the slope transform generalizes them all. The application of those earlier transforms to morphological problems tends to require rather cumbersome administration of admittedness and domains of definition during deductions, obscuring the actual (geometrical) arguments. The proof of the *additivity of radii of curvature* (Section 2.5) shows that slope transform allows such details to be postponed to the projection of the final result onto classical morphology, and this facilitates the analysis. Therefore, we prefer to use the slope transform.

Whatever transform one finds more advantageous, however, one fact remains: all these recently emerged insights on the central importance of slope for morphology point towards the establishment of a quantitative analytical view of mathematical morphology. Such a view will complement the classical set-based qualitative view, and it will lead to a *theory of morphological signal processing*. Such a theory will find immediate application in physical morphological machines like the scanning tunnelling microscope.

Appendix A. Legendre transform, Young–Fenchel conjugate, $\mathcal{A}$ -transform

The slope transform is new, but we found it to be a generalization of some lesser known transforms

from mathematics and physics. The basic idea of dually representing a function by its tangent lines was formalized in the Legendre transform. It is only defined for smooth convex functions f which grow faster than linearly (see [6]), and is defined as

$$\mathcal{L}[f](\omega) = f(\bar{x}) - \omega\bar{x}, \quad (55)$$

with $\bar{x}$ such that $f'(\bar{x}) = \omega$.

Since in the case considered f' is invertible, we may write

$$\mathcal{L}[f](\omega) = f(f'^{-1}(\omega)) - \omega f'^{-1}(\omega). \quad (56)$$

In the theory on convex functions, the Young–Fenchel conjugate is a generalization of the Legendre transform to arbitrary functions:

$$\mathcal{Y}[f](\omega) = \sup_x [\langle \omega, x \rangle - f(x)] \quad (57)$$

(there is a sign change, related to consistent treatment of convex functions giving special interpretations to $\pm \infty$; see [6]). Maragos [9] recently introduced this transform (differing by a minus sign) to analyze the morphological operations on discrete-time signals. He called it the $\mathcal{A}$ -transform, and noted its use as a ‘Fourier transform’ for morphology.

This Young–Fenchel- $\mathcal{A}$ transform is not invertible – the closest one can come to inverting the Young–Fenchel conjugate is the convex closure of the original function:

$$\mathcal{Y}[\mathcal{Y}[f]](x) = (\overline{\text{conv}} f)(x) \leq f(x). \quad (58)$$

As a consequence, one has awkward properties of dilation, just falling short of complete duality (see [6]):

$$\mathcal{Y}[f \oplus g] = \mathcal{Y}[f] + \mathcal{Y}[g],$$

but $\mathcal{Y}[f + g] \leq \mathcal{Y}[f] \oplus \mathcal{Y}[g]. \quad (59)$

The slope transform is invertible. Its definition,

$$\mathcal{S}[f](\omega) = \text{stat}_x [f(x) - \langle \omega, x \rangle], \quad (60)$$

is set-valued; this set-valuedness permits the retrievable occurrence of the same slope at different values of the abscissa. We have seen that there is

a generalization of dilation, the tangential dilation $\check{\oplus}$, for which we have the fully symmetrical

$$\mathcal{S}[f \check{\oplus} g] = \mathcal{S}[f] + \mathcal{S}[g] \quad (61)$$

and

$$\mathcal{S}[f + g] = \mathcal{S}[f] \check{\oplus} \mathcal{S}[g].$$

We thus have a choice in developing an analytical theory of morphology: we can use the Young–Fenchel conjugate (or the equivalent $\mathcal{A}$ -transform) and have inequalities, or we can use the more symmetrical slope transform and have equalities, but at the expense of having to perform a ‘sup’ projection afterwards to interpret the results (see Eq. (7)). The derivation of the additivity of radii (Eq. (30)) suggests that this may be advantageous.

It should be pointed out that the Young–Fenchel conjugate has been developed rigorously for functions on separable locally convex spaces, and that so far the slope transform lacks such rigor. Such rigor can possibly be achieved by adapting the corresponding results for the Young–Fenchel conjugate. Further developments of the theory will make it necessary to know exactly for what class of functions the slope transform is tractable, how to define the slope transform of functions over a limited interval consistently (the ‘umbra problem’), and under what conditions taking limits is permissible. In the present paper, we have let ourselves be guided by our intuition, attempting to establish the usefulness of the slope transform before establishing its rigor.

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