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## Scaling invariance of the strict KP hierarchy

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## Масштабная инвариантность строгой иерархии Кадо姆цева–Петвиашвили

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**Abstract.** In this paper we show first of all that for solutions of the strict KP hierarchy it is sufficient to work in a standard setting. Further we discuss a minimal realization of the hierarchy and present the scaling invariance of the Lax equations of the hierarchy.

**Keywords:** pseudo differential operators; Lax equations; strict KP hierarchy, minimal realization; scaling transformations

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**Аннотация.** В работе показано, прежде всего, что для решений строгой иерархии Кадо姆цева–Петвиашвили (КП) достаточно использовать стандартное окружение. Далее рассмотрена минимальная реализация этой иерархии, представлена масштабная инвариантность уравнений Лакса строгой иерархии КП.

**Ключевые слова:** псевдодифференциальные операторы; уравнения Лакса; строгой иерархии КП; минимальная реализация; преобразования масштабирования

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## Introduction

The main goal of this paper is to present the scaling transformations that map solutions of the strict KP hierarchy from one setting to another one. Besides that we also show that as far as solutions are concerned, it is sufficient to consider standard settings. We illustrate the scaling invariance at the hand of the Khadomtsev–Petviashvili (KP) equation from plasma physics [1]. This nonlinear equation for a function  $u = u(x, y, t)$  is a two-dimensional variant of the KdV equation and reads

$$\frac{3}{4}u_{yy} := \frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3} - \frac{3}{2}u\frac{\partial u}{\partial x} \right) = (u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x)_x.$$

The numerical coefficients of the various terms in the KP equation are not so relevant. Consider namely the following transformation of the dependent and independent variables

$$\hat{x} = \alpha_1 x, \hat{y} = \alpha_2 y, \hat{t} = \alpha_3 t, u = \beta \hat{u}, \text{ with } \alpha_1 \neq 0, \alpha_3 \neq 0 \text{ and } \beta \neq 0. \quad (0.1)$$

Then a direct computation shows that the KP equation in the new coordinates becomes

$$\frac{3}{4}\beta\alpha_2^2\hat{u}_{\hat{y}\hat{y}} = \alpha_1(\beta\alpha_3\hat{u}_{\hat{t}} - \frac{1}{4}\alpha_1^3\beta\hat{u}_{\hat{x}\hat{x}\hat{x}} - \frac{3}{2}\alpha_1\beta^2\hat{u}\hat{u}_{\hat{x}})_{\hat{x}}$$

and this illustrates the wide range of coefficients that can be obtained. We call transformations as described in formula (0.1), where the dependent and independent variables change by a constant nonzero factor *scaling transformations*. Note that the choice  $\beta = \alpha_2 = \alpha_1^2$  and  $\alpha_3 = \alpha_1^3$  transforms a solution of the KP equation in the old coordinates to a solution in the new. The equations of the strict KP hierarchy also possess a scaling invariance which can conveniently be shown with the help of the minimal realization of the equations presented here. The contents of the various sections is as follows: Section 1. describes the necessary prerequisites of the strict KP hierarchy. The next section treats the standartization of the notion of setting and the last section presents the minimal realization of the hierarchy and its scaling invariance.

### 1. The strict KP hierarchy

In this section we shortly recall the results needed from [2] about the strict KP-hierarchy in the pseudo differential operators Psd. The algebra Psd is built up as follows: one starts with a commutative algebra  $R$  over a field  $k$  of characteristic zero and a privileged  $k$ -linear derivation  $\partial : R \mapsto R$ . Given  $R$  and  $\partial$ , one forms the algebra  $R[\partial]$  of differential operators in  $\partial$  with coefficients from  $R$ . It consists of  $k$ -linear endomorphisms of  $R$  of the form  $\sum_{i=0}^n a_i \partial^i, a_i \in R$ . For simplicity, we assume that the powers of  $\partial$  are  $R$ -linear independent, otherwise one has to pass to a cover of  $R[\partial]$  [3]. Next one extends the algebra  $R[\partial]$  by adding the inverses of all the powers of  $\partial$  and by allowing infinite sums of these negative powers. Thus one arrives at the algebra  $\text{Psd} = R[\partial, \partial^{-1}]$  of all formal series

$$p = \sum_{j=-\infty}^N p_j \partial^j, p_j \in R.$$

If one uses for each  $n \in \mathbb{Z}$ , the notation

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!},$$

then the product of two series  $a = \sum_j a_j \partial^j$  and  $b = \sum_i b_i \partial^i$  is given by

$$ab := \sum_j \sum_i \sum_{s=0}^{\infty} \binom{j}{s} a_j \partial^s (b_i) \partial^{i+j-s}.$$

The algebra  $\text{Psd}$  possesses various decompositions. For  $s \in \mathbb{Z}$ , any pseudo differential operator  $P = \sum_j p_j \partial^j \in \text{Psd}$  can be split as

$$P = P_{>s} + P_{\leq s}, \text{ where } P_{>s} = \sum_{j>s} p_j \partial^j \text{ and } P_{\leq s} = \sum_{j\leq s} p_j \partial^j. \tag{1.1}$$

For  $s = 0$ , this corresponds to writing  $P$  as the sum of its pure differential operator part  $P_{>0}$  and its integral operator part  $P_{\leq 0}$ . This decomposition is important for the strict KP hierarchy, the one for  $s = -1$  for the KP hierarchy.

As any associative  $k$ -algebra, also  $\text{Psd}$  is w.r.t. the commutator a Lie algebra over  $k$ . From the multiplication rules in  $\text{Psd}$  follows that for  $s = 0$  the decomposition (1.1) yields a splitting of the Lie algebra  $\text{Psd}$  into the direct sum of two Lie subalgebras, namely

$$\text{Psd} = \{P \in \text{Psd}, P = P_{\leq 0}\} \oplus \{P \in \text{Psd}, P = P_{>0}\} := \text{Psd}_{\leq 0} \oplus \text{Psd}_{>0}.$$

We write  $\pi_{>0}$  for the projection from  $\text{Psd}$  on  $\text{Psd}_{>0}$  consisting of taking the strict differential operator part of an element in  $\text{Psd}$ . Similarly, one defines the projections of  $\text{Psd}$  on respectively  $\text{Psd}_{\leq 0}$ ,  $\text{Psd}_{>-1}$  and  $\text{Psd}_{\leq -1}$ , by respectively  $\pi_{\leq 0}$ ,  $\pi_{>0}$  and  $\pi_{<0}$ . To the Lie algebra  $\text{Psd}_{\leq 0}$  we associated the group

$$D(0) = \{p_0 + \sum_{j<0} p_j \partial^j \mid p_0 \in R^*\}.$$

Inside  $\text{Psd}$  we consider perturbations  $M$  of the basic derivation  $\partial$  that have the form

$$M = \partial + \sum_{j=0}^{\infty} m_{j+1} \partial^{-j}. \tag{1.2}$$

They are prototypes of deforming the operator  $\partial$  by conjugating or *dressing* with an element from  $D(0)$ .

Let  $k[\partial]_0$  be the  $\{\sum_{i=1}^N a_i \partial^i \mid \text{all } a_i \in k\}$ . Then  $k[\partial]_0$  is a commutative Lie subalgebra of  $\text{Psd}_{>0}$  and we see  $k[M]_0 = \{\sum_{i=1}^N a_i M^i \mid \text{all } a_i \in k\}$  as a commutative deformation of  $k[\partial]_0$ . Now one searches for deformations  $\{M^m, m \geq 1\}$  of the elements  $\{\partial^m\}$  such that their evolution is given by Lax equations whose form is determined by the projection  $\pi_{>0}$ . More concretely, we assume for the deformations  $\{M^m\}$  that the  $k$ -algebra  $R$  is, besides with a privileged  $k$ -linear derivation  $\partial$ , also equipped with a collection  $\{\partial_r \mid r \geq 1\}$  of  $k$ -linear derivations of  $R$  commuting with  $\partial$ , that form the infinitesimal generators of the various flows of the evolution. The Lax equations each  $M^m$  should satisfy are

$$\partial_r(M^m) = [M^m, \pi_{\leq 0}(M^r)] = [\pi_{>0}(M^r), M^m] = [C_r, M^m], \text{ for all } r \geq 1,$$

where  $C_r$  is a short hand notation for  $\pi_{>0}(M^r)$ . Since  $\partial_r$  and taking the commutator with  $C_r$  are both derivations of Psd, one sees that it suffices to find an  $M$  such that

$$\partial_r(M) = [C_r, M] = [M, \pi_{\leq 0}(M^r)]. \tag{1.3}$$

The equations (1.3) for an operator  $M$  in Psd of the form (1.2), are called the *Lax equations of the strict KP hierarchy* and  $M$  is named a *solution* of the hierarchy. The data  $(R, \partial, \{\partial_r\})$  are called a *setting* for this nonlinear system.

**R e m a r k 1.1.** Note that any setting for the strict KP hierarchy admits the *trivial solution*  $M = \partial$ . Since  $C_1 = \partial$  for any  $M$ , the Lax equation for  $r = 1$  becomes

$$\partial_1(M) = \partial(M).$$

Hence, one often takes  $\partial = \partial_1$  and if moreover all the  $\{\partial_r\}$  commute among each other, we call the setting  $(R, \partial_1, \{\partial_r\})$  *standard* and use the notation  $(R, \{\partial_r\})$ . Both at the solvability of the related Cauchy problem in [4] and at the geometric construction of solutions of the strict KP hierarchy in [5], we only worked with standard settings. The next section shows that this is sufficient.

**R e m a r k 1.2.** Let  $M$  be a solution of the strict KP hierarchy. We denote the differential subalgebra of  $R$  generated by the coefficients of  $M$  by  $R(M)$ . It consists of all polynomial expressions in the  $\{\partial^s(m_{j+1}) \mid j \geq 0, s \geq 0\}$ . The derivation  $\partial$  is clearly an endomorphism of  $R(M)$  and for simplicity we denote the restriction of  $\partial$  to  $R(M)$  by  $\partial_M$ . From the fact that all coefficients of the  $\{C_r\}$  belong to  $R(M)$ , one sees that also all the derivations  $\{\partial_r\}$  are mapping  $R(M)$  into itself. The restriction of each  $\partial_r$  to  $R(M)$  is similarly denoted by  $\partial_{r,M}$ . In particular, it follows from Remark 1.1 that  $\partial_M = \partial_{1,M}$ . The data  $(R(M), \partial_M, \{\partial_{r,M}\})$  form then a setting for the strict KP hierarchy and  $M$  is a solution in this setting.

**R e m a r k 1.3.** The independent and dependent variables relevant for the strict KP hierarchy are the flow parameters of the derivations  $\{\partial_r\}$  and  $\partial$  from the setting and the  $\{m_{j+1} \mid j \geq 0\}$ . Because the action of  $\partial$  and  $\partial_1$  on  $M$  is the same, we will consider in the sequel scaling transformations of the form

$$\partial = \alpha_1 \hat{\partial}, \partial_r = \alpha_r \hat{\partial}_r, m_{j+1} = \beta_j \hat{m}_{j+1}, \tag{1.4}$$

with  $\{\alpha_r \in \mathbb{C}^* \mid r \geq 1\}$  and  $\{\beta_j \in \mathbb{C}^* \mid j \geq 0\}$ . They link with the setting  $(R, \hat{\partial}, \{\hat{\partial}_r\})$  for the strict KP hierarchy.

### 2. Reduction to a standard setting

We want to show in this section that, given a setting  $(R, \partial_1, \{\partial_r\})$  of the strict KP hierarchy, there is a  $k$ -subalgebra  $R_1$  of  $R$  such that  $\partial|R_1$  and all the  $\{\partial_r|R_1\}$  are derivations of  $R_1$  and the setting  $(R_1, \partial|R_1, \{\partial_r|R_1\})$  is standard. To do that we need another form of the strict KP hierarchy. It was shown in [2] that the strict differential operators  $\{C_r\}$  in Psd corresponding to a solution  $M$  of the strict KP-hierarchy, satisfy

$$\partial_{r_1}(C_{r_2}) - \partial_{r_2}(C_{r_1}) - [C_{r_1}, C_{r_2}] = 0. \tag{2.1}$$

We call the equations (2.1) the *zero curvature relations* for the strict cut-off's  $\{C_r\}$  of the solution  $M$  of the strict KP-hierarchy. The zero curvature relations are also sufficient to get the Lax equations for  $M$ , for there holds

**Theorem 2.1.** *Let  $(R, \partial, \{\partial_r\})$  be a setting for the strict KP hierarchy and let  $M$  be an element in  $\text{Psd}$  of the form (1.2). Then  $M$  satisfies the Lax equations of the strict KP hierarchy if and only if the zero curvature relations (2.1) for the  $\{C_r \mid r \geq 1\}$  hold.*

By using this zero curvature form of the strict KP hierarchy, we can make an important step towards the realization of our goal. For there holds

**Proposition 2.1.** *The setting  $(R(M), \partial_M, \{\partial_{r,M}\})$  is standard.*

**Proof.** We already saw in Remark 1.2 that  $\partial_M = \partial_{1,M}$ , so we merely have to show that the derivations  $\{\partial_{r,M}\}$  commute. Note that if  $a$  and  $b$  are in  $R(M)$ , then there hold for all  $r_1 \geq 1$  and  $r_2 \geq 1$  the identities

$$\begin{aligned} \partial_{r_1} \partial_{r_2}(ab) &= \partial_{r_1} \partial_{r_2}(a)b + \partial_{r_2}(a) \partial_{r_1}(b) + \partial_{r_1}(a) \partial_{r_2}(b) + a \partial_{r_1} \partial_{r_2}(b), \\ \partial_{r_2} \partial_{r_1}(ab) &= \partial_{r_2} \partial_{r_1}(a)b + \partial_{r_1}(a) \partial_{r_2}(b) + \partial_{r_2}(a) \partial_{r_1}(b) + a \partial_{r_2} \partial_{r_1}(b). \end{aligned}$$

Hence, if  $a$  and  $b$  are in the kernel of the commutator of  $\partial_{r_1}$  and  $\partial_{r_2}$ , then their product also belongs to this kernel. The algebra  $R(M)$  is the polynomial algebra generated by the  $\{\partial^s(m_{j+1}) \mid j \geq 0, s \geq 0\}$  so it suffices to show that their actions on these elements commute. Further, all the  $\{\partial_{r,M}\}$  commute with  $\partial_M$  and that reduces the problem to demonstrating for all  $r_1 \geq 1$  and  $r_2 \geq 1$  that

$$\partial_{r_1} \partial_{r_2}(M) - \partial_{r_2} \partial_{r_1}(M) = 0. \tag{2.2}$$

Using the Lax equations for  $M$  we get

$$\partial_{r_1} \partial_{r_2}(M) = \partial_{r_1}([C_{r_2}, M]) = [\partial_{r_1}(C_{r_2}), M] + [C_{r_2}, [C_{r_1}, M]]$$

and likewise

$$\partial_{r_2} \partial_{r_1}(M) = [\partial_{r_2}(C_{r_1}), M] + [C_{r_1}, [C_{r_2}, M]].$$

Since  $\text{ad}([C_{r_1}, C_{r_2}]) = [\text{ad}(C_{r_1}), \text{ad}(C_{r_2})]$  we see that the left hand side of equation (2.2) is equal to

$$[\partial_{r_1}(C_{r_2}) - \partial_{r_2}(C_{r_1}) - [C_{r_1}, C_{r_2}], M]$$

and, because  $M$  is a solution of the strict KP hierarchy, the left component of this commutator is zero by Theorem 2.1. This proves the statement in the proposition.

Let  $R_{sol}$  be the subalgebra of  $R$  consisting of the polynomial expressions in the  $\{\partial^s(m_{j+1})\}$  for all solutions  $M = \partial + \sum_{j=0}^{\infty} m_{j+1} \partial^{-j}$  of the strict KP hierarchy in the setting  $(R, \partial, \{\partial_r\})$ .  $R_{sol}$  is exactly the subalgebra of  $R$  that is of interest for finding solutions of this hierarchy. The derivations  $\partial$  and  $\partial_1$  are equal on  $R_{sol}$ , all the  $\{\partial_r\}$  map  $R_{sol}$  to itself and it follows by the same argument as in the proof of Proposition 2.1 that all the  $\{\partial_r\}$  commute on  $R_{sol}$ . Hence, by restricting  $R$  to  $R_{sol}$  one does not lose relevance for the strict KP hierarchy and the setting becomes standard. So we have

**Theorem 2.2.** *The setting  $(R_{sol}, \partial|R_{sol}, \{\partial_r|R_{sol}\})$  is standard.*

One advantage of a standard setting is that it allows another characterization of the equations of the hierarchy. Note thereto first of all that there is associated to a solution  $M$  of the strict KP-hierarchy still another set of pseudo differential operators that satisfy zero curvature relations. For, if we write for each  $r \geq 1$

$$D_r := -(M^r)_{\leq 0},$$

then we know that there hold respectively the Lax equations

$$\partial_r(M) = [C_r, M] = [D_r, M]$$

and in that light it is not surprising that the collection  $\{D_r\}$  satisfies

$$\partial_{r_1}(D_{r_2}) - \partial_{r_2}(D_{r_1}) - [D_{r_1}, D_{r_2}] = 0.$$

To show this, one takes the zero curvature equations for the  $\{C_r\}$ , one substitutes  $C_r = M^r + D_r$  and uses the Lax equations for the relevant powers of  $M$ . This yields

$$\begin{aligned} & \partial_{r_1}(D_{r_2} + M^{r_2}) - \partial_{r_2}(D_{r_1} + M^{r_1}) - [D_{r_1} + M^{r_1}, D_{r_2} + M^{r_2}] = \\ & \partial_{r_1}(D_{r_2}) - \partial_{r_2}(D_{r_1}) + \partial_{r_1}(M^{r_2}) - [D_{r_1}, M^{r_2}] - \partial_{r_2}(M^{r_1}) - [M^{r_1}, D_{r_2}] - [D_{r_1}, D_{r_2}] = \\ & \partial_{r_1}(D_{r_2}) - \partial_{r_2}(D_{r_1}) - [D_{r_1}, D_{r_2}] = 0. \end{aligned}$$

In a standard setting there holds also the reverse of the statement

**Theorem 2.3.** *Let  $M$  be a candidate solution to the strict KP-hierarchy in a standard setting. Then there holds that  $M$  is a solution of the strict KP-hierarchy if and only if all the  $\{D_r\}$  satisfy the zero curvature equations*

**P r o o f.** For each  $P \in \text{Psd}$ , we have  $\partial(P) = [\partial, P]$ . Hence, in a standard setting we have  $\partial_1(P) = [\partial, P]$ . Now we only need to show still that the zero curvature equations are sufficient. Thereto we use these equations for the case  $r_1 = 1$  respectively  $r_2 = r$  and we substitute  $D_1 = \partial - M$ . This yields

$$\begin{aligned} & \partial_1(D_r) - \partial_r(\partial - M) - [\partial - M, D_r] = \\ & \partial_1(D_r) - [\partial, D_r] + \partial_r(M) + [M, D_r] = \partial_r(M) - [D_r, M] = 0, \end{aligned}$$

which are the Lax equations one is looking for. This proves the result.

### 3. A minimal realization and scaling invariance

In this section we want to discuss a minimal realization of the equations (1.3) in the sense that there are a minimal number of relations between the coefficients of the potential solution  $M$  and their derivatives w.r.t.  $\partial$ . Therefore we start with a proper complex coefficient algebra  $\tilde{R}$  and a privileged  $k$ -linear derivation  $\tilde{\partial}$  of  $\tilde{R}$ . Keep in mind that any  $k$ -linear derivation  $\Delta$  of a polynomial ring  $k[x_s, s \in S]$  in any number of variables  $S$ , is determined uniquely by prescribing the images  $\Delta(x_s)$  of all the  $\{x_s\}$  thanks to the derivation property

$$\Delta(fg) = \Delta(f)g + f\Delta(g), \text{ for all } f \text{ and } g \in k[x_s].$$

Moreover, one can choose the  $\Delta(x_s)$  arbitrarily. This brings us to the choice

$$\tilde{R} := k[\tilde{m}_{j+1}^{(s)} \mid j \geq 0, s \geq 0]$$

of all polynomials in the unknown  $\{\tilde{m}_{j+1}^{(s)} \mid j \geq 0, s \geq 0\}$  with coefficients from  $k$  and the  $k$ -linear derivation  $\tilde{\partial}$  of  $\tilde{R}$  defined by

$$\tilde{\partial}(\tilde{m}_{j+1}^{(s)}) := \tilde{m}_{j+1}^{(s+1)}, \text{ all } j, j \geq 0, \text{ all } s \geq 0.$$

It is convenient to put a multiplicative grading on the monomials in the unknown of  $\tilde{R}$ , according to the prescription on their building blocks

$$\text{deg}(\tilde{m}_{j+1}^{(s)}) = j + 1 + s$$

and that gives a decomposition

$$\tilde{R} = \bigoplus_{s \geq 0} \tilde{R}^{(s)}, \text{ where } \tilde{R}^{(s)} \text{ is the span of the monomials of degree } s.$$

Then  $\tilde{\partial}$  is a  $k$ -linear map of order one w.r.t. the grading and hence each  $k$ -linear map  $r\tilde{\partial}^m, m \geq 0$ , with  $r$  a homogeneous element of degree  $p$  in  $\tilde{R}$ , maps  $\tilde{R}^{(s)}$  to  $\tilde{R}^{(s+p+m)}$ . Using this property, it is a straightforward verification that the pair  $(\tilde{R}, \tilde{\partial})$  is a proper starting point in the sense that

**Lemma 3.1.** *The action of  $\tilde{R}[\tilde{\partial}]$  on  $\tilde{R}$  is faithful.*

The grading on  $\tilde{R}$  extends to a grading on  $\tilde{R}[\tilde{\partial}]$  by assigning the order  $m + p$  to the  $\mathbb{C}$ -linear maps  $r\tilde{\partial}^m, r \in \tilde{R}^{(p)}$ . Likewise, we can call an element  $P$  of  $\tilde{R}[\tilde{\partial}, \tilde{\partial}^{-1}]$  *homogeneous of degree  $m$* , if  $P$  can be written as

$$P = \sum_{i \leq N} p_{m-i} \tilde{\partial}^i, \text{ with all } p_{m-i} \in \tilde{R}^{(m-i)}.$$

The multiplication rules in  $\tilde{R}[\tilde{\partial}, \tilde{\partial}^{-1}]$  are such that the product of two homogeneous elements of degrees  $m_1$  resp.  $m_2$  yields a homogeneous element of degree  $m_1 + m_2$ . This implies that all strict cut-off's  $\tilde{C}_i := (\tilde{M}^i)_{>0}, i \geq 1$ , are homogeneous as well. Note that  $\tilde{C}_1 = \tilde{\partial}$ . Next we try to find  $k$ -linear derivations  $\{\tilde{\partial}_i \mid i \geq 1\}$  of  $\tilde{R}$  that all commute with  $\tilde{\partial}$  and such that  $\tilde{M}$  becomes for the setting  $(\tilde{R}, \tilde{\partial}, \{\tilde{\partial}_i\})$  a solution of the Lax equations (1.3). Since they have to commute with  $\tilde{\partial}$ , there has to hold for all  $r \geq 1$ , all  $s \geq 0$  and all  $j, j \geq 0$  that

$$\tilde{\partial}_r(\tilde{m}_{j+1}^{(s)}) = \tilde{\partial}^s(\tilde{\partial}_r(\tilde{m}_{j+1}^{(0)})),$$

so that one merely has to prescribe the action of the derivation  $\tilde{\partial}_r$  on the set of coefficients  $\{\tilde{m}_{j+1}^{(0)} \mid j \geq 0\}$  of the differential operator  $\tilde{M}$ . Keeping our goal in mind, we have to define the action of  $\tilde{\partial}_r$  on  $\tilde{M}$  by

$$\tilde{\partial}_r(\tilde{M}) := [\tilde{C}_r, \tilde{M}]. \tag{3.1}$$

In this way  $\tilde{M}$  becomes by definition a solution of the strict KP hierarchy w.r.t. the setting  $(\tilde{R}, \tilde{\partial}, \{\tilde{\partial}_r\})$  and  $\tilde{M}$  together with the setting  $(\tilde{R}, \tilde{\partial}, \{\tilde{\partial}_r\})$  we call a *minimal realization of*



the strict KP hierarchy. Note that, since  $\tilde{R} = \tilde{R}(\tilde{M})$  the setting  $(\tilde{R}, \tilde{\partial}, \{\tilde{\partial}_r\})$  is standard by Proposition 2.1.

Next we describe in an algebraic way how other realizations of solutions of the strict KP hierarchy are related to this minimal realization. Consider any setting  $(R, \partial, \{\partial_r\})$  for this hierarchy and a potential solution  $M \in R[\partial, \partial^{-1}]$  of the form (1.2). Each pseudo differential operator  $M$  determines uniquely a  $k$ -algebra morphism

$$i_M : \tilde{R} \rightarrow R$$

by the prescription

$$i_M(\tilde{m}_{j+1}^{(s)}) = \partial^s(m_{j+1})$$

and this  $k$ -algebra morphism satisfies by definition

$$i_M \circ \tilde{\partial} = \partial \circ i_M. \tag{3.2}$$

The map  $i_M$  extends to a  $k$ -algebra morphism from the pseudo differential operators  $\tilde{R}[\tilde{\partial}, \tilde{\partial}^{-1}]$  to  $R[\partial, \partial^{-1}]$  such that

$$i_M(\tilde{M}) = M \text{ and } i_M(\tilde{C}_r) = C_r.$$

Assume now that  $M$  is a solution of the Lax equations of the strict  $n$ -KdV hierarchy, then we have for all  $r \geq 1$  that

$$\begin{aligned} \partial_r(M) &= \partial_r \circ i_M(\tilde{M}) = [C_r, M] = [i_M(\tilde{C}_r), i_M(\tilde{M})] = \\ &= i_M([\tilde{C}_r, \tilde{M}]) = i_M \circ \tilde{\partial}_r(\tilde{M}) \end{aligned}$$

Thus the  $k$ -linear maps  $\partial_r \circ i_M$  and  $i_M \circ \tilde{\partial}_r$  are equal on the coefficients of  $\tilde{M}$ , but, because of relation (3.2) and the fact that the derivations  $\{\partial_r\}$  commute with  $\partial$ , we get on  $\tilde{R}[\tilde{\partial}, \tilde{\partial}^{-1}]$  the compatibilities

$$\partial_r \circ i_M = i_M \circ \tilde{\partial}_r, \text{ for all } r \geq 1. \tag{3.3}$$

On the other hand, if the compatibilities (3.3) hold for the map  $i_M$ , then one applies these identities to  $\tilde{M}$  and, as  $i_M$  is a  $k$ -algebra morphism, we obtain the Lax equations for  $M$ . So we may conclude

**Lemma 3.2.** *The relations (3.3) for the map  $i_M$  are equivalent to  $M$  being a solution of the strict KP hierarchy w.r.t. the setting  $(R, \partial, \{\partial_r\})$ .*

Next we discuss the effect of the scaling transformations (1.4) in  $\tilde{R}[\tilde{\partial}]$ . Thereto we first make the substitutions

$$\tilde{\partial} = \alpha_1 \hat{\partial} \text{ and } \tilde{m}_{j+1} = \beta_j \hat{m}_{j+1}.$$

If  $\hat{R} = k[\hat{m}_{j+1}^{(s)} \mid j \geq 0, s \geq 0]$  with  $\hat{m}_{j+1}^{(s)} = (\hat{\partial})^s(\hat{m}_{j+1})$ , then  $\hat{R} = R$  and this substitution determines an isomorphism between  $\tilde{R}[\tilde{\partial}, \tilde{\partial}^{-1}]$  and  $\hat{R}[\hat{\partial}, \hat{\partial}^{-1}]$ . The element  $\tilde{M}$  expresses as follows in  $\hat{R}[\hat{\partial}, \hat{\partial}^{-1}]$

$$\tilde{M} = \alpha_1(\hat{\partial}) + \sum_{j=0}^{\infty} \beta_j \alpha_1^{-j} \hat{m}_{j+1}(\hat{\partial})^{-j}.$$

Hence, if we take from now on  $\beta_j = \alpha_1^{j+1}$  for all  $j \geq 0$ , then  $\tilde{M} = \alpha_1 \hat{M}$  with  $\hat{M} = (\hat{\partial}) + \sum_{j=0}^{\infty} \hat{m}_{j+1} (\hat{\partial})^j$  of the form (1.2). So, we get  $\tilde{C}_r = \alpha_1^r \hat{C}_r = \alpha_1^r (\hat{M}^r)_{>0}$  for all  $r \geq 1$ . Hence, under the present scaling transformation the Lax equations for  $\tilde{M}$  become

$$\tilde{\partial}_r(\tilde{M}) = \alpha_r \alpha_1 \hat{\partial}_r(\hat{M}) = [\tilde{C}_r, \tilde{M}] = \alpha_1^{1+r} [\hat{C}_r, \hat{M}]. \quad (3.4)$$

If we choose, moreover, all  $\alpha_r = \alpha_1^r$ , then the equations (3.4) show that  $\hat{M}$  is a solution of the strict KP hierarchy in the setting  $(\hat{R}, \hat{\partial}, \hat{\partial}_r)$ . Combining the considerations above leads to the following scaling invariance for solutions of the strict KP hierarchy

**Theorem 3.1.** *Let  $M$  solve the strict KP hierarchy in the setting  $(R, \partial, \{\partial_r\})$ . For  $\alpha \in \mathbb{C}^*$ , we consider the scaling transformation (1.4) with  $\alpha_r = \alpha^r$ ,  $r \geq 1$  and  $\beta_j = \alpha^{j+1}$ ,  $j \geq 0$ . Then substitution of this transformation into  $M$  yields an  $\hat{M} = \alpha^{-1} M$  in  $R\{\hat{\partial}, \hat{\partial}^{-1}\}$  that is a solution of the strict KP hierarchy in the setting  $(R, \hat{\partial}, \{\hat{\partial}_r\})$ . Hence, if the flow parameters in the original setting were  $s = \{s_r\}$ , then in the new setting  $(R, \hat{\partial}, \{\hat{\partial}_r\})$  the flow parameters become the  $\hat{s} = \{\hat{s}_r = \alpha^{-r} s_r\}$ .*

**Remark 3.1.** The scaling invariance of the strict KP hierarchy, as described in Theorem 3.1, offers the possibility to construct wave functions of the hierarchy corresponding to various spaces of boundary values on circles around the origin. In [5] one can find the construction for the  $L^2$ -boundary values on the unit circle. In a forthcoming paper we will treat various examples.

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