Supplemental Material for "Quantum many-body scars in transverse field Ising ladders and beyond"

Bart van Voorden,* Jiří Minář, and Kareljan Schoutens
Institute for Theoretical Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, the Netherlands

I. DEGENERACY OF THE EIGENSTATES

A. Direct counting

In this section we evaluate the degeneracies of the eigenstates of the Ising ladder without the transverse field. Consider the Hamiltonian

$$H = H_z = \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z \tag{S1}$$

acting on the basis states (2) given in terms of local occupation numbers ($|0\rangle$ and $|1\rangle$) on each site. As noted in the main text, two neighbouring sites with equal (unequal) occupation number will contribute energy $+1$ (-1). It is convenient to introduce the following symbols for the four possible configurations of a rung, see Fig. S1, namely

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ A' = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ B' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{S2}$$

with respective energies -2,-2,+2,+2 due to periodic boundary conditions (PBC). Each basis state thus corresponds to a string formed by letters from the set $\{A, A', B, B'\}$ so that we have reduced the problem of counting the degeneracies of the state of a given energy to enumerating the number of strings yielding that energy subject to the following rules. We denote the set $\{A, A'\}$ as the $A-$subset and $\{B, B'\}$ as the $B-$subset. On top of the energies associated with the rungs, there are also energies associated to the links connecting two letters. When two letters are the same, the link contributes $E = +2$ (e.g. $A - A$), two different letters of the same subset contribute $E = -2$ (e.g. $A - A'$) and links between letters from different subsets contribute $E = 0$ (e.g. $A - B$).

The extremal energies take values $E = \pm 4L$ and the spectrum is symmetric. Focusing on $E = 4L$ states, they are two-fold degenerate and, as stated in the main text, they correspond to all sites occupied by either 0 or 1 spins. Using the letter notation, they correspond to $B - B - \ldots - B-$ and $B' - B' - \ldots - B'-$ respectively. Here, the last link represents the link connecting the last and first site due to the PBC. Each basis state thus corresponds to a one-dimensional graph, where both the vertices (the letters) and the edges (the links) carry energy. This motivates us to introduce a symbolic two-line notation, where the lower line describes the energy of the vertices and the upper line that of the edges. For instance, considering $L = 4$ we have for the $E = 4L$ states

$$B - B - B - B- \ \cong\ B' - B' - B' - B'- \ \cong\ \begin{pmatrix} +_{12} & +_{23} & +_{34} & +_{41} \\ +_1 & +_2 & +_3 & +_4 \end{pmatrix}, \tag{S3}$$

where the indices i in $+_i$ label the sites and $+_{ij}$ denotes the energy of the link connecting sites i and j. In what follows we drop these indices for simplicity and just use the symbols $\{+, -, 0\}$ to denote an energy contribution of respectively $\{+2, -2, 0\}$. We note that the assignment in Eq. (S3) is not unique and the matrix notation represents an equivalence class of graphs, whose energy pattern is identical up to the translations $(T_x)^k$. The notation (S3) is useful mainly for bookkeeping purposes. To demonstrate this, we next evaluate the degeneracies of the first de-excited state of $E = 4L - 8$.

There are two different ways in which a state with $E = 4L - 8$ can be created starting from Eq. (S3). Firstly, replacing one B by an A and secondly by changing a string of l consecutive B-letters by a string of its complement in the B-subset (i.e. $B \to B'$ or $B' \to B$ within the string of length l). Taking $L = 4$ as an example, the states with $E = 4L - 8$ are

$$\begin{pmatrix} 0 & + & + & 0 \\ - & + & + & + \end{pmatrix}\ \deg = 16, \qquad \begin{pmatrix} - & - & + & + \\ + & + & + & + \end{pmatrix}\ \deg = 8, \qquad \begin{pmatrix} - & + & - & + \\ + & + & + & + \end{pmatrix}\ \deg = 4, \tag{S4}$$

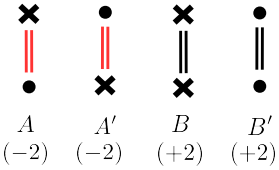

FIG. S1. Possible configurations of the rungs of the ladder with the respective energies indicated in parenthesis. The black dot (cross) stands for 0 (1) spin, the double lines are due to the periodic boundary conditions and their colour denote the contribution to the energy - each red (black) line contributes energy -1 (+1).

where the numbers denote the respective degeneracies. For instance, the first class has L-fold degeneracy due to translation and there are two possibilities to choose the A-letter and two possibilities for the string of the + vertices, i.e. $L \times 2 \times 2 = 4L$. Similar considerations for the two remaining classes lead to a general formula for the degeneracies of the first (de-)excited state

$$\begin{aligned} \deg_{E=4L-8} &= \deg_{E=-4L+8} \\ &= 2\left[2L + L\left(\frac{L}{2} - 1\right) + \frac{L}{2}\right] \\ &= L^2 + 3L. \end{aligned} \tag{S5}$$

B. Generating function

In principle one can extend the above described direct counting to other energy manifolds, but the method becomes quickly cumbersome with increasing system size. Here we describe a recursive construction of a generating function (partition sum)

$$Z(x) = \sum_n a_n x^n, \tag{S6}$$

where the coefficients a_n are the degeneracies of a manifold of energy n. Lets first consider an *open* chain which starts with a letter A of energy -2. We thus attribute an element x^{-2} to this configuration. One can now add one of the four possible letters to the right of A: A (0), A' (-4), B (+2), B' (+2) and similarly for having either A', B or B' at the beginning of the chain. Here the numbers in the brackets denote the energy contribution from adding the given letter, i.e. the power of x. This method can be continued recursively by the following prescription

$$Z_{A\alpha}^{(L+1)} = \left(1 + x^{-4}\right) Z_{A\alpha}^{(L)} + 2x^{-2} Z_{A\beta}^{(L)} \tag{S7a}$$

$$Z_{A\beta}^{(L+1)} = 2x^2 Z_{A\alpha}^{(L)} + \left(1 + x^4\right) Z_{A\beta}^{(L)}, \tag{S7b}$$

with the seed $(Z_{A\alpha}, Z_{A\beta}) = (x^{-2}, 0)$. Here $Z_{A\alpha} = Z_{AA} + Z_{AA'}$, $Z_{A\beta} = Z_{AB} + Z_{AB'}$ and $Z_{A\mu}^{(L)}$ is a generating function of order L in the recursion corresponding to adding a letter $\mu = A, A', B, B'$ to the right of the chain. Considering only A-letters, expressions analogous to Eqs. (S7) can be obtained for the respective generating functions

$$Z_{AA}^{(L+1)} = Z_{AA}^{(L)} + x^{-4} Z_{AA'}^{(L)} \tag{S8a}$$

$$Z_{AA'}^{(L+1)} = x^{-4} Z_{AA}^{(L)} + Z_{AA'}^{(L)} \tag{S8b}$$

For later convenience we define

$$\delta Z_{A\alpha}^{(L)} \equiv Z_{AA}^{(L)} - Z_{AA'}^{(L)} = x^{-2}(1 - x^{-4})^{L-1}. \tag{S9}$$

The periodic boundary condition is then implemented by the mappings

$$Z_{AA} \to x^2 Z_{AA},\ Z_{AA'} \to x^{-2} Z_{AA'},\ Z_{AB} \to Z_{AB},\ Z_{AB'} \to Z_{AB'}. \tag{S10}$$

Analogous mappings hold for A' instead of A at the beginning of the chain. Starting with B, B', analogous mapping to (S10) holds again, with $x \to x^{-1}$, provided L is *even*, which is the case of our interest. This results in the expression for the generating function of a chain of length L

$$\begin{aligned} Z^{(L)} &= 2\left(f(x) + f(x^{-1})\right) \\ f(x) &= x^2 Z^{(L)}_{AA} + x^{-2} Z^{(L)}_{AA'} + Z^{(L)}_{A\beta}, \end{aligned} \tag{S11}$$

which can be readily evaluated with the help of the relations (S7) and (S9). This result can be however further manipulated as follows. First we note, that casting the relation (S7) in the matrix form, it can be diagonalized as

$$\vec{Z}^{(L+1)} = M\vec{Z}^{(L)} = \begin{pmatrix} 1+x^{-4} & 2x^{-2} \\ 2x^2 & 1+x^4 \end{pmatrix} \vec{Z}^{(L)} \;\to\; v_\pm^{(L+1)} = \lambda_\pm v_\pm^{(L)}, \tag{S12}$$

where $\vec{Z}^{(L)} = (Z^{(L)}_{A\alpha}, Z^{(L)}_{A\beta})^{\mathrm{T}}$,

$$\lambda_\pm = 2x^{-4}\left(1 + 2x^4 + x^8 \pm s\right),\; s = \sqrt{1 + 14x^8 + x^{16}} \tag{S13}$$

are the eigenvalues of M and $v_\pm$ the corresponding eigenvectors. Expressing $Z_{A\alpha}, Z_{A\beta}$ in terms of $v_\pm$, substituting to (S11) and using again (S9), after some algebraic manipulations we obtain a closed-form expression for the generating function

$$Z^{(L)} = \frac{s}{2}\left(\lambda_+^{L-1} - \lambda_-^{L-1}\right) + \frac{1}{2}\left(x^{-4} + 2 + x^4\right)\left(\lambda_+^{L-1} + \lambda_-^{L-1}\right) + \left(x^{-2L} + x^{2L}\right)\left(x^2 - x^{-2}\right)^L. \tag{S14}$$

Applying first (S14) to the example of the ladder of $L = 4$ we find

$$Z^{(L=4)} = 2\left(x^{-16} + 14x^{-8} + 24x^{-4} + 50 + 24x^4 + 14x^8 + x^{16}\right), \tag{S15}$$

which gives $\deg_{E=\pm 4L} = 2$ and $\deg_{E=\pm(4L-8)} = 28$ as it should, cf. Eq. (S5). Furthermore, we can now evaluate the degeneracy of the $E = 0$ manifold, which gives $\deg_{E=0} = 100$ and for $L = 6, 8$ studied in this work we get $\deg_{E=0} = 1188, 15876$.

II. AUTOCORRELATION OF THE (A)TYPICAL STATES

The difference between a typical state and an atypical $|\mathbb{Z}_2\rangle$ state results in revivals of the autocorrelation $A_{|\psi_0\rangle}(t) = |\langle\psi_0|e^{-iHt}|\psi_0\rangle|^2$, Eq. (6), for the latter in contrast to the former. Writing the initial state as a superposition of the eigenstates $|\psi_0\rangle = \sum_j c_j |v_j\rangle$, the autocorrelation can be expressed as

$$A(t)_{|\psi_0\rangle} = \sum_j |c_j|^4 + 2\sum_{j<k} |c_j|^2 |c_k|^2 \cos\left(E_{jk} t\right), \tag{S16}$$

where $E_{jk} = E_j - E_k$ is the energy difference between eigenenergies of the eigenstates v_j, v_k. The properties of the autocorrelation thus immediately follow from the weights $w_j = |c_j|^2$ as well as the energy differences E_{jk} between the states with non-zero weights. First we analyze the distribution of the weights for each basis state $|\psi_0\rangle = |b_i\rangle\,, \forall i$ by ordering them from high to low values such that $w_j \geq w_{j+1}$. We can then quantify the corresponding distribution of the weights using the variance

$$\mathrm{var}_{|b_i\rangle} = \sum_j (j - \bar{n})^2 w_j^{(i)} \text{ with } \bar{n} = \sum_j j w_j^{(i)}, \tag{S17}$$

where $|b_i\rangle = \sum_j c_j^{(i)} |v_j\rangle$. In Fig. S2 we show the ordered variances for $L = 8$ and for basis states with potential $|V| = |\langle b_i|H_z|b_i\rangle| \leq 8$ as we are interested in the mid-spectrum states. As can be seen, the $\mathbb{Z}_2$ state and the typical state (7) that were used in Fig. 3, have respectively a low and high variance. The basis states with the lowest variance belong to the (twin) peak states, including the ones used in the main text [Eq. (15) and (16)]. We note that the low variance of the weights alone is not sufficient to guarantee quasiperiodic revivals of the autocorrelation (so that not all low-variance states would be classified as QMBS) and has to be supplemented by the quasi equidistant energy spacings between the eigenstates with the highest weight as in the case of the $|\mathbb{Z}_2\rangle$ state.

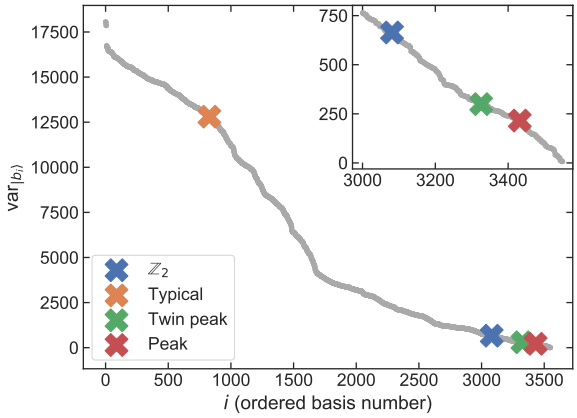

FIG. S2. The weight variances for all basis states with $V = 0, \pm 4, \pm 8$ of the $L = 8$ and $h_x = 0.1$ system. The four specific states considered in the text are denoted by a cross. The inset shows the lowest variance states.

A. Autocorrelation vs. L and h_x

Here we study numerically the dependence of the average autocorrelation $\langle A \rangle = \lim_{\tau\to\infty} \tau^{-1} \int_{t_0}^{\tau} A(t)\, dt$ on the system size L and the transverse field h_x, where we choose $t_0 > 0$ such that the initial decay of $A(t)$ is not included in the average and τ large enough that $\langle A \rangle$ becomes τ-independent. In Fig. S3a we show $\langle A \rangle$ vs. h_x for $L = 4, 6, 8$ (blue, orange, green) for both an atypical ($|\mathbb{Z}_2\rangle$, solid lines) and a typical ($|\psi_{\text{typ}}\rangle$, Eq. (7), dashed lines) initial state. We see the decrease of the average autocorrelation with h_x for all system sizes, which is compatible with the decrease of the fidelity of the SMA states, describing idealized scar behaviour, with respect to the system eigenstates, cf. Fig. 4.

Comparing $\langle A \rangle$ for the $|\mathbb{Z}_2\rangle$ and the typical state, we see that $\langle A_{|\mathbb{Z}_2\rangle} \rangle > \langle A_{|\psi_{\text{typ}}\rangle} \rangle$ for all values of h_x and L (except for $L = 4$ and $h_x = 0.1$, which is likely a finite-size effect). In Fig. S3b we plot $\langle A_{|\mathbb{Z}_2\rangle} \rangle / \langle A_{|\psi_{\text{typ}}\rangle} \rangle$ vs. $1/L$ for $h_x = 0.1, 0.5, 0.9$ and see a clear increase of the ratio with the system size. We have verified that the higher value of $\langle A_{|\mathbb{Z}_2\rangle} \rangle$ can be indeed attributed to the revivals of $A_{|\mathbb{Z}_2\rangle}(t)$. At the same time, as can be seen in Fig. S3c, $\langle A_{|\mathbb{Z}_2\rangle} \rangle$ decreases with L, similar to results on the PXP model[18]. In conclusion, while the quantitative details depend on the choice of the typical state, the numerical simulations suggest that the atypical behaviour of the $|\mathbb{Z}_2\rangle$ states is robust in the sense that the ratio $\langle A_{|\mathbb{Z}_2\rangle} \rangle / \langle A_{\text{typ}} \rangle \gg 1$ for all h_x and system sizes L and in fact increases with system size. Further studies, including e.g. the use of SMA construction or Mazur inequalities[41,85–87] are required to assess the nature of the autocorrelations in the thermodynamic limit.

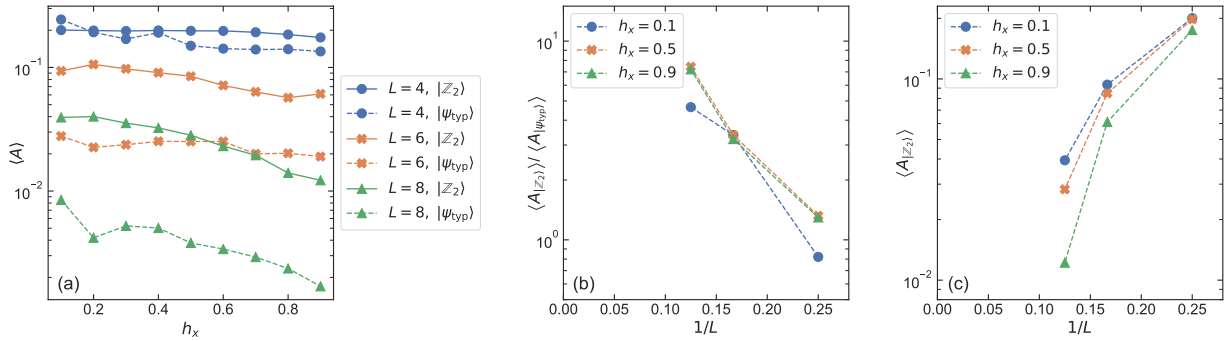

FIG. S3. (a) The average autocorrelation for a $|\mathbb{Z}_2\rangle$ state (solid line) and the typical state $|\psi_{\text{typ}}\rangle$ Eq. (7) (dotted line) for $L = 4, 6, 8$ (blue dots, orange crosses, green triangles) as a function of h_x. (b) The ratio $\langle A_{|\mathbb{Z}_2\rangle} \rangle / \langle A_{|\psi_{\text{typ}}\rangle} \rangle$ and (c) $\langle A_{|\mathbb{Z}_2\rangle} \rangle$ as a function of $1/L$ for $h_x = 0.1, 0.5, 0.9$. In all three subplots the same data is used with $t_0 = 133$ and $\tau = 500$.

III. PROOF OF ZERO ENERGY STATE

To show that $|\psi'_{E=0}\rangle$, Eq. (8), is a zero energy eigenvector independent of h_x, i.e. $H|\psi'_{E=0}\rangle = 0$, we first consider the action of H_z by expanding it as $H_z = \sum_i H_i^z$ with

$$H_i^z \equiv \sigma_{i,1}^z \sigma_{i+1,1}^z + 2\,\sigma_{i+1,0}^z \sigma_{i+1,1}^z + \sigma_{i+1,0}^z \sigma_{i+2,0}^z. \tag{S18}$$

H_i^z is a local operator that only acts on the sites occupied by two neighbouring singlets s_i and s_{i+1} such that

$$H_z|\psi'_{E=0}\rangle = \sum_i H_i^z |\psi'_{E=0}\rangle = \sum_i \left(\prod_{j\neq i,i+1} s_j\right) H_i^z s_i s_{i+1}|\emptyset\rangle\,. \tag{S19}$$

The product of the two singlets written out in basis states is

$$\begin{aligned} s_i s_{i+1}|\emptyset\rangle &= \left(c_{i,1}^\dagger - c_{i+1,0}^\dagger\right)\left(c_{i+1,1}^\dagger - c_{i+2,0}^\dagger\right)|\emptyset\rangle \\ &= \left|\begin{smallmatrix}1&1&\cdot\\ \cdot&0&0\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&0&\cdot\\ \cdot&0&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}0&1&\cdot\\ \cdot&1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&\cdot\\ \cdot&1&1\end{smallmatrix}\right\rangle, \end{aligned} \tag{S20}$$

where in the last line only the occupation numbers of the relevant positions (i.e. those occupied by the singlets s_i and s_{i+1}) are denoted. In each of these basis states there are two neighbouring pairs with the same and two pairs with a different occupation number, such that the action of H_i^z on all these basis states is zero: $H_i^z s_i s_{i+1}|\emptyset\rangle = 0$. Entering this result in Eq. (S19) then shows that $H_z|\psi'_{E=0}\rangle = 0$. Secondly, consider the action of H_x in a similar way by using the expansion $H_x = \sum_i H_i^x$ with $H_i^x = \sigma_{i,1}^x + \sigma_{i+1,0}^x$ that only acts on the sites of a single singlet. Then

$$H_x|\psi'_{E=0}\rangle = h_x \sum_i \left(\prod_{j\neq i} s_j\right) H_i^x s_i|\emptyset\rangle \tag{S21}$$

and

$$\begin{aligned} H_i^x s_i|\emptyset\rangle &= (\sigma_{i,1}^x + \sigma_{i+1,0}^x)(c_{i,1}^\dagger - c_{i+1,0}^\dagger)|\emptyset\rangle \\ &= (1 - 1 + c_{i,1}^\dagger c_{i+1,0}^\dagger - c_{i,1}^\dagger c_{i+1,0}^\dagger)|\emptyset\rangle \\ &= 0. \end{aligned} \tag{S22}$$

Because of Eq. (S22) it follows that $H_x|\psi'_{E=0}\rangle = 0$ and therefore $H|\psi'_{E=0}\rangle = 0$. This concludes the proof that the state (8) is indeed a zero energy state, independent of h_x.

IV. OTHER h_x-INDEPENDENT EIGENSTATES

While the $|\psi'_{E=0}\rangle$ state is an exact eigenstate of the Hamiltonian (1), we could identify other eigenstates, which are independent of the transverse field even for $E \neq 0$. For example, for $L = 4$, non-degenerate eigenstates exist with energy $E = \pm 4$, which are explicitly

$$|\psi_{E=-4}\rangle = \sum_{i,j,r,q} T_x^i T_y^j R^r Q^q (-1)^r \left|\begin{smallmatrix}1&1&1&0\\ 1&0&0&1\end{smallmatrix}\right\rangle, \tag{S23}$$

$$|\psi_{E=4}\rangle = \sum_{i,j,r,q} T_x^i T_y^j R^r Q^q (-1)^r \left|\begin{smallmatrix}1&1&0&0\\ 1&0&0&0\end{smallmatrix}\right\rangle, \tag{S24}$$

with T_x and T_y the translation operator, R the reflection operator ($c_{x,y}^\dagger \to c_{L-x,y}^\dagger$) and Q the particle-hole inversion operator ($n_{x,y} \to 1 - n_{x,y}$), with $r \in \{0,1\}$ and $q \in \{0,1\}$. Additionally, we could identify h_x-independent $E = \pm 4$ ($E = -4$) eigenstates for a $L = 3$ ladder ($L_x = 3, L_y = 4$ system). However, analogous $E = \pm 4$ eigenstates do not seem to exist for larger system sizes. Moreover, we do not consider these eigenstates any further, because in contrast to the $|\psi_{E=0}\rangle$ state, single mode approximation excitations on top of these states do not lead to non-thermalizing behaviour.

V. SYSTEMATICS FOR COMPUTING ENERGIES OF THE SMA STATES

In this section we apply the forward scattering approximation to systematically construct approximate eigenstates $|w_{(j)}\rangle$, with j denoting the order of the approximation, to the exact eigenstates $|v'_{\mathrm{QMBS},n}\rangle$ of H', Eq. (3), $H' = \sum_{\langle i,j\rangle} \sigma_i^z \sigma_j^z + h_x \sum_i P\sigma_i^x P$. We are interested on the action of H' on the space that is projected onto the states with $V = 0$ in order to analyze the energy difference between the SMA states on this space. For simplicity we will set $h_x = 1$, such that $H' \to \sum_i P\sigma_i^x P$. It is useful to rewrite the basis states in terms of the building blocks

$$\begin{aligned} s^- &= \left|\begin{smallmatrix} 0 & \cdot \\ \cdot & 1 \end{smallmatrix}\right\rangle - \left|\begin{smallmatrix} 1 & \cdot \\ \cdot & 0 \end{smallmatrix}\right\rangle, \quad s^+ = \left|\begin{smallmatrix} 0 & \cdot \\ \cdot & 1 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1 & \cdot \\ \cdot & 0 \end{smallmatrix}\right\rangle, \\ t^- &= \left|\begin{smallmatrix} 0 & \cdot \\ \cdot & 0 \end{smallmatrix}\right\rangle - \left|\begin{smallmatrix} 1 & \cdot \\ \cdot & 1 \end{smallmatrix}\right\rangle, \quad t^+ = \left|\begin{smallmatrix} 0 & \cdot \\ \cdot & 0 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1 & \cdot \\ \cdot & 1 \end{smallmatrix}\right\rangle, \end{aligned} \tag{S25}$$

where the normalization constant is omitted for clarity. The state $|\psi'_{\mathrm{SMA},1}\rangle$ (11) is built up starting from the reference state $|\psi'_{E=0}\rangle$ (8), which is

$$|\psi'_{E=0}\rangle = \left[\ldots s^- s^- \ldots\right]. \tag{S26}$$

Acting with the $O^{\pm}$ operators on $|\psi'_{E=0}\rangle$ in a translationally invariant way produces the first SMA states

$$|\psi'_{\mathrm{SMA},\pm1}\rangle = \frac{1}{\sqrt{2}} P\left[\ldots s^- s^+ s^+ s^- \ldots\right]_{k_x=0} \pm \frac{1}{2} P\left[\ldots s^-(s^+t^+ + t^+s^+)s^- \ldots\right]_{k_x=0}. \tag{S27}$$

This is independent of the length of the ladder L, such that the dots indicate that on all other sites is a s^-. Note that this state does not have $k_y = 0$. In the remainder we will omit the explicit $k_x = 0$ after each state, but all further states in this appendix will be with $k_x = 0$. We can make use of the lemma

$$P\left[s^- t^+ s^+\right] = \frac{1}{2}\left[s^- t^+ s^+ + s^+ t^+ s^-\right] \quad \text{and} \quad P\left[s^- t^+ s^-\right] = \frac{1}{2}\left[s^- t^+ s^- + s^+ t^+ s^+\right] \tag{S28}$$

to rewrite the state (S27) as

$$\begin{aligned} |\psi'_{\mathrm{SMA},\pm1}\rangle &= \pm\frac{1}{4}\left[\ldots s^-(s^-s^+t^+s^- + s^-s^-t^+s^+ + s^-t^+s^+s^- + s^+t^+s^-s^-)s^- \ldots\right] + \frac{1}{\sqrt{2}}\left[\ldots s^-s^+s^+s^- \ldots\right] \\ &= \pm\frac{1}{2}\left[\ldots s^-(s^+t^+ + t^+s^+)s^- \ldots\right] + \frac{1}{\sqrt{2}}\left[\ldots s^-s^+s^+s^- \ldots\right]. \end{aligned} \tag{S29}$$

Working out the terms results in

$$\begin{aligned} |\psi'_{\mathrm{SMA},\pm1}\rangle &= \frac{\pm1}{\sqrt{8}}\left[\ldots s^-\left(\left|\begin{smallmatrix} 1 & 0 & 1 & \cdot \\ \cdot & 0 & 0 & 0 \end{smallmatrix}\right\rangle - \left|\begin{smallmatrix} 0 & 1 & 0 & \cdot \\ \cdot & 1 & 1 & 1 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1 & 1 & 1 & \cdot \\ \cdot & 0 & 1 & 0 \end{smallmatrix}\right\rangle - \left|\begin{smallmatrix} 0 & 0 & 0 & \cdot \\ \cdot & 1 & 0 & 1 \end{smallmatrix}\right\rangle\right)s^- \ldots\right] \\ &\quad + \frac{1}{\sqrt{2}}\left[\ldots s^-s^+s^+s^- \ldots\right] \\ &= \frac{1}{\sqrt{2}}(\pm|w_{st+ts}\rangle + |w_{ss}\rangle). \end{aligned} \tag{S30}$$

This wave function is an approximation to the exact eigenstates of the Hamiltonian. In order to get the energy of the excited state from the approximation, we will use the forward scattering approximation (FSA). We will take the state $|\psi'_{\mathrm{SMA},1}\rangle$ as the initial vector of the FSA and which we call $|w_{(0)}\rangle$, where the index denotes the order of the FSA. In leading order, the energy of the excited state $|w_{(0)}\rangle$ is $E_{(0)} = \pm2\sqrt{2} = \pm2.828...$. We will now calculate the first corrections for the vectors and energies, which are independent of L.

A. Finding $|w_{(1)}\rangle$

As a first step we act on $|w_{(0)}\rangle$ with H' and find $H'|w_{(0)}\rangle = 2\sqrt{2}|w_{(0)}\rangle + |w_{(1)}\rangle$, with

$$\begin{aligned} |w_{(1)}\rangle &= \frac{1}{2}\left[\ldots s^-\left(\left|\begin{smallmatrix} 0 & 1 & 0 & 0 & \cdot \\ \cdot & 1 & 1 & 0 & 1 \end{smallmatrix}\right\rangle + \left|\begin{smallmatrix} 1 & 0 & 1 & 1 & \cdot \\ \cdot & 0 & 0 & 1 & 0 \end{smallmatrix}\right\rangle - \left|\begin{smallmatrix} 1 & 1 & 1 & 0 & \cdot \\ \cdot & 0 & 1 & 1 & 1 \end{smallmatrix}\right\rangle - \left|\begin{smallmatrix} 0 & 0 & 0 & 1 & \cdot \\ \cdot & 1 & 0 & 0 & 0 \end{smallmatrix}\right\rangle\right)s^- \ldots\right] \\ &= |w_{tt}\rangle. \end{aligned} \tag{S31}$$

The FSA matrix M_{FSA} with matrix elements $M_{i,j} = \langle w_{(i)}|H'|w_{(j)}\rangle$ becomes up to this order

$$M^{(1)}_{\mathrm{FSA}} = \begin{pmatrix} 2\sqrt{2} & 1 \\ 1 & 0 \end{pmatrix}. \tag{S32}$$

It has as the largest eigenvalue $E_{(1)} = \sqrt{2}+\sqrt{3} = 3.146...$, which is the first-order approximation of the energy of the state $|v'_{\mathrm{QMBS},1}\rangle$.

B. Finding $|w_{(2)}\rangle$

We now apply H' on $\left|w_{(1)}\right\rangle$ to obtain the next order correction. Acting on the inner indices of the blocks of four gives, after subtracting $\left|w_{(0)}\right\rangle$,

$$H'\left|w_{(1)}\right\rangle - \left|w_{(0)}\right\rangle = \sqrt{\frac{3}{2}}\left|w'_{(0)}\right\rangle - \frac{1}{\sqrt{2}}|w_{ss}\rangle$$

with

$$\begin{aligned}\left|w'_{(0)}\right\rangle = \frac{1}{4\sqrt{3}}\Bigg[\ldots s^-\left(3\left|\begin{smallmatrix}0&0&0&0&1&\cdot\\ \cdot&1&1&0&1&0\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}0&0&0&0&0&\cdot\\ \cdot&1&1&0&1&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&0&0&0&1&\cdot\\ \cdot&0&1&0&1&0\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&0&0&0&0&\cdot\\ \cdot&0&1&0&1&1\end{smallmatrix}\right\rangle\right)s^-\ldots\Bigg]\\ - Q - (\nwarrow\!\!\searrow) + (Q, \nwarrow\!\!\searrow),\end{aligned} \tag{S33}$$

with Q the particle-hole conjugation operator and $(\nwarrow\!\!\searrow)$ denoting interchanging the diagonals. Acting on the outer indices and projecting back to the $E = 0$ space adds a new irreducible combination, named $|w_{ttt}\rangle$,

$$H'\left|w_{(1)}\right\rangle - \left|w_{(0)}\right\rangle = \sqrt{\frac{3}{2}}\left|w'_{(0)}\right\rangle - \frac{1}{\sqrt{2}}|w_{ss}\rangle + \sqrt{2}\,|w_{ttt}\rangle \tag{S34}$$

with

$$|w_{ttt}\rangle = \frac{1}{2}\left[\ldots s^-\left(\left|\begin{smallmatrix}1&1&1&0&0&\cdot\\ \cdot&0&1&1&0&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}0&0&0&1&1&\cdot\\ \cdot&1&0&0&1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1&0&0&1&\cdot\\ \cdot&1&1&0&0&0\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&0&1&1&0&\cdot\\ \cdot&0&0&1&1&1\end{smallmatrix}\right\rangle\right)s^-\ldots\right]. \tag{S35}$$

So we obtain

$$\left|w_{(2)}\right\rangle = \frac{1}{2}\left[\sqrt{\frac{3}{2}}\left|w'_{(0)}\right\rangle - \frac{1}{\sqrt{2}}|w_{ss}\rangle + \sqrt{2}\,|w_{ttt}\rangle\right] \tag{S36}$$

and

$$H'\left|w_{(1)}\right\rangle - \left|w_{(0)}\right\rangle = 2\left|w_{(2)}\right\rangle. \tag{S37}$$

Some further algebra shows that

$$\begin{aligned}\langle w_{(2)}|H'|w_{(2)}\rangle &= -\frac{\sqrt{3}}{4}\left\langle w'_{(0)}|H'|w_{ss}\right\rangle \\ &= -\frac{\sqrt{6}}{2}\left\langle w'_{(0)}|H'|w_{ts+st}\right\rangle \\ &= -\frac{1}{\sqrt{2}},\end{aligned} \tag{S38}$$

where we used that $\left\langle w'_{(0)}|w_{st+ts}\right\rangle = 1/\sqrt{3}$. To this order, the FSA matrix becomes (assuming $L > 4$)

$$M^{(2)}_{\mathrm{FSA}} = \begin{pmatrix} 2\sqrt{2} & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{S39}$$

with largest eigenvalue $E_{(2)} = 3.270...$.

C. Finding $|w_{(3)}\rangle$

Again, we act with H' on $|w_{(2)}\rangle$ and subtract the previous vectors to determine

$$|u\rangle = H'|w_{(2)}\rangle - 2|w_{(1)}\rangle + \frac{1}{\sqrt{2}}|w_{(2)}\rangle. \tag{S40}$$

We collect the following terms in $|u\rangle$:

- terms with zero $t^\pm$ combine into $\frac{3}{4}|w_{ss}\rangle$
- terms with a single $t^\pm$ give $\sqrt{\frac{3}{16}}\left|w'_{(0)}\right\rangle - |w_{st+ts}\rangle = \sqrt{\frac{11}{16}}|w'_t\rangle$, with

$$\begin{aligned}|w'_t\rangle = \frac{1}{4\sqrt{11}}\Bigg[\ldots s^-\left(-\left|\begin{smallmatrix}0&0&0&0&1&\cdot\\ \cdot&1&1&0&1&0\end{smallmatrix}\right\rangle + 3\left|\begin{smallmatrix}0&0&0&0&0&\cdot\\ \cdot&1&1&0&1&1\end{smallmatrix}\right\rangle + 3\left|\begin{smallmatrix}1&0&0&0&1&\cdot\\ \cdot&0&1&0&1&0\end{smallmatrix}\right\rangle - 5\left|\begin{smallmatrix}1&0&0&0&0&\cdot\\ \cdot&0&1&0&1&1\end{smallmatrix}\right\rangle\right)s^-\ldots\Bigg]\\ -(Q)-(\nwarrow\!\!\searrow)+(Q,\nwarrow\!\!\searrow).\end{aligned} \tag{S41}$$

- terms with two adjacent $t^\pm$ give $\sqrt{\frac{9}{8}}|w'_{tt}\rangle$, with

$$\begin{aligned}|w'_{tt}\rangle =& \frac{1}{\sqrt{8}}\left[\ldots s^-s^+\left(\left|\begin{smallmatrix}0&1&0&0&\cdot\\ \cdot&1&1&0&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&0&1&1&\cdot\\ \cdot&0&0&1&0\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&1&1&0&\cdot\\ \cdot&0&1&1&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&0&0&1&\cdot\\ \cdot&1&0&0&0\end{smallmatrix}\right\rangle\right)s^-\ldots\right]\\ &+\frac{1}{\sqrt{8}}\left[\ldots s^-\left(\left|\begin{smallmatrix}0&1&0&0&\cdot\\ \cdot&1&1&0&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&0&1&1&\cdot\\ \cdot&0&0&1&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&1&0&\cdot\\ \cdot&0&1&1&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}0&0&0&1&\cdot\\ \cdot&1&0&0&0\end{smallmatrix}\right\rangle\right)s^+s^-\ldots\right].\end{aligned} \tag{S42}$$

- terms with two $t^\pm$ separated by an $s^\pm$ give $\frac{1}{\sqrt{2}}|w_{tst}\rangle$, with

$$|w_{tst}\rangle = \frac{1}{2}\left[\ldots s^-\left(\left|\begin{smallmatrix}0&0&0&1&0&\cdot\\ \cdot&1&0&1&1&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}0&1&0&0&0&\cdot\\ \cdot&1&1&1&0&1\end{smallmatrix}\right\rangle - \left|\begin{smallmatrix}1&1&1&0&1&\cdot\\ \cdot&0&1&0&0&0\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&0&1&1&1&\cdot\\ \cdot&0&0&0&1&0\end{smallmatrix}\right\rangle\right)s^-\ldots\right]. \tag{S43}$$

- terms with 3 adjacent $t^\pm$ give $\frac{1}{2}|w_{ttt}\rangle$.
- terms with 4 adjacent $t^\pm$ give $|w_{tttt}\rangle$, with

$$\begin{aligned}|w_{tttt}\rangle =& \frac{1}{2}\left[\ldots s^-\left(\left|\begin{smallmatrix}1&0&1&1&0&0&\cdot\\ \cdot&0&0&1&1&0&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}0&1&0&0&1&1&\cdot\\ \cdot&1&1&0&0&1&0\end{smallmatrix}\right\rangle\right.\\ &\left.+\left|\begin{smallmatrix}0&0&0&1&1&0&\cdot\\ \cdot&1&0&0&1&1&1\end{smallmatrix}\right\rangle + \left|\begin{smallmatrix}1&1&1&0&0&1&\cdot\\ \cdot&0&1&1&0&0&0\end{smallmatrix}\right\rangle\right)s^-\ldots\right].\end{aligned} \tag{S44}$$

The total result is

$$|u\rangle = \frac{3}{4}|w_{ss}\rangle + \sqrt{\frac{11}{16}}|w'_t\rangle + \sqrt{\frac{9}{8}}|w'_{tt}\rangle + \frac{1}{\sqrt{2}}|w_{tst}\rangle + \frac{1}{2}|w_{ttt}\rangle + |w_{tttt}\rangle \tag{S45}$$

so that finally

$$H'|w_{(2)}\rangle = 2|w_{(1)}\rangle - \frac{1}{\sqrt{2}}|w_{(2)}\rangle + \sqrt{\frac{33}{8}}|w_{(3)}\rangle \tag{S46}$$

with

$$|w_{(3)}\rangle = \sqrt{\frac{8}{33}}\left[\frac{3}{4}|w_{ss}\rangle + \sqrt{\frac{11}{16}}|w'_t\rangle + \sqrt{\frac{9}{8}}|w'_{tt}\rangle + \frac{1}{\sqrt{2}}|w_{tst}\rangle + \frac{1}{2}|w_{ttt}\rangle + |w_{tttt}\rangle\right]. \tag{S47}$$

It is useful to note that

$$\begin{aligned} \left\langle w_{st+ts}|w'_{(0)}\right\rangle &= \frac{1}{\sqrt{3}}, \qquad & \langle w_{st+ts}|w'_t\rangle &= \frac{-3}{\sqrt{11}}, \\ \left\langle w'_{(0)}|w'_t\right\rangle &= \frac{-1}{\sqrt{33}}, & \langle w_{tt}|w'_{tt}\rangle &= 0. \end{aligned} \tag{S48}$$

With this one easily checks that all vectors $\{\left|w_{(0)}\right\rangle, \left|w_{(1)}\right\rangle, \left|w_{(2)}\right\rangle, \left|w_{(3)}\right\rangle\}$ are orthonormal, as they should be. In order to calculate $\left\langle w_{(3)}|H'|w_{(3)}\right\rangle$, we make use of

$$\begin{aligned} &\langle w_{ss}|H'|w'_t\rangle = -\frac{6\sqrt{2}}{\sqrt{11}}, \quad \langle w'_t|H'|w'_{tt}\rangle = \frac{1}{\sqrt{11}}, \quad \langle w_{tttt}|H'|w_{ttt}\rangle = \sqrt{2}, \\ &\langle w'_t|H'|w_{tst}\rangle = \frac{2}{\sqrt{11}}, \qquad \langle w'_{tt}|H'|w_{ttt}\rangle = 1. \end{aligned} \tag{S49}$$

This leads to

$$\left\langle w_{(3)}|H'|w_{(3)}\right\rangle = \frac{\sqrt{2}}{11}\,. \tag{S50}$$

To this order, the FSA matrix becomes (assuming $L > 4$)

$$M^{(3)}_{\mathrm{FSA}} = \begin{pmatrix} 2\sqrt{2} & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & \frac{-1}{\sqrt{2}} & \sqrt{\frac{33}{8}} \\ 0 & 0 & \sqrt{\frac{33}{8}} & \frac{\sqrt{2}}{11} \end{pmatrix} \tag{S51}$$

with largest eigenvalue $E_{(3)}$ such that we got for the successive approximations $(E_{(1)}, E_{(2)}, E_{(3)}) = (3.146\ldots, 3.270\ldots, 3.351\ldots)$. As can be seen the energies of the excited states keep increasing with higher orders of the FSA and are crucially not dependent on L, except that the FSA stops after a finite number of steps for every L. Continuing the FSA without cut off will lead to the energy of the excited state in the limit $L \to \infty$, that was also approximated in the main text by a fit to the energies of $L = 4, 6, 8$ to be $E \approx 3.49 h_x$.

VI. ENTANGLEMENT ENTROPY OF THE $|\psi_{E=0}\rangle$ STATE

In this section we evaluate the entanglement entropy of the horizontal half-ladder by explicitly finding the reduced density matrix. In analogy to the left leaning singlets s^-, we define the right leaning singlets as

$$r^- = \left|\begin{smallmatrix} \cdot & 0 \\ 1 & \cdot \end{smallmatrix}\right\rangle - \left|\begin{smallmatrix} \cdot & 1 \\ 0 & \cdot \end{smallmatrix}\right\rangle = \left|\begin{smallmatrix} \cdot & \\ & \cdot \end{smallmatrix}\right\rangle, \tag{S52}$$

where in the last equality we have introduced a graphical notation for the singlets. In this way we can write the density matrix of the symmetrized zero energy state as

$$\rho = |\mathcal{R}\rangle\langle\mathcal{R}| + |\mathcal{R}\rangle\langle\mathcal{S}| + |\mathcal{S}\rangle\langle\mathcal{R}| + |\mathcal{S}\rangle\langle\mathcal{S}|\,. \tag{S53}$$

Here

$$|\mathcal{R}\rangle = |/\ \ldots /\rangle\,, \tag{S54a}$$
$$|\mathcal{S}\rangle = |\backslash\ \ldots \backslash\rangle\,, \tag{S54b}$$

where the periodic boundary conditions are assumed on the last singlet, which connects sites with x-coordinate L and 1. In the following we refer to sites simply by their x-coordinate, unless stated otherwise. Since we consider only even lengths of the ladder, we parametrize it as $L = 2l$. We divide the ladder into two subsystems by breaking the singlets between sites $l, l+1$ and $L, 1$, such that the states (S54) can be written as

$$|\mathcal{R}\rangle = -|^{00}R\rangle\,|^{11}R'\rangle + |^{01}R\rangle\,|^{01}R'\rangle + |^{10}R\rangle\,|^{10}R'\rangle - |^{11}R\rangle\,|^{00}R'\rangle\,, \tag{S55a}$$
$$|\mathcal{S}\rangle = -|^{00}S\rangle\,|^{11}S'\rangle + |^{01}S\rangle\,|^{01}S'\rangle + |^{10}S\rangle\,|^{10}S'\rangle - |^{11}S\rangle\,|^{00}S'\rangle\,, \tag{S55b}$$

where we have labeled the right subsystem with a prime and introduced a notation

$$|^{xy}R\rangle = \Big| \overset{\mathrm{x}}{\diagup}\cdots\diagup\overset{\mathrm{y}}{} \Big\rangle, \tag{S56a}$$

$$|^{xy}S\rangle = \Big| \overset{}{\diagdown}\cdots\diagdown\overset{\mathrm{y}}{}\ \Big\rangle. \tag{S56b}$$

In order to perform the trace over the primed subsystem of (S53), we first need to find a suitable basis. To this end we note, that while $|^{xy}R'\rangle$ and $|^{x'y'}R'\rangle$ are orthogonal for all $x \neq x'$, $y \neq y'$, $|^{xy}S'\rangle$ and $|^{x'y'}R'\rangle$ are not. In other words, we are looking for a suitable decomposition of $|^{xy}S'\rangle$ on $|^{x'y'}R'\rangle$. To proceed, we will distinguish two cases, l even and l odd.

A. l odd

To find the decomposition of $|^{xy}S'\rangle$ on $|^{x'y'}R'\rangle$, we note that two sites belonging to a singlet can feature occupation numbers (0,1) or (1,0), but never (0,0) or (1,1). Writing a specific example of

$$|^{00}S'\rangle = \Big| \overset{}{0}\diagdown\cdots\diagdown\overset{0}{}\ \Big\rangle, \tag{S57a}$$

$$|^{xy}R'\rangle = \Big| \overset{\mathrm{x}}{\diagup}\cdots\diagup\overset{\mathrm{y}}{} \Big\rangle, \tag{S57b}$$

we see that the state components of $|^{xy}R'\rangle$ with non-zero overlap with $|^{00}S'\rangle$ have to contain (0,1) on the first right leaning singlet

$$\Big| 0\ \overset{1}{\diagup}\cdots\diagup \Big\rangle.$$

The presence of the occupation "1" then forces the configuration (1,0) on the first left-leaning singlet and so on, until we reach the right end which fixes $y = 0$. Performing the same procedure starting from the right, which fixes $x = 0$, we find that the only component of $|^{xy}R'\rangle$ with non-vanishing overlap with $|^{00}S'\rangle$ is the $|\mathbb{Z}_2\rangle$ state, so that we can write

$$|^{00}S'\rangle = \frac{1}{\sqrt{w}}\left(|^{00}R'\rangle + \sqrt{w-1}\,|^{00}R'_\perp\rangle\right), \tag{S58}$$

where $|^{00}R'_\perp\rangle$ is the orthogonal complement of $|^{00}R'\rangle$, $\left\langle {}^{00}R' \,\middle|\, {}^{00}R'_\perp \right\rangle = 0$. Importantly, while so far we have considered unnormalized states, it is now necessary to include the proper normalization in order to get the correct structure for the reduced density matrix. Expanding $|^{00}S'\rangle$ in the basis states yields the coefficient of each basis state of magnitude $1/2^{(l-1)/2}$ and similarly for $|^{00}R'\rangle$, so that $\left\langle {}^{00}R' \,\middle|\, {}^{00}S' \right\rangle = 1/2^{l-1}$. At the same time, from (S58) we have $\left\langle {}^{00}R' \,\middle|\, {}^{00}S' \right\rangle = 1/\sqrt{w}$, which implies $w = 2^{2(l-1)}$.

Continuing the same procedure for the remaining S' states yields

$$|^{01}S'\rangle = \frac{1}{\sqrt{w}}\left(|^{10}R'\rangle + \sqrt{w-1}\,|^{10}R'_\perp\rangle\right), \tag{S59a}$$

$$|^{10}S'\rangle = \frac{1}{\sqrt{w}}\left(|^{01}R'\rangle + \sqrt{w-1}\,|^{01}R'_\perp\rangle\right), \tag{S59b}$$

$$|^{11}S'\rangle = \frac{1}{\sqrt{w}}\left(|^{11}R'\rangle + \sqrt{w-1}\,|^{11}R'_\perp\rangle\right). \tag{S59c}$$

Since $\left\langle {}^{xy}S' \,\middle|\, {}^{x'y'}S' \right\rangle = \delta_{xx'}\delta_{yy'}$, we conclude that the set $\{|^{00}R'\rangle, |^{01}R'\rangle, |^{10}R'\rangle, |^{11}R'\rangle, |^{00}R'_\perp\rangle, |^{01}R'_\perp\rangle, |^{10}R'_\perp\rangle, |^{11}R'_\perp\rangle\}$ constitutes the necessary orthonormal basis (together with its non-primed counterpart) for the decomposition of ρ. With the relations (S58),(S59) we can now perform the partial trace of (S53) and after some algebra we find an

expression for the reduced density matrix of the half-ladder in the non-primed basis

$$\rho_{\rm red} = \begin{pmatrix} 1+\frac{3}{w} & 0 & 0 & 0 & \frac{2\sqrt{w-1}}{w} & 0 & 0 & 0 \\ 0 & 1+\frac{3}{w} & 0 & 0 & 0 & \frac{2\sqrt{w-1}}{w} & 0 & 0 \\ 0 & 0 & 1+\frac{3}{w} & 0 & 0 & 0 & \frac{2\sqrt{w-1}}{w} & 0 \\ 0 & 0 & 0 & 1+\frac{3}{w} & 0 & 0 & 0 & \frac{2\sqrt{w-1}}{w} \\ \frac{2\sqrt{w-1}}{w} & 0 & 0 & 0 & \frac{w-1}{w} & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{w-1}}{w} & 0 & 0 & 0 & \frac{w-1}{w} & 0 & 0 \\ 0 & 0 & \frac{2\sqrt{w-1}}{w} & 0 & 0 & 0 & \frac{w-1}{w} & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{w-1}}{w} & 0 & 0 & 0 & \frac{w-1}{w} \end{pmatrix}. \tag{S60}$$

As we are interested in the evaluation of the second Rényi entanglement entropy, we find (with the overall normalization)

$$\frac{\mathrm{Tr}\left(\rho_{\rm red}^2\right)}{\mathrm{Tr}\left(\rho_{\rm red}\right)^2} = \frac{1+w(6+w)}{8(1+w)^2}. \tag{S61}$$

B. l even

Proceeding along similar lines as in the l-odd case, we first find that $\left\langle {}^{00}S' \middle| {}^{xy}R' \right\rangle = \left\langle {}^{11}S' \middle| {}^{xy}R' \right\rangle = 0$ so that $|{}^{00}S'\rangle, |{}^{11}S'\rangle$ are part of the basis on which we seek to decompose the $|\psi_{E=0}\rangle$ state. Next, we find

$$|{}^{01}S'\rangle = \frac{1}{\sqrt{w}}\left(-|{}^{01}R'\rangle + |{}^{01}R'\rangle + \sqrt{w-2}\,|{}^{01}R'_\perp\rangle\right), \tag{S62a}$$

$$|{}^{10}S'\rangle = \frac{1}{\sqrt{w}}\left(|{}^{01}R'\rangle - |{}^{10}R'\rangle + \sqrt{w-2}\,|{}^{10}R'_\perp\rangle\right), \tag{S62b}$$

from where it follows that

$$\left\langle {}^{01}R'_\perp \middle| {}^{10}R'_\perp \right\rangle = \frac{2}{w}, \tag{S63}$$

which forces us to further decompose $|{}^{01}R'_\perp\rangle, |{}^{10}R'_\perp\rangle$ as a sum of a state we denote as $|\delta R'\rangle$, which is common to both, and the remainders $|{}^{01}\Delta R'\rangle, |{}^{10}\Delta R'\rangle$. The Eqs. (S62) become

$$|{}^{01}S'\rangle = \frac{1}{\sqrt{w}}\left(-|{}^{01}R'\rangle + |{}^{01}R'\rangle + \sqrt{2}\,|\delta R'\rangle + \sqrt{w-4}\,|{}^{01}\Delta R'\rangle\right), \tag{S64a}$$

$$|{}^{10}S'\rangle = \frac{1}{\sqrt{w}}\left(|{}^{01}R'\rangle - |{}^{10}R'\rangle + \sqrt{2}\,|\delta R'\rangle + \sqrt{w-4}\,|{}^{10}\Delta R'\rangle\right). \tag{S64b}$$

We thus have the orthonormal basis $\{|{}^{00}R\rangle, |{}^{01}R\rangle, |{}^{10}R\rangle, |{}^{11}R\rangle, |\delta R\rangle, |{}^{01}\Delta R'\rangle, |{}^{10}\Delta R'\rangle, |{}^{00}S\rangle, |{}^{11}S\rangle\}$ in which the reduced density matrix is evaluated to

$$\rho_{\rm red} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+\frac{6}{w} & -\frac{6}{w} & 0 & 0 & -\frac{2\sqrt{w-4}}{w} & \frac{2\sqrt{w-4}}{w} & 0 & 0 \\ 0 & -\frac{6}{w} & 1+\frac{6}{w} & 0 & 0 & \frac{2\sqrt{w-4}}{w} & -\frac{2\sqrt{w-4}}{w} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{w} & \frac{\sqrt{2}\sqrt{w-4}}{w} & \frac{\sqrt{2}\sqrt{w-4}}{w} & 0 & 0 \\ 0 & -\frac{2\sqrt{w-4}}{w} & \frac{2\sqrt{w-4}}{w} & 0 & \frac{\sqrt{2}\sqrt{w-4}}{w} & \frac{w-4}{w} & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{w-4}}{w} & -\frac{2\sqrt{w-4}}{w} & 0 & \frac{\sqrt{2}\sqrt{w-4}}{w} & 0 & \frac{w-4}{w} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{S65}$$

and

$$\frac{\mathrm{Tr}\left(\rho_{\rm red}^2\right)}{\mathrm{Tr}\left(\rho_{\rm red}\right)^2} = \frac{4+w(6+w)}{8(1+w)^2}. \tag{S66}$$

Finally, we remark that in the thermodynamic limit $l \to \infty$, the second Rényi entanglement entropy evaluates to $3\ln(2)$ for both l even and odd.

VII. COUNTING OF PEAK STATES

In this section we show that the number of peak states is of the order ϕ^{2L} where $\phi = (1+\sqrt{5})/2 \approx 1.618$ is the golden ratio. A peak state on the Ising ladder is a basis state that is only connected by H_x with basis states that have a different eigenvalue of H_z, i.e. the potential V. Since H_x only changes the value of one occupation number at a time, this means that flipping any spin in the basis state changes the potential. The only local configuration that conserves the potential is when the number of equal and unequal neighbours is the same and is given in Eq. (14), which we here repeat for reader's convenience

$$\left| \begin{matrix} \cdots & n & \underline{n_a} & n & \cdots \\ \cdots & \cdot & \underline{1-n} & \cdot & \cdots \end{matrix} \right\rangle .$$

We can then raise the question: given a ladder of length L, how many basis states exist that do not have the configuration Eq. (14) anywhere? The important quantity to look at are the sites that are diagonally placed from each other, i.e. the next-nearest neighbour sites. For an easier analysis we denote this chain of occupation numbers as one string as follows

$$\left| \begin{matrix} \cdots & n_{i,1} & \cdot & n_{i+2,1} & \cdots \\ \cdots & \cdot & n_{i+1,0} & \cdot & \cdots \end{matrix} \right\rangle \rightarrow \cdots n_{i,1}\, n_{i+1,0}\, n_{i+2,1} \cdots . \tag{S67}$$

Each basis state consists of two non-overlapping strings, one that starts at $n_{0,0}$ and one starting at $n_{0,1}$. For a basis state to lead to a peak state, both strings should not contain the substrings 101 and 010, because we want to exclude the configurations (14). This leads to the constraint that in the string after a 10, the next site is not allowed to be a 1 and similarly after a 01 the next one is not allowed to be a 0. We can use these constraints to calculate the total number of allowed strings by making use of a transfer matrix. This matrix shows which configuration on sites $i+1$ and $i+2$ are allowed, based on the occupation of sites i and $i+1$. Taking as basis states $(00, 01, 10, 11)$, the transfer matrix is

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} . \tag{S68}$$

Because periodic boundary conditions are assumed, applying the transfer matrix L times should return to the initial values of site 0 and 1. Therefore, the total number of allowed configurations of a string is $\mathrm{Tr}(T^L)$, because the diagonal entries are the number of possibilities that a valid string starts and ends with the same contribution. The eigenvalues of (S68) are $(1 \pm \sqrt{5})/2, \mathrm{e}^{\pm i\pi/3}$ so that for large L $\mathrm{Tr}(T^L) \approx \phi^L$ will be dominated by the maximum magnitude eigenvalue, which is $\phi = (1+\sqrt{5})/2$. Because every basis state consists of two of those strings, the number of peak states scales as ϕ^{2L}.