Finite-size effects in the nonphononic density of states in computer glasses

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The universal form of the density of nonphononic, quasilocalized vibrational modes of frequency $\omega$ in structural glasses, $D(\omega)$, was predicted theoretically decades ago, but only recently revealed in numerical simulations. In particular, it has been recently established that, in generic computer glasses, $D(\omega)$ increases from zero frequency as $\omega^4$, independent of spatial dimension and of microscopic details. However, it has been shown [Lerner and Bouchbinder, Phys. Rev. E 96, 020104(R) (2017)] that the preparation protocol employed to create glassy samples may affect the form of their resulting $D(\omega)$: glassy samples rapidly quenched from high-temperature liquid states were shown to feature $D(\omega) \sim \omega^4$ with $\beta < 4$, presumably limiting the degree of universality of the $\omega^4$ law. Here we show that exponents $\beta < 4$ are seen only in small glassy samples quenched from high-temperature liquid states—whose sizes are comparable to or smaller than the size of the disordered core of soft quasilocalized vibrations—while larger glassy samples made with the same protocol feature the universal $\omega^4$ law. Our results demonstrate that observations of $\beta < 4$ in the nonphononic density of states stem from finite-size effects, and we thus conclude that the $\omega^4$ law should be featured by any sufficiently large glass quenched from a melt.

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I. INTRODUCTION

The form of low-frequency spectra of glassy solids remains a subject of lively debates, despite decades of theoretical [1–7], numerical [8–15], and experimental research [16–19]. Low-frequency excitations are believed to be responsible for several important glassy phenomena, ranging from yielding [20–23] and shear banding [24–27] to anomalous wave attenuation [28]. Relations between low-frequency excitations and both relaxation in supercooled liquids [29–31] and low-temperature thermodynamics of glassy solids [32–34] have also been suggested.

Numerical investigations of computer glass models have contributed significantly to our understanding of many problems in glass physics and, in particular, to revealing the structural and statistical properties of soft, quasilocalized modes (QLMs) in glassy solids. Pioneered by Schober and coworkers decades ago [8,9], computational investigations have revealed that the lowest-frequency vibrational modes in computer glasses are quasilocalized: they consist of a disordered core, decorated by an algebraically decaying field away from the core [13,35,36]. Furthermore, it was discovered that the distribution $D(\omega)$ of these nonphononic modes follows a universal $\sim \omega^4$ law (with $\omega$ denoting the frequency), independent of spatial dimension [14] or microscopic details [13,15]. Remarkably, these findings are consistent with theoretical predictions put forward some time ago [1–3].

Universality of the nonphononic $D(\omega) \sim \omega^4$ density of states with respect to glass preparation protocol was also reported [37–39], but with some caveats; in particular, in Ref. [37] it was shown that glassy samples created by an infinitely fast quench from high-temperature liquid states, or by continuous quenches at very high rates, feature $D(\omega) \sim \omega^\beta$ with $\beta < 4$. In the same reference it was further suggested that the observation of $\beta < 4$ could be consistent with the Soft Potential Model [1,2], which states that very unstable states—presumably similar to those obtained by instantaneous overdamped quenches of high-temperature liquid states—could feature $\beta < 4$.

In this paper we show that the observations of $D(\omega) \sim \omega^\beta$ with $\beta < 4$ made in Ref. [37] are, in fact, the result of too-small glassy samples employed. The results presented here for $D(\omega)$ were obtained using the same computer glass model as employed in that reference. We prepared large ensembles of glassy samples by instantaneous quenches from high-temperature liquids states, and by gradually varying the system size $N$, we show in what follows that the exponent $\beta$ that characterizes the distribution of soft, nonphononic QLMs varies from $\beta < 4$ in small systems to $\beta = 4$ in larger systems. We further show that the characteristic length $\xi_g$ of soft QLMs—recently shown to be well represented by the spatial form of linear responses to localized force dipoles [36]—is comparable to the system sizes in which $\beta < 4$ is observed, suggesting that finite-size effects in $D(\omega)$ depend on the ratio $\xi_g/L$, with $L$ denoting the linear system size. We expect our results to serve as constraints on the formulation of future theories about the statistical mechanics of mesoscale elasticity in glassy solids.

II. MODELS, METHODS, AND UNITS

The model employed here is described in detail in, e.g., Ref. [38]; it is a binary mixture of “large” and “small” particles interacting via a radially symmetric, purely repulsive $\propto r^{-10}$ pairwise potential, with $r$ denoting the distance...
between pairs of particles. In what follows, we refer to the parent temperature \( T_p \) of an ensemble of glassy samples as the equilibrium temperature from which they were instantaneously quenched; see, e.g., Refs. [36,38]. Parent temperatures \( T_p \) are expressed in terms of the onset parent temperature \( T_{p \text{onset}} \) above which the sample-to-sample mean of the athermal shear modulus \( G \) [38] of underlying inherent states plateaus, as shown in Fig. 1. We express lengths in terms of \( L \equiv (V/N)^{1/3} \) where \( V = L^3 \) is the volume. The mass \( m \) is the same for all particles and is chosen as our microscopic units of mass. Elastic moduli are expressed in terms of the high-parent-temperature plateau \( G_{\infty} \) of the sample-to-sample mean athermal shear modulus (as seen in Fig. 1), frequencies in terms of \( \omega_0 \equiv a_0\sqrt{G_{\infty}/\rho} \) where \( a_0 = \sqrt{G_{\infty}/\rho} \) represents the speed of sound of glasses quenched from \( T_p > T_{p \text{onset}} \), and \( \rho = mN/V \) is the mass density.

Glassy samples were made by first equilibrating in an ensemble of glassy samples as seen in Fig. 1, frequencies associated with vibrational modes \( \omega \), which are solutions to the eigenvalue problem

\[
\mathcal{M} \cdot \psi_\omega = m\omega^2 \psi_\omega,
\]

where \( \mathcal{M} = \frac{\partial^2 U}{\partial x^2} \) is the Hessian of the potential \( U(x) \) evaluated at a local minimum, and \( x \) denotes the particles’ coordinates. For each member of our ensembles of glassy samples we calculated the lowest 100 vibrational frequencies using ARPACK [40]. A discussion regarding the conditions under which \( D(\omega) \) can be cleanly observed, without hindrance by hybridizations with phonons, can be found in Ref. [41].

### III. RESULTS

As mentioned above, in Ref. [37] it was shown that glassy samples quenched instantaneously from high parent temperatures \( T_p \) feature a density of nonphononic quasilocalized modes \( D(\omega) \sim \omega^\beta \) with \( \beta < 4 \). In Fig. 2 we show the low-frequency regime of the total density of states \( D(\omega) \) measured in our glassy samples. Importantly, we note that below the lowest phonon frequency \( 2\pi c_\infty/L \), the nonphononic and total density of states are the same, i.e., \( D(\omega) = D(\omega) \). Consistent with Ref. [37], we find \( \beta \approx 3.65 \) for \( N = 2048 \); however, upon increasing the system size we find \( \beta \approx 3.8 \) for \( N = 8192 \), and \( \beta \approx 4 \) for \( N \geq 32,768 \), in agreement with previous results [15,38,42].

What gives rise to this finite-size effect in the exponent \( \beta \)? We assert below that the finite-size effect seen in the exponent \( \beta \) is the result of soft QLMs not “fitting” well in the simulation box. To establish this, we build on the recent result [36] that the length that characterizes the linear response to local force dipoles agrees well with the core size of QLMs. We thus first aim at measuring the length that characterizes the response to local force dipoles in our glassy samples.

![Graph](image-url)
We expect to see a crossover to the continuum 

(b) Scaling the decay functions by 

distance to local force dipoles, as defined in the text, plotted against the
employed here are too small to exhibit the crossover to the expected
normalized the fields 

To this aim, we carry out the following procedure, also 

FIG. 3. (a) Decay functions 

to local force dipoles, as defined in the text, plotted against the
distance 

from the applied force dipole, for different system sizes as
reported in the legend. We expect to see a crossover to the continuum 

induced by deeper supercooling was attributed

to the suppression of phononic (plane-wave) modes.

Our findings strongly suggest that the universality of the 

IV. SUMMARY, DISCUSSION, AND OUTLOOK

In this work we have shown that generic computer glasses quenched quickly from high-parent-temperature liquid states feature the universal exponent \( \beta = 4 \) in their nonphononic density of states \( \mathcal{D}(\omega) \sim \omega^{-\beta} \)—if large-enough glassy samples are considered—and that smaller exponents \( \beta < 4 \) result from a finite-size effect. We further showed that \( \beta < 4 \) is seen when the system size is comparable to the core size of QLMs (represented by the response to local force dipoles [36]). Our findings correct the misinterpretation of numerical results presented and discussed in Ref. [37], where it was incorrectly concluded that exponents \( \beta < 4 \) are not a finite-size effect, but rather have to do with the physical process of glass formation by rapid quenches from high-temperature liquid states. Our results also cast doubts on the claims made in Ref. [47], where the change in \( \beta \) induced by deeper supercooling was attributed to the suppression of phononic (plane-wave) modes.

Our findings strongly suggest that the universality of the \( \omega^4 \) law extends to glasses formed from any parent temperature, conditioned that those glasses are sufficiently larger than the core size of QLMs. Importantly, in Ref. [36] it was shown that the core size of QLMs decreases with decreasing parent temperature, explaining why the \( \omega^4 \) law of the nonphononic density of states can be measured in smaller glassy samples quenched from low parent temperatures, or quenched at slow rates into the glassy phase.
Further support for the physical picture proposed in this work was put forward in Ref. [35], where it was shown that the far, algebraically decaying field of QLMs stabilizes their energetically unstable core. It is therefore reasonable to expect that if QLMs are deprived from their far fields—as expected to occur in small glassy samples—those QLMs will be less stable and have lower frequencies, possibly leading to $\beta < 4$ in their nonphononic density of states.

Finally, we propose that the finite-size effect reported here, in which a system constrains itself by virtue of its small size, giving rise to reduced stability manifested by the appearance of soft modes, might be an instance of a more general phenomenon. For example, in Ref. [48] it was reported that small packings of frictionless hard spheres possess relatively fewer small gaps between nearby particles that are not in contact, compared to the abundance of small gaps in larger packings. Since smaller gaps are known to stabilize hard-sphere packings [48], that phenomenon bears similarities to the reduced stability that small computer glasses possess, as reported here.

In light of the findings reported here, it seems important to explore which additional finite-size effects on QLMs appear in computer glasses, and how those effects influence our insights and conclusions regarding the properties of macroscopic structural glasses.

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