Supplemental Material for: Off-axis trapping in optical tweezers by an optical analog of the Magnus effect

Robert J.C. Spreeuw
Van der Waals-Zeeman Institute, Institute of Physics, University of Amsterdam, PO Box 94485, 1090 GL Amsterdam, The Netherlands
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SPATIAL VS. ANGULAR k-SPACE COORDINATES

In this paper we express all fields by their angular spectrum $\mathbf{E}(\Omega) = \mathbf{E}(\theta, \phi)$. This is usually defined for fields propagating out into a half space $z \geq 0$ (see Ch. 3.2 in [S1]), as is clearly the case for the incident beams. The relationship with the field in the plane $z = 0$ is given by

$$\tilde{\mathbf{E}}(x, y, 0) = k^2 \iint_{\theta \leq \pi/2} \mathbf{E}(\Omega) \sin\theta \, d\theta d\phi \qquad \text{(S1)}$$

with $k = \omega/c$ the laser wave vector. For the Gaussian beam with angular waist w_θ the above equation yields the familiar Gaussian beam cross section, with minimum waist $w_0 = \lambda/\pi w_\theta$, see also Ch. 5 in [S1]. The angular tophat beam approximates the output of a uniformly illuminated circular lens, and Eq. (S1) yields the resulting Airy pattern in the focal plane $z = 0$.

Although the emission by a dipole is not confined to $z \geq 0$, the angular representation of the radiation pattern of a dipole $\mathbf{p}$ in a direction $\mathbf{u}_\Omega$ is well known to be given by Eq. (10) (main text), see for example Ch. 9 in [S2]. Only the radiating, or 'far field', terms ($\sim r^{-1}$) are relevant in our case, because one can evaluate the beam deflection at arbitrarily large distance of the dipole, where the near fields ($\sim r^{-2}, r^{-3}$) have become negligible.

In the plane ($\phi = 0$) of a $\mathbf{u}_\pm$ dipole,

$$(\mathbf{u}_\Omega \times \mathbf{u}_\pm) \times \mathbf{u}_\Omega = \frac{e^{\pm i\theta}}{\sqrt{2}} (\cos\theta, 0, -\sin\theta) \qquad \text{(S2)}$$

shows the spiral wave character in the prefactor $e^{\pm i\theta}$.

The factor i in Eq. (10) (main text) is a crucial detail. It is a consequence of expressing the spherical waves e^{ikr}/r of the dipole field as an angular spectrum of plane waves. The same factor i can be recognized in the Weyl representation of a diverging spherical wave [S1]. In the case at hand, one can readily see that it also ensures that a resonant beam is attenuated (absorbed) in the forward direction, due to destructive interference of incident and scattered waves.

The phase factor $e^{i\alpha}$ in Eq. (10) (main text) follows from the steady state of the optical Bloch equations [S3]. In a two-level atom with states e, g, the induced dipole moment is given by the off-diagonal density matrix element ρ_{eg}. If the atom is driven at detuning Δ by a monochromatic field with (real) amplitude $\mathcal{E}_0$ the steady state (for $s \ll 1$) is given by

$$\rho_{eg} = \frac{i}{2} \frac{D\mathcal{E}_0/\hbar}{\gamma - i\Delta} \qquad \text{(S3)}$$

which has a complex argument $\alpha = \arg \rho_{eg}$ given by $\cot\alpha = -\Delta/\gamma$. Here, since we choose an x polarized incident wave, α is the phase of the p_x component of the dipole, relative to the incident field.

LOW SATURATION LIMIT

In the definition of the saturation parameter s we include the detuning, following [S3],

$$s = \frac{I/I_0}{1 + \Delta^2/\gamma^2} \qquad \text{(S4)}$$

with I the intensity and $I_0 = 2\pi hc\gamma/3\lambda^3$ the saturation intensity. In the low-saturation limit, characterized by $s \ll 1$, the scattered light is almost entirely coherent, with a small incoherent fraction equal to $s/(1+s)$. In optical tweezer experiments, using far off-resonant laser beams, typical values for s are in the range $10^{-6} - 10^{-8}$, so that $s \ll 1$ is indeed well fulfilled and the incoherent scattering rate is low.

FIELD AMPLITUDES

The peak amplitudes $\mathcal{E}_0^{(\mathrm{G})}, \mathcal{E}_0^{(\Pi)}$ are related to the total power in the incident beam by

$$P = \int J_{\mathrm{in}}(\Omega) \, d\Omega = \begin{cases} \approx \frac{(\mathcal{E}_0^{(\mathrm{G})})^2}{2Z_0} \times \frac{\pi w_\theta^2}{2} \\ \frac{(\mathcal{E}_0^{(\Pi)})^2}{2Z_0} \times 2\pi\,(1 - \cos r_\theta) \end{cases} \qquad \text{(S5)}$$

for the Gaussian and angular tophat beam, respectively. The integrals were performed using Mathematica software [S4]. For the Gaussian, the equality is only approximate, we give here the leading term of a power series in w_θ. The above expressions have been written as a product of the forward ($\theta = 0$) radiant intensity $\mathcal{E}_0^2/2Z_0$ and an effective solid angle.

The average wave vector of the incident beams is shorter than the corresponding value for a plane wave,

$$\langle \mathbf{k} \rangle_{\rm in} = k\hat{\mathbf{z}} \times \begin{cases} 1 - \frac{w_\theta^2}{4} + \mathcal{O}(w_\theta^4) & \text{(Gauss)} \\ \cos^2\left(\frac{r_\theta}{2}\right) & \text{(tophat)} \end{cases} \tag{S6}$$

The amplitude ratios $\mathcal{E}_{\rm sc}/\mathcal{E}_0^{(\rm G)}$ and $\mathcal{E}_{\rm sc}/\mathcal{E}_0^{(\Pi)}$ can be obtained from the energy conservation condition

$$\int \left[J_{\rm if}(\Omega) + J_{\rm sc}(\Omega)\right] d\Omega = 0 \tag{S7}$$

The scattering term $J_{\rm sc}(\Omega) > 0$ would increase the outflowing power, which must be cancelled by the interference term $J_{\rm if}(\Omega)$. As a result,

$$\frac{\mathcal{E}_{\rm sc}}{\mathcal{E}_0^{(\rm G)}} \approx \frac{3\sin\alpha}{4\sqrt{2}}\, w_\theta^2 \tag{S8}$$

$$\frac{\mathcal{E}_{\rm sc}}{\mathcal{E}_0^{(\Pi)}} = \frac{3\sin\alpha}{4\sqrt{2}}\, \sin^2\left(\frac{r_\theta}{2}\right)(\cos r_\theta + 3) \tag{S9}$$

where in the Gaussian case the leading order in w_θ is given.

With these ratios the interference terms in the radiant intensity, Eq. (5) (main text) can be obtained as

$$J_{\rm if}(\Omega) = -\frac{\mathcal{E}_{\rm sc}}{\sqrt{2}Z_0}\, f(\Omega,\Delta) \times \begin{cases} \sim \mathcal{E}_0^{(\rm G)} e^{-\theta^2/w_\theta^2} \\ \mathcal{E}_0^{(\Pi)}\, \Pi(\theta/2r_\theta) \end{cases} \tag{S10}$$

with

$$f(\Omega,\Delta) = \frac{\gamma\left(\cos\theta\cos^2\phi + \sin^2\phi\right) - \Delta\sin\theta\cos\phi}{\sqrt{\gamma^2+\Delta^2}} \tag{S11}$$

For the deflection, expressed as $\delta\langle\mathbf{k}\rangle = \langle\mathbf{k}\rangle - \langle\mathbf{k}\rangle_{\rm in}$ we evaluate the integral of Eq. (7) (main text) to obtain

$$\delta\langle\mathbf{k}\rangle = \frac{3k}{4}\frac{\gamma\Delta}{\gamma^2+\Delta^2}\left(w_\theta^4 + \mathcal{O}(w_\theta^5), 0, -2\frac{\gamma}{\delta}w_\theta^2 + \mathcal{O}(w_\theta^4)\right) \tag{S12}$$

for the Gaussian beam, and

$$\delta\langle\mathbf{k}\rangle = \frac{3k}{16}\frac{\gamma\Delta}{\gamma^2+\Delta^2}\left(r_\theta^4 + \mathcal{O}(r_\theta^5), 0, -4\frac{\gamma}{\delta}r_\theta^2 + \mathcal{O}(r_\theta^4)\right) \tag{S13}$$

for the angular tophat beam.

Note that in both cases the y component is absent. To leading order, the deflection angle is just given by

$$\delta\theta \approx \frac{(\delta\langle\mathbf{k}\rangle)_x}{k} \tag{S14}$$

which leads to Eq. (14) of the main text.

CALCULATION FOR A DISPLACED ATOM

When the atom is located at a position $\mathbf{d}$ away from the origin, the angular components of the scattered wave are phase shifted by an amount $\exp(-ik\mathbf{u}_\Omega\cdot\mathbf{d})$, so that the interference term, Eq. (5) (main text), is modified to

$$J_{\rm if}(\Omega) = \frac{1}{2Z_0}\left[\mathbf{E}_{\rm in}^*(\Omega)\cdot\mathbf{E}_{\rm sc}^{(\rm coh)}(\Omega)e^{-ik\mathbf{u}_\Omega\cdot\mathbf{d}} + c.c.\right] \tag{S15}$$

For a displacement along x, we have $\mathbf{d} = d\hat{\mathbf{x}}$ so that

$$k\mathbf{u}_\Omega\cdot\mathbf{d} = kd\,\mathbf{u}_\Omega\cdot\hat{\mathbf{x}} = kd\sin\theta\cos\phi \tag{S16}$$

In the integrals $\int J_{\rm if}(\Omega)\,d\Omega$ and $\int \mathbf{u}_\Omega\, J_{\rm if}(\Omega)\,d\Omega$, we develop the integrand in a power series of kd, up to fourth order and integrate the terms separately.

For the amplitude ratios $\mathcal{E}_{\rm sc}/\mathcal{E}_0^{(\rm G)}$ and $\mathcal{E}_{\rm sc}/\mathcal{E}_0^{(\Pi)}$ we find that their lowest order ($\sim w_\theta^2$ and $\sim r_\theta^2$) is not affected by kd. For the deflection angle the leading order in w_θ, r_θ is still fourth order, and up to order $(kd)^4$ the angle is

$$\delta\theta \approx \frac{3}{4}(1 \mp kd)\frac{\gamma\Delta}{(\gamma^2+\Delta^2)} \times \begin{cases} w_\theta^4 & \text{(Gauss)} \\ r_\theta^4/4 & \text{(tophat)} \end{cases} \tag{S17}$$

for a $\mathbf{u}_\pm$ dipole, respectively. This shows that the deflection angle, and thus also the transverse force, vanishes if the atom is displaced by an amount $d = \pm k^{-1} = \pm\lambda$.

[S1] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995) Ch. 3 and 5.

[S2] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, NY, 1999) Ch. 9 and 10.

[S3] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Process and Applications* (Wiley-VCH, New York, 1998).

[S4] Wolfram Research, Inc., Mathematica, Version 12.0, Champaign, IL (2020).