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DOI

[10.1016/0550-3213\(96\)00228-3](https://doi.org/10.1016/0550-3213(96)00228-3)

Publication date

1996

Document Version

Final published version

Published in

Nuclear Physics B

[Link to publication](#)

Citation for published version (APA):

Buza, M., Matiounine, Y., Smith, J., Migneron, R., & van Neerven, W. L. (1996). Heavy quark coefficient functions at asymptotic values $Q^2 \gg m^2$. *Nuclear Physics B*, 472(3), 611-658. [https://doi.org/10.1016/0550-3213\(96\)00228-3](https://doi.org/10.1016/0550-3213(96)00228-3)

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ELSEVIER

Nuclear Physics B 472 (1996) 611–658

NUCLEAR
PHYSICS B

Heavy quark coefficient functions at asymptotic values $Q^2 \gg m^2$

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Received 18 January 1996; revised 22 March 1996; accepted 23 April 1996

Abstract

In this paper we present the analytic form of the heavy quark coefficient functions for deep inelastic lepton–hadron scattering in the kinematical regime $Q^2 \gg m^2$. Here Q^2 and m^2 stand for the masses squared of the virtual photon and heavy quark, respectively. The calculations have been performed up to next-to-leading order in the strong coupling constant α_s using operator product expansion techniques. Apart from a check on earlier calculations, which however are only accessible via large computer programs, the asymptotic forms of the coefficient functions are useful for charm production at HERA when the condition $Q^2 \gg m_c^2$ is satisfied. Furthermore, the analytical expressions can also be used when one applies the variable flavour number scheme up to next-to-leading order in α_s .

PACS: 12.39.H; 13.60.H; 11.15; 12.38.B

Keywords: Electroproduction; Charm; Structure function; Operator product

1. Introduction

The study of deep inelastic electroproduction has led to important information on the structure of the proton. This information is extracted from the structure functions

¹ Supported by the Foundation for Fundamental Research on Matter (FOM).

² Partially supported under the contract NSF 93-09888.

³ Partially supported by the Netherlands Organization for Scientific Research (NWO) and by the National Sciences and Engineering Research Counsel of Canada (NSERC).

$F_2(x, Q^2)$ and $F_L(x, Q^2)$ which appear in the cross section for deep inelastic lepton-hadron scattering. Here x denotes the Bjorken scaling variable and Q^2 is the mass squared of the virtual photon exchanged between the lepton and the hadron. In the framework of perturbative Quantum Chromodynamics (QCD) these structure functions can be described by a convolution of parton densities with coefficient functions. The latter are calculable order by order in perturbation theory. However, the parton densities cannot be computed yet since they are of a non-perturbative origin and have to be extracted from the data. Starting from the late sixties these densities have been obtained from many experiments. Until recently the analysis was carried out in the kinematical range $0.01 < x < 0.95$ and $Q^2 < 300 \text{ (GeV/c)}^2$. However, since the advent of the HERA accelerator the kinematical region has been extended to much smaller values of x ($x > 10^{-4}$) and much larger values of Q^2 ($Q^2 < 2 \times 10^4 \text{ (GeV/c)}^2$). The most recent results come from the H1 and ZEUS experiments at HERA, see Refs. [1] and [2], respectively.

The low- x region is of great experimental as well as theoretical interest. The structure function $F_2(x, Q^2)$ rises very steeply when $x \rightarrow 0$ which can be mainly attributed to a corresponding increase in the gluon density. Therefore this density is a very important issue in the investigation of the small- x structure of the proton. Since the gluon density appears together with the other parton densities in most cross sections, one has to look for those specific reactions in which it plays a dominant role so that it can be isolated from its partners. One of these processes is extrinsic charm production in deep inelastic electron-proton scattering, which was investigated for the first time in [3]. Here the dominant production mechanism is represented by the photon-gluon fusion process which is indeed the only one in the Born approximation. Next-to-leading order (NLO) calculations [4], to which also other processes contribute, reveal that this picture remains unaltered.

Apart from the interest in the gluon density, charm production also revived the important issue of how to treat the charm quark in deep inelastic scattering processes. Here one can distinguish between intrinsic [5] and extrinsic charm production. In the former case the charm is considered to be a part of the hadronic wave function and it is described by a parton density in the hadron like the other light flavours (u, d, s) and the gluon (g). However, this prescription is only correct if the charm quark can be treated as a massless particle which is certainly not the case at small- Q^2 values where threshold effects become important. In this region the charm quark has to be treated as a massive particle. On the other hand, when $Q^2 \gg m_c^2$, where m_c is the charm mass, extrinsic charm production via the photon-gluon fusion process and its higher order QCD corrections reveal large logarithms of the type $\alpha_s^k \ln^k(Q^2/m_c^2)$. These large corrections bedevil the perturbation series and have therefore to be resummed via the renormalization group equations. This resummation entails the definition of a charm parton density in the hadron. Although a recent investigation [6] shows that the above logarithms lead to a rather stable cross section for charm production with respect to variations in the renormalization and mass factorization scale, the size of these large corrections warrants a special treatment. This is provided by the so-called variable flavour number scheme

(VFNS) [7,8]. In this scheme the treatment of the charm quark depends on the values chosen for Q^2 . At low- Q^2 values the deep inelastic structure functions are described by the light parton densities (u, d, s, g). The charm contribution is given by the photon–gluon fusion process and its higher order QCD corrections. At large Q^2 the charm is treated in the same way as the other light quarks and it is represented by a charm parton density in the hadron, which evolves in Q^2 . In the intermediate Q^2 region one has to make a smooth connection between the two different prescriptions. In Refs. [9–11] this was done by adding and subtracting certain mass factorization terms. Notice that the above considerations also apply to bottom production when Q^2 gets extremely large.

Up to now the VFNS scheme has only been applied to heavy flavour electroproduction using the Born approximation to the photon–gluon fusion process [9–11]. It is our aim to extend this scheme to next-to-leading order in α_s . For that purpose we will calculate the full two-loop operator matrix elements containing one heavy quark loop. This calculation provides us with the terms containing the logarithms $\ln^k(m_c^2/\mu^2)$ which have to be subtracted from the charm cross section so that the final result becomes independent of the charm mass m_c . Here μ denotes the renormalization scale which can be identified with the mass factorization scale. In addition to their application to the VFNS scheme one can also use the two-loop operator matrix elements to construct an analytic form of the NLO heavy quark coefficient functions in the limit $Q^2 \gg m_c^2$ by combining them with the NLO light parton coefficient functions in [12]. We will refer to these results as the asymptotic heavy quark coefficient functions. These expressions serve as a check on the exact calculation in [4] which is only available in a large computer program involving numerical integrations over several variables. Furthermore, it enables us to see at which Q^2 -values the asymptotic heavy quark coefficient functions coincide numerically with the exact ones, which gives an indication when the charm quark can be treated as a massless quark.

The content of this paper can be summarized as follows. In Section 2 we introduce our notations and give an outline how the heavy quark coefficient functions can be determined in the asymptotic limit $Q^2 \gg m_c^2$. In Section 3 we present the calculation of the full two-loop operator matrix elements needed for the computation of the asymptotic form of the heavy quark coefficient functions. The latter will be calculated in Section 4. Finally, in Section 5 we will show at which Q^2 -values the asymptotic form coincides with the exact one given in [4]. In Appendix A an exact analytic expression for the heavy quark coefficient function, which is valid for any Q^2 and m_c^2 , is presented in the case of the Compton subprocess. An important trick how to compute an operator matrix element integral containing five different propagators is given in Appendix B. The long formulae obtained for the full operator matrix elements and the asymptotic heavy quark coefficient functions are presented in Appendices C and D, respectively.

2. Heavy quark electroproduction

In this section we will show the connection between the heavy flavour coefficient functions computed in the asymptotic limit $Q^2 \gg m^2$ and the operator matrix elements (OME's). Here Q^2 denotes the mass squared of the virtual photon with momentum q ($q^2 = -Q^2 < 0$) and m stands for the heavy flavour mass. The variable x is defined by $x = Q^2/2p \cdot q$ (Bjorken scaling variable) where p stands for the momentum of the proton. The OME's arise when the local operators, which show up in the operator product expansion (OPE) of two electromagnetic currents, are sandwiched between the proton states indicated by $|P\rangle$. In the limit that x is fixed and $Q^2 \gg M^2$, where M denotes the mass of the proton, this OPE dominates the integrand of the hadronic structure tensor $W_{\mu\nu}$, which is defined by

$$W_{\mu\nu}(p, q) = \frac{1}{4\pi} \int d^4z e^{iq \cdot z} \langle P | [J_\mu(z), J_\nu(0)] | P \rangle. \quad (2.1)$$

This structure tensor arises when one computes the cross section for deep inelastic electroproduction of heavy flavours

$$e^-(\ell_1) + P(p) \rightarrow e^-(\ell_2) + Q(p_1)(\bar{Q}(p_2)) + 'X', \quad (2.2)$$

provided the above reaction is completely inclusive with respect to the hadronic state 'X' as well as the heavy flavours $Q(\bar{Q})$. When the virtuality Q^2 ($q = \ell_1 - \ell_2$) of the exchanged photon is not too large ($Q^2 \ll M_Z^2$) the reaction in (2.2) is dominated by the one-photon exchange mechanism and we can neglect any weak effects caused by the exchange of the Z-boson. In this case $W_{\mu\nu}(p, q)$ in (2.1) can be written as

$$W_{\mu\nu}(p, q) = \frac{1}{2x} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_L(x, Q^2) + \left(p_\mu p_\nu - \frac{p \cdot q}{q^2} (p_\mu q_\nu + p_\nu q_\mu) \right. \\ \left. + g_{\mu\nu} \frac{(p \cdot q)^2}{q^2} \right) \frac{F_2(x, Q^2)}{p \cdot q}. \quad (2.3)$$

The above formula follows from Lorentz covariance of $W_{\mu\nu}(p, q)$, parity invariance and the conservation of the electromagnetic current J_μ . Besides the longitudinal structure function $F_L(x, Q^2)$ and the structure function $F_2(x, Q^2)$ one can also define the transverse structure function $F_1(x, Q^2)$ which, however, depends on the two previous ones and is given by

$$F_1(x, Q^2) = \frac{1}{2x} [F_2(x, Q^2) - F_L(x, Q^2)]. \quad (2.4)$$

The above structure functions show up in the deep inelastic cross section of the process in (2.2) provided we integrate over the whole final state

$$\frac{d^2\sigma}{dx dy} = \frac{2\pi\alpha^2}{(Q^2)^2} S \left[\{1 + (1-y)^2\} F_2(x, Q^2) - y^2 F_L(x, Q^2) \right], \quad (2.5)$$

where S denotes the square of the c.m. energy of the electron proton system and the variables x and y are defined (see above (2.1)) as

$$x = \frac{Q^2}{2p \cdot q} \quad (0 < x \leq 1), \quad y = \frac{p \cdot q}{p \cdot \ell_1} \quad (0 < y < 1), \quad (2.6)$$

with

$$-q^2 = Q^2 = xyS. \quad (2.7)$$

In the QCD improved parton model the heavy flavour contribution to the hadronic structure functions can be expressed as integrals over the partonic scaling variable $z = Q^2/(s + Q^2)$, where s is the square of the photon-parton centre-of-mass energy ($s \geq 4m^2$). This yields the following result:

$$\begin{aligned} F_i(x, Q^2, m^2) = & x \int_x^{z_{\max}} \frac{dz}{z} \left[\frac{1}{n_f} \sum_{k=1}^{n_f} e_k^2 \left\{ \Sigma\left(\frac{x}{z}, \mu^2\right) L_{i,q}^S\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \right. \right. \\ & + G\left(\frac{x}{z}, \mu^2\right) L_{i,g}^S\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \left. \left. + \Delta\left(\frac{x}{z}, \mu^2\right) L_{i,q}^{\text{NS}}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \right\} \right] \\ & + x e_Q^2 \int_x^{z_{\max}} \frac{dz}{z} \left\{ \Sigma\left(\frac{x}{z}, \mu^2\right) H_{i,q}^{\text{PS}}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \right. \\ & \left. + G\left(\frac{x}{z}, \mu^2\right) H_{i,g}^S\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \right\}, \quad (2.8) \end{aligned}$$

where $i = 2, L$ and the upper boundary of the integration is given by $z_{\max} = Q^2/(4m^2 + Q^2)$. The function $G(z, \mu^2)$ stands for the gluon density. The flavour singlet combination of the quark densities is defined by

$$\Sigma(z, \mu^2) = \sum_{i=1}^{n_f} \left(f_i(z, \mu^2) + \bar{f}_i(z, \mu^2) \right), \quad (2.9)$$

where f_i and $\bar{f}_i$ stand for the light quark and anti-quark densities of species i , respectively. The non-singlet flavour combination of the quark densities is given by

$$\Delta(z, \mu^2) = \sum_{i=1}^{n_f} \left(e_i^2 - \frac{1}{n_f} \sum_{k=1}^{n_f} e_k^2 \right) \left(f_i(z, \mu^2) + \bar{f}_i(z, \mu^2) \right). \quad (2.10)$$

In the above expressions the charges of the light quark and the heavy quark in process (2.2) are denoted by e_i and e_Q , respectively. Furthermore, n_f stands for the number of light quarks and μ denotes the mass factorization scale, which we choose to be equal to the renormalization scale. The latter shows up in the running coupling constant contained in $\alpha_s(\mu^2)$. Like the parton densities the heavy quark coefficient functions $L_{i,j}$ ($i = 2, L$; $j = q, g$) can also be divided into flavour singlet and non-singlet parts which are indicated by the superscripts S and NS in Eq. (2.8).

Furthermore, the singlet quark coefficient function can be split into

$$L_{i,q}^S = L_{i,q}^{\text{NS}} + L_{i,q}^{\text{PS}}. \quad (2.11)$$

The above relation originates from the light flavour decomposition of the Feynman graphs contributing to the structure function $F_i(x, Q^2)$ in (2.8). One class of graphs gives the same contribution to $L_{i,q}^S$ as to $L_{i,q}^{NS}$, whereas another class, which we call purely singlet (PS), only contributes to $L_{i,q}^S$. The latter class is characterized by those diagrams which have a gluon in the t -channel and can therefore only contribute to the singlet quark coefficient function $L_{i,q}^S$. It turns out that up to order α_s^2 $L_{i,q}^S = L_{i,q}^{NS}$ and $L_{i,g}^S = 0$. In the case of the heavy quark coefficient functions $H_{i,j}$ there is no non-singlet part and therefore $H_{i,q}^S = H_{i,q}^{PS}$. The latter gets contributions for the first time in order α_s^2 , whereas the perturbation series in $H_{i,g}^S$ starts in order α_s . The distinction between the heavy quark coefficient functions $L_{i,j}$ and $H_{i,j}$ can be traced back to the different photon-parton production processes from which they originate. The functions $L_{i,j}$ are attributed to the reactions where the virtual photon couples to the light quark whereas $H_{i,j}$ originates from the reactions where the virtual photon couples to the heavy quark. Hence $L_{i,j}$ and $H_{i,j}$ in (2.8) are multiplied by e_i^2 and e_Q^2 , respectively. Moreover, when the reaction where the photon couples to the heavy quark contains a light quark in the initial state, then it can only proceed via the exchange of a gluon in the t -channel between the light and the heavy quark. Therefore this subprocess is purely singlet and there does not exist a non-singlet contribution to $H_{i,q}^S$. Here below we will discuss the various subprocesses which contribute to the heavy quark coefficient functions up to order α_s^2 and which have been exactly calculated in [4].

In the Born approximation (first order of α_s or LO) one has the photon-gluon fusion process

$$\gamma^*(q) + g(k_1) \rightarrow Q(p_1) + \bar{Q}(p_2), \quad (2.12)$$

which leads to the lowest order contribution to $H_{i,g}^S$ denoted by $H_{i,g}^{S,(1)}$. Here the coefficients $H_{i,j}^{(k)}$ are defined as follows:

$$H_{i,j}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \sum_{k=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k H_{i,j}^{(k)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right), \quad (2.13)$$

with a similar definition for $L_{i,j}^{(k)}$.

The next order is obtained by including the virtual gluon corrections to process (2.12) and the gluon bremsstrahlung process

$$\gamma^*(q) + g(k_1) \rightarrow g(k_2) + Q(p_1) + \bar{Q}(p_2), \quad (2.14)$$

both of which contribute to the second order term in $H_{i,g}^S$ denoted by $H_{i,g}^{S,(2)}$. In addition to the above reaction we also have the subprocess where the gluon in (2.14) is replaced by a light (anti-) quark, i.e.

$$\gamma^*(q) + q(\bar{q})(k_1) \rightarrow q(\bar{q})(k_2) + Q(p_1) + \bar{Q}(p_2). \quad (2.15)$$

This process has however two different production mechanisms, which both start in second order of α_s . The first one is given by the Bethe-Heitler process (see Figs. 5a,b in

Ref. [4]) and the second one can be attributed to the Compton reaction (see Figs. 5c,d in Ref. [4]). In the case of the Bethe–Heitler process the virtual photon couples to the heavy quark, where the latter exchanges a gluon with the incoming light quark, providing us with the coefficient function $H_{i,q}^{\text{PS},(2)}$. In the Compton reaction the virtual photon couples to the light (anti-) quark and leads to $L_{i,q}^{\text{NS},(2)} = L_{i,q}^{\text{S},(2)}$ (see (2.11)). Finally we want to make the remark that there are no interference terms between the Bethe–Heitler and Compton reactions in (2.15) if one integrates over all final state momenta.

The complexity of the second order heavy quark coefficient functions prohibits publishing them in an analytic form, except for $L_{i,q}^{\text{NS},(2)}$, which is given in Appendix A, so that they are only available in large computer programs [4], involving two-dimensional integrations. To shorten the long running time for these programs we have previously tabulated the coefficient functions in the form of a two-dimensional array in the variables η and ξ in a different computer program [13]. These variables are defined by (using $s = (q + k_1)^2$, $z = Q^2/(s + Q^2)$, and $\eta = (s - 4m^2)/(4m^2)$)

$$\eta = \frac{(1 - z)}{4z} \xi - 1, \quad \xi = \frac{Q^2}{m^2}. \quad (2.16)$$

This new program has shortened the computation of the heavy flavour structure functions $F_i(x, Q^2, m^2)$ considerably as one only requires one integral over the variable z in (2.8), therefore making our results for the NLO corrections more amenable for phenomenological applications. However, when $Q^2 \gg m^2$ it is possible to get complete analytic forms for the heavy quark coefficient functions which are similar to the ones presented for the light quark and gluon coefficient functions given in [12]. To get the analytic form in the above asymptotic regime one can follow two approaches. The first one is to go back to the original calculation of the exact coefficient functions in [4] and repeat the computation of the Feynman graphs and the phase space integrals in the limit $Q^2 \gg m^2$. An example of such a calculation can be found in [14], where all photonic corrections to the initial state of the process $e^- + e^+ \rightarrow \mu^- + \mu^+$ in the limit $S \gg m_e^2$ were computed. However, this procedure is still quite complicated because one cannot neglect the fermion masses at too premature a stage which results in rather messy calculations. Fortunately, as one can find in [14], there exists an alternative method which we will use for the heavy quark coefficient functions.

In the limit $Q^2 \gg m^2$ the heavy quark coefficient functions behave logarithmically as

$$H_{i,j}^{(k)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \sum_{l=0}^k a_{i,j}^{(k,l)}\left(z, \frac{m^2}{\mu^2}\right) \ln^l \frac{Q^2}{m^2}, \quad (2.17)$$

with a similar expression for $L_{i,j}^{(k)}$. As has been already mentioned in the introduction these large logarithms $\ln^l(Q^2/m^2)$ dominate the radiative corrections. This is in particular the case for charm production in the large Q^2 region which is accessible to HERA experiments. The above large logarithms also arise when Q^2 is kept fixed and $m^2 \rightarrow 0$ so that they originate from collinear singularities. These collinear divergences

can be removed via mass factorization. The latter proceeds in the following way. In the non-singlet case we have

$$C_{i,q}^{\text{NS}}\left(\frac{Q^2}{\mu^2}, n_f\right) + L_{i,q}^{\text{NS}}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \Gamma_{qq}^{\text{NS}}\left(\frac{m^2}{\mu^2}\right) \otimes C_{i,q}^{\text{NS}}\left(\frac{Q^2}{\mu^2}, n_f + 1\right). \quad (2.18)$$

For the singlet case the mass factorization becomes

$$\begin{aligned} C_{i,l}^{\text{S}}\left(\frac{Q^2}{\mu^2}, n_f\right) + L_{i,l}^{\text{S}}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) + H_{i,l}^{\text{S}}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) \\ = \sum_k \Gamma_{kl}^{\text{S}}\left(\frac{m^2}{\mu^2}\right) \otimes C_{i,k}^{\text{S}}\left(\frac{Q^2}{\mu^2}, n_f + 1\right), \end{aligned} \quad (2.19)$$

with $k, l = q, g$. Notice that in the above expression we have suppressed the z -dependence for simplicity. The convolution symbol is defined by

$$(f \otimes g)(z) = \int_0^1 dz_1 \int_0^1 dz_2 \delta(z - z_1 z_2) f(z_1) g(z_2). \quad (2.20)$$

The quantities Γ_{kl} and $C_{i,k}$ ($k, l = q, g$) which appear in (2.18), (2.19) stand for the transition functions and the light parton coefficient functions, respectively. Notice that all mass dependence is transferred to the transition function Γ_{kl} . The removal of the logarithmic terms $\ln^l(m^2/\mu^2)$ from the heavy flavour coefficient functions leads to an enhancement by one of the number of light flavours n_f in the light parton coefficient functions $C_{i,k}$ – see Eqs. (2.18), (2.19). The latter have been calculated up to second order in α_s in [12]. Since these coefficient functions do not depend on the way one has regularized the collinear divergences in the parton cross sections (for a discussion see Ref. [15]) one can also use them in Eqs. (2.18) and (2.19). Now if one also knows the transition functions Γ_{kl} one can reconstruct the asymptotic behaviour of the heavy flavour coefficient functions $L_{i,k}$ and $H_{i,k}$. This is possible because the transition functions Γ_{kl} are identical to the operator matrix elements (OME) denoted by A_{kl} . The latter appear in the operator product expansion (OPE) of the commutator of the two electromagnetic currents in (2.1) near the light cone. Suppressing the Lorentz index of the electromagnetic current the expansion can be written as

$$\lim_{z^2 \rightarrow 0} [J(z), J(0)] = \sum_k \sum_m C_k^{(m)}(z^2) z_{\mu_1} \dots z_{\mu_m} O_k^{\mu_1 \dots \mu_m}(0), \quad (2.21)$$

where the distributions $C_k^{(m)}(z^2)$ ($k = q, g$) are the Fourier transforms of the light parton coefficient functions $C_k^{(m)}(Q^2/\mu^2)$ contained in (2.18), (2.19). Notice that the latter has been Mellin transformed according to

$$C_k^{(m)}\left(\frac{Q^2}{\mu^2}\right) = \int_0^1 dy y^{m-1} C_k\left(y, \frac{Q^2}{\mu^2}\right). \quad (2.22)$$

In the above expressions the index i ($i = 2, L$) in $C_{i,k}^{(m)}$ has been omitted since we suppressed the Lorentz index of the current $J_\mu(z)$. Inclusion of the latter index implies that one has two independent structure functions F_2 and F_L . The quantity m refers to the spin of the local operators $O_k^{\mu_1 \dots \mu_m}$ which appear in the OPE (2.21). The latter are given by the non-singlet quark operator

$$O_{q,r}^{\mu_1 \dots \mu_m}(x) = \frac{1}{2} i^{m-1} S \left[\bar{\psi}(x) \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_m} \frac{\lambda_r}{2} \psi(x) \right] + \text{trace terms}, \quad (2.23)$$

and the singlet operators

$$O_q^{\mu_1 \dots \mu_m}(x) = \frac{1}{2} i^{m-1} S \left[\bar{\psi}(x) \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_m} \psi(x) \right] + \text{trace terms}, \quad (2.24)$$

$$O_g^{\mu_1 \dots \mu_m}(x) = \frac{1}{2} i^{m-2} S \left[F_\alpha^{a, \mu_1}(x) D^{\mu_2} \dots D^{\mu_{m-1}} F^{a, \alpha \mu_m}(x) \right] + \text{trace terms}. \quad (2.25)$$

Notice that $O_q^{\mu_1 \dots \mu_m}$ in (2.23), (2.24) can stand for the light quark as well as for the heavy quark operator. Where possible we put $q = Q$ when the heavy quark operator is considered. Furthermore, S denotes the symmetrization of the operators in their Lorentz indices μ_i and the trace terms are needed to make them traceless. The λ_r in (2.23) represent the generators of the flavour group $SU(n_f)$, whereas the index a in (2.25) is the color index. The objects $\psi(x)$, $F_{\mu\nu}^a$ represent the quark field and the gluon field tensor respectively and D_μ denotes the covariant derivative.

The OPE expansion in (2.21) can be applied in the limit $Q^2 \gg m^2$ at fixed x so that the integrand in (2.1) gets its dominant contribution from the light-cone behavior of the current-current commutator. Inserting (2.21) in (2.1) and replacing the hadron state $|P\rangle$ by a light quark or gluon state $|\ell\rangle$ one gets the relations

$$\tilde{\mathcal{F}}_{i,q}^{\text{NS}}\left(\frac{Q^2}{\mu^2}, \epsilon\right) + \tilde{L}_{i,q}^{\text{NS}}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}, \epsilon\right) = \tilde{A}_{qq}^{\text{NS}}\left(\frac{m^2}{\mu^2}, \epsilon\right) \otimes C_{i,q}^{\text{NS}}\left(\frac{Q^2}{\mu^2}\right), \quad (2.26)$$

$$\tilde{\mathcal{F}}_{i,l}^{\text{S}}\left(\frac{Q^2}{\mu^2}, \epsilon\right) + \tilde{L}_{i,l}^{\text{S}}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}, \epsilon\right) + \tilde{H}_{i,l}^{\text{S}}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}, \epsilon\right) = \tilde{A}_{kl}^{\text{S}}\left(\frac{m^2}{\mu^2}, \epsilon\right) \otimes C_{i,k}^{\text{S}}\left(\frac{Q^2}{\mu^2}\right), \quad (2.27)$$

where $i = 2, L$; $k, l = q, g$ and $\tilde{\mathcal{F}}_{i,k}$ denote the light parton structure functions. The OME's $\tilde{A}_{kl}$ are defined by

$$\tilde{A}_{kl}\left(\frac{m^2}{\mu^2}, \epsilon\right) = \langle \ell | O_k | \ell \rangle. \quad (2.28)$$

Since the operators in (2.21) are already renormalized, the quantities in (2.26), (2.27) can be only collinearly divergent. The collinear divergences due to the presence of light partons are regularized by the method of n -dimensional regularization so that they show up in the form of pole terms $(1/\epsilon)^i$ ($\epsilon = n - 4$). In this and the next section all quantities like $\tilde{\mathcal{F}}_{i,k}$ and $\tilde{A}_{kl}$ which only depend on the collinear pole terms will be indicated by a tilde.

Fig. 1. (a,b) One-loop graphs contributing to the OME $A_{Qg}^{(1)}$. The solid line indicates the heavy quark Q . (c) External gluon self-energy contributions to $A_{gg}^{(1)}$. In the quark loop a sum over all heavy quark species indicated by H ($m_H^2 \geq m^2$) is understood. Here the solid line represents the heavy quark H .

Before we can compare (2.26), (2.27) with (2.18), (2.19) one has first to remove the collinear divergences due to the light partons via mass factorization. In this way the parton structure functions $\tilde{F}_{i,k}$ turn into the light parton coefficient functions $C_{i,k}$ and the heavy quark coefficient functions change into their collinearly finite analogues in (2.18), (2.19). Also the $\tilde{A}_{kl}$ in (2.28) turn into the collinearly finite OME's A_{kl} so that one can make the following identification:

$$\Gamma_{qq}^{\text{NS}}\left(\frac{m^2}{\mu^2}\right) = A_{qq}^{\text{NS}}\left(\frac{m^2}{\mu^2}\right), \quad (2.29)$$

$$\Gamma_{kl}^{\text{S}}\left(\frac{m^2}{\mu^2}\right) = A_{kl}^{\text{S}}\left(\frac{m^2}{\mu^2}\right). \quad (2.30)$$

For the computation of the asymptotic behaviour of the heavy quark coefficient functions corresponding to the processes (2.12), (2.14) and (2.15) one needs the following quantities. For processes (2.12) and (2.14) we have to calculate the one-loop OME $A_{Qg}^{\text{S},(1)}$ and the two-loop OME $A_{Qg}^{\text{S},(2)}$, respectively. They are represented by the Feynman graphs in Figs. 1 and 2. The Bethe–Heitler process given by reaction (2.15) corresponds to the two-loop OME $A_{Qq}^{\text{PS},(2)}$ in Fig. 3 whereas the Compton process (2.15) is related to $A_{qq}^{\text{NS},(2)}$ with the two-loop OME's in Fig. 4.

Notice that for the OME's we can make the same distinction in non-singlet (NS), singlet (S) and purely singlet (PS) as done for the coefficient functions. In particular we have the relation

$$A_{qq}^{\text{S}} = A_{qq}^{\text{NS}} + A_{qq}^{\text{PS}}, \quad (2.31)$$

and the origin of A_{qq}^{PS} is the same as mentioned below (2.11) for $L_{i,q}^{\text{PS}}$ and $H_{i,q}^{\text{PS}}$. This becomes clear when one examines Fig. 3. Here A_{qq}^{PS} is determined by the graphs where two gluons are exchanged between the heavy and the light quark so that only the singlet channel can contribute.

The calculation of the graphs in Figs. 1–4 and the derivation of the asymptotic form of the heavy quark coefficient functions will be the aim of the next two sections.

(a)

Fig. 2. Two-loop graphs contributing to the OME $A_{Qg}^{(2)}$. The solid line indicates the heavy quark Q . Diagrams s and t with the external ghosts (dashed lines) are needed to cancel the unphysical polarizations which appear in the sum in (3.43). Graphs u and v contain the external gluon self-energy with the heavy quark loop indicated by H which satisfies the same conditions as mentioned in Fig. 1c.

3. Calculation of the two-loop operator matrix elements

Before presenting the results of our calculation of the OME's we will first derive the general structure of the OME's discussed in the last section. If we insert the OPE (2.21) into the structure tensor $W_{\mu\nu}$ in (2.1) the OME's which are derived from the Feynman graphs in Figs. 1–4 will be computed in the forward direction. The latter means that the momentum leaving the operator vertex equals zero. Further if one puts the momentum,

(b)

Fig. 2 — continued.

Fig. 3. Two-loop graphs contributing to the OME $A_{Qq}^{PS,(2)}$. The solid line represents the heavy quark Q whereas the dashed line stands for the light quark q .

Fig. 4. Two-loop graphs contributing to the OME $A_{qq,Q}^{NS,(2)}$. The gluon self-energy contains the heavy quark Q in the quark loop only which is indicated by the solid line. The dashed line stands for the light quark q .

indicated by p , of the external light quark and gluon off-shell ($p^2 < 0$) only ultraviolet (UV) singularities appear in the OME's. Using this off-mass-shell assignment one can express the renormalized as well as the unrenormalized OME's into the renormalization group coefficients as is done in [16]. However, for our computations we have to put the external momentum on shell ($p^2 = 0$) so that the OME's turn into genuine S -matrix elements. This mass assignment implies that in addition to UV divergences one also encounters collinear (C) divergences which originate from the coupling of the external on-shell massless quanta to internal massless quanta. In the computation of the Feynman graphs both types of divergences will be regularized using the technique of n -dimensional regularization. However, since both singularities in the OME's will manifest themselves in the form of pole terms of the type ϵ^{-k} ($\epsilon = n - 4$) it is very hard to trace back their origins. Nevertheless, one can express the OME's into the renormalization group coefficients in a similar way as has been derived for the off-shell case in [16]. Where possible we will make a distinction between UV-pole terms and C-pole terms, which are indicated by ϵ_{UV}^{-k} and ϵ_C^{-k} respectively, and identify them $\epsilon_{UV} = \epsilon_C$ when it is appropriate.

In the subsequent part of this section we will construct the OME's, corresponding to the graphs in Figs. 1–4, in such a way that the coefficients of the pole terms are given by the renormalization group. These coefficients are products of the terms appearing in the beta function and the Altarelli–Parisi (AP) splitting functions (anomalous dimensions) [17]. The purpose of this presentation is threefold. First we need these coefficients for the construction of the heavy quark coefficient functions in the next section. Second, since the renormalization group coefficients are known in the literature we can predict the residues of the pole terms so that these expressions serve as a check on our calculations. Third, it is much easier to show the renormalization and mass factorization for the algebraic expressions, which are short, than for the analytic formulae in our calculations because the latter are rather long.

The OME's A_{ij} can be expanded in a perturbation series as follows:

$$A_{ij} = \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^k A_{ij}^{(k)}. \quad (3.1)$$

In the following discussion we distinguish three different types of OME's. First we have the unrenormalized ones indicated by $\hat{A}_{ij}$. They contain UV singularities as well as C-singularities. Second, after renormalization the UV divergences are removed and

we are left with the OME's defined by $\tilde{A}_{ij}$ which still contain C-divergences. The latter have to be removed via mass factorization so that the $\tilde{A}_{ij}$ turn into the finite OME's indicated by A_{ij} . Notice that the expansion in (3.1) holds for all three different types of OME's.

The renormalization of $\hat{A}_{ij}$ proceeds in three steps. First we will perform mass renormalization for which we choose the on-mass-shell scheme. This implies that the bare mass $\hat{m}$, which occurs in $\hat{A}_{ij}$ has to be replaced by

$$\hat{m} = m + \frac{\hat{\alpha}_s}{4\pi} \delta m, \quad \delta m = C_F S_\epsilon m \left(\frac{m^2}{\mu^2} \right)^{\epsilon/2} \left\{ \frac{6}{\epsilon_{UV}} - 4 \right\}, \quad (3.2)$$

so that the mass renormalized $\hat{A}_{ij}$ reads up to order α_s^2

$$\hat{A}_{ij} = \hat{A}_{ij}^{(0)} + \left(\frac{\hat{\alpha}_s}{4\pi} \right) \hat{A}_{ij}^{(1)} + \left(\frac{\hat{\alpha}_s}{4\pi} \right)^2 \left\{ \hat{A}_{ij}^{(2)} + \delta m \frac{d}{dm} \hat{A}_{ij}^{(1)} \right\}. \quad (3.3)$$

Notice that the zeroth order term $\hat{A}_{ij}^{(0)} = \delta_{ij}$ is mass independent. In the above perturbation series the quantities μ^2 and S_ϵ originate from n -dimensional regularization. The mass parameter μ is due to the dimensionality of the gauge coupling constant g ($\alpha_s = g^2/(4\pi)$) in n dimensions and should not be confused with the renormalization and mass-factorization scale. However, if one only subtracts the pole terms like in the $\overline{\text{MS}}$ scheme the mass parameter μ turns into the afore-mentioned scales. The spherical factor S_ϵ is defined as

$$S_\epsilon = \exp \left\{ \frac{\epsilon}{2} (\gamma_E - \ln 4\pi) \right\}, \quad (3.4)$$

where γ_E is the Euler constant and the colour factor C_F in $\text{SU}(N)$ is

$$C_F = \frac{N^2 - 1}{2N}. \quad (3.5)$$

Further, $\hat{\alpha}_s$ denotes the bare coupling constant which will be renormalized as follows:

$$\hat{\alpha}_s = \alpha_s(\mu^2) \left(1 + \frac{\alpha_s(\mu^2)}{4\pi} \delta\alpha_s \right), \quad (3.6)$$

$$\delta\alpha_s = S_\epsilon \left\{ \frac{2\beta_0}{\epsilon_{UV}} + \sum_{H=Q}^I \frac{2\beta_{0,H}}{\epsilon_{UV}} \left(\frac{m_H^2}{\mu^2} \right)^{\epsilon/2} \left(1 + \frac{1}{8} \epsilon_{UV}^2 \zeta(2) \right) \right\}. \quad (3.7)$$

In the equation above β_0 is the lowest order term in the series expansion of the beta-function which is given by

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_f n_f, \quad (3.8)$$

where C_A and T_f stand for the colour factors of $\text{SU}(N)$ which are

$$C_A = N, \quad T_f = \frac{1}{2}. \quad (3.9)$$

Further, n_f denotes the number of light flavours which enter via the quark-loop contributions to the gluon self-energy. The heavy quarks of species H with mass m_H also contribute to the latter. The lightest heavy quark occurring in the sum of (3.7) is represented by $H = Q$ with mass $m_Q = m$. It is the same quark as the one produced in the final state of process (2.2) and it also appears in the heavy quark operator whose OME's are shown in Figs. 1–3. The sum in (3.7) starts with $H = Q$ and ends with $H = t$ (top quark). For instance for charm production in (2.2) we have $Q = c$, $m_c = m$ and the sum in (3.7) runs over $H = c, b, t$. In the case of bottom (b) production we have $Q = b$, $m_b = m$ and the sum runs over $H = b, t$. The contribution to the beta function from a heavy quark H is given by

$$\beta_{0,H} = -\frac{4}{3}T_f, \quad (3.10)$$

for all species H , which implies that $\beta_{0,Q} = \beta_{0,H}$. The coupling constant renormalization in (3.7) is chosen in such a way that the heavy quarks H decouple from the running coupling constant $\alpha_s(\mu^2)$ for $\mu^2 < m_H^2$ and from the renormalized OME's. This is achieved by requiring that the light quark loop contributions to the gluon self-energy are renormalized in the $\overline{\text{MS}}$ scheme whereas the heavy quark loop contributions indicated by $\Pi(p^2, m_H^2)$ are renormalized in such a way that $\Pi_R(0, m_H^2) = 0$. The last condition leads to the factor $1 + \frac{1}{8}\epsilon_{\text{UV}}^2\zeta(2)$ in (3.7) where $\zeta(n)$ is the Riemann zeta function and $\zeta(2) = \pi^2/6$.

After coupling constant renormalization $\hat{A}_{ij}$ is represented up to $O(\alpha_s^2)$ by

$$\hat{A}_{ij} = \delta_{ij} + \left(\frac{\alpha_s}{4\pi}\right)\hat{A}_{ij}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ \hat{A}_{ij}^{(2)} + \delta m \frac{d}{dm} \hat{A}_{ij}^{(1)} + \delta\alpha_s \hat{A}_{ij}^{(1)} \right\}. \quad (3.11)$$

The remaining UV divergences are removed by operator renormalization, which is achieved by

$$\hat{A}_{ij}(\epsilon_{\text{UV}}, \epsilon_C, \frac{m^2}{\mu^2}, \alpha_s) = Z_{ik}(\epsilon_{\text{UV}}, \alpha_s) \otimes \tilde{A}_{kj}(\epsilon_C, \frac{m^2}{\mu^2}, \alpha_s), \quad (3.12)$$

where Z_{ij} ($i, j = q, g$) are the operator renormalization constants corresponding to the operators in (2.23)–(2.25). Notice that for the non-singlet operator in (2.23) Z_{qq}^{NS} is a real number, whereas for the singlet operators in (2.24), (2.25) Z_{ij} becomes a matrix. The operator matrix elements can be expanded in α_s analogous to A_{ij} in (3.1) as follows:

$$Z_{ij} = \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^k Z_{ij}^{(k)}, \quad (3.13)$$

so that the renormalized OME's $\tilde{A}_{ij}$ in (3.12) read up to $O(\alpha_s^2)$, (using $Z_{ij}^{(0)} = \delta_{ij}$)

$$\tilde{A}_{ij} = \delta_{ij} + \left(\frac{\alpha_s}{4\pi}\right) [\hat{A}_{ij}^{(1)} + Z_{ij}^{-1, (1)}]$$

$$+ \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ \hat{A}_{ij}^{(2)} + \delta m \frac{d}{dm} \hat{A}_{ij}^{(1)} + \delta \alpha_s \hat{A}_{ij}^{(1)} + Z_{ik}^{-1,(1)} \hat{A}_{kj}^{(1)} + Z_{ij}^{-1,(2)} \right\}, \quad (3.14)$$

where $Z_{ij}^{-1,(k)}$ are the expansion coefficients of the inverse matrix Z^{-1} . Choosing the $\overline{\text{MS}}$ scheme one obtains the following expression (see Ref. [16]) up to $O(\alpha_s^2)$:

$$\begin{aligned} Z_{ij}(\epsilon_{\text{UV}}, \alpha_s) = & \delta_{ij} + \left(\frac{\alpha_s}{4\pi} \right) S_\epsilon \left[-\frac{1}{\epsilon_{\text{UV}}} P_{ij}^{(0)} \right] \\ & + \left(\frac{\alpha_s}{4\pi} \right)^2 S_\epsilon^2 \left[\frac{1}{\epsilon_{\text{UV}}^2} \left\{ \frac{1}{2} P_{ik}^{(0)} \otimes P_{kj}^{(0)} + \left(\beta_0 + \sum_{H=Q}^t \beta_{0,H} \right) P_{ij}^{(0)} \right\} \right. \\ & \left. - \frac{1}{\epsilon_{\text{UV}}} \frac{1}{S_\epsilon} \delta \alpha_s P_{ij}^{(0)} - \frac{1}{2\epsilon_{\text{UV}}} P_{ij}^{(1)} \right]. \end{aligned} \quad (3.15)$$

Here $P_{ij}^{(0)}$ and $P_{ij}^{(1)}$ stand for the first and second order AP splitting functions [17] respectively which have been calculated in the literature (see Refs. [17–21]). Notice that in our notation A_{ij} and Z_{ij} depend on the partonic Bjorken scaling variable z in (2.8) which for convenience has been suppressed in our formulae. Hence, instead of multiplications we have to deal with convolutions denoted by the symbol $\otimes$ and defined in (2.20). Furthermore, the AP splitting functions are related to the anomalous dimensions $\gamma_{ij}^{(m)}$ of the composite operators in (2.23)–(2.25) via the Mellin transform

$$\gamma_{ij}^{(m),(k)} = - \int_0^1 dz z^{m-1} P_{ij}^{(k)}(z). \quad (3.16)$$

Finally we have to remove the C-divergences appearing in $\tilde{A}_{ij}$ (3.12). This is achieved by performing mass factorization, which proceeds as

$$\tilde{A}_{ij}(\epsilon_C, \frac{m^2}{\mu^2}, \alpha_s) = A_{ik}(\frac{m^2}{\mu^2}, \alpha_s) \otimes \Gamma_{kj}(\epsilon_C, \alpha_s), \quad (3.17)$$

where A_{ij} denote the finite OME's which do not have UV or C-divergences anymore. The quantities Γ_{ij} stand for the transition functions which have the same properties as Z_{ij} mentioned above (3.13). Like A_{ij} and Z_{ij} the Γ_{ij} can be expanded as a power series in α_s ,

$$\Gamma_{ij} = \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^k \Gamma_{ij}^{(k)}, \quad (3.18)$$

so that the mass factorized OME's A_{ij} read up to $O(\alpha_s^2)$ (using $\Gamma_{ij}^{(0)} = \delta_{ij}$)

$$\begin{aligned} A_{ij} = & \delta_{ij} + \left(\frac{\alpha_s}{4\pi} \right) [\hat{A}_{ij}^{(1)} + Z_{ij}^{-1,(1)} + \Gamma_{ij}^{-1,(1)}] \\ & + \left(\frac{\alpha_s}{4\pi} \right)^2 \left[\hat{A}_{ij}^{(2)} + \delta m \frac{d}{dm} \hat{A}_{ij}^{(1)} + \delta \alpha_s \hat{A}_{ij}^{(1)} + Z_{ik}^{-1,(1)} \hat{A}_{kj}^{(1)} + Z_{ij}^{-1,(2)} \right. \\ & \left. + \left\{ \hat{A}_{ik}^{(1)} + Z_{ik}^{-1,(1)} \right\} \otimes \Gamma_{kj}^{-1,(1)} + \Gamma_{ij}^{-1,(2)} \right], \end{aligned} \quad (3.19)$$

where $\Gamma_{ij}^{-1,(k)}$ are the expansion coefficients of the inverse matrix Γ^{-1} . Choosing the $\overline{\text{MS}}$ scheme one obtains the following expression (see Ref. [12]) up to $O(\alpha_s^2)$:

$$\Gamma_{ij}(\epsilon_C, \alpha_s) = \delta_{ij} + \left(\frac{\alpha_s}{4\pi}\right) S_\epsilon \left[\frac{1}{\epsilon_C} P_{ij}^{(0)} \right] + \left(\frac{\alpha_s}{4\pi}\right)^2 S_\epsilon^2 \left[\frac{1}{\epsilon_C^2} \left\{ \frac{1}{2} P_{ik}^{(0)} \otimes P_{kj}^{(0)} + \beta_0 P_{ij}^{(0)} \right\} + \frac{1}{2\epsilon_C} P_{ij}^{(1)} \right]. \quad (3.20)$$

If all quarks were massless we would have the identity $\Gamma_{ij} = Z_{ij}^{-1}$. However, since the heavy quark Q is massive it does not contribute to those splitting functions $P_{ij}^{(k)}$ which appear in the transition functions. Hence in this case i and j only represent the light quarks and the gluon. This is in contrast to the operator renormalization constant Z_{ij} where i and j can also stand for the heavy quark. The same assertion holds for the heavy flavour contributions to the beta-function given by $\beta_{0,H}$ ($m_H^2 \geq m^2$) which shows up in (3.15) but not in (3.20). The reason is that the mass m_H of the heavy quark H acts as a regulator for the C-divergences but not for the UV singularities.

Using the master formula in (3.19) one can now construct the general form of the finite OME A_{ij} as well as the unrenormalized OME $\hat{A}_{ij}$ expressed in their renormalization group coefficients β_0 , $\beta_{0,H}$ and $P_{ij}^{(k)}$ ($k = 0, 1$; $i, j = q, g$). For convenience we will take for $\hat{A}_{ij}$ the representation (3.3), where the mass renormalization has been already carried out.

Suppressing the index S on A_{Qg}^S we start with the one-loop OME $\hat{A}_{Qg}^{(1)}$, which receives contributions from the graphs in Fig. 1a,b. It becomes equal to

$$\hat{A}_{Qg}^{(1)} = S_\epsilon \left(\frac{m^2}{\mu^2}\right)^{\epsilon/2} \left\{ -\frac{1}{\epsilon_{UV}} P_{qg}^{(0)} + a_{Qg}^{(1)} + \epsilon_{UV} \bar{a}_{Qg}^{(1)} \right\}. \quad (3.21)$$

The renormalization group coefficients are given by

$$\begin{aligned} P_{qg}^{(0)} &= 8T_f [z^2 + (1-z)^2], \\ a_{Qg}^{(1)} &= 0, \\ \bar{a}_{Qg}^{(1)} &= -\frac{1}{8} \zeta(2) P_{qg}^{(0)}. \end{aligned} \quad (3.22)$$

The expression for the two-loop contribution to $\hat{A}_{Qg}$ (see Fig. 2) can be written as

$$\begin{aligned} \hat{A}_{Qg}^{(2)} &= S_\epsilon^2 \left(\frac{m^2}{\mu^2}\right)^\epsilon \left[\frac{1}{\epsilon^2} \left\{ \frac{1}{2} P_{qg}^{(0)} \otimes (P_{qq}^{(0)} - P_{gg}^{(0)}) + \beta_0 P_{qg}^{(0)} \right\} \right. \\ &\quad \left. + \frac{1}{\epsilon} \left\{ -\frac{1}{2} P_{qg}^{(1)} - 2\beta_0 a_{Qg}^{(1)} - a_{Qg}^{(1)} \otimes (P_{qq}^{(0)} - P_{gg}^{(0)}) \right\} + a_{Qg}^{(2)} \right] \\ &\quad - \frac{2}{\epsilon} S_\epsilon \sum_{H=Q}^t \beta_{0,H} \left(\frac{m_H^2}{\mu^2}\right)^{\epsilon/2} \left(1 + \frac{1}{8} \epsilon^2 \zeta(2) \right) \hat{A}_{Qg}^{(1)}. \end{aligned} \quad (3.23)$$

The last term in the above equation originates from Figs. 2u,v and the meaning of the sum is the same as discussed below (3.9).

Notice that in the above expression the pole terms ϵ^{-k} stand for the UV divergencies as well as the C-divergences so that we have put $\epsilon_{UV} = \epsilon_C$. Further we infer from the literature that

$$\begin{aligned}
 P_{qq}^{(0)} &= 4C_F \left[2 \left(\frac{1}{1-z} \right)_+ - 1 - z + \frac{3}{2} \delta(1-z) \right], \\
 P_{gg}^{(0)} &= 8C_A \left[\left(\frac{1}{1-z} \right)_+ + \frac{1}{z} - 2 + z - z^2 \right] + 2\beta_0 \delta(1-z), \\
 P_{qg}^{(1)} &= 8C_F T_f [2(1-2z+2z^2) \{ \ln^2(1-z) - 2 \ln z \ln(1-z) - 2\zeta(2) \} \\
 &\quad + (1-2z+4z^2) \ln^2 z + 8z(1-z) \ln(1-z) + (3-4z+8z^2) \ln z \\
 &\quad + 14 - 29z + 20z^2] \\
 &\quad + 8C_A T_f \left[(1+2z+2z^2) \{ \ln^2 z - 4 \ln z \ln(1+z) - 4\text{Li}_2(-z) - 2\zeta(2) \} \right. \\
 &\quad + 2(1-2z+2z^2) [\zeta(2) - \ln^2(1-z)] - (3+6z+2z^2) \ln^2 z \\
 &\quad - 8z(1-z) \ln(1-z) + \left(2 + 16z + \frac{88}{3} z^2 \right) \ln z \\
 &\quad \left. + \frac{2}{9} \left(\frac{20}{z} - 18 + 225z - 218z^2 \right) \right], \tag{3.24}
 \end{aligned}$$

and $a_{Qg}^{(2)}$ (3.23) has to be computed in this paper. In the above and subsequent expressions the functions $\text{Li}_n(z)$ denote the polylogarithms which can be found in [22]. The finite OME's follow from (3.19) where we have

$$A_{Qg}^{(1)} = \hat{A}_{Qg}^{(1)} + Z_{qg}^{-1,(1)} = -\frac{1}{2} P_{qg}^{(0)} \ln \frac{m^2}{\mu^2} + a_{Qg}^{(1)}, \tag{3.25}$$

and

$$\begin{aligned}
 A_{Qg}^{(2)} &= \hat{A}_{Qg}^{(2)} + \delta\alpha_s \hat{A}_{Qg}^{(1)} + Z_{qq}^{-1,(1)} \otimes \hat{A}_{Qg}^{(1)} + Z_{qg}^{-1,(1)} \otimes \hat{A}_{gg}^{(1)} \\
 &\quad + Z_{qg}^{-1,(2)} + (\hat{A}_{Qg}^{(1)} + Z_{qg}^{-1,(1)}) \otimes \Gamma_{gg}^{-1,(1)} \\
 &= \left\{ \frac{1}{8} P_{qg}^{(0)} \otimes P_{qq}^{(0)} - \frac{1}{8} P_{qg}^{(0)} \otimes P_{gg}^{(0)} + \frac{1}{4} \beta_0 P_{qg}^{(0)} \right\} \ln^2 \frac{m^2}{\mu^2} \\
 &\quad + \left\{ -\frac{1}{2} P_{qg}^{(1)} - \beta_0 a_{Qg}^{(1)} - \frac{1}{2} P_{qq}^{(0)} \otimes a_{Qg}^{(1)} + \frac{1}{2} P_{gg}^{(0)} \otimes a_{Qg}^{(1)} \right\} \ln \frac{m^2}{\mu^2} \\
 &\quad + a_{Qg}^{(2)} + 2\beta_0 \bar{a}_{Qg}^{(1)} + P_{qq}^{(0)} \otimes \bar{a}_{Qg}^{(1)} - P_{gg}^{(0)} \otimes \bar{a}_{Qg}^{(1)}. \tag{3.26}
 \end{aligned}$$

In (3.25), (3.26) we have included the heavy flavour contributions to the one-loop OME $A_{gg}^{(1)}$ in Fig. 1c which can be written as

$$\hat{A}_{gg}^{(1)} = -\frac{2}{\epsilon_{UV}} S_\epsilon \sum_{H=Q}^t \beta_{0,H} \left(\frac{m_H^2}{\mu^2} \right)^{\epsilon/2} \left(1 + \frac{1}{8} \epsilon_{UV}^2 \zeta(2) \right), \tag{3.27}$$

where the meaning of the sum is the same as discussed for the coupling constant renormalization in (3.7) (see the discussion below (3.9)). Notice that the choice of scheme for α_s in (3.7) implies that the last term in (3.23) is completely removed from the renormalized $A_{Qg}^{(2)}$ in (3.26). The constants in (3.25) and (3.26) are given by

$$\begin{aligned} Z_{qg}^{-1,(1)} &= S_\epsilon \left[\frac{1}{\epsilon_{UV}} P_{qg}^{(0)} \right], \\ Z_{qg}^{-1,(2)} &= S_\epsilon^2 \left[\frac{1}{\epsilon_{UV}^2} \left\{ \frac{1}{2} P_{qg}^{(0)} \otimes (P_{qq}^{(0)} + P_{gg}^{(0)}) - \beta_0 P_{qg}^{(0)} \right\} \right. \\ &\quad \left. + \frac{1}{\epsilon_{UV}} \frac{1}{S_\epsilon} \delta\alpha_s P_{qg}^{(0)} + \frac{1}{2\epsilon_{UV}} P_{qg}^{(1)} \right], \\ Z_{qq}^{-1,(1)} &= S_\epsilon \left[\frac{1}{\epsilon_{UV}} P_{qq}^{(0)} \right], \\ \Gamma_{gg}^{-1,(1)} &= S_\epsilon \left[-\frac{1}{\epsilon_C} P_{gg}^{(0)} \right]. \end{aligned} \quad (3.28)$$

The renormalization and the mass factorization of the OME's $\hat{A}_{Qq}^{\text{PS},(2)}$ in Fig. 3 proceeds in the same way as done for $\hat{A}_{Qg}$ (3.26). The unrenormalized OME reads

$$\begin{aligned} \hat{A}_{Qq}^{\text{PS},(2)} &= S_\epsilon^2 \left(\frac{m^2}{\mu^2} \right)^\epsilon \left[\frac{1}{\epsilon^2} \left\{ -\frac{1}{2} P_{qg}^{(0)} \otimes P_{gq}^{(0)} \right\} \right. \\ &\quad \left. + \frac{1}{\epsilon} \left\{ -\frac{1}{2} P_{qq}^{\text{PS},(1)} + a_{Qg}^{(1)} \otimes P_{gq}^{(0)} \right\} + a_{Qq}^{\text{PS},(2)} \right]. \end{aligned} \quad (3.29)$$

Like in the case of $\hat{A}_{Qg}^{(2)}$ we did not make any distinction between UV-singular and C-singular pole terms ϵ^{-k} ($\epsilon_{UV} = \epsilon_C$). The renormalization group coefficients are given by (see also (3.22))

$$\begin{aligned} P_{gq}^{(0)} &= 4C_F \left[\frac{2}{z} - 2 + z \right], \\ P_{qq}^{\text{PS},(1)} &= 8C_F T_f \left[-2(1+z) \ln^2 z + \left(2 + 10z + \frac{16}{3} z^2 \right) \ln z \right. \\ &\quad \left. + \frac{40}{9z} - 4 + 12z - \frac{112}{9} z^2 \right], \end{aligned} \quad (3.30)$$

and $a_{Qq}^{\text{PS},(2)}$ will be calculated in this paper. The finite OME $A_{Qq}^{\text{PS},(2)}$ can be again derived from (3.19) which yields

$$\begin{aligned} A_{Qq}^{\text{PS},(2)} &= \hat{A}_{Qq}^{\text{PS},(2)} + \left(Z_{qq}^{\text{PS}} \right)^{-1,(2)} + \left(\hat{A}_{Qg}^{(1)} + Z_{qg}^{-1,(1)} \right) \otimes \Gamma_{gq}^{-1,(1)} \\ &= \left\{ -\frac{1}{8} P_{qg}^{(0)} \otimes P_{gq}^{(0)} \right\} \ln^2 \frac{m^2}{\mu^2} + \left\{ -\frac{1}{2} P_{qq}^{\text{PS},(1)} + \frac{1}{2} a_{Qg}^{(1)} \otimes P_{gq}^{(0)} \right\} \ln \frac{m^2}{\mu^2} \\ &\quad + a_{Qq}^{\text{PS},(2)} - \bar{a}_{Qg}^{(1)} \otimes P_{gq}^{(0)}. \end{aligned} \quad (3.31)$$

Here we have used the renormalization group constants (see also (3.22), (3.30))

$$\begin{aligned} \left(Z_{qq}^{\text{PS}}\right)^{-1,(2)} &= S_\epsilon^2 \left[\frac{1}{\epsilon_{\text{UV}}^2} \left\{ \frac{1}{2} P_{qg}^{(0)} \otimes P_{gq}^{(0)} \right\} + \frac{1}{2\epsilon_{\text{UV}}} P_{qq}^{\text{PS},(1)} \right], \\ \Gamma_{gq}^{-1,(1)} &= S_\epsilon \left[-\frac{1}{\epsilon_{\text{C}}} P_{gq}^{(0)} \right]. \end{aligned} \quad (3.32)$$

The last OME which we have to deal with is represented by the heavy quark loop contribution to $A_{qq}^{\text{NS},(2)}$ in Fig. 4. The unrenormalized expression reads

$$\hat{A}_{qq,Q}^{\text{NS},(2)} = S_\epsilon^2 \left(\frac{m^2}{\mu^2} \right)^\epsilon \left[\frac{1}{\epsilon_{\text{UV}}^2} \{ -\beta_{0,Q} P_{qq}^{(0)} \} + \frac{1}{\epsilon_{\text{UV}}} \{ -\frac{1}{2} P_{qq,Q}^{\text{NS},(1)} \} + a_{qq,Q}^{\text{NS},(2)} \right]. \quad (3.33)$$

Contrary to $\delta\alpha_s$ (3.7) and $\hat{A}_{Qg}^{(2)}$ (3.23) the gluon self-energy appearing in the graphs of Fig. 4 only gets a contribution from the heavy quark Q which is the one produced in the final state of reaction (2.2). Furthermore, the above OME only contains UV divergences since the mass of the heavy quark Q given by m prevents $\hat{A}_{qq,Q}^{\text{NS},(2)}$ from being C-singular. However, this is only true if the heavy quark contribution to the gluon self-energy, denoted by $\Pi(p^2, m^2)$, is renormalized in such a way that $\Pi_R(0, m^2) = 0$.

The renormalization group coefficients in (3.33) are given by (see also (3.10) and below)

$$\begin{aligned} \beta_{0,Q} &= -\frac{4}{3} T_f, \\ P_{qq,Q}^{\text{NS},(1)} &= C_F T_f \left[-\frac{160}{9} \left(\frac{1}{1-z} \right)_+ + \frac{176}{9} z - \frac{16}{9} - \frac{16}{3} \frac{1+z^2}{1-z} \ln z \right. \\ &\quad \left. + \delta(1-z) \left(-\frac{4}{3} - \frac{32}{3} \zeta(2) \right) \right]. \end{aligned} \quad (3.34)$$

Since $\hat{A}_{qq,Q}^{\text{NS},(2)}$ does not contain C-divergences we only need operator renormalization to render it finite. From (3.19) we infer

$$\begin{aligned} A_{qq,Q}^{\text{NS},(2)} &= \hat{A}_{qq,Q}^{\text{NS},(2)} + \left(Z_{qq,Q}^{\text{NS}} \right)^{-1,(2)} \\ &= \left\{ -\frac{1}{4} \beta_{0,Q} P_{qq}^{(0)} \right\} \ln^2 \frac{m^2}{\mu^2} + \left\{ -\frac{1}{2} P_{qq,Q}^{\text{NS},(1)} \right\} \ln \frac{m^2}{\mu^2} \\ &\quad + a_{qq,Q}^{\text{NS},(2)} + \frac{1}{4} \beta_{0,Q} \zeta(2) P_{qq}^{(0)}, \end{aligned} \quad (3.35)$$

where we have used

$$\begin{aligned} \left(Z_{qq,Q}^{\text{NS}} \right)^{-1,(2)} &= S_\epsilon^2 \left[-\frac{1}{\epsilon_{\text{UV}}^2} \beta_{0,Q} P_{qq}^{(0)} + \frac{2}{\epsilon_{\text{UV}}^2} \beta_{0,Q} \left(\frac{m^2}{\mu^2} \right)^{\epsilon/2} \left(1 + \frac{1}{8} \epsilon_{\text{UV}}^2 \zeta(2) \right) P_{qq}^{(0)} \right. \\ &\quad \left. + \frac{1}{2\epsilon_{\text{UV}}} P_{qq,Q}^{\text{NS},(1)} \right]. \end{aligned} \quad (3.36)$$

Notice that expressions (3.33) and (3.35) also contribute to the singlet part of the OME represented by $A_{qq,Q}^{\text{S},(2)}$.

Since we can infer the coefficients of the double and single pole terms of the unrenormalized OME $\hat{A}_{ij}$ from the two-loop corrected AP splitting functions and the beta-function published in the literature [17–21], the above expressions serve as a check on our calculations. The non-pole terms defined by $a_{Qg}^{(2)}$ (3.23), $a_{Qq}^{\text{PS},(2)}$ (3.31) and $a_{Qq}^{\text{NS},(2)}$ (3.35), which cannot be predicted, have to be calculated in this paper. They are needed to compute the heavy quark coefficient functions (2.17) up to the non-logarithmic terms, which will be done in the next section.

Before finishing this section we will give an outline of our calculation of the OME's depicted in Figs. 1–4. We have computed the OME's using the standard QCD Feynman rules and the operator vertices corresponding to (2.23), (2.24) which can be found in the literature (see Ref. [18]). Since in our case the latter are S -matrix elements we have to consider the connected Green functions where the external quark and gluon propagators are amputated. Notice that one has to include the external self-energies. The connected Green function needed for A_{Qg} (Figs. 1 and 2) is given by

$$\langle 0 | T(A_\mu^a(x) O_Q^{\mu_1 \dots \mu_m}(0) A_\nu^b(y)) | 0 \rangle_c, \quad (3.37)$$

where O_Q is the heavy quark composite operator defined in (2.24). It also appears in the connected Green function for A_{Qq}^{PS} (Fig. 3), which reads

$$\langle 0 | T(\bar{\psi}_i(x) O_Q^{\mu_1 \dots \mu_n}(0) \psi_j(y)) | 0 \rangle_c. \quad (3.38)$$

Finally we need the connected Green function for the non-singlet operator $A_{qq,Q}^{\text{NS}}$ (Fig. 4). Here the light quark operator $O_{q,r}$ (2.23) has to be inserted so that we get

$$\langle 0 | T(\bar{\psi}_i(x) O_{q,r}^{\mu_1 \dots \mu_n}(0) \psi_j(y)) | 0 \rangle_c. \quad (3.39)$$

Notice that the field ψ occurring in (3.38), (3.39) stands for the external light quark only. Further, a and i, j are the colour indices of the gluon field A_μ and the quark fields $\bar{\psi}$, ψ , respectively. Since the above operators $O^{\mu_1 \dots \mu_m}(0)$ are traceless symmetric tensors under the Lorentz group, the computation of the connected Green functions reveals the presence of many trace terms which are not essential for the determination of the anomalous dimensions or splitting functions. Therefore we will project them out by multiplying the operators by an external source $J_{\mu_1 \dots \mu_m} = \Delta_{\mu_1} \dots \Delta_{\mu_m}$ with $\Delta^2 = 0$. Performing the Fourier transform into momentum space and sandwiching the connected Green functions by the external gluon polarizations and quark spinors Eqs. (3.37)–(3.39) become

$$\begin{aligned} & \epsilon^\mu(p) G_{Q,\mu\nu}^{ab} \epsilon^\nu(p), \\ & \bar{u}(p, s) G_Q^{ij} u(p, s), \\ & \bar{u}(p, s) G_q^{ij} \lambda_r u(p, s), \end{aligned} \quad (3.40)$$

respectively, where λ_r denote the generators of the flavour group $\text{SU}(n_f)$. The tensors $G_{Q,\mu\nu}^{ab}$ and G_f^{ij} with $f = q, Q$ have the form

$$G_{Q,\mu\nu}^{ab} = \hat{A}_{Qg}^{(m)} \left(\epsilon, \frac{m^2}{\mu^2}, \alpha_s \right) \delta^{ab} (\Delta \cdot p)^m \left(-g_{\mu\nu} + \frac{\Delta_\mu p_\nu + \Delta_\nu p_\mu}{\Delta \cdot p} \right), \quad (3.41)$$

and

$$G_f^{ij} = \hat{A}_{fq}^{(m)} \left(\epsilon, \frac{m^2}{\mu^2}, \alpha_s \right) \delta^{ij} (\Delta \cdot p)^{m-1} \not{A}, \quad (3.42)$$

with $\hat{A}_{Qq} = \hat{A}_{Qq}^{\text{PS}}$ and $\hat{A}_{qq} = \hat{A}_{qq,Q}^{\text{NS}}$. By projecting these tensors out one obtains finally the OME's

$$\hat{A}_{Qg}^{(m)} \left(\epsilon, \frac{m^2}{\mu^2}, \alpha_s \right) = \frac{1}{N^2 - 1} \frac{1}{n - 2} (-g_{\mu\nu}) \delta^{ab} (\Delta \cdot p)^{-m} G_{Q,\mu\nu}^{ab}, \quad (3.43)$$

$$\hat{A}_{fq}^{(m)} \left(\epsilon, \frac{m^2}{\mu^2}, \alpha_s \right) = \frac{1}{N} \delta^{ij} \frac{1}{4} (\Delta \cdot p)^{-m} \text{Tr}(\not{p} G_f^{ij}). \quad (3.44)$$

Notice that in (3.43) the summation over the dummy indices μ and ν includes unphysical (non-transverse) gluon polarizations which have to be compensated by adding graphs containing the external ghost lines represented in Figs. 2s and 2t. Instead of $-g^{\mu\nu}/(n-2)$ one can also use the physical polarization sum $[-g_{\mu\nu} + (\Delta_\mu p_\nu + \Delta_\nu p_\mu)/\Delta \cdot p]/(n-2)$ and omit the graphs with the external ghost lines. However, in this case the individual Feynman graphs lead to integrals with higher powers of the momenta in the numerator so that they become more difficult to compute. Moreover, the number of independent integrals is artificially increased. The advantage of constructing the Green function in the way shown in (3.43), (3.44) is that one does not have to resort to complicated tensorial reduction programs as had to be used for example in [16]. Therefore one has only Lorentz scalars in the numerators of the Feynman integrals which can be partially cancelled by terms in the denominators. The traces of the quark loops in Figs. 1 and 2 and the contraction with the metric tensor in (3.43) have been performed by using the algebraic manipulation program FORM [23]. We did the same for the graphs in Figs. 3 and 4 where we had to compute the trace in (3.44). The computation of the scalar integrals is straightforward as long as the number of propagators does not exceed four. In the case of five different propagators the calculation becomes more cumbersome but here one can use the trick of integration by parts [24]. Examples are given in Appendix B. The results for the unrenormalized OME's are too long to put them in this section and we will defer them to Appendix C.

Apart from the check on the pole terms mentioned above we can also check the finite term of the Abelian part of $\hat{A}_{Qg}$. Removing the overall constant $(1 + (-1)^m)/2$ the Mellin transform reads up to order α_s^2 (see Appendix C)

$$\hat{A}_{Qg}^{(m)} = S_\epsilon \left(\frac{\alpha_s}{4\pi} \right) T_f \left[\hat{A}_{Qg}^{(1),(m)} \right] + S_\epsilon^2 \left(\frac{\alpha_s}{4\pi} \right)^2 T_f \left[C_F \hat{A}_{Qg,F}^{(2),(m)} + C_A \hat{A}_{Qg,A}^{(2),(m)} \right]. \quad (3.45)$$

Notice that coupling constant renormalization has already been carried out in the above equation so that we get rid of the term proportional to T_f^2 in (C.1). If we now take the first moment $m = 1$, $\hat{A}_{Qg,A}^{(2),(m)} \approx (m-1)^{-1}$ but $\hat{A}_{Qg}^{(1),(1)}$ and $\hat{A}_{Qg,F}^{(2),(1)}$ are finite. One can

now easily show that there exists a relation between the Abelian terms and the heavy quark loop contribution to the gluon self-energy which we will denote by

$$\hat{\Pi}_V(p^2, m^2) = S_\epsilon \left(\frac{\alpha_s}{4\pi} \right) T_f \hat{\Pi}_V^{(1)}(p^2, m^2) + S_\epsilon^2 \left(\frac{\alpha_s}{4\pi} \right)^2 C_F T_f \hat{\Pi}_V^{(2)}(p^2, m^2). \quad (3.46)$$

The relation is given by

$$\hat{\Pi}_V^{(1)}(0, m^2) = \frac{1}{2} \hat{A}_{Q_8}^{(1), (1)}, \quad (3.47)$$

$$\hat{\Pi}_V^{(2)}(0, m^2) = \frac{1}{2} \hat{A}_{Q_8, F}^{(2), (1)}. \quad (3.48)$$

$\hat{\Pi}_V^{(1)}(0, m^2)$ and $\hat{\Pi}_V^{(2)}(0, m^2)$ can be inferred from Eqs. (2.11) and (5.1) in [25]. In the latter reference the heavy fermion loop contribution to the self-energy of the Z-boson and the photon were calculated. The self-energy contribution to the gluon and the photon are related as follows:

$$\hat{\Pi}_V^{(1)}(p^2, m^2) = 4 \frac{1}{p^2} \Pi_T^V(p^2), \quad (3.49)$$

$$\hat{\Pi}_V^{(2)}(p^2, m^2) = 4^2 \frac{3}{4} \frac{1}{3} \frac{1}{p^2} \Pi_T^V(p^2), \quad (3.50)$$

where $\Pi_T^V(p^2)$ in Eqs. (3.49), (3.50) are equal to the expressions quoted in (2.11) and (5.1) of [25], respectively. The factors of 4 originate from the convention that in [25] the quantities are expanded in α_s/π instead of $\alpha_s/(4\pi)$ as is done in this paper. The factors of 4/3 and 3 refer to the colour factors C_F and C_A in SU(3). From Ref. [25] we infer that

$$\hat{\Pi}_V^{(1)}(0, m^2) = \left(\frac{m^2}{\mu^2} \right)^{\epsilon/2} \left[-\frac{8}{3\epsilon} \right], \quad (3.51)$$

$$\hat{\Pi}_V^{(2)}(0, m^2) = \left(\frac{m^2}{\mu^2} \right)^\epsilon \left[-\frac{4}{\epsilon} + 15 \right], \quad (3.52)$$

where $n = 4 + \epsilon$. (Notice that in [25] $n = 4 - 2\epsilon$). Calculation of the first moment of (3.45) (see also (C.1), (C.2)) reveals that the relations in (3.47), (3.48) are satisfied.

4. Heavy quark coefficient functions

In this section we will compute the heavy quark coefficient functions $H_{i,l}$ and $L_{i,l}$ defined in (2.13) up to order α_s^2 in the asymptotic limit $Q^2 \gg m^2$. For this computation we will use the mass factorization theorems as represented by Eqs. (2.18) and (2.19), where the transition functions $\Gamma_{kl}(m^2/\mu^2)$ stand for the finite OME's $A_{kl}(m^2/\mu^2)$ computed in Section 3.

Let us start with the heavy quark coefficient functions which originate from the subprocesses where the virtual photon couples to the heavy quark. Here the mass factorization theorem implies

$$H_{i,l}^S\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{kl}^S\left(\frac{m^2}{\mu^2}\right) \otimes C_{i,k}^S\left(\frac{Q^2}{\mu^2}\right), \quad (4.1)$$

where $H_{i,l}$ ($i = 2, L; l = q, g$) denote the heavy quark coefficient functions in the limit $Q^2 \gg m^2$ and A_{kl} ($k, l = q, g$) are the finite OME's computed in the last section. The coefficient functions $C_{i,l}(Q^2/\mu^2)$ have been calculated in [12]. They are obtained from the massless parton structure functions as defined in Eqs. (2.26), (2.27), after having performed mass factorization in the $\overline{\text{MS}}$ scheme. In the last section also the OME's A_{kl} have been calculated in the $\overline{\text{MS}}$ scheme. In the product on the right-hand side of (4.1) the scheme dependence is only partially cancelled which is revealed by the fact that $H_{i,l}$ is still scheme dependent. This dependence, which is indicated by m^2/μ^2 in $H_{i,l}$, originates from the coupling of a light parton (gluon or quark) to an internal light parton characteristic of the production mechanisms of the processes in (2.14) and (2.15).

Dropping the superscript S on $H_{i,g}$ we have for the lowest order photon–gluon fusion process (see (2.12) and (2.13)) the following result:

$$H_{L,g}^{(1)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = C_{L,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right), \quad (4.2)$$

$$H_{2,g}^{(1)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right) + C_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right), \quad (4.3)$$

where $C_{L,g}^{(1)}, C_{2,g}^{(1)}$ are the order α_s gluonic coefficient functions which, e.g., can be found in [12]. Like the OME's they can be expressed in the renormalization group coefficients as

$$C_{L,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) = c_{L,g}^{(1)}, \quad (4.4)$$

$$C_{2,g}^{(1)}\left(\frac{Q^2}{\mu^2}\right) = \frac{1}{2} P_{qg}^{(0)} \ln \frac{Q^2}{\mu^2} + c_{2,g}^{(1)}. \quad (4.5)$$

From (3.25) and (4.2)–(4.5) one obtains the asymptotic forms for the order α_s heavy quark coefficient functions

$$H_{L,g}^{(1)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = c_{L,g}^{(1)}, \quad (4.6)$$

$$H_{2,g}^{(1)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \frac{1}{2} P_{qg}^{(0)} \ln \frac{Q^2}{m^2} + a_{Qg}^{(1)} + c_{2,g}^{(1)}. \quad (4.7)$$

In order α_s^2 the coefficients of the photon–gluon fusion process (2.14) become

$$H_{L,g}^{(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right) \otimes C_{L,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + C_{L,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right), \quad (4.8)$$

$$H_{2,g}^{(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{Qg}^{(2)}\left(\frac{m^2}{\mu^2}\right) + A_{Qg}^{(1)}\left(\frac{m^2}{\mu^2}\right) \otimes C_{2,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) + C_{2,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right), \quad (4.9)$$

where $C_{i,q}^{(1)}, C_{i,g}^{(2)}$, ($i = 2, L$) become (see Ref. [12])

$$C_{L,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) = c_{L,q}^{(1)}, \quad (4.10)$$

$$C_{2,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) = \frac{1}{2}P_{qq}^{(0)} \ln \frac{Q^2}{\mu^2} + c_{2,q}^{(1)}, \quad (4.11)$$

$$C_{L,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right) = \left\{ -\beta_0 c_{L,g}^{(1)} + \frac{1}{2}P_{gg}^{(0)} \otimes c_{L,g}^{(1)} + \frac{1}{2}P_{qg}^{(0)} \otimes c_{L,q}^{(1)} \right\} \ln \frac{Q^2}{\mu^2} + c_{L,g}^{(2)}, \quad (4.12)$$

$$C_{2,g}^{(2)}\left(\frac{Q^2}{\mu^2}\right) = \left\{ \frac{1}{8}P_{qg}^{(0)} \otimes (P_{gg}^{(0)} + P_{qq}^{(0)}) - \frac{1}{4}\beta_0 P_{qg}^{(0)} \right\} \ln^2 \frac{Q^2}{\mu^2} \\ + \left\{ \frac{1}{2}P_{qg}^{(1)} - \beta_0 c_{2,g}^{(1)} + \frac{1}{2}P_{gg}^{(0)} \otimes c_{2,g}^{(1)} + \frac{1}{2}P_{qg}^{(0)} \otimes c_{2,q}^{(1)} \right\} \ln \frac{Q^2}{\mu^2} + c_{2,g}^{(2)}. \quad (4.13)$$

From (3.25), (3.26) and (4.10)–(4.13) one infers the asymptotic form

$$H_{L,g}^{(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \left\{ -\beta_0 c_{L,g}^{(1)} + \frac{1}{2}P_{gg}^{(0)} \otimes c_{L,g}^{(1)} + \frac{1}{2}P_{qg}^{(0)} \otimes c_{L,q}^{(1)} \right\} \ln \frac{Q^2}{m^2} \\ + \left\{ -\beta_0 c_{L,g}^{(1)} + \frac{1}{2}P_{gg}^{(0)} \otimes c_{L,g}^{(1)} \right\} \ln \frac{m^2}{\mu^2} + c_{L,g}^{(2)} + a_{Qg}^{(1)} \otimes c_{L,q}^{(1)}, \quad (4.14)$$

$$H_{2,g}^{(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \left\{ \frac{1}{8}P_{qg}^{(0)} \otimes (P_{gg}^{(0)} + P_{qq}^{(0)}) - \frac{1}{4}\beta_0 P_{qg}^{(0)} \right\} \ln^2 \frac{Q^2}{m^2} \\ + \left\{ \frac{1}{2}P_{qg}^{(1)} - \beta_0 c_{2,g}^{(1)} + \frac{1}{2}P_{qg}^{(0)} \otimes a_{Qg}^{(1)} + \frac{1}{2}P_{gg}^{(0)} \otimes c_{2,g}^{(1)} + \frac{1}{2}P_{qg}^{(0)} \otimes c_{2,q}^{(1)} \right\} \\ \times \ln \frac{Q^2}{m^2} + \left\{ \frac{1}{4}P_{qg}^{(0)} \otimes P_{gg}^{(0)} - \frac{1}{2}\beta_0 P_{qg}^{(0)} \right\} \ln \frac{Q^2}{m^2} \ln \frac{m^2}{\mu^2} \\ + \left\{ -\beta_0 (c_{2,g}^{(1)} + a_{Qg}^{(1)}) + \frac{1}{2}P_{gg}^{(0)} \otimes (c_{2,g}^{(1)} + a_{Qg}^{(1)}) \right\} \ln \frac{m^2}{\mu^2} \\ + c_{2,g}^{(2)} + a_{Qg}^{(2)} + 2\beta_0 \bar{a}_{Qg}^{(1)} + c_{2,q}^{(1)} \otimes a_{Qg}^{(1)} + P_{qq}^{(0)} \otimes \bar{a}_{Qg}^{(1)} - P_{gg}^{(0)} \otimes \bar{a}_{Qg}^{(1)}. \quad (4.15)$$

Notice that the above formulae are still dependent on the mass factorization and renormalization scale μ^2 . The same scale dependence was also found for the exact expressions for the heavy quark coefficient functions where Q^2 and m^2 can be arbitrarily chosen. The μ^2 dependence can be attributed to coupling constant renormalization represented by the lowest order coefficient β_0 in the beta function and the lowest order splitting function $P_{gg}^{(0)}$ standing for the transition $g \rightarrow g + g$.

The computation of the asymptotic expression of the heavy quark coefficient function corresponding to the Bethe–Heitler process (2.15) proceeds in the same way. From (4.1) we derive

$$H_{L,q}^{\text{PS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = C_{L,q}^{\text{PS},(2)}\left(\frac{Q^2}{\mu^2}\right), \quad (4.16)$$

$$H_{2,q}^{\text{PS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{Qq}^{\text{PS},(2)}\left(\frac{m^2}{\mu^2}\right) + C_{2,q}^{\text{PS},(2)}\left(\frac{Q^2}{\mu^2}\right). \quad (4.17)$$

The coefficient functions $C_{i,q}^{\text{PS},(2)}$ ($i = 2, L$) are computed in [12] and they read

$$C_{L,q}^{\text{PS},(2)}\left(\frac{Q^2}{\mu^2}\right) = \left\{ \frac{1}{2} P_{gq}^{(0)} \otimes c_{L,g}^{(1)} \right\} \ln \frac{Q^2}{\mu^2} + c_{L,q}^{\text{PS},(2)}, \quad (4.18)$$

$$C_{2,q}^{\text{PS},(2)}\left(\frac{Q^2}{\mu^2}\right) = \left\{ \frac{1}{8} P_{qg}^{(0)} \otimes P_{gq}^{(0)} \right\} \ln^2 \frac{Q^2}{\mu^2} + \left\{ \frac{1}{2} P_{qq}^{\text{PS},(1)} + \frac{1}{2} P_{gq}^{(0)} \otimes c_{2,g}^{(1)} \right\} \ln \frac{Q^2}{\mu^2} + c_{2,q}^{\text{PS},(2)}. \quad (4.19)$$

Using (3.31) and (4.18), (4.19) the heavy quark coefficient functions become

$$H_{L,q}^{\text{PS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \left\{ \frac{1}{2} P_{gq}^{(0)} \otimes c_{L,g}^{(1)} \right\} \ln \frac{Q^2}{m^2} + \left\{ \frac{1}{2} P_{gq}^{(0)} \otimes c_{L,g}^{(1)} \right\} \ln \frac{m^2}{\mu^2} + c_{L,q}^{\text{PS},(2)}, \quad (4.20)$$

$$\begin{aligned} H_{2,q}^{\text{PS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= \left\{ \frac{1}{8} P_{qg}^{(0)} \otimes P_{gq}^{(0)} \right\} \ln^2 \frac{Q^2}{m^2} + \left\{ \frac{1}{2} P_{qq}^{\text{PS},(1)} + \frac{1}{2} P_{gq}^{(0)} \otimes c_{2,g}^{(1)} \right\} \\ &\quad \times \ln \frac{Q^2}{m^2} + \left\{ \frac{1}{4} P_{qg}^{(0)} \otimes P_{gq}^{(0)} \right\} \ln \frac{Q^2}{m^2} \ln \frac{m^2}{\mu^2} \\ &\quad + \left\{ \frac{1}{2} P_{gq}^{(0)} \otimes (c_{2,g}^{(1)} + a_{Qg}^{(1)}) \right\} \ln \frac{m^2}{\mu^2} \\ &\quad + c_{2,q}^{\text{PS},(2)} + a_{Qq}^{\text{PS},(2)} - P_{gq}^{(0)} \otimes \bar{a}_{Qg}^{(1)}. \end{aligned} \quad (4.21)$$

Notice that the above μ^2 dependence can be only attributed to the mass factorization scheme which enters via the transition $q \rightarrow g + q$ represented by the splitting function $P_{gq}^{(0)}$. The mass factorization for the heavy quark coefficient functions where the photon couples to the light quark proceeds in a similar way as given in Eq. (4.1), i.e.

$$L_{i,l}^r\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{kl}^r\left(\frac{m^2}{\mu^2}\right) \otimes C_{i,k}^r\left(\frac{Q^2}{\mu^2}\right), \quad (4.22)$$

with $r = \text{NS}, \text{S}$. In the case of the Compton process (2.15) where up to order α_s^2 $L_{i,q}^{\text{NS},(2)} = L_{i,q}^{\text{S},(2)}$ one can derive

$$L_{L,q}^{\text{NS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = C_{L,q,Q}^{\text{NS},(2)}\left(\frac{Q^2}{\mu^2}\right), \quad (4.23)$$

$$L_{2,q}^{\text{NS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = A_{qq,Q}^{\text{NS},(2)}\left(\frac{m^2}{\mu^2}\right) + C_{2,q,Q}^{\text{NS},(2)}\left(\frac{Q^2}{\mu^2}\right). \quad (4.24)$$

Notice that in [12] all coefficient functions are calculated for massless quarks. In particular $C_{i,q,Q}^{\text{NS},(2)}$ originates from the Compton process (2.15) and now the heavy quark is taken to be massless. Therefore this coefficient function also gets virtual corrections among which is the heavy quark (Q) loop contribution to the gluon self-energy (for more details see Ref. [12]). The coefficient functions appearing above take the form

$$C_{L,q,Q}^{\text{NS},(2)}\left(\frac{Q^2}{\mu^2}\right) = \left\{ -\beta_{0,Q} c_{L,q}^{(1)} \right\} \ln \frac{Q^2}{\mu^2} + c_{L,q,Q}^{\text{NS},(2)}, \quad (4.25)$$

$$C_{2,q,Q}^{\text{NS},(2)}\left(\frac{Q^2}{\mu^2}\right) = \left\{ -\frac{1}{4}\beta_{0,Q} P_{qq}^{(0)} \right\} \ln^2 \frac{Q^2}{\mu^2} + \left\{ \frac{1}{2} P_{qq,Q}^{\text{NS},(1)} - \beta_{0,Q} c_{2,q}^{(1)} \right\} \ln \frac{Q^2}{\mu^2} + c_{2,q,Q}^{\text{NS},(2)}. \quad (4.26)$$

In the above equations the heavy quark loop contribution to the gluon self-energy was renormalized in the $\overline{\text{MS}}$ scheme since Q was taken to be massless. However, in $A_{qq,Q}^{\text{NS},(2)}$ this renormalization was carried out in a different scheme (see below (3.35)). Choosing the latter scheme also for the coefficient functions we obtain, instead of (4.25), (4.26),

$$C_{i,q,Q}^{\text{NS},(2)}\left(\frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{i,q,Q}^{\text{NS},(2)}\left(\frac{Q^2}{\mu^2}\right) + \left\{ \beta_{0,Q} C_{i,q}^{(1)}\left(\frac{Q^2}{\mu^2}\right) \right\} \ln \frac{m^2}{\mu^2}, \quad (4.27)$$

with $C_{i,q}^{(1)}$ defined in (4.10), (4.11). Substitution of (4.27) and (3.35) in Eqs. (4.23), (4.24) yields the following results:

$$L_{L,q}^{\text{NS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \left\{ -\beta_{0,Q} c_{L,q}^{(1)} \right\} \ln \frac{Q^2}{m^2} + c_{L,q,Q}^{\text{NS},(2)}, \quad (4.28)$$

$$L_{2,q}^{\text{NS},(2)}\left(\frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = \left\{ -\frac{1}{4}\beta_{0,Q} P_{qq}^{(0)} \right\} \ln^2 \frac{Q^2}{m^2} + \left\{ \frac{1}{2} P_{qq,Q}^{\text{NS},(1)} - \beta_{0,Q} c_{2,q}^{(1)} \right\} \ln \frac{Q^2}{m^2} \\ + c_{2,q,Q}^{\text{NS},(2)} + a_{qq,Q}^{\text{NS},(2)} + \frac{1}{4} \beta_{0,Q} \zeta(2) P_{qq}^{(0)}. \quad (4.29)$$

Notice that in the above expressions the μ^2 dependence has completely disappeared due to the special choice of the renormalization scheme for the heavy quark (Q) loop contribution to the gluon self-energy.

Since we have now all renormalization group coefficients at hand we can calculate the heavy quark coefficient functions $H_{k,i}$ and $L_{k,i}$ in the asymptotic limit $Q^2 \gg m^2$ for arbitrary z . The splitting functions $P_{ij}^{(0)}$, $P_{ij}^{(1)}$ and the non-pole terms in the OME's $a_{ij}^{(1)}$, $a_{ij}^{(2)}$ ($i, j = q, g$) were presented in Section 3. The coefficients $c_{L,i}^{(k)}$, $c_{2,i}^{(k)}$ ($k = 1, 2$; $i = q, g$) appearing in the light quark and gluon deep inelastic coefficient functions can be found in Appendix B of Ref. [12] (see also Appendix B in Ref. [26]). The analytic expressions for Eqs. (4.14), (4.15), (4.20), (4.21), (4.28), (4.29) are too long to be presented here and one can find them in Appendix D.

Before finishing this section we want to study the behaviour of the coefficient functions $H_{k,i}(z, Q^2/m^2, m^2/\mu^2)$ ($k = 2, L$; $i = q, g$) in the limit $z \rightarrow 0$, or $\eta \rightarrow \infty$. The behaviour of these functions for $z = 0$ at arbitrary Q^2 has been predicted in [27] based on the method of k_T factorization. Using the notations in [13] we have the following predictions from Ref. [27]:

$$\lim_{z \rightarrow 0} H_{L,i}^{(2)}\left(z, \xi, \frac{m^2}{\mu^2}\right) = \frac{1}{z} 16\pi C_f T_f \xi \left[G_L(\eta, \xi) - \bar{G}_L(\eta, \xi) \ln \frac{m^2}{\mu^2} \right], \quad (4.30)$$

$$\lim_{z \rightarrow 0} H_{2,i}^{(2)}\left(z, \xi, \frac{m^2}{\mu^2}\right) = \frac{1}{z} 16\pi C_f T_f \xi \left[G_T(\eta, \xi) + G_L(\eta, \xi) \right]$$

$$-\left\{\bar{G}_T(\eta, \xi) + \bar{G}_L(\eta, \xi)\right\} \ln \frac{m^2}{\mu^2}, \quad (4.31)$$

with $i = q, g$ and $C_q = C_F$, $C_g = C_A$ and η, ξ are defined in (2.16). Further we drop the superscript PS on $H_{k,q}^{\text{PS},(2)}$ (4.20), (4.21). The functions $G_k(\eta, \xi)$, $\bar{G}_k(\eta, \xi)$ ($k = L, T$) are given by Eqs. (19)–(24) in [13] for arbitrary ξ . In the limit $\xi \rightarrow \infty$ they behave like

$$\lim_{\xi \rightarrow \infty} G_L(\eta, \xi) = \frac{1}{6\pi} \frac{1}{\xi} \left[4 \ln \xi - \frac{4}{3} \right], \quad (4.32)$$

$$\lim_{\xi \rightarrow \infty} G_T(\eta, \xi) = \frac{1}{6\pi} \frac{1}{\xi} \left[2 \ln^2 \xi + \frac{14}{3} \ln \xi - 4\zeta(2) + \frac{14}{3} \right], \quad (4.33)$$

$$\lim_{\xi \rightarrow \infty} \bar{G}_L(\eta, \xi) = \frac{1}{6\pi} \frac{1}{\xi} [-4], \quad (4.34)$$

$$\lim_{\xi \rightarrow \infty} \bar{G}_T(\eta, \xi) = \frac{1}{6\pi} \frac{1}{\xi} [-4 \ln \xi + 2]. \quad (4.35)$$

Substitution of the above equations into (4.30) and (4.31) yields the asymptotic behaviour

$$\lim_{z \rightarrow 0} H_{L,i}^{(2)}(z, \xi, \frac{m^2}{\mu^2}) = \frac{1}{z} C_i T_f \left[\frac{32}{3} \ln \xi + \frac{32}{3} \ln \frac{m^2}{\mu^2} - \frac{32}{9} \right], \quad (4.36)$$

$$\begin{aligned} \lim_{z \rightarrow 0} H_{2,i}^{(2)}(z, \xi, \frac{m^2}{\mu^2}) = & \frac{1}{z} C_i T_f \left[\frac{16}{3} \ln^2 \xi + \left(\frac{32}{3} \ln \frac{m^2}{\mu^2} + \frac{208}{9} \right) \ln \xi \right. \\ & \left. + \frac{16}{3} \ln \frac{m^2}{\mu^2} - \frac{32}{3} \zeta(2) + \frac{80}{9} \right], \end{aligned} \quad (4.37)$$

which agrees with expressions (D.3)–(D.6) in the limit $z \rightarrow 0$.

5. Results

In this section we want to make a comparison between the exact heavy quark coefficient functions in [4] and the asymptotic ones derived in this paper. The exact coefficient functions which are either available in computer programs [4] or in tables [13] will be denoted by $H_{k,i}^{\text{exact}}$ and $L_{k,i}^{\text{exact}}$ ($k = 2, L, i = q, g$). The coefficient functions calculated in this paper, which are only valid for $Q^2 \gg m^2$, will be referred to as $H_{k,i}^{\text{asympt}}$ and $L_{k,i}^{\text{asympt}}$ and their analytic expressions can be found in Appendix D. These functions depend on the scaling variables z ($z = Q^2/(2p \cdot q)$), ξ ($\xi = Q^2/m^2$) and m^2/μ^2 , where μ is the mass factorization and renormalization scale (see the definitions in (2.16)). We are now interested to find out at which values of ξ (or Q^2) the asymptotic coefficient functions approach the exact ones⁴ so that the latter can be replaced by the former. For that purpose we define the ratios $R_{k,i}^{(l)}$ and $T_{k,i}^{(l)}$ which are given by

⁴ The order α_s^2 heavy quark coefficient functions in Refs. [4,13] have to be multiplied by $16\pi\xi/z$ in order to obtain the $H_{k,i}^{\text{exact},(2)}$ and $L_{k,i}^{\text{exact},(2)}$ in this section.

$$R_{k,i}^{(l)}(z, \xi, \frac{m^2}{\mu^2}) = \frac{H_{k,i}^{\text{exact},(l)}(z, \xi, m^2/\mu^2)}{H_{k,i}^{\text{asympt},(l)}(z, \xi, m^2/\mu^2)}, \quad (5.1)$$

$$T_{k,i}^{(l)}(z, \xi, \frac{m^2}{\mu^2}) = \frac{L_{k,i}^{\text{exact},(l)}(z, \xi, m^2/\mu^2)}{L_{k,i}^{\text{asympt},(l)}(z, \xi, m^2/\mu^2)}. \quad (5.2)$$

In the following we will only study both ratios for $\mu^2 = m^2$. It turns out that the minimal ξ value for which the asymptotic and exact coefficient functions coincide is not very sensitive to the choice of μ . Here we want to emphasize that the conclusions drawn on the ξ -behaviour are independent of the heavy flavour species. However, bearing in mind the experimental Q^2 -values available it is clear that at this moment our studies are only of interest for charm production at HERA.

Choosing the range $5 < \xi < 10^5$ we will present the above ratios $R_{k,i}^{(l)}$ (5.1) and $T_{k,i}^{(l)}$ (5.2) for $z = 10^{-2}$ and 10^{-4} , although we also studied other values. In particular we are interested in the minimal ξ -value called $\xi_{\min}$ above which the ratios in (5.1), (5.2) tend to unity. These two z -values are chosen because $z = 10^{-4}$ is the minimum value reached by the HERA experiments and $z = 10^{-2}$ is representative for our study. It turns out that in the region $10^{-1} > z > 10^{-2}$ the plots made for $R_{k,i}^{(2)}$ are similar to those made at $z = 10^{-2}$. Also the plots made for $10^{-2} > z > 10^{-4}$ are between those made for $z = 10^{-2}$ and $z = 10^{-4}$. It appears that for $R_{k,i}^{(2)}$ $\xi_{\min}$ is almost independent of z . In the case of $T_{k,i}^{(2)}$ the sensitivity of the plots on z is larger than it is for $R_{k,i}^{(2)}$. There is a dependence on $\xi_{\min}$ for the different z -values which increases when z tends to unity. Note that we will not study the Born contribution $R_{k,g}^{(1)}$ corresponding to the photon–gluon fusion process (2.12) since the asymptotic as well as the exact expressions for the heavy quark coefficient functions are very simple and they can be found in the literature.

In Figs. 5 and 6 we have plotted $R_{L,g}^{(2)}$ and $R_{2,g}^{(2)}$, respectively. Both ratios are determined by the exact and asymptotic coefficient functions $H_{k,g}^{(2)}$ which originate from the NLO photon–gluon fusion process (2.14). The exact coefficient functions are numerically computed in [4,13], whereas the asymptotic ones can be found in (4.14), (D.3) for $H_{L,g}^{\text{asympt},(2)}$ and in (4.15), (D.4) for $H_{2,g}^{\text{asympt},(2)}$. From Fig. 5 we infer that $H_{L,g}^{\text{exact},(2)}$ and $H_{L,g}^{\text{asympt},(2)}$ coincide when $\xi \geq \xi_{\min} = 10^3$ and there is essentially no difference between the ratios for $z = 10^{-2}$ and $z = 10^{-4}$. In the case of $H_{2,g}^{\text{exact},(2)}$ and $H_{2,g}^{\text{asympt},(2)}$ the above ξ -value is much smaller and both coefficient functions coincide when $\xi \geq \xi_{\min} = 10$. In Fig. 5 as well as Fig. 6 the exact coefficient functions show computer instabilities when ξ gets very large and z becomes very small. This numerical uncertainty is one of the reasons why we derived the asymptotic formulae for the heavy quark coefficient functions in this paper.

In Figs. 7 and 8 we made the same plots but now for the Bethe–Heitler process (2.15). The ratios $R_{L,q}^{(2)}$ (Fig. 7) and $R_{2,q}^{(2)}$ (Fig. 8) are determined by $H_{k,q}^{\text{exact},(2)}$ in [4,13] and the asymptotic expressions $H_{L,q}^{\text{asympt},(2)}$ in (4.20), (D.5) and $H_{2,q}^{\text{asympt},(2)}$ in (4.21), (D.6). Like in the case of the photon–gluon fusion process we observe that $H_{L,q}^{\text{exact},(2)}$ coincides with $H_{L,q}^{\text{asympt},(2)}$ for $\xi \geq \xi_{\min} = 10^3$ and $H_{2,q}^{\text{exact},(2)}$ approaches $H_{2,q}^{\text{asympt},(2)}$ for $\xi \geq \xi_{\min} = 10$.

Fig. 5. $R_{L,g}^{(2)}$ plotted as a function of ξ for fixed $z = 10^{-2}$ (solid line) and $z = 10^{-4}$ (dashed line).

Finally we turn to the ratios $T_{k,q}^{(2)}$ (5.2) which originate from the Compton process in (2.15). Contrary to $H_{k,i}^{\text{exact},(2)}$, for which only numerical results exist, we were able to compute $L_{k,q}^{\text{exact},(2)}$ analytically and the expressions can be found in Appendix A. The asymptotic formulae are presented in (4.28), (D.7) for $L_{L,q}^{\text{asympt},(2)}$ and in (4.29), (D.8) for $L_{2,q}^{\text{asympt},(2)}$. Using these coefficient functions we have plotted $T_{L,q}^{(2)}$ and $T_{2,q}^{(2)}$ in Fig. 9 and Fig. 10, respectively. From these figures we infer that at small z (here $z = 10^{-4}$) the asymptotic formulae coincide with the exact ones over the whole ξ -range. At larger z -values (e.g. $z = 10^{-2}$) the approximation gets worse and it becomes only good when $\xi \geq \xi_{\min} = 10^2$.

Translating to the case of charm electroproduction we find the following results. Choosing the value of $m_c = 1.5 \text{ GeV}/c^2$ it appears that at $Q^2 \geq Q_{\min}^2 = 2250 \text{ (GeV}/c)^2$ ($\xi_{\min} = 10^3$) the exact longitudinal coefficient function $H_{L,i}^{(2)}$ coincides with the asymptotic one. For the coefficient functions $H_{2,i}^{(2)}$ this value is much smaller and it becomes equal to $Q_{\min}^2 = 22.5 \text{ (GeV}/c)^2$ ($\xi_{\min} = 10$). The reason why the convergence of $R_{L,i}^{(2)}$

Fig. 6. $R_{2,g}^{(2)}$ plotted as a function of ξ for fixed $z = 10^{-2}$ (solid line) and $z = 10^{-4}$ (dashed line).

(Figs. 5 and 7) to one is so slow in comparison to $R_{2,i}^{(2)}$ (Figs. 6 and 8) is unclear. Apparently the logarithmic terms appearing in $H_{L,i}^{\text{exact},(2)}$ start to dominate the coefficient function at much larger values of ξ (or Q^2) than is the case for $H_{2,i}^{\text{exact},(2)}$. To see why this happens one needs exact analytical expressions for $H_{k,i}^{\text{exact},(2)}$ which are extremely hard to compute in view of the many mass scales (Q^2 and m^2) on which these coefficient functions depend. In the case of the coefficient functions $L_{k,q}$ one gets $Q_{\min}^2 = 225$ (GeV/c)² ($\xi_{\min} = 100$ at $z = 10^{-2}$) which lies between the values given for $H_{L,i}^{(2)}$ and $H_{2,i}^{(2)}$.

Summarizing the above we have seen that above a certain ξ -value called $\xi_{\min}$ the exact coefficient functions can be replaced by the asymptotic ones. In this way we can avoid the computer instabilities occurring in the exact coefficient functions, which are numerically calculated in [4,13]. However, these $\xi_{\min}$ -values vary depending on the production mechanisms and the projection $k = L$ or $k = 2$ under consideration.

Fortunately it turns out that not all production mechanisms are of equal importance

Fig. 7. $R_{L,q}^{(2)}$ plotted as a function of ξ for fixed $z = 10^{-2}$ (solid line) and $z = 10^{-4}$ (dashed line).

when we compute the heavy flavour component of the deep inelastic structure functions $F_k(x, Q^2, m^2)$ (2.8). As has already been mentioned in the introduction it appears that the photon–gluon fusion subprocesses (2.12), (2.14) dominate the other production mechanisms, which means that the coefficient functions $H_{k,g}^{(2)}$ constitute the bulk of the NLO corrections. Since the $\xi_{\min}$ found for $H_{L,g}^{(2)}$ did not depend on the chosen z -value we can conclude that for $\xi > 10^3$, $H_{L,g}^{\text{exact},(2)}$ in $F_L(x, Q^2, m^2)$ can be replaced by $H_{L,g}^{\text{asympt},(2)}$. Similarly we can replace $H_{2,g}^{\text{exact},(2)}$ in $F_2(x, Q^2, m^2)$ by $H_{2,g}^{\text{asympt},(2)}$, when $\xi > 10$. Translating these values to charm production we find that for $Q^2 > 2250 \text{ (GeV/c)}^2$ the exact coefficient functions in $F_L(x, Q^2, m_c^2)$ can be replaced by their asymptotic ones. In the case of $F_2(x, Q^2, m_c^2)$ however, this replacement can be done for $Q^2 > 22.5 \text{ (GeV/c)}^2$, which is much more interesting from the physical point of view because $F_2(x, Q^2, m_c^2)$ can be measured with much higher accuracy than $F_L(x, Q^2, m_c^2)$. Furthermore, one can argue that in view of the experimental and theoretical errors on the charm component of the structure functions the exact coefficient functions can be probably replaced by

Fig. 8. $R_{2,q}^{(2)}$ plotted as a function of ξ for fixed $z = 10^{-2}$ (solid line) and $z = 10^{-4}$ (dashed line).

their asymptotic ones at much lower values of ξ than $\xi_{\min}$. This holds for $F_L(x, Q^2, m_c^2)$ in particular, where the experimental errors will be very large. The question what the most reasonable value is for $\xi_{\min}$ can only be answered after a thorough investigation of $F_k(x, Q^2, m_c^2)$ which will be left for a future publication.

Apart from the usefulness of the asymptotic heavy quark coefficient functions to avoid computer instabilities at large ξ -values one can use them in the context of the variable flavour number scheme (VFNS) advocated in [9]. This also includes the renormalized operator matrix elements (OME's) A_{kl} calculated in Section 3. As already mentioned above and said below (2.17), for $Q^2 \geq Q_{\min}^2$ ($\xi \geq \xi_{\min}$) the logarithmic terms $\ln^i(m^2/\mu^2) \ln^k(Q^2/m^2)$ dominate the heavy quark coefficient functions so that according to VFNS the heavy flavour behaves like a massless quark. The large logarithmic terms have to be resummed which is done in the following way. First one has to factorize the mass (m)-dependent terms out of the heavy quark coefficient functions according to (2.26), (2.27). The latter turn then into the massless quark coefficient

Fig. 9. $T_{L,q}^{(2)}$ plotted as a function of ξ for fixed $z = 10^{-2}$ (solid line) and $z = 10^{-4}$ (dashed line).

functions calculated in [12]. The mass dependent terms represented by the renormalized OME's A_{kl} computed in Section 3 satisfy a renormalization group equation which provides us with an asymptotic solution representing the resummation of the above large logarithms. This procedure has been used for charm production in [9] in leading order (LO). In particular this procedure entails the definition of the charm quark density which is related via the renormalized OME's A_{kl} to the light parton densities for u , d , s and g . Our calculations allow us to work out this extension to NLO, which will be shown in a future publication.

To summarize our work above we have used the OPE techniques and the renormalization group equation to find analytic formulae for the asymptotic behaviour $Q^2 \gg m^2$ ($\xi \gg 1$) of the heavy quark coefficient functions which enter in deep inelastic electroproduction. We have tested these asymptotic formulae against the exact order α_s^2 coefficient functions available via our rather complicated computer programs and found agreement when $\xi \geq 10$ for $F_2(x, Q^2, m^2)$ and $\xi \geq 10^3$ in the case of $F_L(x, Q^2, m^2)$. In the case of charm production these values are translated into $Q^2 \geq 22.5 \text{ (GeV/c)}^2$

Fig. 10. $T_{2,q}^{(2)}$ plotted as a function of ξ for fixed $z = 10^{-2}$ (solid line) and $z = 10^{-4}$ (dashed line).

for $F_2(x, Q^2, m_c^2)$ and $Q^2 \geq 2250 \text{ (GeV/c)}^2$ for $F_L(x, Q^2, m_c^2)$, respectively. However, taking the experimental uncertainties into account one can argue that much smaller values can be quoted. This will be investigated in the future. Further we computed the order α_s^2 contributions to the renormalized OME's A_{kl} . They can be used in the context of the variable flavour number scheme to determine the charm quark density up to next-to-leading order as a function of the light parton densities u , d , s and g , so that they become much better known than before. This is left to a future publication.

Appendix A

In this appendix we present the exact expressions for the heavy quark coefficient functions $L_{i,q}^{(2)}$ (2.13) corresponding to the Compton reaction (2.15). The calculation is straightforward because one can first integrate over the heavy quark momenta in the final state without affecting the momentum of the remaining light quark (see Figs. 5c,d

in [4]). The phase space integrals are then the same as the ones obtained for the process $\gamma^* + q \rightarrow g^* + q$ ($g^* \rightarrow Q + \bar{Q}$) where the gluon g^* becomes virtual. After integration over the virtual mass of the gluon one gets the expressions

$$L_{L,q}^{r,(2)}(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}) = C_F T_f \left[96z \left(\frac{z}{\xi} \right)^2 \left\{ \ln \left(\frac{1-z}{z^2} \right) L_1 + L_1 L_2 + 2(-DIL_1 + DIL_2 + DIL_3 - DIL_4) \right\} + \left(\frac{z}{(1-z)\xi} \right)^2 (64 - 288z + 192z^2) L_1 + z \left\{ \frac{16}{3} - \frac{416}{3} \left(\frac{z}{\xi} \right) + \frac{1408}{3} \left(\frac{z}{\xi} \right)^2 \right\} \frac{L_3}{sq_2} + \left\{ \frac{16}{3} - \frac{400}{18} z + \left(\frac{z}{(1-z)\xi} \right) \times \left(-\frac{160}{3} + \frac{3824}{9} z - \frac{992}{3} z^2 \right) \right\} sq_1 \right], \quad (A.1)$$

$$L_{2,q}^{r,(2)}(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}) = C_F T_f \left[\left\{ \frac{4}{3} \frac{1+z^2}{1-z} - \frac{16}{1-z} \left(\frac{z}{\xi} \right)^2 (1-9z+9z^2) \right\} \times \left\{ \ln \left(\frac{1-z}{z^2} \right) L_1 + L_1 L_2 + 2(-DIL_1 + DIL_2 + DIL_3 - DIL_4) \right\} + \left\{ -\frac{8}{3} + \frac{4}{1-z} + \left(\frac{z}{(1-z)\xi} \right)^2 \left(128 - 432z + 288z^2 - \frac{8}{1-z} \right) \right\} L_1 + \left\{ \frac{88}{9} + \frac{136}{9} z - \frac{152}{9} \frac{1}{1-z} + \left(\frac{z}{(1-z)\xi} \right) \left(\frac{464}{9} - \frac{512}{3} z + \frac{2048}{9} z^2 \right) + \left(\frac{z}{(1-z)\xi} \right)^2 \left(-\frac{832}{9} + \frac{6208}{9} z - \frac{11392}{9} z^2 + \frac{6016}{9} z^3 \right) \right\} \frac{L_3}{sq_2} + \left\{ -\frac{272}{27} - \frac{1244}{27} z + \frac{718}{27} \frac{1}{1-z} + \left(\frac{z}{(1-z)\xi} \right) \left(-\frac{3424}{27} + \frac{15608}{27} z - \frac{4304}{9} z^2 + \frac{20}{27} \frac{1}{1-z} \right) \right\} sq_1 \right], \quad (A.2)$$

with $r = \text{NS, S}$, $z = Q^2/(2p \cdot q)$ and $\xi = Q^2/m^2$ (2.16). Further we have defined

$$sq_1 = \sqrt{1 - 4 \frac{z}{(1-z)\xi}}, \quad sq_2 = \sqrt{1 - 4 \frac{z}{\xi}}, \quad (A.3)$$

$$L_1 = \ln \left(\frac{1+sq_1}{1-sq_1} \right), \quad L_2 = \ln \left(\frac{1+sq_2}{1-sq_2} \right), \quad L_3 = \ln \left(\frac{sq_2+sq_1}{sq_2-sq_1} \right), \quad (A.4)$$

$$DIL_1 = \text{Li}_2 \left(\frac{(1-z)(1+sq_1)}{1+sq_2} \right), \quad DIL_2 = \text{Li}_2 \left(\frac{1-sq_2}{1+sq_1} \right), \quad (A.5)$$

$$DIL_3 = \text{Li}_2 \left(\frac{1-sq_1}{1+sq_2} \right), \quad DIL_4 = \text{Li}_2 \left(\frac{1+sq_1}{1+sq_2} \right), \quad (A.6)$$

with $0 < z < \xi/(\xi+4)$.

Appendix B

Here we apply the method of integration by parts which enables us to reduce scalar two-loop OME integrals with five different propagators to expressions containing only four different propagators. The method is a generalization of the trick invented in [24] where it was used to reduce two-loop self-energy integrals with five different propagators to integrals which only contain four different propagators.

Feynman integrals with five different propagators emerge from the computation of the graphs e, f, h, l, m, n, o in Fig. 2. Let us first start with graphs 2l, 2o which lead to the following expression:

$$I^{(m)} = \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot q)^m}{D_k D_q^a D_{kp} D_{qp} D_{kq}}, \quad (\text{B.1})$$

where a is 1 or 2. The denominators are defined as

$$D_k = k^2 - m_1^2, \quad (\text{B.2})$$

$$D_q = q^2 - m_2^2, \quad (\text{B.3})$$

$$D_{kp} = (k - p)^2 - m_3^2, \quad (\text{B.4})$$

$$D_{qp} = (q - p)^2 - m_4^2, \quad (\text{B.5})$$

$$D_{kq} = (k - q)^2 - m_5^2. \quad (\text{B.6})$$

In n -dimensional regularization the following integral is zero:

$$0 = \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{\partial}{\partial k_\mu} \left[\frac{k_\mu (\Delta \cdot q)^m}{D_k D_q^a D_{kp} D_{qp} D_{kq}} \right]. \quad (\text{B.7})$$

Differentiating the right-hand side of (B.7) we obtain

$$0 = nI^{(m)} + \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot q)^m}{D_k D_q^a D_{kp} D_{qp} D_{kq}} \times \left\{ -\frac{2k^2}{D_k} - \frac{2k \cdot (k - p)}{D_{kp}} - \frac{2k \cdot (k - q)}{D_{kq}} \right\}. \quad (\text{B.8})$$

In the case that the following conditions are satisfied, i.e. $m_1 = 0$, $p^2 = m_1^2 + m_3^2$, $m_2^2 = m_1^2 + m_5^2$, one can write

$$-2k \cdot (k - p) = -D_k - D_{kp}, \quad (\text{B.9})$$

$$-2k \cdot (k - q) = -D_k - D_{kq} + D_q. \quad (\text{B.10})$$

Substitution of (B.9) and (B.10) into (B.8) yields

$$I^{(m)} = \frac{1}{n-4} \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} (\Delta \cdot q)^m \left\{ \frac{1}{D_q^a D_{kp}^2 D_{qp} D_{kq}} + \frac{1}{D_q^a D_{kp} D_{qp} D_{kq}^2} - \frac{1}{D_k D_q^{a-1} D_{kp} D_{qp} D_{kq}^2} \right\}. \quad (\text{B.11})$$

If $a = 1$ we have already reached our goal. When $a = 2$ we have to repeat the trick for the last term in (B.11) so that we finally get

$$\begin{aligned}
 I^{(m)} = & \frac{1}{n-4} \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} (\Delta \cdot q)^m \left[\frac{1}{D_q^a D_{kp}^2 D_{qp} D_{kq}} \right. \\
 & + \frac{1}{D_q^a D_{kp} D_{qp} D_{kq}^2} - \frac{1}{n-5} \left\{ \frac{1}{D_q^{a-1} D_{kp}^2 D_{qp} D_{kq}^2} \right. \\
 & \left. \left. + \frac{2}{D_q^{a-1} D_{kp} D_{qp} D_{kq}^3} - \frac{2}{D_k D_q^{a-2} D_{kp} D_{qp} D_{kq}^3} \right\} \right]. \quad (\text{B.12})
 \end{aligned}$$

For graphs 2e, 2h we have to differentiate as follows:

$$\begin{aligned}
 0 = & \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{\partial}{\partial k_\mu} \left[\frac{(k-q)_\mu (\Delta \cdot q)^m}{D_k D_q^a D_{kp} D_{qp} D_{kq}} \right] \\
 = & n I^{(m)} + \int \frac{d^n q}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot q)^m}{D_k D_q^a D_{kp} D_{qp} D_{kq}} \\
 & \times \left\{ -\frac{2k \cdot (k-q)}{D_k} - \frac{2(k-q) \cdot (k-p)}{D_{kp}} - \frac{2(k-q)^2}{D_{kq}} \right\}. \quad (\text{B.13})
 \end{aligned}$$

Imposing the following conditions $m_5 = 0$, $m_2^2 = m_1^2 + m_3^2$, $m_4^2 = m_3^2 + m_5^2$ one gets identity (B.10) and

$$-2(k-q) \cdot (k-p) = D_{qp} - D_{kq} - D_{kp}, \quad (\text{B.14})$$

so that $I^{(m)}$ equals

$$\begin{aligned}
 I^{(m)} = & \frac{1}{n-4} \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} (\Delta \cdot q)^m \left[\frac{1}{D_k^2 D_q^a D_{kp} D_{qp}} \right. \\
 & \left. + \frac{1}{D_k D_q^a D_{kp}^2 D_{qp}} - \frac{1}{D_k D_q^a D_{kp}^2 D_{kq}} - \frac{1}{D_k^2 D_q^{a-1} D_{kp} D_{qp} D_{kq}} \right]. \quad (\text{B.15})
 \end{aligned}$$

For $a = 1$ we have already four different propagators. If $a = 2$ one has to repeat the trick again analogous to (B.11).

The trick of integration by parts also applies to graph 2m. For this graph we have the integral

$$J^{(m)} = \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot q)^m}{D_k D_q^a D_{kp} D_{kq} D_{kqp}}, \quad (\text{B.16})$$

with

$$D_{kqp} = (k-q-p)^2 - m_6^2. \quad (\text{B.17})$$

Performing the differentiation with respect to k in the same way as in (B.7) one obtains

$$J^{(m)} = \frac{1}{n-4} \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} (\Delta \cdot q)^m \left[\frac{1}{D_q^a D_{kp}^2 D_{kq} D_{kpq}} + \frac{1}{D_q^a D_{kp} D_{kq}^2 D_{kpq}} \right. \\ \left. + \frac{1}{D_k D_q^a D_{kp} D_{kpq}^2} + \frac{1}{D_k D_q^a D_{kq} D_{kpq}^2} \right. \\ \left. - \frac{1}{D_k D_q^{a-1} D_{kp} D_{kq}^2 D_{kpq}} - \frac{1}{D_k D_q^{a-1} D_{kq} D_{kp} D_{kpq}^2} \right]. \quad (\text{B.18})$$

In the above expression we have used the relations in (B.9), (B.10) and

$$-2k \cdot (k - q - p) = D_q - D_{kp} - D_{kp}, \quad (\text{B.19})$$

which only holds under the conditions $p^2 = m_3^2$, $m_1^2 = 0$, $m_2^2 = m_5^2$.

The integrals corresponding to graphs 2f and 2n need some special treatment because of the sum $\sum_{i=0}^m (\Delta \cdot k)^i (\Delta \cdot q)^{m-i}$ which appears in the operator vertex. Notice that this sum also appears in graphs 2h, 2o but after some rearrangement of terms it drops out of the integrals. For graph 2n we have the expression

$$K^{(m)} = \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot k)^m - (\Delta \cdot q)^m}{(\Delta \cdot k - \Delta \cdot q)} \frac{1}{D_q D_k D_{kp} D_{kq} D_{kpq}}, \quad (\text{B.20})$$

with $m_1^2 = m_2^2 = m_3^2 = m^2$ and $m_5^2 = m_6^2 = 0$. The integral $K^{(m)}$ can be split into $K^{(m)} = K_1^{(m)} - K_2^{(m)}$ with

$$K_1^{(m)} = \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot k)^m}{(\Delta \cdot k - \Delta \cdot q)} \frac{1}{D_q D_k D_{kp} D_{kq} D_{kpq}}, \quad (\text{B.21})$$

$$K_2^{(m)} = \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot q)^m}{(\Delta \cdot k - \Delta \cdot q)} \frac{1}{D_q D_k D_{kp} D_{kq} D_{kpq}}. \quad (\text{B.22})$$

Differentiating the integrand in (B.21) with respect to q and (B.22) with respect to k analogous to (B.13) we obtain

$$K_1^{(m)} = \frac{1}{n-5} \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot k)^m}{(\Delta \cdot k - \Delta \cdot q)} \left[\frac{1}{D_k D_q^2 D_{kp} D_{kpq}} \right. \\ \left. + \frac{1}{D_k D_q D_{kp} D_{kpq}^2} - \frac{1}{D_q^2 D_{kp} D_{kq} D_{kpq}} \right], \quad (\text{B.23})$$

$$K_2^{(m)} = \frac{1}{n-5} \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot q)^m}{(\Delta \cdot k - \Delta \cdot q)} \left[\frac{1}{D_k^2 D_q D_{kp} D_{kpq}} \right. \\ \left. + \frac{1}{D_q D_{kp}^2 D_{kq} D_{kpq}} + \frac{1}{D_k D_q D_{kp}^2 D_{kq}} + \frac{1}{D_k D_q D_{kp} D_{kpq}^2} \right. \\ \left. - \frac{1}{D_k^2 D_{kp} D_{kq} D_{kpq}} - \frac{1}{D_k D_{kp}^2 D_{kq} D_{kpq}} \right]. \quad (\text{B.24})$$

As $p^2 = 0$ the last two terms in (B.24) are equal to zero.

Finally we apply the method of partial integration to graph 2f. The corresponding integral becomes

$$L^{(m)} = \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot k)^m - (\Delta \cdot q)^m}{(\Delta \cdot k - \Delta \cdot q)} \frac{1}{D_k D_q D_{kp} D_{qp} D_{kq}}, \quad (\text{B.25})$$

with $m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2$ and $m_5^2 = 0$. $L^{(m)}$ can be split in the same way as done for $K^{(m)}$ in (B.20). However, because of the symmetry in $k \leftrightarrow q$ one can simplify the calculation. Here we have

$$L^{(m)} = 2L_1^{(m)} = 2 \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot k)^m}{(\Delta \cdot k - \Delta \cdot q)} \frac{1}{D_k D_q D_{kp} D_{qp} D_{kq}}. \quad (\text{B.26})$$

Differentiating the integrand with respect to q analogous to (B.13) we obtain

$$L_1^{(m)} = \frac{1}{n-5} \int \frac{d^n q}{(2\pi)^n} \int \frac{d^n k}{(2\pi)^n} \frac{(\Delta \cdot k)^m}{(\Delta \cdot k - \Delta \cdot q)} \left[\frac{1}{D_k D_q^2 D_{kp} D_{qp}} + \frac{1}{D_k D_q D_{kp} D_{qp}^2} - \frac{1}{D_q^2 D_{kp} D_{qp} D_{kq}} - \frac{1}{D_k D_q D_{qp}^2 D_{kq}} \right]. \quad (\text{B.27})$$

Appendix C

Here we present the unrenormalized operator matrix elements $\hat{A}_{ij}^{(2)}$ whose general structure expressed in renormalization group coefficients was derived in Section 3. After having carried out mass renormalization the two-loop OME in Fig. 2 is given by the following expression (see also (3.23)):

$$\begin{aligned} \hat{A}_{Q8}^{(2)}\left(\frac{m^2}{\mu^2}, \epsilon\right) &= S_\epsilon^2\left(\frac{m^2}{\mu^2}\right)^\epsilon \left[\frac{1}{\epsilon^2} \left\{ C_F T_f [(32 - 64z + 64z^2) \ln(1-z) \right. \right. \\ &\quad \left. \left. - (16 - 32z + 64z^2) \ln z - (8 - 32z)] \right. \right. \\ &\quad \left. \left. + C_A T_f \left[- (32 - 64z + 64z^2) \ln(1-z) - (32 + 128z) \ln z - \frac{64}{3z} \right. \right. \right. \\ &\quad \left. \left. \left. - 16 - 128z + \frac{496}{3} z^2 \right] \right\} \right. \\ &\quad \left. + \frac{1}{\epsilon} \left\{ C_F T_f \left[(8 - 16z + 16z^2) [2 \ln z \ln(1-z) - \ln^2(1-z) + 2\zeta(2)] \right. \right. \right. \\ &\quad \left. \left. - (4 - 8z + 16z^2) \ln^2 z - 32z(1-z) \ln(1-z) \right. \right. \\ &\quad \left. \left. - (12 - 16z + 32z^2) \ln z - 56 + 116z - 80z^2 \right] \right. \\ &\quad \left. + C_A T_f \left[(16 + 32z + 32z^2) [\text{Li}_2(-z) + \ln z \ln(1+z)] \right. \right. \\ &\quad \left. \left. + (8 - 16z + 16z^2) \ln^2(1-z) + (8 + 16z) \ln^2 z \right. \right. \\ &\quad \left. \left. + 32z\zeta(2) + 32z(1-z) \ln(1-z) - \left(8 + 64z + \frac{352}{3} z^2 \right) \ln z \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{160}{9z} + 16 - 200z + \frac{1744}{9}z^2 \Big] \Big\} + a_{Q_8}^{(2)}(z) \Big] \\
& + \sum_{H=Q}^I S_\epsilon^2 \left(\frac{m_H^2}{\mu^2} \right)^{\epsilon/2} \left(\frac{m^2}{\mu^2} \right)^{\epsilon/2} \left[\frac{1}{\epsilon^2} T_f^2 \left(-\frac{64}{3} + \frac{128}{3}z - \frac{128}{3}z^2 \right) \right. \\
& \times \left. \left(1 + \frac{\epsilon^2}{4} \zeta(2) \right) \right], \tag{C.1}
\end{aligned}$$

with

$$\begin{aligned}
a_{Q_8}^{(2)}(z) = & C_F T_f \Big\{ (1 - 2z + 2z^2) [8\zeta(3) + 8\zeta(2) \ln(1 - z) + \frac{4}{3} \ln^3(1 - z) \\
& - 8 \ln(1 - z) \text{Li}_2(1 - z) + 4\zeta(2) \ln z - 4 \ln z \ln^2(1 - z) \\
& + \frac{2}{3} \ln^3 z - 8 \ln z \text{Li}_2(1 - z) + 8 \text{Li}_3(1 - z) - 24 \text{S}_{1,2}(1 - z)] \\
& + z^2 \Big[-24\zeta(2) \ln z + \frac{4}{3} \ln^3 z + 16 \ln z \text{Li}_2(1 - z) + 32 \text{S}_{1,2}(1 - z) \Big] \\
& - (4 + 96z - 64z^2) \text{Li}_2(1 - z) - (6 - 56z + 40z^2) \zeta(2) \\
& - (8 + 48z - 24z^2) \ln z \ln(1 - z) + (4 + 8z - 12z^2) \ln^2(1 - z) \\
& - (1 + 12z - 20z^2) \ln^2 z - (52z - 48z^2) \ln(1 - z) \\
& - (16 + 18z + 48z^2) \ln z + 26 - 82z + 80z^2 \Big\} \\
& + C_A T_f \Big\{ (1 - 2z + 2z^2) \Big[-8\zeta(2) \ln(1 - z) - \frac{4}{3} \ln^3(1 - z) \\
& + 8 \ln(1 - z) \text{Li}_2(1 - z) - 8 \text{Li}_3(1 - z) \Big] + (1 + 2z + 2z^2) \\
& \times \Big[-8\zeta(2) \ln(1 + z) - 16 \ln(1 + z) \text{Li}_2(-z) - 8 \ln z \ln^2(1 + z) \\
& + 4 \ln^2 z \ln(1 + z) + 8 \ln z \text{Li}_2(-z) - 8 \text{Li}_3(-z) - 16 \text{S}_{1,2}(-z) \Big] \\
& + (16 + 64z) [2 \text{S}_{1,2}(1 - z) + \ln z \text{Li}_2(1 - z)] - \left(\frac{4}{3} + \frac{8}{3}z \right) \ln^3 z \\
& + (8 - 32z + 16z^2) \zeta(3) - (24 + 96z) \zeta(2) \ln z + (16 + 16z^2) \\
& \times [\text{Li}_2(-z) + \ln z \ln(1 + z)] + \left(\frac{32}{3z} + 12 + 64z - \frac{272}{3}z^2 \right) \text{Li}_2(1 - z) \\
& - \left(16 + 80z - 128z^2 + \frac{16}{z} \right) \zeta(2) - 4z^2 \ln z \ln(1 - z) \\
& - (2 + 8z - 10z^2) \ln^2(1 - z) + \left(2 + 8z + \frac{46}{3}z^2 \right) \ln^2 z \\
& + (4 + 16z - 16z^2) \ln(1 - z) - \left(\frac{56}{3} + \frac{172}{3}z + \frac{1600}{9}z^2 \right) \ln z \\
& - \frac{448}{27z} - \frac{4}{3} - \frac{628}{3}z + \frac{6352}{27}z^2 \Big\}. \tag{C.2}
\end{aligned}$$

The unrenormalized OME corresponding to Fig. 3 (see (3.29)) is given by

$$\begin{aligned} \hat{A}_{Qq}^{\text{PS},(2)}\left(\frac{m^2}{\mu^2}, \epsilon\right) = & S_\epsilon^2\left(\frac{m^2}{\mu^2}\right)^\epsilon C_F T_f \left\{ \frac{1}{\epsilon^2} \left(-32(1+z) \ln z - \frac{64}{3z} - 16 \right. \right. \\ & + 16z + \frac{64}{3}z^2 \Big) + \frac{1}{\epsilon} \left(8(1+z) \ln^2 z - \left(8 + 40z + \frac{64}{3}z^2 \right) \ln z - \frac{160}{9z} \right. \\ & \left. \left. + 16 - 48z + \frac{448}{9}z^2 \right) + a_{Qq}^{\text{PS},(2)}(z) \right\}, \end{aligned} \quad (\text{C.3})$$

with

$$\begin{aligned} a_{Qq}^{\text{PS},(2)}(z) = & \left[(1+z) \left(32S_{1,2}(1-z) + 16 \ln z \text{Li}_2(1-z) - 24\zeta(2) \ln z \right. \right. \\ & \left. - \frac{4}{3} \ln^3 z \right) + \left(\frac{32}{3z} + 8 - 8z - \frac{32}{3}z^2 \right) \text{Li}_2(1-z) \\ & + \left(-\frac{16}{z} - 12 + 12z + 16z^2 \right) \zeta(2) + \left(2 + 10z + \frac{16}{3}z^2 \right) \ln^2 z \\ & \left. - \left(\frac{56}{3} + \frac{88}{3}z + \frac{448}{9}z^2 \right) \ln z - \frac{448}{27z} - \frac{4}{3} - \frac{124}{3}z + \frac{1600}{27}z^2 \right]. \end{aligned} \quad (\text{C.4})$$

The unrenormalized OME corresponding to Fig. 4 is (see (3.33))

$$\begin{aligned} \hat{A}_{qq,Q}^{\text{NS},(2)}\left(\frac{m^2}{\mu^2}, \epsilon\right) = & S_\epsilon^2\left(\frac{m^2}{\mu^2}\right)^\epsilon C_F T_f \left\{ \frac{1}{\epsilon^2} \left[\frac{32}{3} \left(\frac{1}{1-z} \right)_+ - \frac{16}{3} - \frac{16}{3}z + 8\delta(1-z) \right] \right. \\ & + \frac{1}{\epsilon} \left[\frac{80}{9} \left(\frac{1}{1-z} \right)_+ + \frac{8}{3} \frac{1+z^2}{1-z} \ln z + \frac{8}{9} - \frac{88}{9}z + \delta(1-z) \left(\frac{16}{3} \zeta(2) + \frac{2}{3} \right) \right] \\ & \left. + a_{qq,Q}^{\text{NS},(2)}(z) \right\}, \end{aligned} \quad (\text{C.5})$$

with

$$\begin{aligned} a_{qq,Q}^{\text{NS},(2)}(z) = & \left(\frac{8}{3} \left(\frac{1}{1-z} \right)_+ - \frac{4}{3} - \frac{4}{3}z \right) \zeta(2) + \frac{1+z^2}{1-z} \left(\frac{2}{3} \ln^2 z + \frac{20}{9} \ln z \right) \\ & + \frac{8}{3} (1-z) \ln z + \frac{224}{27} \left(\frac{1}{1-z} \right)_+ + \frac{44}{27} - \frac{268}{27}z \\ & + \delta(1-z) \left(-\frac{8}{3} \zeta(3) + \frac{58}{9} \zeta(2) + \frac{73}{18} \right). \end{aligned} \quad (\text{C.6})$$

Here the term $1/(1-z)_+$ has to be defined in the distribution sense

$$\int_0^1 dz \left(\frac{1}{1-z} \right)_+ f(z) = \int_0^1 dz \frac{1}{1-z} [f(z) - f(1)]. \quad (\text{C.7})$$

Notice that as long as $z < Q^2/(Q^2 + 4m^2)$ with $m^2 \neq 0$ the terms proportional to $\delta(1-z)$ do not contribute and the subscript $+$ in $(1/(1-z))_+$ can be dropped. We will comment on this in Appendix D after having predicted the asymptotic expression for $L_{2,q}^{\text{NS},(2)}$. The OME's which emerge from the calculation of the graphs in Figs. 1–4 usually appear in the Mellin transformed representation

$$A_{ij}^{(m)} = \frac{1}{2} [1 + (-1)^m] \int_0^1 dz z^{m-1} A_{ij}(z). \quad (\text{C.8})$$

This implies that the anomalous dimensions $\gamma_{ij}^{(m)}$ correspond to the physical operators listed in Eqs. (2.23)–(2.25) only for even m .

Appendix D

In this appendix we present the heavy quark coefficient functions $H_{i,j}^{(2)}$ and $L_{i,j}^{(2)}$ ($i = L, 2; j = q, g$) in the asymptotic limit $Q^2 \gg m^2$. Starting with the leading order photon–gluon fusion process (2.12) the heavy quark coefficient functions read (see Eqs. (4.6), (4.7))

$$H_{L,g}^{(1)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = T_f [16z(1-z)], \quad (\text{D.1})$$

$$H_{2,g}^{(1)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = T_f [(4 - 8z + 8z^2) \left(\ln \frac{Q^2}{m^2} + \ln(1-z) - \ln z \right) - 4 + 32z - 32z^2]. \quad (\text{D.2})$$

In next-to-leading order the coefficient functions are given by the corrections to the Born reaction (2.12) and the bremsstrahlung process (2.14). They read (see Eqs. (4.14), (4.15))

$$\begin{aligned} H_{L,g}^{(2)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = & \left[C_F T_f \{ 32z \ln z + 16 + 16z - 32z^2 \} \right. \\ & + C_A T_f \left\{ 64z(1-z) \ln(1-z) - 128z \ln z + \frac{32}{3z} - 32 \right. \\ & \left. \left. - 160z + \frac{544}{3} z^2 \right\} \right] \ln \frac{Q^2}{m^2} \\ & + C_A T_f \left\{ 64z(1-z) \ln(1-z) - 128z \ln z + \frac{32}{3z} - 32 \right. \\ & \left. \left. - 160z + \frac{544}{3} z^2 \right\} \ln \frac{m^2}{\mu^2} \right. \\ & + C_F T_f \left\{ \left(\frac{64}{15z^2} - \frac{64}{3} z + \frac{128}{5} z^3 \right) [\text{Li}_2(-z) + \ln z \ln(1+z)] \right. \\ & + 32z [\text{Li}_2(1-z) + \ln z \ln(1-z)] - \left(\frac{64}{3} z - \frac{128}{5} z^3 \right) \zeta(2) \\ & - \left(\frac{64}{3} z + \frac{64}{5} z^3 \right) \ln^2 z + (16 + 48z - 64z^2) \ln(1-z) \\ & + \left(-\frac{64}{15z} - \frac{208}{15} - \frac{416}{5} z + \frac{192}{5} z^2 \right) \ln z + \frac{64}{15z} - \frac{256}{15} - \frac{608}{5} z + \frac{672}{5} z^2 \left. \right\} \\ & + C_A T_f \left\{ (64z + 64z^2) [\text{Li}_2(-z) + \ln z \ln(1+z)] - 128z \text{Li}_2(1-z) \right\} \end{aligned}$$

$$\begin{aligned}
& +64z^2\zeta(2) - (192z - 64z^2) \ln z \ln(1-z) + (32z - 32z^2) \ln^2(1-z) \\
& +96z \ln^2 z + \left(\frac{32}{3z} - 32 - 288z + \frac{928}{3}z^2\right) \ln(1-z) \\
& + (32 + 256z - 416z^2) \ln z - \frac{32}{9z} + \frac{32}{3} + \frac{544}{3}z - \frac{1696}{9}z^2 \Big\}, \quad (D.3) \\
H_{2,g}^{(2)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = & \left[C_F T_f \left\{ (8 - 16z + 16z^2) \ln(1-z) \right. \right. \\
& - (4 - 8z + 16z^2) \ln z - 2 + 8z \Big\} \\
& + C_A T_f \left\{ (8 - 16z + 16z^2) \ln(1-z) + (8 + 32z) \ln z + \frac{16}{3z} \right. \\
& \left. \left. + 4 + 32z - \frac{124}{3}z^2 \right\} \right] \ln^2 \frac{Q^2}{m^2} \\
& + \left[C_F T_f \left\{ (8 - 16z) \text{Li}_2(1-z) - (32 - 64z + 64z^2) \zeta(2) \right. \right. \\
& - (24 - 48z + 64z^2) \ln z \ln(1-z) + (16 - 32z + 32z^2) \ln^2(1-z) \\
& + (8 - 16z + 32z^2) \ln^2 z - (28 - 96z + 80z^2) \ln(1-z) \\
& + (8 - 48z + 80z^2) \ln z + 36 - 68z + 16z^2 \Big\} \\
& + C_A T_f \left\{ - (16 + 32z + 32z^2) [\text{Li}_2(-z) + \ln z \ln(1+z)] \right. \\
& + (16 + 64z) \text{Li}_2(1-z) - (16 + 32z^2) \zeta(2) \\
& + (96z - 32z^2) \ln z \ln(1-z) + (8 - 16z + 16z^2) \ln^2(1-z) \\
& - (16 + 48z) \ln^2 z + \left(\frac{32}{3z} - 8 + 160z - \frac{536}{3}z^2\right) \ln(1-z) \\
& \left. \left. - (192z - 200z^2) \ln z + \frac{208}{9z} - \frac{220}{3} - \frac{368}{3}z + \frac{1628}{9}z^2 \right\} \right] \ln \frac{Q^2}{m^2} \\
& + C_A T_f \left[\left\{ (16 - 32z + 32z^2) \ln(1-z) + (16 + 64z) \ln z + \frac{32}{3z} + 8 \right. \right. \\
& \left. \left. + 64z - \frac{248}{3}z^2 \right\} \ln \frac{Q^2}{m^2} \right. \\
& + (16 + 64z) \text{Li}_2(1-z) - (16 - 32z + 32z^2) \zeta(2) \\
& + (96z - 32z^2) \ln z \ln(1-z) + (16 - 32z + 32z^2) \ln^2(1-z) \\
& - (8 + 32z) \ln^2 z + \left(\frac{32}{3z} - 8 + 192z - \frac{632}{3}z^2\right) \ln(1-z) \\
& \left. \left. - \left(8 + 256z - \frac{248}{3}z^2\right) \ln z + \frac{16}{3z} - \frac{172}{3} - \frac{968}{3}z + \frac{1124}{3}z^2 \right] \ln \frac{m^2}{\mu^2} \right. \\
& \left. + C_F T_f \left[16(1+z)^2 \left(-4\text{S}_{1,2}(-z) - 4 \ln(1+z) \text{Li}_2(-z) \right) \right] \right]
\end{aligned}$$

$$\begin{aligned}
& -2\zeta(2) \ln(1+z) - 2 \ln z \ln^2(1+z) + \ln^2 z \ln(1+z) - \frac{3}{2} \text{Li}_2(1-z) \Big) \\
& + 8(1-z)^2 \Big(-4S_{1,2}(1-z) - 3\text{Li}_3(1-z) + 12\text{Li}_3(-z) + \ln^3(1-z) \\
& - \frac{5}{2} \ln z \ln^2(1-z) + 2 \ln^2 z \ln(1-z) + \ln(1-z) \text{Li}_2(1-z) \\
& - 4 \ln z \text{Li}_2(-z) + 4\zeta(2) \ln z - \frac{1}{3} \ln^3 z + 3 \ln z \ln(1-z) \Big) \\
& + 8z^2 \Big(4S_{1,2}(1-z) - \text{Li}_3(1-z) + \ln^3(1-z) \\
& - \frac{7}{2} \ln z \ln^2(1-z) + 4 \ln^2 z \ln(1-z) - \ln(1-z) \text{Li}_2(1-z) \\
& - 4\zeta(2) \ln(1-z) + 4 \ln z \text{Li}_2(1-z) + 4\zeta(2) \ln z - \ln^3 z + 11\text{Li}_2(1-z) \Big) \\
& + 128z \text{Li}_3(-z) + (112 - 96z + 192z^2) \zeta(3) - (112z - 144z^2) \ln z \ln(1-z) \\
& + \left(\frac{16}{15z^2} + 96 + \frac{128}{3} z + \frac{192}{5} z^3 \right) [\text{Li}_2(-z) + \ln z \ln(1+z)] \\
& + \left(48 - \frac{208}{3} z + 104z^2 + \frac{192}{5} z^3 \right) \zeta(2) - (22 - 88z + 84z^2) \ln^2(1-z) \\
& - \left(4 - \frac{8}{3} z + 52z^2 + \frac{96}{5} z^3 \right) \ln^2 z + (28 - 132z + 96z^2) \ln(1-z) \\
& - \left(\frac{16}{15z} + \frac{712}{15} - \frac{136}{5} z + \frac{672}{5} z^2 \right) \ln z + \frac{16}{15z} - \frac{904}{15} + \frac{68}{5} z + \frac{328}{5} z^2 \Big] \\
& + C_A T_f \Big[16(1+2z+2z^2) \Big(\text{Li}_3\left(\frac{1-z}{1+z}\right) - \text{Li}_3\left(-\frac{1-z}{1+z}\right) \\
& - \ln z \ln(1-z) \ln(1+z) - \ln(1-z) \text{Li}_2(-z) \\
& + \frac{3}{4} \ln^2 z \ln(1+z) + \frac{3}{2} \ln z \text{Li}_2(-z) \Big) \\
& + 8(1+2z) \{ 5S_{1,2}(1-z) + 2S_{1,2}(-z) - 3\text{Li}_3(-z) + \ln z \ln^2(1+z) \\
& + 2 \ln(1+z) \text{Li}_2(-z) + \zeta(2) \ln(1+z) \} \\
& + 32z [2S_{1,2}(1-z) + \ln z \text{Li}_2(1-z)] \\
& - (16 + 128z) \text{Li}_3(1-z) + 8z^2 [\ln^2 z \ln(1+z) \\
& - 2 \ln z \text{Li}_2(-z)] + (48z - 16z^2) \ln z \ln^2(1-z) \\
& - (8 + 64z - 16z^2) \ln^2 z \ln(1-z) + (16 + 64z) \ln(1-z) \text{Li}_2(1-z) \\
& - (40 - 48z + 64z^2) \zeta(2) \ln(1-z) + \left(\frac{16}{3} + 16z \right) \ln^3 z \\
& - (16 + 160z - 32z^2) \zeta(2) \ln z - (12 + 56z + 8z^2) \zeta(3) \\
& - \left(\frac{32}{3z} + 48 - 16z - \frac{208}{3} z^2 \right) [\text{Li}_2(-z) + \ln z \ln(1+z)] \\
& + \left(\frac{64}{3z} + 20 - 64z + \frac{80}{3} z^2 \right) \text{Li}_2(1-z) - \left(\frac{32}{z} + 4 + 208z - \frac{796}{3} z^2 \right) \zeta(2)
\end{aligned}$$

$$\begin{aligned}
& + (16 - 288z + 292z^2) \ln z \ln(1-z) + \left(\frac{16}{3z} - 6 + 64z - \frac{214}{3}z^2 \right) \ln^2(1-z) \\
& + (184z - 114z^2) \ln^2 z + \left(\frac{208}{9z} - \frac{112}{3} - \frac{860}{3}z + \frac{2996}{9}z^2 \right) \ln(1-z) \\
& + \left(\frac{292}{3} + 332z - \frac{5780}{9}z^2 \right) \ln z + \frac{80}{9z} + \frac{466}{9} + \frac{260}{9}z - \frac{878}{9}z^2 \Big]. \quad (\text{D.4})
\end{aligned}$$

The coefficient functions corresponding to the Bethe–Heitler process (2.15) read (see (4.20), (4.21))

$$\begin{aligned}
H_{L,q}^{\text{PS},(2)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= C_F T_f \left[\left\{ -32z \ln z + \frac{32}{3z} - 32 + \frac{64}{3}z^2 \right\} \right. \\
&\times \left(\ln \frac{Q^2}{m^2} + \ln \frac{m^2}{\mu^2} \right) - 32z [\text{Li}_2(1-z) + \ln z \ln(1-z) - \ln^2 z] \\
&+ \left(\frac{32}{3z} - 32 + \frac{64}{3}z^2 \right) \ln(1-z) + (32 - 32z - 64z^2) \ln z \\
&\left. - \frac{32}{9z} + \frac{32}{3} - \frac{128}{3}z + \frac{320}{9}z^2 \right], \quad (\text{D.5})
\end{aligned}$$

$$\begin{aligned}
H_{2,q}^{\text{PS},(2)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= C_F T_f \left[\left\{ 8(1+z) \ln z + \frac{16}{3z} + 4 - 4z - \frac{16}{3}z^2 \right\} \ln^2 \frac{Q^2}{m^2} \right. \\
&+ \left\{ \left[16(1+z) \ln z + \frac{32}{3z} + 8 - 8z - \frac{32}{3}z^2 \right] \ln \frac{Q^2}{m^2} \right. \\
&+ 8(1+z) [2\text{Li}_2(1-z) + 2 \ln z \ln(1-z) - \ln^2 z] \\
&+ \left(\frac{32}{3z} + 8 - 8z - \frac{32}{3}z^2 \right) \ln(1-z) \\
&\left. - \left(8 + 40z - \frac{32}{3}z^2 \right) \ln z + \frac{16}{3z} - \frac{160}{3} + \frac{16}{3}z + \frac{128}{3}z^2 \right\} \ln \frac{m^2}{\mu^2} \\
&+ \left\{ 16(1+z) [\text{Li}_2(1-z) + \ln z \ln(1-z) - \ln^2 z] \right. \\
&+ \left(\frac{32}{3z} + 8 - 8z - \frac{32}{3}z^2 \right) \ln(1-z) + 32z^2 \ln z + \frac{208}{9z} - \frac{208}{3} \\
&+ \frac{160}{3}z - \frac{64}{9}z^2 \Big\} \ln \frac{Q^2}{m^2} \\
&+ (1+z) \left(32\text{S}_{1,2}(1-z) - 16\text{Li}_3(1-z) + 8 \ln z \ln^2(1-z) \right. \\
&- 16 \ln^2 z \ln(1-z) + 16 \ln(1-z) \text{Li}_2(1-z) - 32\zeta(2) \ln z + \frac{16}{3} \ln^3 z \Big) \\
&- \left(\frac{32}{3z} + 32 + 32z + \frac{32}{3}z^2 \right) [\text{Li}_2(-z) + \ln z \ln(1+z)] \\
&+ \left(\frac{64}{3z} + 16 - 16z + \frac{32}{3}z^2 \right) \text{Li}_2(1-z) - \left(\frac{32}{z} + 16 + 16z - \frac{64}{3}z^2 \right) \zeta(2) \\
&\left. + 32z^2 \ln z \ln(1-z) + \left(\frac{16}{3z} + 4 - 4z - \frac{16}{3}z^2 \right) \ln^2(1-z) \right]
\end{aligned}$$

$$\begin{aligned}
& + (40z - 16z^2) \ln^2 z + \left(\frac{208}{9z} - \frac{208}{3} + \frac{160}{3}z - \frac{64}{9}z^2 \right) \ln(1-z) \\
& + \left(\frac{280}{3} - 88z - \frac{704}{9}z^2 \right) \ln z + \frac{80}{9z} + \frac{304}{9} - \frac{1216}{9}z + \frac{832}{9}z^2 \Big]. \quad (\text{D.6})
\end{aligned}$$

Finally we present the coefficient functions corresponding to the Compton process (2.15). They are given by Eqs. (4.28), (4.29) which become equal to

$$L_{L,q}^{\text{NS},(2)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) = C_F T_f \left[\frac{16}{3} z \left(\ln \frac{Q^2}{m^2} + \ln(1-z) - 2 \ln z \right) + \frac{16}{3} - \frac{400}{18} z \right], \quad (\text{D.7})$$

$$\begin{aligned}
L_{2,q}^{\text{NS},(2)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= C_F T_f \left[\frac{4}{3} \left(\frac{1+z^2}{1-z} \right) \ln^2 \frac{Q^2}{m^2} \right. \\
&+ \left\{ \frac{1+z^2}{1-z} \left(\frac{8}{3} \ln(1-z) - \frac{16}{3} \ln z - \frac{58}{9} \right) + \frac{2}{3} + \frac{26}{3} z \right\} \ln \frac{Q^2}{m^2} \\
&+ \left(\frac{1+z^2}{1-z} \right) \left(-\frac{8}{3} \text{Li}_2(1-z) - \frac{8}{3} \zeta(2) - \frac{16}{3} \ln z \ln(1-z) \right. \\
&+ \left. \frac{4}{3} \ln^2(1-z) + 4 \ln^2 z - \frac{58}{9} \ln(1-z) + \frac{134}{9} \ln z + \frac{359}{27} \right) \\
&+ \left. \left(\frac{2}{3} + \frac{26}{3} z \right) \ln(1-z) - \left(2 + \frac{46}{3} z \right) \ln z + \frac{29}{9} - \frac{295}{9} z \right]. \quad (\text{D.8})
\end{aligned}$$

In the above expressions one should bear in mind that the singularity at $z = 1$ will never show up because of the kinematical constraint $z < Q^2/(Q^2 + 4m^2)$. However, after convoluting $L_{2,q}^{\text{NS}}$ by the parton densities the structure function $F_2(x, Q^2, m^2)$ in (2.8) will diverge as $\ln^3(Q^2/m^2)$ in the limit $Q^2 \gg m^2$. In the above limit the upper boundary in (2.8) $z_{\text{max}} \rightarrow 1$ and the virtual gluon which decays into the heavy quark pair becomes soft. The soft gluon contribution which causes the cubic logarithm above has to be added to the two-loop vertex corrections calculated in Appendix A of [28]. These corrections contain the heavy quark (Q) contribution to the gluon self-energy. The cubic logarithm is then cancelled. The final result will be that in (D.8) the singular terms at $z = 1$ have to be replaced by the distributions $(\ln^k(1-z)/(1-z))_+$ defined by

$$\int_0^1 dz \left(\frac{\ln^k(1-z)}{1-z} \right)_+ f(z) = \int_0^1 dz \left(\frac{\ln^k(1-z)}{1-z} \right) (f(z) - f(1)), \quad (\text{D.9})$$

and one has to add the following delta function contribution:

$$\begin{aligned}
L_{2,q}^{\text{NS},S+V(2)}\left(z, \frac{Q^2}{m^2}, \frac{m^2}{\mu^2}\right) &= C_F T_f \delta(1-z) \left\{ 2 \ln^2 \frac{Q^2}{m^2} - \left[\frac{32}{3} \zeta(2) + \frac{38}{3} \right] \ln \frac{Q^2}{m^2} \right. \\
&+ \left. \frac{268}{9} \zeta(2) + \frac{265}{9} \right\}. \quad (\text{D.10})
\end{aligned}$$

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