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SPLITTING TYCHONOFF CUBES INTO HOMEOMORPHIC AND HOMOGENEOUS PARTS

A. V. ARHANGEL’SKIĬ AND J. VAN MILL

Abstract. Let $\tau$ be an infinite cardinal. We prove that the Tychonoff cube $I^\tau$ can be split into two homeomorphic and homogeneous parts. If $\tau$ is uncountable, such a partition cannot consist of spaces homeomorphic to topological groups.

1. Introduction

It is known that the real line $\mathbb{R}$ can be partitioned into two homeomorphic and homogeneous parts, [11]. Although it is not mentioned in [11], this was an answer to a question posed by the late Maarten Maurice. Since then, various similar results were obtained. Shelah [15] and, independently, van Engelen [7], showed that $\mathbb{R}$ can be partitioned into two homeomorphic rigid parts. Here a space is called rigid if the identity is its only homeomorphism. See also [8] and [14] for other results in the same spirit.

It was asked by the second author of the present paper whether the closed unit interval $I = [0, 1]$ can be partitioned into two homogeneous and homeomorphic parts. The aim of this paper is to answer this question in the affirmative. It immediately leads to the following result:

**Theorem 1.1.** Let $\tau$ be any infinite cardinal. Then the Tychonoff cube $I^\tau$ can be partitioned into two homogeneous and homeomorphic parts.

We do not know whether a similar result holds for the finite dimensional cubes $I^n$, where $1 < n < \omega$. Theorem 1.1 suggests the question whether the homeomorphic parts can actually be chosen to be (homeomorphic to) a topological group. For uncountable $\tau$, the answer is in the negative.

**Theorem 1.2.** Let $\tau$ be any uncountable cardinal. Then for every subspace $A$ of $I^\tau$ which is (homeomorphic to) a topological group, we have that $I^\tau \setminus A$ and $A$ are not homeomorphic.

2. The closed unit interval can be conveniently split

We begin by reviewing the construction from van Mill [11]. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$.

**Lemma 2.1.** [11, 2.3] If $X \subseteq \mathbb{R}$ is such that $X = X + \mathbb{Q}$, then $X$ is homogeneous.
In [11, §3], a subset $A \subseteq \mathbb{R}$ was constructed having the following properties:

1. $A$ is dense in $\mathbb{R}$, and so is $B = \mathbb{R} \setminus A$,
2. $Q \subseteq A$ and $A + Q = A$ (hence $B + Q = B$),
3. the map $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x + \pi$ sends $A$ onto $B$.

Let $D = \pi + Q$. Then $D$ is dense in $B$, and $\phi(Q) = D$. If $s, t \in D$ and $s < t$, then $[s, t]_A = [s, t] \cap A$ is called a clopen arc in $A$. Moreover, if $p, q \in Q$ and $p < q$, then $[p, q]_B = [p, q] \cap B$ is called a clopen arc in $B$. Observe that clopen arcs in $A$ respectively $B$ are clopen subsets of $A$ respectively $B$. If $C = [s, t]_A$ is a clopen arc in $A$, then $\lambda(C) = t - s$ denotes its length. Observe that $\lambda(C) \in Q$. If $\mathcal{C}$ is a pairwise disjoint family of clopen arcs in $A$, then $\lambda(\bigcup \mathcal{C}) = \sum_{C \in \mathcal{C}} \lambda(C)$. Similarly for $B$.

We use some ideas in van Mill [12].

**Lemma 2.2.** If $C_0$ and $C_1$ are clopen arcs in $A$ such that $\lambda(C_0) = \lambda(C_1)$, then $C_0$ and $C_1$ are homeomorphic. Similarly for $B$. Moreover, if $C$ is a clopen arc in $A$ and $D$ is a clopen arc in $B$ such that $\lambda(C) = \lambda(D)$, then $C$ and $D$ are homeomorphic.

**Proof.** Let $C_0 = [r_0, t_0]_A$ and $C_1 = [r_1, t_1]_A$. Define $f : C_0 \to C_1$ by $f(t) = (t - r_0) + r_1$. Since $r_1 - r_0 \in Q$ and $A + Q = A$, it easily follows that $f$ is a homeomorphism. Similarly for $B$.

Assume that $C = [r, t]_A$ and $D = [p_1, q_1]_B$. Let $r = \pi + p_0$ and $t = \pi + q_0$. Then $\phi^{-1}$ sends $C$ homeomorphically onto the clopen arc $[p_0, q_0]_B$ of $B$. By the above, $[p_0, q_0]_B$ and $[p_1, q_1]_B$ are homeomorphic, hence we are done. \hfill \Box

**Lemma 2.3.** Let $\mathcal{C}$ be a pairwise disjoint collection of clopen arcs in $A$ such that $\varepsilon = \lambda(\bigcup \mathcal{C}) \in Q$. Then $\bigcup \mathcal{C}$ is homeomorphic to the clopen arc $[\pi, \pi + \varepsilon]_A$. Similarly, let $\mathcal{D}$ be a pairwise disjoint collection of clopen arcs in $D$ such that $\delta = \lambda(\bigcup \mathcal{D}) \in Q$, then $\bigcup \mathcal{D}$ is homeomorphic to the clopen arc $[0, \delta]_B$.

**Proof.** We assume that $\mathcal{C}$ is infinite. The proof when $\mathcal{C}$ is finite is entirely similar. Assume that

$$\mathcal{C} = \{[\pi + r_0, \pi + t_0]_A, [\pi + r_1, \pi + t_1]_A, \ldots, [\pi + r_n, \pi + t_n]_A, \ldots\}.$$ 

By Lemma 2.2,

$$[\pi + r_0, \pi + t_0]_A \approx [\pi, \pi + (t_0 - r_0)]_A,$$

$$[\pi + r_1, \pi + t_1]_A \approx [\pi + (t_0 - r_0), \pi + (t_0 - r_0) + (t_1 - r_1)]_A,$$

$$\vdots$$

$$[\pi + r_n, \pi + t_n]_A \approx [\pi + \sum_{j \leq n-1} (t_j - r_j), \pi + \sum_{j \leq n} (t_j - r_j)]_A,$$

$$\vdots$$

Since all sets involved are clopen, the union of these homeomorphisms gives us that

$$\bigcup \mathcal{C} \approx [\pi, \pi + \sum_{j \leq \omega} (t_j - r_j)]_A = [\pi, \pi + \varepsilon]_A.$$
The proof for $B$ is entirely similar. \hfill \Box

**Corollary 2.4.** Let $C$ and $D$ be pairwise disjoint collections of clopen arcs in $A$ respectively $B$ such that $\lambda(\bigcup C) = \lambda(\bigcup D) \in \mathbb{Q}$. Then $\bigcup C$ and $\bigcup D$ are homeomorphic.

**Proof.** Let $\gamma = \lambda(\bigcup C) = \lambda(\bigcup D)$. By Lemma 2.3,

$$\bigcup C \approx [\pi, \pi + \lambda]_A, \quad \bigcup D \approx [0, \lambda]_B.$$ 

Hence we are done by Lemma 2.2. \hfill \Box

In the proof of the next result, we use the well-known result from Calculus, that for every $t \in \mathbb{I}$ there is a subset $A$ of $\mathbb{N}$ such that $\sum_{n \in A} 2^{-n} = t$. For more on this topic, see Ferdinands [9].

**Lemma 2.5.** Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{0\} \cup [0, q]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0, q]_B$.

**Proof.** Put $q_0 = q$. For every $n \geq 1$, put $q_n = 2^{-n}q$. Moreover, put $t_0 = q$ and for $n \geq 1$, $t_n = t_{n-1} - q_n$. Let $x \in B \cap (2, 3)$. Pick $r \in \mathbb{Q}$ such that $r < x < r + q$. Let $F \subseteq \mathbb{N}$ be such that $\sum_{n \in F} q_n = x - r$. Observe that $F$ has to be infinite since $x$ is irrational. Put $G = \mathbb{N} \setminus F$. Then $\sum_{n \in G} q_n = r + q - x$. It also follows that $G$ is infinite.

Put $r_0 = r$. There clearly is a sequence $(r_n)_{n \geq 1}$ of rational numbers in $(r, x)$ such that $(r_n)_n \nearrow x$ while moreover for every $n \geq 1$ we have

$$r_n - r_{n-1} = q_{\mu(n)},$$

where $\mu(n)$ is the $n$-the element of $F$ (ordered as a subset of $\mathbb{N}$). Put $s_0 = r + q$. There similarly is a sequence $(s_n)_{n \geq 1}$ of rational numbers in $(x, r + q)$ such that $(s_n)_n \searrow x$ while moreover for every $n \geq 1$ we have

$$s_{n-1} - s_n = q_{\nu(n)},$$

where $\nu(n)$ is the $n$-the element of $G$ (ordered as a subset of $\mathbb{N}$).

Let $\mu(n) \in A$. By Lemma 2.2 we may pick a homeomorphism

$$g_n : [t_{\mu(n)}, t_{\mu(n)-1}]_B \rightarrow [r_{n-1}, r_n]_B.$$ 

Similarly, if $\nu(n) \in B$, we may pick a homeomorphism

$$h_n : [t_{\nu(n)}, t_{\nu(n)-1}]_B \rightarrow [s_n, s_{n-1}]_B.$$ 

Since all sets involved are clopen, the function $f : \{0\} \cup [0, q]_B \rightarrow [r, r + q]_B$ defined by

$$f(x) = \begin{cases} 
g_n(x) & (t_{\mu(n)} < x < t_{\mu(n)-1}), \\
h_n(x) & (t_{\nu(n)} < x < t_{\nu(n)-1}), \\
x & (t = 0), \end{cases}$$

is a homeomorphism. Hence we are done by Lemma 2.2. \hfill \Box

The following can be proved with the same method.
Lemma 2.6. Let \( q \in \mathbb{Q} \) be such that \( 0 < q < 1 \). Then \( \{1\} \cup [1-q,1)_B \) (with the subspace topology it inherits from \( \mathbb{R} \)) is homeomorphic to the clopen arc \([0,q)_B\).

We now come to the main result in this section.

Theorem 2.7. The closed unit interval \( I = [0,1] \) can be partitioned into two homogeneous and homeomorphic sets.

Proof. Put \( E = (0,1) \cap A \) and \( F = [0,1)_B = (0,1) \cap B \), respectively. Observe that \( E \) and \( F \) are homogeneous being both open subsets of zero-dimensional homogeneous spaces. Also, both \( E \) and \( F \) are the union of a pairwise disjoint family clopen arcs in \( A \) respectively \( B \) and have the same rational ‘length’. Hence \( E \approx F \) by Corollary 2.4.

The partition \( \{E,F\cup\{0,1\}\} \) of \( I \) is consequently the one we are after. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Keller’s Theorem [10] (see also [13]), \( \mathbb{I}^\tau \) is homogeneous. This implies that \( I^\tau \approx I \times I^\tau \) is homogeneous for every infinite cardinal \( \tau \). Hence we are done by Theorem 2.7. □

3. Topological groups

We show here that Theorem 1.1 for uncountable cardinals cannot be improved to the case of a splitting into homeomorphic topological groups. For information on topological groups, see Arhangel’skii and Tkachenko [4].

The following result is well-known, its proof is included for completeness sake.

Lemma 3.1. Let \( G \) be a topological group. If \( S \) is a \( G_\delta \)-subset of \( G \) containing the neutral element \( e \) of \( G \), then there is a closed subgroup \( N \) of \( G \) such that

\[ 1. \] \( N \subseteq S \),
\[ 2. \] \( N \) is a \( G_\delta \)-subset of \( G \).

Proof. Write \( S \) as \( \bigcap_{n<\omega} U_n \), where each \( U_n \) is open in \( G \). Recursively, pick open symmetric neighborhoods \( V_n \) of \( e \) such that \( V^2_{n+1} \subseteq V_n \subseteq U_n \), and let \( N = \bigcap_{n<\omega} V_n \). □

Theorem 3.2. If \( G \) is a dense subset of \( \mathbb{I}^\tau \), where \( \tau \) is uncountable, such that \( \mathbb{I}^\tau \setminus G \) is Lindelöf, then \( G \) is not a topological group.

Proof. Striving for a contradiction, assume that \( G \) is a topological group.

We may assume by homogeneity that the element of \( \mathbb{I}^\tau \) with constant coordinates 0 is the neutral element \( e \) of \( G \). Since \( \mathbb{I}^\tau \setminus G \) is Lindelöf, there is a compact \( G_\delta \)-subset \( S_0 \) of \( \mathbb{I}^\tau \) such that \( e \in S_0 \subseteq G \).

There is a countable subset \( A_0 \) of \( \tau \) such that

\[ S_1 = \{x \in \mathbb{I}^\tau : (\forall \alpha \in A_0)(x_\alpha = 0)\} \subseteq S_0 . \]
By Lemma 3.1, we may pick a closed subgroup \( N_1 \) of \( G \) which is a \( G_\delta \)-subset of \( G \) such that \( N_1 \subseteq S_1 \). Clearly, \( N_1 \) is a \( G_\delta \)-subset of \( S_1 \) and hence is a compact \( G_\delta \)-subset of \( I^\tau \).

There is a countable subset \( A_1 \) of \( \tau \) such that for every \( n \),

1. \( A_n \subseteq A_{n+1} \),
2. \( S_{n+1} = (\forall \alpha \in A_{n+1})(x_\alpha = 0) \subseteq N_n \subseteq S_n \).

Put \( A = \bigcup_{n<\omega} A_n \). Then since \( \tau \) is uncountable,

\[
\bigcap_{n<\omega} N_n = \{ x \in I^\tau : (\forall \alpha \in A)(x_\alpha = 0) \} \approx I^\tau.
\]

Hence \( I^\tau \) is a topological group, which contradicts the Brouwer Fixed-Point Theorem. \( \Box \)

We are now in the position to present a proof of Theorem 1.2. We use a factorization result of Arhangel’skii [1], the key feature of which is that it concerns continuous functions on dense subspaces of products of separable metrizable spaces [4, Corollary 1.7.8 (see also Theorem 1.7.7)]. Arhangel’skii’s result is also stated and applied in his book [2, Lemma 0.2.3]. It implies that every continuous real-valued function on a dense subset of a Tychonoff cube depends on countably many coordinates. Therefore, if \( A \) is a dense pseudocompact subset of some Tychonoff cube \( I^\tau \), then \( I^\tau \) is the Čech-Stone-compactification \( \beta A \) of \( A \). Indeed, for every continuous function \( f : A \to \mathbb{R} \) there is by Corollary 1.7.8 in [4], a countable subset \( L \) of \( \tau \) and a continuous function \( g : \pi_L(A) \to \mathbb{R} \), where \( \pi_L : I^\tau \to I^L \) is the projection, such that \( g(\pi_L(a)) = f(a) \) for all \( a \in A \). However, since \( A \) is pseudocompact, \( \pi_L(A) = I^L \), which evidently implies that \( f \) can be extended over \( I^\tau \).

**Proof of Theorem 1.2.** Assume the contrary. First observe that \( A \) is nowhere locally compact. Indeed, if \( A \) would be somewhere locally compact, it would be locally compact at all points by homogeneity and so its complement would be compact implying that \( A \) would be compact; this is clearly impossible. This also gives us that \( A \) is dense. For if \( A \) would not be dense, \( I^\tau \setminus A \) would be somewhere locally compact, and so \( A \) would be somewhere locally compact.

The Dichotomy Theorem from Arhangel’skii [3] implies that \( B = I^\tau \setminus A \) is pseudocompact or Lindelöf. But it cannot be Lindelöf by Theorem 3.2. Hence \( B \) is pseudocompact and so \( A \) is pseudocompact. Since \( A \) is dense in \( I^\tau \), it follows by the above that \( I^\tau = \beta A \).

We complete the proof now in two ways. The first proof is as follows. Since \( A \) is a pseudocompact topological group, \( \beta A \) is a topological group by the Comfort-Ross theorem [6]. But \( I^\tau \) is not a topological group, for example because it has the Fixed-Point Property by Brouwer’s Theorem.

The second proof is more direct and avoids the use of the complicated Comfort-Ross Theorem. Indeed, we first claim that \( A \) does not contain any nonempty compact \( G_\delta \)-subset. For if it would contain such a compact \( G_\delta \)-subset \( S \), then \( S \) has a countable base of open neighbourhoods in \( A \), since the space \( A \) is pseudocompact. Since \( A \) is a topological
group, it follows from this that $A$ is paracompact [4, Corollary 4.3.21](see also page 314 there).

Since $A$ is also pseudocompact, it consequently follows that $A$ is compact, - a contradiction.

Fix a homeomorphism $f$ of $A$ onto $B$. Clearly, $f$ can be extended to a homeomorphism $h$ of $\mathbb{I}^\tau$ onto $\mathbb{I}^\tau$. Since $h(A) = B$ and $h(B) = A$, it follows that $h$ has no fixed-points. This is a contradiction with the Brouwer Fixed-Point Theorem. □

In the zero-dimensional case, the case of Cantor cubes instead of Tychonoff cubes, Theorem 1.2 does not hold. Indeed, let $\kappa$ be an infinite cardinal, and let $p$ be a free ultrafilter on $\kappa$. The set

$$A = \{ x \in \{0, 1\}^\tau : \{ \alpha : x_\alpha = 1 \} \in p \}$$

is a subgroup of $\{0, 1\}^\tau$ of index 2. Hence $A$ as well as its complement are homeomorphic to topological groups.

We do not know whether every compact topological group can be split into two homeomorphic and homogeneous parts.

References
