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# SPLITTING TYCHONOFF CUBES INTO HOMEOMORPHIC AND HOMOGENEOUS PARTS

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Abstract. Let $\tau$ be an infinite cardinal. We prove that the Tychonoff cube $\mathbb{I}^\tau$ can be split into two homeomorphic and homogeneous parts. If $\tau$ is uncountable, such a partition cannot consist of spaces homeomorphic to topological groups.

## 1. Introduction

It is known that the real line $\mathbb{R}$ can be partitioned into two homeomorphic and homogeneous parts, [11]. Although it is not mentioned in [11], this was an answer to a question posed by the late Maarten Maurice. Since then, various similar results were obtained. Shelah [15] and, independently, van Engelen [7], showed that $\mathbb{R}$ can be partitioned into two homemorphic rigid parts. Here a space is called *rigid* if the identity is its only homeomorphism. See also [8] and [14] for other results in the same spirit.

It was asked by the second author of the present paper whether the closed unit interval $\mathbb{I} = [0, 1]$ can be partitioned into two homogeneous and homeomorphic parts. The aim of this paper is to answer this question in the affirmative. It immediately leads to the following result:

**Theorem 1.1.** *Let $\tau$ be any infinite cardinal. Then the Tychonoff cube $\mathbb{I}^\tau$ can be partitioned into two homogeneous and homeomorphic parts.*

We do not know whether a similar result holds for the finite dimensional cubes $\mathbb{I}^n$, where $1 < n < \omega$. Theorem 1.1 suggests the question whether the homeomorphic parts can actually be chosen to be (homeomorphic to) a topological group. For uncountable $\tau$, the answer is in the negative.

**Theorem 1.2.** *Let $\tau$ be any uncountable cardinal. Then for every subspace $A$ of $\mathbb{I}^\tau$ which is (homeomorphic to) a topological group, we have that $\mathbb{I}^\tau \setminus A$ and $A$ are not homeomorphic.*

## 2. The closed unit interval can be conveniently split

We begin by reviewing the construction from van Mill [11]. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$.

**Lemma 2.1.** [11, 2.3] *If $X \subseteq \mathbb{R}$ is such that $X = X + \mathbb{Q}$, then $X$ is homogeneous.*

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In [11, §3], a subset $A \subseteq \mathbb{R}$ was constructed having the following properties:

(1) $A$ is dense in $\mathbb{R}$, and so is $B = \mathbb{R} \setminus A$,
(2) $\mathbb{Q} \subseteq A$ and $A + \mathbb{Q} = A$ (hence $B + \mathbb{Q} = B$),
(3) the map $\phi\colon \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x+\pi$ sends $A$ onto $B$.

Let $\mathbb{D} = \pi+\mathbb{Q}$. Then $\mathbb{D}$ is dense in $B$, and $\phi(\mathbb{Q}) = \mathbb{D}$. If $s, t \in \mathbb{D}$ and $s < t$, then $[s,t]_A = [s,t] \cap A$ is called a *clopen arc* in $A$. Moreover, if $p, q \in \mathbb{Q}$ and $p < q$, then $[p,q]_B = [p,q] \cap B$ is called a *clopen arc* in $B$. Observe that clopen arcs in $A$ respectively $B$ are clopen subsets of $A$ respectively $B$. If $C = [s,t]_A$ is a clopen arc in $A$, then $\lambda(C) = t-s$ denotes its length. Observe that $\lambda(C) \in \mathbb{Q}$. If $\mathscr{C}$ is a pairwise disjoint family of clopen arcs in $A$, then $\lambda(\bigcup \mathscr{C}) = \sum_{C\in\mathscr{C}} \lambda(C)$. Similarly for $B$.

We use some ideas in van Mill [12].

**Lemma 2.2.** *If $C_0$ and $C_1$ are clopen arcs in $A$ such that $\lambda(C_0) = \lambda(C_1)$, then $C_0$ and $C_1$ are homeomorphic. Similarly for $B$. Moreover, if $C$ is a clopen arc in $A$ and $D$ is a clopen arc in $B$ such that $\lambda(C) = \lambda(D)$, then $C$ and $D$ are homeomorphic.*

*Proof.* Let $C_0 = [r_0, t_0]_A$ and $C_1 = [r_1, t_1]_A$. Define $f\colon C_0 \to C_1$ by $f(t) = (t-r_0) + r_1$. Since $r_1-r_0 \in \mathbb{Q}$ and $A + \mathbb{Q} = A$, it easily follows that $f$ is a homeomorphism. Similarly for $B$.

Assume that $C = [r,t]_A$ and $D = [p_1, q_1]_B$. Let $r = \pi + p_0$ and $t = \pi + q_0$. Then $\phi^{-1}$ sends $C$ homeomorphically onto the clopen arc $[p_0, q_0]_B$ of $B$. By the above, $[p_0, q_0]_B$ and $[p_1, q_1]_B$ are homeomorphic, hence we are done. □

**Lemma 2.3.** *Let $\mathscr{C}$ be a pairwise disjoint collection of clopen arcs in $A$ such that $\varepsilon = \lambda(\bigcup \mathscr{C}) \in \mathbb{Q}$. Then $\bigcup \mathscr{C}$ is homeomorphic to the clopen arc $[\pi, \pi+\varepsilon]_A$. Similarly, let $\mathscr{D}$ be a pairwise disjoint collection of clopen arcs in $D$ such that $\delta = \lambda(\bigcup \mathscr{D}) \in \mathbb{Q}$, then $\bigcup \mathscr{D}$ is homeomorphic to the clopen arc $[0, \delta]_B$.*

*Proof.* We assume that $\mathscr{C}$ is infinite. The proof when $\mathscr{C}$ is finite is entirely similar. Assume that

$$\mathscr{C} = \{[\pi+r_0, \pi+t_0]_A, [\pi+r_1, \pi+t_1]_A, \ldots, [\pi+r_n, \pi+t_n]_A, \ldots\}.$$

By Lemma 2.2,

$$\begin{aligned}
[\pi+r_0, \pi+t_0]_A &\approx [\pi, \pi+(t_0 - r_0)]_A, \\
[\pi+r_1, \pi+t_1]_A &\approx [\pi+(t_0 - r_0), \pi+(t_0 - r_0)+(t_1 - r_1)]_A, \\
&\vdots \\
[\pi+r_n, \pi+t_n]_A &\approx [\pi+\sum_{j\le n-1}(t_j - r_j), \pi+\sum_{j\le n}(t_j - r_j)]_A, \\
&\vdots
\end{aligned}$$

Since all sets involved are clopen, the union of these homeomorphisms gives us that

$$\bigcup \mathscr{C} \approx [\pi, \pi + \sum_{j<\omega}(t_j - r_j)]_A = [\pi, \pi+\varepsilon]_A.$$

The proof for $B$ is entirely similar. $\square$

**Corollary 2.4.** *Let $\mathscr{C}$ and $\mathscr{D}$ be pairwise disjoint collections of clopen arcs in $A$ respectively $B$ such that $\lambda(\bigcup\mathscr{C}) = \lambda(\bigcup\mathscr{D}) \in \mathbb{Q}$. Then $\bigcup\mathscr{C}$ and $\bigcup\mathscr{D}$ are homeomorphic.*

*Proof.* Let $\gamma = \lambda(\bigcup\mathscr{C}) = \lambda(\bigcup\mathscr{D})$. By Lemma 2.3,

$$\bigcup\mathscr{C} \approx [\pi, \pi+\lambda]_A, \quad \bigcup\mathscr{D} \approx [0,\lambda]_B.$$

Hence we are done by Lemma 2.2. $\square$

In the proof of the next result, we use the well-known result from Calculus, that for every $t \in \mathbb{I}$ there is a subset $A$ of $\mathbb{N}$ such that $\sum_{n\in A} 2^{-n} = t$. For more on this topic, see Ferdinands [9].

**Lemma 2.5.** *Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{0\} \cup [0,q]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0,q]_B$.*

*Proof.* Put $q_0 = q$. For every $n \geq 1$, put $q_n = 2^{-n}q$. Moreover, put $t_0 = q$ and for $n \geq 1$, $t_n = t_{n-1} - q_n$.

Let $x \in B \cap (2,3)$. Pick $r \in \mathbb{Q}$ such that $r < x < r+q$. Let $F \subseteq \mathbb{N}$ be such that $\sum_{n\in F} q_n = x - r$. Observe that $F$ has to be infinite since $x$ is irrational. Put $G = \mathbb{N} \setminus F$. Then $\sum_{n\in G} q_n = r + q - x$. It also follows that $G$ is infinite.

Put $r_0 = r$. There clearly is a sequence $(r_n)_{n\geq 1}$ of rational numbers in $(r,x)$ such that $(r_n)_n \nearrow x$ while moreover for every $n \geq 1$ we have

$$r_n - r_{n-1} = q_{\mu(n)},$$

where $\mu(n)$ is the $n$-the element of $F$ (ordered as a subset of $\mathbb{N}$). Put $s_0 = r+q$. There similarly is a sequence $(s_n)_{n\geq 1}$ of rational numbers in $(x, r+q)$ such that $(s_n)_n \searrow x$ while moreover for every $n \geq 1$ we have

$$s_{n-1} - s_n = q_{\nu(n)},$$

where $\nu(n)$ is the $n$-the element of $G$ (ordered as a subset of $\mathbb{N}$).

Let $\mu(n) \in A$. By Lemma 2.2 we may pick a homeomorphism

$$g_n \colon [t_{\mu(n)}, t_{\mu(n)-1}]_B \to [r_{n-1}, r_n]_B.$$

Similarly, if $\nu(n) \in B$, we may pick a homeomorphism

$$h_n \colon [t_{\nu(n)}, t_{\nu(n)-1}]_B \to [s_n, s_{n-1}]_B.$$

Since all sets involved are clopen, the function $f \colon \{0\} \cup [0,q]_B \to [r, r+q]_B$ defined by

$$f(x) = \begin{cases} g_n(x) & (t_{\mu(n)} < x < t_{\mu(n)-1}), \\ h_n(x) & (t_{\nu(n)} < x < t_{\nu(n)-1}), \\ x & (t = 0), \end{cases}$$

is a homeomorphism. Hence we are done by Lemma 2.2. $\square$

The following can be proved with the same method.

**Lemma 2.6.** *Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{1\} \cup [1-q, 1]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0, q]_B$.*

We now come to the main result in this section.

**Theorem 2.7.** *The closed unit interval $\mathbb{I} = [0, 1]$ can be partitioned into two homogeneous and homeomorphic sets.*

*Proof.* Put $E = (0, 1) \cap A$ and $F = [0, 1]_B = (0, 1) \cap B$, respectively. Observe that $E$ and $F$ are homogeneous being both open subsets of zero-dimensional homogeneous spaces. Also, both $E$ and $F$ are the union of a pairwise disjoint family clopen arcs in $A$ respectively $B$ and have the same rational 'length'. Hence $E \approx F$ by Corollary 2.4.

Let us now consider the space $F$, and let $0 < q < 1/2$ be rational. Then by Lemmas 2.5, 2.6 and 2.2 we have that $\{0\} \cup [0, q]_B \approx [0, q]_B$ and $\{1\} \cup [1, 1-q]_B \approx [q, 2q]_B$. Moreover, $[q, 1-q]_B$ is homeomorphic to $[2q, 1]_B$, again by Lemma 2.2. Hence we conclude that $\{0\} \cup F \cup \{1\}$ is homeomorphic to $F$.

The partition $\{E, F \cup \{0, 1\}\}$ of $\mathbb{I}$ is consequently the one we are after. □

We are now in a position to prove Theorem 1.1.

*Proof of Theorem 1.1.* By Keller's Theorem [10] (see also [13]), $\mathbb{I}^\omega$ is homogeneous. This implies that $\mathbb{I}^\tau \approx \mathbb{I} \times \mathbb{I}^\tau$ is homogeneous for every infinite cardinal $\tau$. Hence we are done by Theorem 2.7. □

## 3. Topological groups

We show here that Theorem 1.1 for uncountable cardinals cannot be improved to the case of a splitting into homeomorphic topological groups. For information on topological groups, see Arhangel'skii and Tkachenko [4].

The following result is well-known, its proof is included for completeness sake.

**Lemma 3.1.** *Let $G$ be a topological group. If $S$ is a $G_\delta$-subset of $G$ containing the neutral element $e$ of $G$, then there is a closed subgroup $N$ of $G$ such that*

*(1) $N \subseteq S$,*
*(2) $N$ is a $G_\delta$-subset of $G$.*

*Proof.* Write $S$ as $\bigcap_{n<\omega} U_n$, where each $U_n$ is open in $G$. Recursively, pick open symmetric neighborhoods $V_n$ of $e$ such that $V_{n+1}^2 \subseteq V_n \subseteq U_n$, and let $N = \bigcap_{n<\omega} V_n$. □

**Theorem 3.2.** *If $G$ is a dense subset of $\mathbb{I}^\tau$, where $\tau$ is uncountable, such that $\mathbb{I}^\tau \setminus G$ is Lindelöf, then $G$ is not a topological group.*

*Proof.* Striving for a contradiction, assume that $G$ is a topological group.

We may assume by homogeneity that the element of $\mathbb{I}^\tau$ with constant coordinates 0 is the neutral element $e$ of $G$. Since $\mathbb{I}^\tau \setminus G$ is Lindelöf, there is a compact $G_\delta$-subset $S_0$ of $\mathbb{I}^\tau$ such that $e \in S_0 \subseteq G$.

There is a countable subset $A_0$ of $\tau$ such that

$$S_1 = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A_0)(x_\alpha = 0)\} \subseteq S_0.$$

By Lemma 3.1, we may pick a closed subgroup $N_1$ of $G$ which is a $G_\delta$-subset of $G$ such that $N_1 \subseteq S_1$. Clearly, $N_1$ is a $G_\delta$-subset of $S_1$ and hence is a compact $G_\delta$-subset of $\mathbb{I}^\tau$. There is a countable subset $A_1$ of $\tau$ such that $A_0 \subseteq A_1$ while moreover

$$S_2 = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A_1)(x_\alpha = 0)\} \subseteq N_1.$$

Continuing in this way, it is easy to construct by recursion countable subsets $A_n$ of $\tau$ and closed subgroups $N_n$ of $G$ such that for every $n$,

(1) $A_n \subseteq A_{n+1}$,
(2) $S_{n+1} = (\forall\, \alpha \in A_{n+1})(x_\alpha = 0)\} \subseteq N_n \subseteq S_n$.

Put $A = \bigcup_{n<\omega} A_n$. Then since $\tau$ is uncountable,

$$\bigcap_{n<\omega} N_n = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A)(x_\alpha = 0)\} \approx \mathbb{I}^\tau.$$

Hence $\mathbb{I}^\tau$ is a topological group, which contradicts the Brouwer Fixed-Point Theorem. $\square$

We are now in the position to present a proof of Theorem 1.2. We use a factorization result of Arhangel'skii [1], the key feature of which is that it concerns continuous functions on dense subspaces of products of separable metrizable spaces [4, Corollary 1.7.8 (see also Theorem 1.7.7)]. Arhangel'skii's result is also stated and applied in his book [2, Lemma 0.2.3]. It implies that every continuous realvalued function on a dense subset of a Tychonoff cube depends on countably many coordinates. Therefore, if $A$ is a dense pseudocompact subset of some Tychonoff cube $\mathbb{I}^\tau$, then $\mathbb{I}^\tau$ is the Čech-Stone-compactification $\beta A$ of $A$. Indeed, for every continuous function $f \colon A \to \mathbb{R}$ there is by Corollary 1.7.8 in [4], a countable subset $L$ of $\tau$ and a continuous function $g \colon \pi_L(A) \to \mathbb{R}$, where $\pi_L \colon \mathbb{I}^\tau \to \mathbb{I}^L$ is the projection, such that $g(\pi_L(a)) = f(a)$ for all $a \in A$. However, since $A$ is pseudocompact, $\pi_L(A) = \mathbb{I}^L$, which evidently implies that $f$ can be extended over $\mathbb{I}^\tau$.

*Proof of Theorem 1.2.* Assume the contrary. First observe that $A$ is nowhere locally compact. Indeed, if $A$ would be somewhere locally compact, it would be locally compact at all points by homogeneity and so its complement would be compact implying that $A$ would be compact; this is clearly impossible. This also gives us that $A$ is dense. For if $A$ would not be dense, $\mathbb{I}^\tau \setminus A$ would be somewhere locally compact, and so $A$ would be somewhere locally compact.

The Dichotomy Theorem from Arhangel'skii [3] implies that $B = \mathbb{I}^\tau \setminus A$ is pseudocompact or Lindelöf. But it cannot be Lindelöf by Theorem 3.2. Hence $B$ is pseudocompact and so $A$ is pseudocompact. Since $A$ is dense in $\mathbb{I}^\tau$, it follows by the above that $\mathbb{I}^\tau = \beta A$.

We complete the proof now in two ways. The first proof is as follows. Since $A$ is a pseudocompact topological group, $\beta A$ is a topological group by the Comfort-Ross theorem [6]. But $\mathbb{I}^\tau$ is not a topological group, for example because it has the Fixed-Point Property by Brouwer's Theorem.

The second proof is more direct and avoids the use of the complicated Comfort-Ross Theorem. Indeed, we first claim that $A$ does not contain any nonempty compact $G_\delta$-subset. For if it would contain such a compact $G_\delta$-subset $S$, then $S$ has a countable base of open neighbourhoods in $A$, since the space $A$ is pseudocompact. Since $A$ is a topological group, it follows from this that $A$ is paracompact [4, Corollary 4.3.21](see also page 314 there).

Since $A$ is also pseudocompact, it consequently follows that $A$ is compact, - a contradiction.

Fix a homeomorphism $f$ of $A$ onto $B$. Clearly, $f$ can be extended to a homeomorphism $h$ of $\mathbb{I}^\tau$ onto $\mathbb{I}^\tau$. Since $h(A) = B$ and $h(B) = A$, it follows that $h$ has no fixed-points. This is a contradiction with the Brouwer Fixed-Point Theorem. $\square$

In the zero-dimensional case, the case of Cantor cubes instead of Tychonoff cubes, Theorem 1.2 does not hold. Indeed, let $\kappa$ be an infinite cardinal, and let $p$ be a free ultrafilter on $\kappa$. The set

$$A = \{x \in \{0,1\}^\tau : \{\alpha : x_\alpha = 1\} \in p\}$$

is a subgroup of $\{0,1\}^\tau$ of index 2. Hence $A$ as well as its complement are homeomorphic to topological groups.

We do not know whether every compact topological group can be split into two homeomorphic and homogeneous parts.

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