SPLITTING TYCHONOFF CUBES INTO HOMEOMORPHIC AND HOMOGENEOUS PARTS

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Abstract. Let $\tau$ be an infinite cardinal. We prove that the Tychonoff cube $I^\tau$ can be split into two homeomorphic and homogeneous parts. If $\tau$ is uncountable, such a partition cannot consist of spaces homeomorphic to topological groups.

1. Introduction

It is known that the real line $\mathbb{R}$ can be partitioned into two homeomorphic and homogeneous parts, [11]. Although it is not mentioned in [11], this was an answer to a question posed by the late Maarten Maurice. Since then, various similar results were obtained. Shelah [15] and, independently, van Engelen [7], showed that $\mathbb{R}$ can be partitioned into two homeomorphic rigid parts. Here a space is called rigid if the identity is its only homeomorphism. See also [8] and [14] for other results in the same spirit.

It was asked by the second author of the present paper whether the closed unit interval $I = [0, 1]$ can be partitioned into two homogeneous and homeomorphic parts. The aim of this paper is to answer this question in the affirmative. It immediately leads to the following result:

Theorem 1.1. Let $\tau$ be any infinite cardinal. Then the Tychonoff cube $I^\tau$ can be partitioned into two homogeneous and homeomorphic parts.

We do not know whether a similar result holds for the finite dimensional cubes $I^n$, where $1 < n < \omega$. Theorem 1.1 suggests the question whether the homeomorphic parts can actually be chosen to be (homeomorphic to) a topological group. For uncountable $\tau$, the answer is in the negative.

Theorem 1.2. Let $\tau$ be any uncountable cardinal. Then for every subspace $A$ of $I^\tau$ which is (homeomorphic to) a topological group, we have that $I^\tau \setminus A$ and $A$ are not homeomorphic.

2. The closed unit interval can be conveniently split

We begin by reviewing the construction from van Mill [11]. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$.

Lemma 2.1. [11, 2.3] If $X \subseteq \mathbb{R}$ is such that $X = X + \mathbb{Q}$, then $X$ is homogeneous.
In [11, §3], a subset $A \subseteq \mathbb{R}$ was constructed having the following properties:

1. $A$ is dense in $\mathbb{R}$, and so is $B = \mathbb{R} \setminus A$,
2. $Q \subseteq A$ and $A + Q = A$ (hence $B + Q = B$),
3. the map $\phi: \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x + \pi$ sends $A$ onto $B$.

Let $D = \pi + Q$. Then $D$ is dense in $B$, and $\phi(Q) = D$. If $s, t \in D$ and $s < t$, then $[s, t]_A = [s, t] \cap A$ is called a clopen arc in $A$. Moreover, if $p, q \in Q$ and $p < q$, then $[p, q]_B = [p, q] \cap B$ is called a clopen arc in $B$. Observe that clopen arcs in $A$ respectively $B$ are clopen subsets of $A$ respectively $B$. If $C = [s, t]_A$ is a clopen arc in $A$, then $\lambda(C) = t - s$ denotes its length. Observe that $\lambda(C) \in Q$. If $\mathcal{C}$ is a pairwise disjoint family of clopen arcs in $A$, then $\lambda(\bigcup \mathcal{C}) = \sum_{C \in \mathcal{C}} \lambda(C)$. Similarly for $B$.

We use some ideas in van Mill [12].

**Lemma 2.2.** If $C_0$ and $C_1$ are clopen arcs in $A$ such that $\lambda(C_0) = \lambda(C_1)$, then $C_0$ and $C_1$ are homeomorphic. Similarly for $B$. Moreover, if $C$ is a clopen arc in $A$ and $D$ is a clopen arc in $B$ such that $\lambda(C) = \lambda(D)$, then $C$ and $D$ are homeomorphic.

**Proof.** Let $C_0 = [r_0, t_0]_A$ and $C_1 = [r_1, t_1]_A$. Define $f: C_0 \to C_1$ by $f(t) = (t - r_0) + r_1$. Since $r_1 - r_0 \in Q$ and $A + Q = A$, it easily follows that $f$ is a homeomorphism. Similarly for $B$.

Assume that $C = [r, t]_A$ and $D = [p_0, q_0]_B$. Let $r = \pi + p_0$ and $t = \pi + q_0$. Then $\phi^{-1}$ sends $C$ homeomorphically onto the clopen arc $[p_0, q_0]_B$ of $B$. By the above, $[p_0, q_0]_B$ and $[p_1, q_1]_B$ are homeomorphic, hence we are done. \square

**Lemma 2.3.** Let $\mathcal{C}$ be a pairwise disjoint collection of clopen arcs in $A$ such that $\varepsilon = \lambda(\bigcup \mathcal{C}) \in Q$. Then $\bigcup \mathcal{C}$ is homeomorphic to the clopen arc $[\pi, \pi + \varepsilon]_A$. Similarly, let $\mathcal{D}$ be a pairwise disjoint collection of clopen arcs in $D$ such that $\delta = \lambda(\bigcup \mathcal{D}) \in Q$, then $\bigcup \mathcal{D}$ is homeomorphic to the clopen arc $[0, \delta]_B$.

**Proof.** We assume that $\mathcal{C}$ is infinite. The proof when $\mathcal{C}$ is finite is entirely similar. Assume that $\mathcal{C} = \{[\pi + r_0, \pi + t_0]_A, [\pi + r_1, \pi + t_1]_A, \ldots, [\pi + r_n, \pi + t_n]_A, \ldots\}$.

By Lemma 2.2,

$[\pi + r_0, \pi + t_0]_A \approx [\pi, \pi + (t_0 - r_0)]_A$,

$[\pi + r_1, \pi + t_1]_A \approx [\pi + (t_0 - r_0), \pi + (t_0 - r_0) + (t_1 - r_1)]_A$,

\vdots

$[\pi + r_n, \pi + t_n]_A \approx [\pi + \sum_{j \leq n-1} (t_j - r_j), \pi + \sum_{j \leq n} (t_j - r_j)]_A$,

\vdots

Since all sets involved are clopen, the union of these homeomorphisms gives us that

$\bigcup \mathcal{C} \approx [\pi, \pi + \sum_{j \leq \omega} (t_j - r_j)]_A = [\pi, \pi + \varepsilon]_A$. 

The proof for $B$ is entirely similar.

**Corollary 2.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be pairwise disjoint collections of clopen arcs in $A$ respectively $B$ such that $\lambda(\bigcup \mathcal{C}) = \lambda(\bigcup \mathcal{D}) \in \mathbb{Q}$. Then $\bigcup \mathcal{C}$ and $\bigcup \mathcal{D}$ are homeomorphic.

**Proof.** Let $\gamma = \lambda(\bigcup \mathcal{C}) = \lambda(\bigcup \mathcal{D})$. By Lemma 2.3,

$$\bigcup \mathcal{C} \approx [\pi, \pi + \lambda]_A, \quad \bigcup \mathcal{D} \approx [0, \lambda]_B.$$  

Hence we are done by Lemma 2.2. □

In the proof of the next result, we use the well-known result from Calculus, that for every $t \in \mathbb{I}$ there is a subset $A$ of $\mathbb{N}$ such that $\sum_{n \in A} 2^{-n} = t$. For more on this topic, see Ferdinands [9].

**Lemma 2.5.** Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{0\} \cup [0,q]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0,q]_B$.

**Proof.** Put $q_0 = q$. For every $n \geq 1$, put $q_n = 2^{-n}q$. Moreover, put $t_0 = q$ and for $n \geq 1$,

$$t_n = t_{n-1} - q_n.$$  

Let $x \in B \cap (2,3)$. Pick $r \in \mathbb{Q}$ such that $r < x < r + q$. Let $F \subseteq \mathbb{N}$ be such that $\sum_{n \in F} q_n = x - r$. Observe that $F$ has to be infinite since $x$ is irrational. Put $G = \mathbb{N} \setminus F$. Then $\sum_{n \in G} q_n = r + q - x$. It also follows that $G$ is infinite.

Put $r_0 = r$. There clearly is a sequence $(r_n)_{n \geq 1}$ of rational numbers in $(r,x)$ such that $(r_n)_n \nearrow x$ while moreover for every $n \geq 1$ we have

$$r_n - r_{n-1} = q_{\mu(n)},$$

where $\mu(n)$ is the $n$-the element of $F$ (ordered as a subset of $\mathbb{N}$). Put $s_0 = r + q$. There similarly is a sequence $(s_n)_{n \geq 1}$ of rational numbers in $(x,r + q)$ such that $(s_n)_n \searrow x$ while moreover for every $n \geq 1$ we have

$$s_{n-1} - s_n = q_{\nu(n)},$$

where $\nu(n)$ is the $n$-the element of $G$ (ordered as a subset of $\mathbb{N}$).

Let $\mu(n) \in A$. By Lemma 2.2 we may pick a homeomorphism

$$g_n : [t_{\mu(n)}, t_{\mu(n)} - 1] \rightarrow [r_{n-1}, r_n].$$

Similarly, if $\nu(n) \in B$, we may pick a homeomorphism

$$h_n : [t_{\nu(n)}, t_{\nu(n)} - 1] \rightarrow [s_n, s_{n-1}].$$

Since all sets involved are clopen, the function $f : \{0\} \cup [0,q]_B \rightarrow [r, r + q]_B$ defined by

$$f(x) = \begin{cases} 
  g_n(x) & (t_{\mu(n)} < x < t_{\mu(n)} - 1), \\
  h_n(x) & (t_{\nu(n)} < x < t_{\nu(n)} - 1), \\
  x & (t = 0),
\end{cases}$$

is a homeomorphism. Hence we are done by Lemma 2.2. □

The following can be proved with the same method.
Lemma 2.6. Let \( q \in \mathbb{Q} \) be such that \( 0 < q < 1 \). Then \( \{1\} \cup [1-q, 1]_B \) (with the subspace topology it inherits from \( \mathbb{R} \)) is homeomorphic to the clopen arc \([0, q]_B\).

We now come to the main result in this section.

Theorem 2.7. The closed unit interval \( I = [0,1] \) can be partitioned into two homogeneous and homeomorphic sets.

Proof. Put \( E = (0,1) \cap A \) and \( F = [0,1]_B = (0,1) \cap B \), respectively. Observe that \( E \) and \( F \) are homogeneous being both open subsets of zero-dimensional homogeneous spaces. Also, both \( E \) and \( F \) are the union of a pairwise disjoint family clopen arcs in \( A \) respectively \( B \) and have the same rational ‘length’. Hence \( E \approx F \) by Corollary 2.4.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Keller’s Theorem [10] (see also [13]), \( I^\tau \) is homogeneous. This implies that \( I^\tau \approx I \times I^\tau \) is homogeneous for every infinite cardinal \( \tau \). Hence we are done by Theorem 2.7.

3. Topological groups

We show here that Theorem 1.1 for uncountable cardinals cannot be improved to the case of a splitting into homeomorphic topological groups. For information on topological groups, see Arhangel’skii and Tkachenko [4].

The following result is well-known, its proof is included for completeness sake.

Lemma 3.1. Let \( G \) be a topological group. If \( S \) is a \( G_5 \)-subset of \( G \) containing the neutral element \( e \) of \( G \), then there is a closed subgroup \( N \) of \( G \) such that

1. \( N \subseteq S \),
2. \( N \) is a \( G_5 \)-subset of \( G \).

Proof. Write \( S \) as \( \bigcap_{n<\omega} U_n \), where each \( U_n \) is open in \( G \). Recursively, pick open symmetric neighborhoods \( V_n \) of \( e \) such that \( V_{n+1}^2 \subseteq V_n \subseteq U_n \), and let \( N = \bigcap_{n<\omega} V_n \).

Theorem 3.2. If \( G \) is a dense subset of \( \mathbb{R}^\tau \), where \( \tau \) is uncountable, such that \( \mathbb{R}^\tau \setminus G \) is Lindelöf, then \( G \) is not a topological group.

Proof. Striving for a contradiction, assume that \( G \) is a topological group.

We may assume by homogeneity that the element of \( \mathbb{R}^\tau \) with constant coordinates 0 is the neutral element \( e \) of \( G \). Since \( \mathbb{R}^\tau \setminus G \) is Lindelöf, there is a compact \( G_5 \)-subset \( S_0 \) of \( \mathbb{R}^\tau \) such that \( e \in S_0 \subseteq G \).

There is a countable subset \( A_0 \) of \( \tau \) such that

\[ S_1 = \{ x \in \mathbb{R}^\tau : (\forall \alpha \in A_0)(x_\alpha = 0) \} \subseteq S_0. \]
By Lemma 3.1, we may pick a closed subgroup $N_1$ of $G$ which is a $G_δ$-subset of $G$ such that $N_1 \subseteq S_1$. Clearly, $N_1$ is a $G_δ$-subset of $S_1$ and hence is a compact $G_δ$-subset of $\mathbb{I}^\tau$. There is a countable subset $A_1$ of $\tau$ such that $A_0 \subseteq A_1$ while moreover

$$S_2 = \{ x \in \mathbb{I}^\tau : (\forall \alpha \in A_1)(x_\alpha = 0) \} \subseteq N_1.$$ 

Continuing in this way, it is easy to construct by recursion countable subsets $A_n$ of $\tau$ and closed subgroups $N_n$ of $G$ such that for every $n$,

1. $A_n \subseteq A_{n+1}$,
2. $S_{n+1} = (\forall \alpha \in A_{n+1})(x_\alpha = 0) \subseteq N_n \subseteq S_n$.

Put $A = \bigcup_{n<\omega} A_n$. Then since $\tau$ is uncountable,

$$\bigcap_{n<\omega} N_n = \{ x \in \mathbb{I}^\tau : (\forall \alpha \in A)(x_\alpha = 0) \} \approx \mathbb{I}^\tau.$$ 

Hence $\mathbb{I}^\tau$ is a topological group, which contradicts the Brouwer Fixed-Point Theorem. □

We are now in the position to present a proof of Theorem 1.2. We use a factorization result of Arhangel’skii [1], the key feature of which is that it concerns continuous functions on dense subspaces of products of separable metrizable spaces [4, Corollary 1.7.8 (see also Theorem 1.7.7)]. Arhangel’skii’s result is also stated and applied in his book [2, Lemma 0.2.3]. It implies that every continuous real-valued function on a dense subset of a Tychonoff cube depends on countably many coordinates. Therefore, if $A$ is a dense pseudocompact subset of some Tychonoff cube $\mathbb{I}^\tau$, then $\mathbb{I}^\tau$ is the Čech-Stone-compactification $\beta A$ of $A$. Indeed, for every continuous function $f: A \to \mathbb{R}$ there is by Corollary 1.7.8 in [4], a countable subset $L$ of $\tau$ and a continuous function $g: \pi_L(A) \to \mathbb{R}$, where $\pi_L: \mathbb{I}^\tau \to \mathbb{I}^L$ is the projection, such that $g(\pi_L(a)) = f(a)$ for all $a \in A$. However, since $A$ is pseudocompact, $\pi_L(A) = \mathbb{I}^L$, which evidently implies that $f$ can be extended over $\mathbb{I}^\tau$.

**Proof of Theorem 1.2.** Assume the contrary. First observe that $A$ is nowhere locally compact. Indeed, if $A$ would be somewhere locally compact, it would be locally compact at all points by homogeneity and so its complement would be compact implying that $A$ would be compact; this is clearly impossible. This also gives us that $A$ is dense. For if $A$ would not be dense, $\mathbb{I}^\tau \setminus A$ would be somewhere locally compact, and so $A$ would be somewhere locally compact.

The Dichotomy Theorem from Arhangel’skii [3] implies that $B = \mathbb{I}^\tau \setminus A$ is pseudocompact or Lindelöf. But it cannot be Lindelöf by Theorem 3.2. Hence $B$ is pseudocompact and so $A$ is pseudocompact. Since $A$ is dense in $\mathbb{I}^\tau$, it follows by the above that $\mathbb{I}^\tau = \beta A$.

We complete the proof now in two ways. The first proof is as follows. Since $A$ is a pseudocompact topological group, $\beta A$ is a topological group by the Comfort-Ross theorem [6]. But $\mathbb{I}^\tau$ is not a topological group, for example because it has the Fixed-Point Property by Brouwer’s Theorem.

The second proof is more direct and avoids the use of the complicated Comfort-Ross Theorem. Indeed, we first claim that $A$ does not contain any nonempty compact $G_δ$-subset. For if it would contain such a compact $G_δ$-subset $S$, then $S$ has a countable base of open neighbourhoods in $A$, since the space $A$ is pseudocompact. Since $A$ is a topological
group, it follows from this that $A$ is paracompact [4, Corollary 4.3.21](see also page 314 there).

Since $A$ is also pseudocompact, it consequently follows that $A$ is compact, - a contradiction.

Fix a homeomorphism $f$ of $A$ onto $B$. Clearly, $f$ can be extended to a homeomorphism $h$ of $I^\tau$ onto $I^\tau$. Since $h(A) = B$ and $h(B) = A$, it follows that $h$ has no fixed-points. This is a contradiction with the Brouwer Fixed-Point Theorem. □

In the zero-dimensional case, the case of Cantor cubes instead of Tychonoff cubes, Theorem 1.2 does not hold. Indeed, let $\kappa$ be an infinite cardinal, and let $p$ be a free ultrafilter on $\kappa$. The set

$$A = \{x \in \{0, 1\}^\tau : \{\alpha : x_\alpha = 1\} \in p\}$$

is a subgroup of $\{0, 1\}^\tau$ of index 2. Hence $A$ as well as its complement are homeomorphic to topological groups.

We do not know whether every compact topological group can be split into two homeomorphic and homogeneous parts.

### References


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