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Optimal reinsurance design with distortion risk measures and asymmetric information

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Abstract

This paper studies a problem of optimal reinsurance design under asymmetric information. The insurer adopts distortion risk measures to quantify his/her risk position, and the reinsurer does not know the functional form of this distortion risk measure. The risk-neutral reinsurer maximizes his/her net profit subject to individual rationality and incentive compatibility constraints. The optimal reinsurance menu is succinctly derived under the assumption that one type of insurer has a larger willingness-to-pay than the other type of insurer for every risk. Some comparative analysis are given as illustrations when the insurer adopts the Value-at-Risk (VaR) or the Tail Value-at-Risk (TVaR) as preferences.

Key words: Optimal reinsurance; Asymmetric information; Individual rationality; Incentive compatibility; Distortion risk measure.

JEL classification: C61, G22, G32.

1 Introduction

This paper presents an asymmetric information model in optimal reinsurance design. The reinsurer is not be able to identify the risk preferences of the insurer. The literature on

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asymmetric information in insurance markets focusses on adverse selection, where the underlying distribution of risk that is endowed by the insurer is unknown to the reinsurer. In such markets, the insurers with more risky endowments are buying reinsurance. See, for instance, Rothschild and Stiglitz (1976), Landsberger and Meilijson (1999), Young (2000), Laffont and Martimort (2009), Chade and Schlee (2012), and Cheung et al. (2020). This leads a selection of only high-risk insurers (*lemons*) buying reinsurance (cf. Akerlof, 1970), and thus the reinsurer needs to charge a higher premium. This paper, on the other hand, studies the situation where the underlying risk preferences of the insurer are unknown by the reinsurer, and the distribution of the reinsurable loss of the insurer is known by the reinsurer. The reinsurer faces a trade-off between high premium/low demand and low premium/high demand. The reinsurer could offer a cheap contract to the insurer, and let the insurer buy this contract no matter what his/her underlying risk preferences are, or the reinsurer could offer an expensive contract to the insurer so that only the insurer with a high willingness-to-pay is buying it. Geruso (2017) shows empirically that there is substantial demand heterogeneity related to heterogeneity in risk preferences that is not related to the underlying risk distribution.

A reinsurance contract (also called treaty) is a couple (π, f), where π is the premium and f is the reinsurance indemnity function (coverage). The reinsurer presents the insurer a menu of reinsurance policies, and the insurer chooses to buy either one of the policies or not. The risk-neutral reinsurer maximizes the net profit, while taking into account the demand of the insurer. Because the risk preferences of the insurer are unknown to the reinsurer, the optimization problem is solved under the incentive compatibility and individual rationality constraints. We assume that there are two types of the insurers, and the risk preferences of both types are assumed to be a distortion risk measure. This allows us to assume an ordering in the risk preferences; the type 1 insurer has a lower willingness-to-pay than the type 2 insurer for every risk.

Distortion risk measures are equivalent to dual utility as introduced by Yaari (1987), and have gained practitioner's interest as the risk measures Value-at-Risk (VaR) and Tail Value-at-Risk (TVaR, also known as expected shortfall) are special cases. The VaR is used in, e.g., Solvency II regulations for insurers in the European Union, and the TVaR is used in Basel III regulation for banks. Preferences given by a distortion risk measure are therefore popular in the context of insurers and financial firms. Therefore, rather than studying a principal-agent model with an insurer and a policyholder, our focus is on a principal-agent model with one reinsurer and one insurer.

Studying distortion risk measures in optimal reinsurance contract theory gained substantial interest in the last decade, that started with Cui et al. (2013). We impose the assumption that indemnity functions are non-decreasing and 1-Lipschitz, which is also called a no-sabotage condition (Carlier and Dana, 2003). This is motivated by *ex post* moral hazard on the side of the ceding insurer, as it creates no incentives for the insurer to report a lower loss, or to create additional (incremental) losses (see, e.g., Huberman et al., 1983; Denuit and Vermandele, 1998). Under the no-sabotage condition, a Marginal Indemnification Function (MIF) approach can be used to solve standard optimal reinsurance problems (Assa, 2015; Zhuang et al., 2016). This generally yields to optimality of layer-type reinsurance indemnities. We follow this approach, and reduce the optimization problem to a problem of optimizing over marginal indemnification functions under one remaining incentive compatibility condition. We show that the remaining incentive compatibility constraint can be removed, and the problem is solved via a quantile optimization approach. We provide solutions by characterizing the marginal indemnities in closed-form.

In most optimal reinsurance design papers that take the perspective of the insurance seller (reinsurer) (e.g., Amarante et al., 2015; Anthropelos and Boonen, 2020; Boonen et al., 2018), the preferences of the insurance buyer (insurer) are common knowledge. The reinsurer exploits this information, and prices a reinsurance contract such that the insurer becomes indifferent between buying or not buying the proposed contract. With asymmetric information, we show that this no longer holds, and the insurer may benefit strictly from the transaction.

The paper that is closest to ours is Landsberger and Meilijson (1994). They study an asymmetric information model, where the insurer is assumed to be a risk-averse, expected utility maximizing agent. Then, when the reinsurer is assumed to be risk-neutral and the types differ substantially in their attitude to heavy losses, they find that there is full insurance. Moreover, Julien (2000) study an asymmetric information model with an expected utility maximizing agent, and assumes that the reservation utility depends on the underlying loss distribution of the insurer. This paper differs from Landsberger and Meilijson (1994) and Julien (2000) by a focus on distortion risk measures for the insurer, who does not need to be averse to mean-preserving spreads because we allow for non-concave distortion functions. In contrast to Landsberger and Meilijson (1994) and Julien (2000), we obtain a full description of the optimal reinsurance solutions. Another paper that is close to our approach is Boonen et al. (2021), who study Bowley solutions under asymmetric information. In Bowley solutions, the reinsurer discloses an entire pricing functional to the insurer. Also, the insurer can select *any* indemnity function given such a pricing functional. Such a pricing functional is assumed to be monotone, and the optimal pricing functional may therefore be tedious to derive (Boonen et al., 2021). In this paper, the insurer is only presented with two different policies (π_1, f_1) and (π_2, f_2), and these two contracts are strategically chosen by the reinsurer. Therefore, Bowley solutions are different from the reinsurance contracts under asymmetric information that are obtained in this paper.

The setting of this paper is also similar to the setting of Cheung et al. (2020), where the key difference is the type of asymmetric information that is assumed. While Cheung et al. (2020) assume asymmetric information with respect to the underlying distribution of the loss of the insurer, our focus in this paper is on asymmetric information with respect to the risk preferences of the insurer. Interestingly, and in contrast to Cheung et al. (2020), we derive the optimal reinsurance menu in closed-form, and we find that the optimal reinsurance indemnities have the form of "tranches". This means that the risk of the insurer is split into layers (tranches), and the layers are then either kept or sold to the reinsurer. Such functional form is popular in reinsurance, as (truncated) stop-loss contracts are special cases. Under symmetric information, tranches are generally not optimal with expected utility (e.g., Raviv, 1979), and proportional reinsurance indemnities are optimal with exponential expected utilities (Barrieu and El Karoui, 2005).

This paper is organized as follows. Section 2 states some preliminaries that are used in Section 3, where we define the main reinsurance problem with asymmetric information that we study in this paper. Section 4 provides the optimal solutions, that are separating equilibrium reinsurance contracts. This section also studies the pooling equilibrium reinsurance contracts, which are shown to be suboptimal. Section 5 provides two examples with the VaR and the TVaR, and finally Section 6 concludes.

2 Preliminaries

The insurer is initially endowed with a bounded, non-negative random loss variable X, which is realized at a given future reference time period. Its distribution function F_X is known by both the insurer and the reinsurer, while the risk preferences of the insurer are unknown to the reinsurer in our study.

The insurer cedes the risk $f(X)$ (the indemnity) to the reinsurer, and in return the reinsurer receives a premium $\pi \geq 0$ from the insurer. We assume that $f \in \mathcal{F}$, with

$$\mathcal{F} = \{f : [0, M] \to [0, M] | f(0) = 0, 0 \leq f(x) - f(y) \leq x - y \text{ for } 0 \leq y \leq x \leq M\},$$

where M is the essential supremum of X. The purpose of restricting the admissible set of indemnity functions to $\mathcal{F}$ is to avoid moral hazard; see for instance Huberman et al. (1983), Denuit and Vermandele (1998), and many more recent papers. This moral hazard condition has a close relationship with the marketability of indemnities. More specifically, Cheung et al. (2014) define the class of *universally marketable* indemnities to ensure that the acceptability by policyholders is universal. They show that an indemnity that is assumed to be non-decreasing a priori is universally marketable if and only if it is a 1-Lipschitz function; see also the related discussions in Lo et al. (2021). The function $f \in \mathcal{F}$ is non-decreasing and 1-Lipschitz and hence absolutely continuous. This implies that f is almost everywhere differentiable on $[0, M]$. Moreover, there exists a Lebesgue integrable function $h : [0, M] \mapsto [0, 1]$ such that

$$f(x) = \int_0^x h(z)\mathrm{d}z, \quad x \in [0, M], \tag{1}$$

where h is the slope of the indemnity function f. Assa (2015) and Zhuang et al. (2016) call this function the *Marginal Indemnification Function* (MIF).

The insurer adopts a *distortion risk measure* to evaluate his/her risk position. A distortion risk measure ρ_g of a non-negative random variable Z is given by

$$\rho_g(Z) = \int_0^\infty g(\overline{F}_Z(z))\mathrm{d}z, \tag{2}$$

whenever the integral exists, where $\overline{F}_Z(z) := 1 - F_Z(z)$ is the survival function of Z, $g \in \mathcal{G}_d$, and

$$\mathcal{G}_d = \{g : [0, 1] \to [0, 1] | g(0) = 0, g(1) = 1, g \text{ is non-decreasing and left-continuous}\}.$$

Two popular examples of a distortion risk measure are the VaR and the TVaR, which will be explicitly defined in Section 5.

Distortion risk measures satisfy comonotonic additivity and translation invariance (Schmeidler, 1986; Wang et al., 1997). A risk measure ρ_g is comonotonic additive when $\rho_g(Y) + \rho_g(Z)$ for all comonotonic random variables Y, Z,[1] and in particular this implies $\rho_g(X - f(X)) + \rho(f(X)) = \rho_g(X)$ for all $f \in \mathcal{F}$. A risk measure ρ_g is translation invariant when $\rho_g(Z + c) = \rho_g(Z) + c$ for all $c \in \mathbb{R}$ and all random variables Z. Since ρ_g is translation invariant, it holds that $\rho_g(Y - \rho_g(Y)) = \rho_g(Y) - \rho_g(Y) = 0 = \rho_g(0)$ for all random variables Y, and thus we can interpret $\rho_g(Y)$ as *willingness-to-pay* for loss Y. Moreover, if $g, g^* \in \mathcal{G}_d$ are such that $g(t) \geq g^*(t)$ for all $t \in [0, 1]$, then it holds that $\rho_g(Y) \geq \rho_{g^*}(Y)$ for all random variables Y, and thus the risk measure ρ_g exhibits a higher willingness-to-pay for loss Y than risk measure ρ_{g^*}.

[1]Random variables Y, Z are called comonotonic if there exists a non-decreasing function k such that $Y = k(Z)$.

We note that a risk-reward trade-off, given by $\mathbb{V}(Z) = \mathbb{E}[Z] + \alpha(\rho_g(Z) - \mathbb{E}[Z])$ for $\alpha \in [0, 1]$ and $g \in \mathcal{G}_d$, can be written as a distortion risk measure: $\mathbb{V}(Z) = \rho_{\hat{g}}(Z)$, with $\hat{g}(t) = (1 - \alpha)t + \alpha g(t)$. Here, $\hat{g} \in \mathcal{G}_d$. The parameter α is then interpreted as a cost-of-capital rate (Chi, 2012). See De Giorgi and Post (2008) for a further study of such preferences.

3 Problem formulation

In this section, we formalize the principal-agent model with asymmetric information studied in this paper. Assume that the identity of the insurer is hidden information to the reinsurer, but the reinsurer knows that the insurer adopts a distortion risk measure ρ_{g_1} with probability p, and adopts another distortion risk measure ρ_{g_2} with probability $1-p$, where $g_1, g_2 \in \mathcal{G}_d$. The distribution of X is assumed to be the same for both types of insurers.[2]

A reinsurance contract is equal to a pair (π, f), where $\pi \geq 0$ is the premium and $f \in \mathcal{F}$ is the indemnity function. We assume that the reinsurer has the monopoly in the reinsurance market. The reinsurer offers a reinsurance menu that consists of two reinsurance contracts to the insurer, and this menu is given by $\{(\pi_1, f_1); (\pi_2, f_2)\}$. This menu is designed such that the insurer of type 1 weakly prefers to buy the reinsurance contract (π_1, f_1) rather than the reinsurance contract (π_2, f_2), while the insurer of type 2 weakly prefers to buy the reinsurance contract (π_2, f_2) rather than the reinsurance contract (π_1, f_1). Once the policy is chosen by the insurer, the reinsurer knows the risk preferences of the insurer. In the optimal solution, π_j is thus the premium charged from the j-th type of the insurer, and f_j is the corresponding indemnity function ceded to the reinsurer by the j-th type of the insurer, $j = 1, 2$. The reinsurer is assumed to be risk-neutral, and the objective for the reinsurer is thus to maximize its expected net profit:

$$\begin{aligned} P &:= \mathbb{E}[(\pi_1 - f_1(X))\, \mathbf{1}_{\{i=1\}} + (\pi_2 - f_2(X))\, \mathbf{1}_{\{i=2\}}] \\ &= p\,(\pi_1 - \mathbb{E}[f_1(X)]) + (1-p)\,(\pi_2 - \mathbb{E}[f_2(X)])\,, \end{aligned} \tag{3}$$

where $\mathbf{1}_{\{i=j\}}$ is the indicator function that is one if the insurer has the j-th type and 0 otherwise, $j = 1, 2$. For the reinsurer, $\mathbf{1}_{\{i=1\}}$ is a Bernoulli random variable with success probability p, which we assumed to be independent of X.

[2]For the sake of presentation, we assumed that the essential supremum M is finite, and thus that X is bounded. However, all the results obtained in the paper still hold even if $M = \infty$, but when we assume $\rho_{g_1}(X) < \infty$, $\rho_{g_2}(X) < \infty$, and $\mathbb{E}[X] < \infty$.

The fundamental Revelation Principle in mechanism design applies, and states that we only need to consider incentive compatibility (IC) and individual rationality (IR). In other words, we study the optimal reinsurance design where the reinsurer maximizes (3) under IC and IR constraints. The IC constraint states that the reinsurer provides a contract aiming at each type of insurers, and the type 1 insurer will not choose the menu designed for type 2 insurer, and vice versa. The IR constraint ensures that the insurer is not worse off through buying the designated reinsurance contract. The problem that we study in this paper is formalized as follows.

Problem 3.1 *Maximize the objective function*

$$\max_{\{(\pi_1,f_1);(\pi_2,f_2)\}} \quad p\left(\pi_1 - \mathbb{E}[f_1(X)]\right) + (1-p)\left(\pi_2 - \mathbb{E}[f_2(X)]\right), \tag{4}$$

subject to the following constraints: $\pi_1, \pi_2 \geq 0$, $f_1, f_2 \in \mathcal{F}$,

$$\mathrm{IR1}: \quad \rho_{g_1}(X - f_1(X) + \pi_1) \leq \rho_{g_1}(X), \tag{5a}$$
$$\mathrm{IR2}: \quad \rho_{g_2}(X - f_2(X) + \pi_2) \leq \rho_{g_2}(X), \tag{5b}$$
$$\mathrm{IC1}: \quad \rho_{g_1}(X - f_1(X) + \pi_1) \leq \rho_{g_1}(X - f_2(X) + \pi_2), \tag{5c}$$
$$\mathrm{IC2}: \quad \rho_{g_2}(X - f_2(X) + \pi_2) \leq \rho_{g_2}(X - f_1(X) + \pi_1). \tag{5d}$$

We refer to the solution of Problem 3.1 as an optimal reinsurance menu or *separating equilibrium*. As a standard assumption in asymmetric information models (cf. Landsberger and Meilijson, 1999; Laffont and Martimort, 2009), we make the following assumptions:

(i) if IC1 holds with an equality, it is implicitly assumed that the type 1 insurer would select contract (π_1, f_1);

(ii) when IC2 is an equality, the type 2 insurer would select contract (π_2, f_2).

Because of the comonotonic additivity and translational invariance of the distortion risk measures, we can simplify the constraints (5a)-(5d) as follows:

$$\mathrm{IR1}: \quad \pi_1 \leq \rho_{g_1}(f_1(X)), \tag{6a}$$
$$\mathrm{IR2}: \quad \pi_2 \leq \rho_{g_2}(f_2(X)), \tag{6b}$$
$$\mathrm{IC1}: \quad \pi_1 + \rho_{g_1}(f_2(X)) \leq \pi_2 + \rho_{g_1}(f_1(X)), \tag{6c}$$
$$\mathrm{IC2}: \quad \pi_2 + \rho_{g_2}(f_1(X)) \leq \pi_1 + \rho_{g_2}(f_2(X)). \tag{6d}$$

Remark 3.1 *We wish to emphasize that we do not study adverse selection. The two types of insurers are endowed with the same reinsurable loss X; there is only asymmetric* *information with respect to the risk preferences of the insurer. Thus, there is no notion of high or low risk types. The reinsurer prices insurance policies only based on the demand: if the reinsurer believes that the insurer is endowed with a large function g_i (a high willingness-to-pay), then the reinsurer may charge a high premium.*

Throughout the paper, we make the following assumption:

Assumption 1: *$g_1(t) \leq g_2(t)$ for all $t \in [0, 1]$.*

It holds that $\rho_{g_1}(Y) \leq \rho_{g_2}(Y)$ for all non-negative random variables Y. This implies that the type 1 insurer has a weakly smaller willingness-to-pay than the type 2 insurer for every risk, and so this allows us to rank the two types of the insurer based on their willingness-to-pay. For instance, Assumption 1 is satisfied if both types of the insurer are endowed with a VaR measure, or if both types of the insurer are endowed with a TVaR measure. This will be further studied in Section 5. First, we argue that the IR1 constraint is binding under this assumption. Suppose that, for the optimal reinsurance menu $\{(\pi_1, f_1); (\pi_2, f_2)\}$, $\pi_1 < \rho_{g_1}(f_1(X))$. For this case, let

$$c := \rho_{g_1}(f_1(X)) - \pi_1 > 0.$$

Then, from IC2 we get

$$\begin{aligned} \rho_{g_2}(f_2(X)) - \pi_2 &\geq \rho_{g_2}(f_1(X)) - \pi_1 \\ &\geq \rho_{g_1}(f_1(X)) - \pi_1 = c > 0, \end{aligned}$$

where the second inequality is due to Assumption 1. Thus,

$$\rho_{g_2}(f_2(X)) \geq \pi_2 + c > \pi_2.$$

Now, let us consider another reinsurance menu $\{(\bar{\pi}_1, f_1); (\bar{\pi}_2, f_2)\}$, where $\bar{\pi}_1 = \pi_1 + c$ and $\bar{\pi}_2 = \pi_2 + c$. The new menu $\{(\bar{\pi}_1, f_1); (\bar{\pi}_2, f_2)\}$ also fulfills with the four constraints (6a)-(6d), from which the reinsurer can make a profit strictly higher than the menu $\{(\pi_1, f_1); (\pi_2, f_2)\}$. Hence, it must hold that $\pi_1 = \rho_{g_1}(f_1(X))$, i.e., IR1 must be binding under Assumption 1. With this observation, we have $\rho_{g_2}(f_2(X)) - \pi_2 \geq \rho_{g_2}(f_1(X)) - \pi_1 \geq \rho_{g_1}(f_1(X)) - \pi_1 = 0$, meaning that IR2 holds naturally under IR1 and IC2. Note that the objective function (4) is increasing in π_2, from which it follows that IC2 must hold with equality. With the help of these observations, Problem 3.1 can be simplified into maximizing the objective function in (4) subject to the constraints

$$\text{IR1}' : \quad \pi_1 = \rho_{g_1}(f_1(X)), \tag{7a}$$

$$
\begin{aligned}
&\mathrm{IC1}': \quad \rho_{g_2}(f_2(X)) - \rho_{g_2}(f_1(X)) \geq \rho_{g_1}(f_2(X)) - \rho_{g_1}(f_1(X)), && \text{(7b)}\\
&\mathrm{IC2}': \quad \pi_2 - \pi_1 = \rho_{g_2}(f_2(X)) - \rho_{g_2}(f_1(X)). && \text{(7c)}
\end{aligned}
$$

Observe that the objective function in (4) can be written as the bivariate function of the indemnity functions f_1 and f_2 by substituting the variables π_1 and π_2 from IR1′ (7a) and IC2′ (7c). Then, Problem 3.1 reduces to the following problem.

Problem 3.2 *Under Assumption 1, solve*

$$
\begin{aligned}
\max_{\{f_1,f_2\}} \quad & p\,[\rho_{g_1}(f_1(X)) - \mathbb{E}[f_1(X)]] + (1-p)[\rho_{g_1}(f_1(X)) \\
& \quad + \rho_{g_2}(f_2(X)) - \rho_{g_2}(f_1(X)) - \mathbb{E}[f_2(X)]] \\
s.t. \quad \mathrm{IC1}': \quad & \rho_{g_2}(f_2(X)) - \rho_{g_2}(f_1(X)) \geq \rho_{g_1}(f_2(X)) - \rho_{g_1}(f_1(X)).
\end{aligned} \tag{8}
$$

From (1) and (2), we note that

$$
\begin{aligned}
\rho_{g_1}(f_1(X)) &= \int_0^M g_1(\overline{F}_X(z))h_1(z)\mathrm{d}z, \quad \rho_{g_2}(f_1(X)) = \int_0^M g_2(\overline{F}_X(z))h_1(z)\mathrm{d}z,\\
\rho_{g_1}(f_2(X)) &= \int_0^M g_1(\overline{F}_X(z))h_2(z)\mathrm{d}z, \quad \rho_{g_2}(f_2(X)) = \int_0^M g_2(\overline{F}_X(z))h_2(z)\mathrm{d}z,\\
\mathbb{E}[f_1(X)] &= \int_0^M \overline{F}_X(z)h_1(z)\mathrm{d}z, \quad \mathbb{E}[f_2(X)] = \int_0^M \overline{F}_X(z)h_2(z)\mathrm{d}z,
\end{aligned}
$$

where h_1 and h_2 are the slopes or marginal indemnity functions (MIFs) of the indemnity functions f_1 and f_2, respectively. In other words,

$$
f_1(x) = \int_0^x h_1(z)\mathrm{d}z \quad \text{and} \quad f_2(x) = \int_0^x h_2(z)\mathrm{d}z, \qquad x \in [0, M].
$$

Then, the objective function (8) can be rewritten as

$$
\begin{aligned}
&\int_0^M \left\{g_1(\overline{F}_X(z)) - \left[p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))\right]\right\} h_1(z)\mathrm{d}z\\
&\qquad + (1-p)\int_0^M \left[g_2(\overline{F}_X(z)) - \overline{F}_X(z)\right] h_2(z)\mathrm{d}z.
\end{aligned}
$$

In a similar manner, the IC1′ condition in Problem 3.2 can be equivalently written as

$$
\int_0^M \left[g_2(\overline{F}_X(z)) - g_1(\overline{F}_X(z))\right][h_1(z) - h_2(z)]\mathrm{d}z \leq 0. \tag{9}
$$

Hence, Problem 3.2 can be reformulated as follows.

Problem 3.3 *Under Assumption 1, solve*

$$
\begin{aligned}
\max_{\{h_1,h_2\}} \quad & \textstyle\int_0^M \left\{g_1(\overline{F}_X(z)) - \left[p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))\right]\right\} h_1(z)\mathrm{d}z \\
& \quad + (1-p)\textstyle\int_0^M \left[g_2(\overline{F}_X(z)) - \overline{F}_X(z)\right] h_2(z)\mathrm{d}z \\
s.t. \quad \mathrm{IC1}': \quad & \textstyle\int_0^M \left[g_2(\overline{F}_X(z)) - g_1(\overline{F}_X(z))\right][h_1(g_1 z) - h_2(z)]\mathrm{d}z \leq 0.
\end{aligned} \tag{10}
$$

4 Main results

In this section, we first study the optimal reinsurance contracts. Thereafter, we study the pooling equilibrium reinsurance contracts.

4.1 Optimal reinsurance contracts under asymmetric information

This subsection studies Problem 3.3, that yields the optimal reinsurance contracts. We will show that condition IC1$'$ can be removed. First, we solve the problem without considering the IC1$'$ condition, which is the following problem.

Problem 4.1 *Under Assumption 1, solve*

$$\begin{aligned}\max_{\{h_1,h_2\}} \ & \int_0^M \left\{g_1(\overline{F}_X(z)) - \left[p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))\right]\right\} h_1(z)\mathrm{d}z \\ & +(1-p)\int_0^M \left[g_2(\overline{F}_X(z)) - \overline{F}_X(z)\right] h_2(z)\mathrm{d}z.\end{aligned} \tag{11}$$

Let

$$\psi_1(t) := g_1(t) - [pt + (1-p)g_2(t)] \quad \text{and} \quad \psi_2(t) := (1-p)[g_2(t) - t], \quad t \in [0,1],$$

and define

$$\mathcal{A}_{\psi_1} = \{z \in [0,M] | \psi_1(\overline{F}_X(z)) > 0\}, \quad \mathcal{B}_{\psi_1} = \{z \in [0,M] | \psi_1(\overline{F}_X(z)) = 0\},$$

$$\mathcal{C}_{\psi_1} = \{z \in [0,M] | \psi_1(\overline{F}_X(z)) < 0\}, \quad \mathcal{A}_{\psi_2} = \{z \in [0,M] | \psi_2(\overline{F}_X(z)) > 0\},$$

$$\mathcal{B}_{\psi_2} = \{z \in [0,M] | \psi_2(\overline{F}_X(z)) = 0\}, \quad \mathcal{C}_{\psi_2} = \{z \in [0,M] | \psi_2(\overline{F}_X(z)) < 0\}.$$

The following lemma summaries the solution of Problem 4.1.

Lemma 4.2 *The optimal values of $\tilde{h}_1$ and $\tilde{h}_2$ that solve Problem 4.1 are given, for $z \in [0,M]$ almost everywhere (a.e.), as follows:*

$$\tilde{h}_1(z) = \begin{cases} 1 & \text{if } z \in \mathcal{A}_{\psi_1}, \\ a_1(z) & \text{if } z \in \mathcal{B}_{\psi_1}, \\ 0 & \text{if } z \in \mathcal{C}_{\psi_1}, \end{cases} \quad \text{and} \quad \tilde{h}_2(z) = \begin{cases} 1 & \text{if } z \in \mathcal{A}_{\psi_2}, \\ a_2(z) & \text{if } z \in \mathcal{B}_{\psi_2}, \\ 0 & \text{if } z \in \mathcal{C}_{\psi_2}, \end{cases}$$

where a_1 and a_2 are any Lebesgue integrable functions between 0 and 1.

Proof. Note that (10) can be written as

$$\max_{\{h_1,h_2\}} \int_0^M \left[\psi_1(\overline{F}(z))h_1(z) + \psi_2(\overline{F}(z))h_2(z)\mathrm{d}z\right]$$

$$
\begin{aligned}
&= \max_{h_1} \int_0^M \psi_1(\overline{F}(z))h_1(z)\mathrm{d}z + \max_{h_2} \int_0^M \psi_2(\overline{F}(z))h_2(z)\mathrm{d}z \\
&= \int_0^M \psi_1(\overline{F}(z))\tilde{h}_1(z)\mathrm{d}z + \int_0^M \psi_2(\overline{F}(z))\tilde{h}_2(z)\mathrm{d}z,
\end{aligned}
$$

where the last equality follows from Eq. (17) in Assa (2015) or Theorem 3.1 in Zhuang et al. (2016). ■

The next lemma shows that the solutions to Problem 4.1 presented in Lemma 4.2 automatically fulfill with the constraint IC1′.

Lemma 4.3 *Every solution to Problem 4.1 satisfies* IC1′.

Proof. Let Assumption 1 hold. We show that any optimal MIF $\tilde{h}_1$ and $\tilde{h}_2$ in Lemma 4.2 is such that

$$
\int_0^M \left[g_2(\overline{F}_X(z)) - g_1(\overline{F}_X(z))\right] [\tilde{h}_1(z) - \tilde{h}_2(z)]\mathrm{d}z \leq 0. \tag{12}
$$

Let $\{\tilde{h}_1, \tilde{h}_2\}$ be as in Lemma 4.2. By Assumption 1, it holds that $g_2(\overline{F}_X(z)) - g_1(\overline{F}_X(z)) \geq 0$. We next identify the $z \in [0, M]$ for which it may hold that $\tilde{h}_1(z) - \tilde{h}_2(z) > 0$. Then, we separate three cases:

- if $z \in \mathcal{C}_{\psi_2}$, then $\psi_2(\overline{F}_X(z)) < 0$, or $g_2(\overline{F}_X(z)) < \overline{F}_X(z)$, or

  $$
  \begin{aligned}
  g_1(\overline{F}_X(z)) - [p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))] &\leq g_2(\overline{F}_X(z)) - [p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))] \\
  &= p(g_2(\overline{F}_X(z)) - \overline{F}_X(z)) < 0,
  \end{aligned}
  $$

  and thus it holds $z \in \mathcal{C}_{\psi_1}$. Hence, $\mathcal{C}_{\psi_2} \subseteq \mathcal{C}_{\psi_1}$;

- if $z \in \mathcal{B}_{\psi_2}$, then $\psi_2(\overline{F}_X(z)) = 0$, or $g_2(\overline{F}_X(z)) = \overline{F}_X(z)$, or

  $$
  \begin{aligned}
  g_1(\overline{F}_X(z)) - [p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))] &\leq g_2(\overline{F}_X(z)) - [p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))] \\
  &= p(g_2(\overline{F}_X(z)) - \overline{F}_X(z)) = 0,
  \end{aligned}
  $$

  and thus it holds $z \in \mathcal{B}_{\psi_1} \cup \mathcal{C}_{\psi_1}$. Hence, $\mathcal{B}_{\psi_2} \subseteq \mathcal{B}_{\psi_1} \cup \mathcal{C}_{\psi_1}$;

- if $z \in \mathcal{A}_{\psi_2}$, then $\tilde{h}_2(z) = 1 \geq \tilde{h}_1(z)$.

Thus, $\tilde{h}_1(z) > \tilde{h}_2(z)$ only happens for $z \in \mathcal{B}_{\psi_1} \cap \mathcal{B}_{\psi_2}$. But, for $z \in \mathcal{B}_{\psi_1} \cap \mathcal{B}_{\psi_2}$, it holds $g_2(\overline{F}_X(z)) = \overline{F}_X(z)$ and $g_1(\overline{F}_X(z)) = p\overline{F}_X(z) + (1-p)g_2(\overline{F}_X(z))$, and thus $g_1(\overline{F}_X(z)) = g_2(\overline{F}_X(z)) = \overline{F}_X(z)$. Hence, for all $z \in [0, M]$,

$$
\left[g_2(\overline{F}_X(z)) - g_1(\overline{F}_X(z))\right] [\tilde{h}_1(z) - \tilde{h}_2(z)] \leq 0,
$$

which implies condition IC1$'$. ■

From this result, we get that the solutions provided in Lemma 4.2 are indeed the solutions to Problem 3.3. This is our main result, and is summarized in the following theorem.

Theorem 4.4 *The optimal values of $\tilde{h}_1$ and $\tilde{h}_2$ that solve Problem 3.3 are given, for $z \in [0, M]$ a.e., as follows:*

$$\tilde{h}_1(z) = \begin{cases} 1 & \text{if } z \in \mathcal{A}_{\psi_1}, \\ a_1(z) & \text{if } z \in \mathcal{B}_{\psi_1}, \\ 0 & \text{if } z \in \mathcal{C}_{\psi_1}, \end{cases} \quad \text{and} \quad \tilde{h}_2(z) = \begin{cases} 1 & \text{if } z \in \mathcal{A}_{\psi_2}, \\ a_2(z) & \text{if } z \in \mathcal{B}_{\psi_2}, \\ 0 & \text{if } z \in \mathcal{C}_{\psi_2}, \end{cases}$$

where a_1 and a_2 are any Lebesgue integrable functions between 0 and 1.

Theorem 4.4 states all solutions to Problem 3.3. For instance, we find that there is a unique solution to Problem 3.3 if and only if the Lebesgue measure of $\mathcal{B}_{\psi_1} \cup \mathcal{B}_{\psi_2}$ is zero.

Based on Theorem 4.4, the following statements can be made on the shape of the optimal indemnity functions.

(i) If $g_2(t) < t$ for all $t \in (0, 1)$, it must hold that $\psi_1(t) < 0$ and $\psi_2(t) < 0$ for $t \in (0, 1)$ since $g_1(t) \leq g_2(t)$ due to Assumption 1. For this case, we have $\mathcal{A}_{\psi_1} = \mathcal{B}_{\psi_1} = \emptyset$, and the Lebesgue measures of $\mathcal{A}_{\psi_2}$ and $\mathcal{B}_{\psi_2}$ are 0, which implies that shut-down policies are ceded by both types of insurers.

(ii) If $g_2(t) > t$ for all $t \in (0, 1)$, it follows that $\psi_2(t) > 0$ for $t \in (0, 1)$, which means that a full reinsurance treaty is ceded by the type 2 insurer. Furthermore,

   (a) if $g_1(t) < t$ for all $t \in (0, 1)$, we have $\psi_1(t) < 0$ for all $t \in (0, 1)$, indicating that the shut-down policy is ceded by the type 1 insurer;

   (b) if $g_1(t) > t$ for all $t \in (0, 1)$, the optimal reinsurance indemnity depends on the sign of the function $\psi_1(t)$ on $t \in (0, 1)$, which can take various layer-type functional forms. For instance, if there exist some $t_1^* \in (0, 1)$ such that $\psi_1(t) > 0$ for $t \in (t_1^*, 1)$ and $\psi_1(t) < 0$ for $t \in (0, t_1^*)$, then this leads to the conclusion that a dual stop-loss reinsurance contract is ceded by the type 1 insurer.

(iii) If there exists some $t_2^* \in (0, 1)$ such that $g_2(t) > t$ for $t \in (t_2^*, 1)$ and $g_2(t) < t$ for $t \in (0, t_2^*)$, it implies that $\psi_2(t) > 0$ for $t \in (t_2^*, 1)$. Then, a dual stop-loss treaty is ceded by the type 2 insurer. Similar to the discussions in case (ii), a variety of layer-type treaties can be optimally ceded by the type 1 insurer depending on the sign of $\psi_1(t)$ on $t \in (0,1)$. We refer interested readers to Section 5 for a more detailed treatment with the VaR and the TVaR.

Next, we discuss the net profit of the reinsurer in the optimal reinsurance menu, where the net profit is defined in (3). To emphasize that this is the profit in a separating equilibrium, we relabel it as P_S. Using Theorem 4.4, the net profit is given by

$$\begin{aligned} P_S &= \int_{\mathcal{A}_{\psi_1}} \left\{ g_1(\overline{F}_X(z)) - \left[p\overline{F}_X(z) + (1-p) g_2(\overline{F}_X(z)) \right] \right\} \mathrm{d}z \\ &\quad + (1-p) \int_{\mathcal{A}_{\psi_2}} \left[g_2(\overline{F}_X(z)) - \overline{F}_X(z) \right] \mathrm{d}z. \end{aligned} \tag{13}$$

Moreover, the welfare gain for the insurer of type 1 is $WG_{1,S} := \rho_{g_1}(X) - (\rho_{g_1}(X - f_1(X)) + \pi_1) = \rho_{g_1}(f_1(X)) - \pi_1 = 0$, where we use comonotonic additivity of ρ_{g_1}.[3] Moreover, the welfare gain for the insurer of type 2 is given by

$$\begin{aligned} WG_{2,S} &:= \rho_{g_2}(X) - (\rho_{g_2}(X - f_2(X)) + \pi_2) \\ &= \rho_{g_2}(f_2(X)) - \pi_2 \\ &= \rho_{g_2}(f_2(X)) - \pi_1 - \rho_{g_2}(f_2(X)) + \rho_{g_2}(f_1(X)) \\ &= \rho_{g_2}(f_1(X)) - \rho_{g_1}(f_1(X)) \geq 0. \end{aligned} \tag{14}$$

We make the following observations:

- *efficiency at the top*: the optimal indemnities for the type 2 insurer coincide with the optimal indemnities in absence of asymmetric information;
- The type 1 insurer is indifferent between buying reinsurance or not buying reinsurance; the type 2 insurer may strictly benefit from buying reinsurance.

From Theorem 4.4, it follows that $\tilde{h}_1(z) > 0$ only if $z \in \mathcal{A}_{\psi_1} \cup \mathcal{B}_{\psi_1}$, and thus only if $g_1(\overline{F}_X(z)) \geq \overline{F}_X(z)$. Moreover, from IR1$'$ it follows that $\pi_1 = \rho_{g_1}(f_1(X))$. Combining these two observations yields $\pi_1 = \rho_{g_1}(f_1(X)) \geq \mathbb{E}[f_1(X)]$. Moreover, since the menu $\{(\pi_1, f_1); (\pi_1, f_1)\}$ is feasible to Problem 3.1, it must also hold that $\pi_2 \geq \mathbb{E}[f_2(X)]$, because otherwise the objective P in (3) is not maximized. This finding not only indicates that the net profit of the reinsurer is non-negative, but also implies a non-negative risk loading for the actuarial premium principle.

[3] While the welfare gains are related to the concept of net profit for the reinsurer, we here refrain from using the terminology "net profit" in relation with risk measures. The welfare gain $WG_{\cdot,\cdot}$ represents the willingness-to-pay for a reinsurance contract (π, f) since it holds by construction that $\rho_g(X - f(X) + \pi + WG_{\cdot,\cdot}) = \rho_g(X)$.

Recall that from IC2′ (7c) it follows that

$$\begin{aligned}\pi_2 &= \pi_1 + \rho_{g_2}(f_2(X)) - \rho_{g_2}(f_1(X)) \\ &= \pi_1 + \int_0^M g_2(\bar{F}_X(z))[h_2(z) - h_1(z)]\mathrm{d}z.\end{aligned}$$

Thus, if the marginal indemnity functions for the two types of insurers h_2 and h_1 are close to each other, it is clear from the above expression that the premium π_2 will be very close to π_1. Hence, if the indemnities f_1 and f_2 are similar, the corresponding prices are similar as well. This excludes price discrimination.

In absence of asymmetric information, the reinsurer only offers the insurer a contract that makes the insurer indifferent between buying and not buying reinsurance (e.g. Amarante et al., 2015). Asymmetric information benefits the insurer if the insurer is of type 2, because the type 2 insurer can mimic to be of type 1, and appears to have a lower willingness-to-pay. On the other hand, the type 1 insurer is still indifferent between buying insurance or not, because this type cannot claim to have a lower willingness-to-pay. We shall highlight the above mentioned observations in Section 5.

4.2 Pooling equilibrium contracts

We again assume that Assumption 1 holds. Let us define the pooling equilibrium contracts provided by the reinsurer. The reinsurer always provides the same contract (π_P, f_P) regardless of the identity of the insurer. A pooling equilibrium is given by the reinsurance contract (π_P, f_P) that solves

$$\max_{(\pi,f)} \{\pi - \mathbb{E}[f(X)]\},$$

subject to $\pi \geq 0$, $f \in \mathcal{F}$, and the following individual rationality conditions:

$$\rho_{g_i}(X - f(X) + \pi) \leq \rho_{g_i}(X), \; i = 1, 2.$$

It is straightforward to see that the individual rationality condition for the type 1 insurer is binding.

Define

$$\begin{aligned}&\mathcal{A}_{p,1} = \{z \in [0, M] | g_1(\overline{F}_X(z)) > \overline{F}_X(z)\}, \quad \mathcal{B}_{p,1} = \{z \in [0, M] | g_1(\overline{F}_X(z)) = \overline{F}_X(z)\}, \\ &\mathcal{C}_{p,1} = \{z \in [0, M] | g_1(\overline{F}_X(z)) < \overline{F}_X(z)\}.\end{aligned}$$

Note that the conditions IR1 and IR2 jointly reduce to IR1, and thus $\pi_P = \rho_{g_1}(f_P(X))$. Then, the optimal indemnity function f_P can be obtained by solving

$$\max_{f_P \in \mathcal{F}} \{\rho_{g_1}(f_P(X)) - \mathbb{E}[f_P(X)]\},$$

which is equivalent to solving

$$\max_{h_P} \int_0^M \left[g_1(\overline{F}_X(z)) - \overline{F}_X(z)\right] h_P(z)\mathrm{d}z.$$

By Eq. (17) in Assa (2015) or the result of Theorem 3.1 in Zhuang et al. (2016), the optimal MIF for f_P can be derived as

$$\tilde{h}_P(z) = \begin{cases} 1 & \text{if } z \in \mathcal{A}_{p,1}, \\ a_P(z) & \text{if } z \in \mathcal{B}_{p,1}, \\ 0 & \text{if } z \in \mathcal{C}_{p,1}, \end{cases}$$

where a_P is any Lebesgue integrable function between 0 and 1. Thus, the net profit acquired by the reinsurer is given by

$$P_P := \int_{\mathcal{A}_{p,1}} \left[g_1(\overline{F}_X(z)) - \overline{F}_X(z)\right] \mathrm{d}z. \tag{15}$$

Obviously, it is easy to see that $P_P \geq 0$, meaning that the net profit acquired by the reinsurer is non-negative. Moreover, the welfare gains of the type 1 and type 2 insurer are given by $WG_{1,P} := \rho_{g_1}(f_P(X)) - \pi_P = 0$ and $WG_{2,P} := \rho_{g_2}(f_P(X)) - \pi_P \geq 0$, respectively.

We next show that the pooling equilibrium optimal reinsurance contracts are never better than the separating equilibrium reinsurance contracts.

Theorem 4.5 *If Assumption 1 holds, then it holds that $P_S \geq P_P$, where P_S and P_P are defined in (13) and (15), respectively.*

Proof. We readily verify that any reinsurance menu $\{(\pi, f); (\pi, f)\}$ satisfying IR1 satisfies IR2, IC1 and IC2 as well. Thus, for the pooling equilibrium reinsurance contract (π_P, f_P), the result is a direct consequence of the fact that the reinsurance menu $\{(\pi_P, f_P); (\pi_P, f_P)\}$ is feasible in Problem 3.1. ■It follows from Theorem 4.5 that the competitive reinsurance market that we study in this paper may only have a separating equilibrium and not a pooling equilibrium. The pooling equilibrium always leads to a (weakly) lower net profit for the reinsurer than the net profit in a separating equilibrium.

Remark 4.1 *If $g_1(t) = g_2(t)$ for all $t \in [0, 1]$, then it holds that $P_P = P_S$. In this case, the two types of insurers are identical, and it is optimal to offer the same insurance contract to the types of insurers.*

5 Numerical studies

In this section, we present two examples to illustrate our findings developed in the previous section. We shall discuss the optimal reinsurance menu when the insurer adopts the VaR measure or the TVaR measure.

5.1 The insurer adopts a VaR measure

In this subsection, we provide an example that serves as an illustration of Theorem 4.4 when the insurer adopts two types of VaR measures.

Definition 5.1 *The VaR of a non-negative random variable Z at a confidence level $\alpha \in (0,1)$ is defined as*

$$VaR_{\alpha}(Z) = \inf\{z \in \mathbb{R}_+ : \mathbb{P}(Z \leq z) \geq \alpha\}.$$

The VaR is a distortion risk measure with distortion function $g(t) = \mathbf{1}_{\{1-\alpha<t\leq 1\}}$ for $t \in [0,1]$, where $\mathbf{1}_A = 1$ if A holds and $\mathbf{1}_A = 0$ otherwise. If Z is a continuous random variable, the VaR of Z coincides with the corresponding the quantile.

Let the type 1 and type 2 insurer both use a VaR. The distortion functions are given by $g_1(t) = \mathbf{1}_{\{1-\alpha_1<t\leq 1\}}$ and $g_2(t) = \mathbf{1}_{\{1-\alpha_2<t\leq 1\}}$, for $t \in [0,1]$ and $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Clearly, we have $g_2(t) \geq g_1(t)$ for all $t \in [0,1]$, and thus Assumption 1 is satisfied. Also, let $p \in (0,1)$.

Note that

$$\psi_2(t) = \begin{cases} -(1-p)t & \text{if } 0 < t \leq 1-\alpha_2, \\ (1-p)(1-t) & \text{if } 1-\alpha_2 < t < 1, \\ 0 & \text{if } t \in \{0,1\}. \end{cases}$$

Thus, $\mathcal{A}_{\psi_2} = \{z \in [0,M] | 1-\alpha_2 < \overline{F}_X(z) < 1\}$, $\mathcal{B}_{\psi_2} = \{z \in [0,M] | \overline{F}_X(z) \in \{0,1\}\}$, and $\mathcal{C}_{\psi_2} = \{z \in [0,M] | 0 < \overline{F}_X(z) \leq 1-\alpha_2\}$. Then, from Theorem 4.4 it follows that

$$\tilde{h}_2(z) = \begin{cases} 1 & \text{if } z \in \mathcal{A}_{\psi_2}, \\ a_2(z) & \text{if } z \in \mathcal{B}_{\psi_2}, \\ 0 & \text{if } z \in \mathcal{C}_{\psi_2}. \end{cases}$$

Further, it can be easily calculated that $f_2(x) = \int_0^x \tilde{h}_2(z)\mathrm{d}z = \int_0^x \mathbf{1}_{\{z\in\mathcal{A}_{\psi_2}\}}\mathrm{d}z = \min\{x, \overline{F}_X^{-1}(1-\alpha_2)\}$, $x \geq 0$, which is a dual stop-loss indemnity contract ceded by the type 2 insurer.

Moreover, it holds that:

$$\psi_1(t) = \begin{cases} -pt & \text{if } 0 < t \leq 1-\alpha_2, \\ -(pt+1-p) & \text{if } 1-\alpha_2 < t \leq 1-\alpha_1, \\ p(1-t) & \text{if } 1-\alpha_1 < t < 1, \\ 0 & \text{if } t \in \{0,1\}, \end{cases}$$

from which we can obtain that $\mathcal{A}_{\psi_1} = \{z \in [0, M] | 1 - \alpha_1 < \overline{F}_X(z) < 1\}$, $\mathcal{B}_{\psi_1} = \{z \in [0, M] | \overline{F}_X(z) \in \{0, 1\}\}$, and $\mathcal{C}_{\psi_1} = \{z \in [0, M] | 0 < \overline{F}_X(z) \leq 1 - \alpha_1\}$. Thus, the result of Theorem 4.4 implies that

$$\tilde{h}_1(z) = \begin{cases} 1 & \text{if } z \in \mathcal{A}_{\psi_1}, \\ a_1(z) & \text{if } z \in \mathcal{B}_{\psi_1}, \\ 0 & \text{if } z \in \mathcal{C}_{\psi_1}. \end{cases}$$

We get that $f_1(x) = \int_0^x \tilde{h}_1(z) \mathrm{d}z = \min\{x, \overline{F}_X^{-1}(1 - \alpha_1)\}$, $x \geq 0$, which is also a dual stop-loss contract ceded by the type 1 insurer. Furthermore, the premium charged from the type 1 insurer is

$$\pi_1 = \rho_{g_1}(f_1(X)) = \overline{F}_X^{-1}(1 - \alpha_1),$$

and the premium charged from the type 2 insurer is

$$\pi_2 = \pi_1 + \rho_{g_2}(f_2(X)) - \rho_{g_2}(f_1(X)) = \overline{F}_X^{-1}(1 - \alpha_2),$$

which follows from $\rho_{g_1}(f_1(X)) = \rho_{g_2}(f_1(X))$. The net profit of the reinsurer is given by

$$P_S = p(\overline{F}_X^{-1}(1 - \alpha_1) - \mathbb{E}[\min\{x, \overline{F}_X^{-1}(1 - \alpha_1)\}]) + (1 - p)(\overline{F}_X^{-1}(1 - \alpha_2) - \mathbb{E}[\min\{x, \overline{F}_X^{-1}(1 - \alpha_2)\}]).$$

Assume that the loss variable X has an exponential distribution with mean 1.[4] Then, $\overline{F}_X(z) = \exp(-z)$. Moreover, let $\alpha_1 = 0.95$ and $\alpha_2 = 0.99$. We derive that $f_1(x) \approx \min\{x, 3.00\}$, $f_2(x) \approx \min\{x, 4.61\}$, $\pi_1 \approx 3.00$, $\pi_2 \approx 4.61$, and $P_S = 3.62 - 1.57p$, $p \in (0, 1)$. This means that the type 2 insurer is also indifferent between buying or not buying the reinsurance contract (π_2, f_2), i.e., $WG_{2,S} = 0$. Moreover, the net profit of the reinsurer is decreasing in the probability that the insurer is of type 1. Thus, dual stop-loss treaties are provided for the insurer regardless of which type he/she belongs to. Moreover, a higher cap is set for the type of insurer with the highest VaR-parameter, and as a result more premium will be charged to that type.

Now, let us consider the pooling equilibrium reinsurance contract proposed in Subsection 4.2. In this case, it is easy to derive that $\mathcal{A}_{p,1} = \mathcal{A}_{\psi_1}$, $\mathcal{B}_{p,1} = \mathcal{B}_{\psi_1}$, and $\mathcal{C}_{p,1} = \mathcal{C}_{\psi_1}$, which yields $f_P(x) = f_1(x) \approx \min\{x, 3.00\}$, $\pi_P = \pi_1 \approx 3.00$, and $P_P = \pi_P - \mathbb{E}[f_P(X)] = -\ln(1 - \alpha_1) - \alpha_1 \approx 2.05$. Clearly, it follows that $P_P \leq P_S$; indeed, they are equal when $p \to 1$. Moreover, the welfare gains of the insurer types 1 and 2 in the pooling equilibrium contract are given by $WG_{1,P} = 0$ and $WG_{2,P} = \rho_{g_2}(f_P(X)) - \pi_P = 0$.

[4] While we assumed boundedness of X, we here slightly abuse our notation by allowing for unbounded risk.

5.2 The insurer adopts a TVaR measure

Next, we present a numerical example when the insurer employs two types of distortion functions corresponding to the TVaR measures.

Definition 5.2 *The TVaR of a non-negative random variable Z at a confidence level $\alpha \in (0,1)$ is defined as*

$$TVaR_\alpha(Z) = \int_\alpha^1 VaR_\tau(Z)\mathrm{d}\tau.$$

The TVaR is a distortion risk measure with distortion function $g(t) = \min\{t/(1-\alpha), 1\}$ for $t \in [0,1]$. If Z is a continuous random variable, then $TVaR_\alpha(Z) = \mathbb{E}[Z|Z \geq VaR_\alpha(Z)]$.

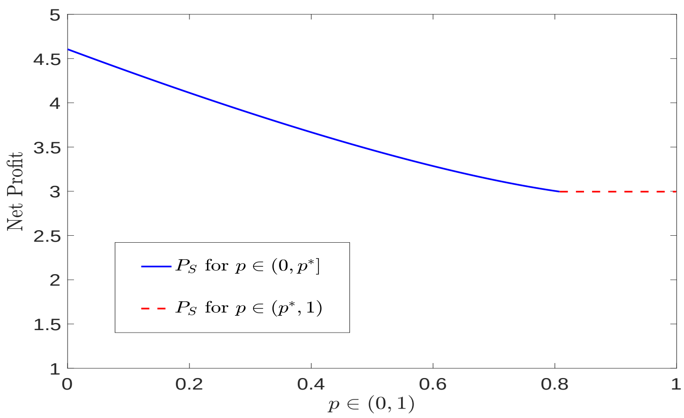

Figure 1: Net profit of the reinsurer in the optimal reinsurance menu, P_S, as function of $p \in (0,1)$, corresponding to Section 5.2.

Let the type 1 and type 2 insurer both use a TVaR measure. The distortion functions are given by $g_1(t) = \min\{t/(1-\alpha_1), 1\}$ and $g_2(t) = \min\{t/(1-\alpha_2), 1\}$, for $t \in [0,1]$ and $0 < \alpha_1 \leq \alpha_2 < 1$. We note that Assumption 1 is satisfied. Let $p \in (0,1)$.

We find that $\psi_2(t) = (1-p)(\min\{t/(1-\alpha), 1\} - t) > 0$ for all $t \in (0,1)$, and $\psi_2(0) = \psi_2(1) = 0$. Thus, $\mathcal{A}_{\psi_2} = (0, M)$, $\mathcal{B}_{\psi_2} = \{0, M\}$, and $\mathcal{C}_{\psi_2} = \emptyset$. By Theorem 4.4, we get that $f_2(x) = x$, $x \geq 0$, is optimal. In other words, full reinsurance is optimal for the type 2 insurer.

Define $p^* = \frac{1-\frac{\alpha_1}{\alpha_2}}{1-\alpha_1}$ and $t^* = \frac{(1-p)(1-\alpha_1)}{1-p(1-\alpha_1)}$. To determine the indemnity f_1, we get from Theorem 4.4 that we need to determine the sign of $\psi_1(t)$ for $t \in [0,1]$. It holds that

$$\psi_1(t) = \min\{t/(1-\alpha_1), 1\} - [pt + (1-p)\min\{t/(1-\alpha_2), 1\}].$$

We separate three different cases of the probability p, that is the probability that the insurer is of type 1.

(a) Case 1: $0 < p < p^*$. For this case, it follows that

$$\psi_1(t)\begin{cases} < 0 & \text{if } 0 < t < t^*, \\ > 0 & \text{if } t^* < t < 1, \\ = 0 & \text{if } t \in \{0, t^*, 1\}, \end{cases}$$

which means that $\mathcal{A}_{\psi_1} = \{z \in [0, M] | t^* < \overline{F}_X(z) < 1\}$, $\mathcal{B}_{\psi_1} = \{z \in [0, M] | \overline{F}_X(z) \in \{0, t^*, 1\}\}$, and $\mathcal{C}_{\psi_1} = \{z \in [0, M] | 0 < \overline{F}_X(z) < t^*\}$. From $0 < p < p^*$ and $\alpha_1 > 0$, we get that $1 - \alpha_2 < t^* < 1 - \alpha_1$.

According to Theorem 4.4, we have $f_1(x) = \int_0^x \mathbf{1}_{\{z\in\mathcal{A}_{\psi_1}\}}\mathrm{d}z = \min\{x, \overline{F}_X^{-1}(t^*)\}$, $x \geq 0$. Therefore, a dual stop-loss contract is optimal for the type 1 insurer, while a full reinsurance policy is optimal for the type 2 insurer as mentioned above. Furthermore, the corresponding premium charged by the reinsurer to the type 1 insurer is given by

$$\begin{aligned}
\pi_1 &= \rho_{g_1}(f_1(X)) = \int_0^M g_1(\overline{F}_X(z))\tilde{h}_1(z)\mathrm{d}z = \int_{\overline{F}(z)>t^*} g_1(\overline{F}_X(z))\mathrm{d}z \\
&= \int_{1-\alpha_1<\overline{F}_X(z)\leq 1} \mathrm{d}z + \int_{t^*<\overline{F}_X(z)\leq 1-\alpha_1} \frac{\overline{F}_X(z)}{1-\alpha_1}\mathrm{d}z \\
&= \overline{F}_X^{-1}(1-\alpha_1) + \frac{1}{1-\alpha_1}\int_{\overline{F}_X^{-1}(1-\alpha_1)}^{\overline{F}_X^{-1}(t^*)} \overline{F}_X(z)\mathrm{d}z.
\end{aligned}$$

The premium charged from the type 2 insurer is given by

$$\begin{aligned}
\pi_2 &= \pi_1 + \rho_{g_2}(f_2(X)) - \rho_{g_2}(f_1(X)) \\
&= \pi_1 + \int_0^M g_2(\overline{F}_X(z))\tilde{h}_2(z)\mathrm{d}z - \int_0^M g_2(\overline{F}_X(z))\tilde{h}_1(z)\mathrm{d}z \\
&= \pi_1 + \int_0^M g_2(\overline{F}_X(z))\mathrm{d}z - \int_{\overline{F}_X(z)>t^*} g_2(\overline{F}_X(z))\mathrm{d}z \\
&= \pi_1 + \int_{\overline{F}_X(z)\leq t^*} g_2(\overline{F}_X(z))\mathrm{d}z \\
&= \pi_1 + \int_{\overline{F}_X(z)\leq 1-\alpha_2} \frac{\overline{F}_X(z)}{1-\alpha_2}\mathrm{d}z + \int_{1-\alpha_2<\overline{F}_X(z)\leq t^*} \mathrm{d}z \\
&= \pi_1 + \frac{1}{1-\alpha_2}\int_{-\ln(1-\alpha_2)/\lambda}^M \overline{F}_X(z)\mathrm{d}z + (\overline{F}_X^{-1}(1-\alpha_2) - \overline{F}_X^{-1}(t^*).
\end{aligned}$$

Suppose again that the risk X has an exponential distribution with mean λ. Then,

$$\pi_1 = -\frac{\ln(1-\alpha_1)}{\lambda} + \frac{1-\alpha_1-t^*}{\lambda(1-\alpha_1)} = -\frac{\ln(1-\alpha_1)}{\lambda} - \frac{t^*}{\lambda(1-\alpha_1)} + \frac{1}{\lambda},$$

$$\pi_2 = \pi_1 + \frac{1}{1-\alpha_2}\int_{-\ln(1-\alpha_2)/\lambda}^{M} e^{-\lambda z}\mathrm{d}z + \int_{-\ln(t^*)/\lambda}^{-\ln(1-\alpha_2)/\lambda} \mathrm{d}z$$

$$= \pi_1 + \frac{1}{\lambda} + \frac{\ln\left(\frac{t^*}{1-\alpha_2}\right)}{\lambda} = \frac{2+\ln\left(\frac{t^*}{(1-\alpha_1)(1-\alpha_2)}\right)}{\lambda} - \frac{t^*}{\lambda(1-\alpha_1)}.$$

The welfare gain of the type 2 insurer is given by

$$WG_{2,S} = \rho_{g_2}(f_2(X)) - \pi_2 = \frac{1}{\lambda} - \frac{\ln(1-\alpha_2)}{\lambda} - \pi_2 = \frac{t^*}{\lambda(1-\alpha_1)} - \frac{1+\ln\left(\frac{t^*}{1-\alpha_1}\right)}{\lambda}.$$

Finally, the net profit of the reinsurer from the menu $\{(\pi_1, f_1); (\pi_2, f_2)\}$ is given by

$$P_S = \pi_1 - \frac{p}{\lambda}(1-t^*) + \frac{1-p}{\lambda}\ln(t^*/(1-\alpha_2)).$$

Now, let $\lambda = 1$, $\alpha_1 = 0.95$ and $\alpha_2 = 0.99$. Then, we find $p^* = 0.81$. Figure 1 displays the plot of the net profit P_S as function of the probability p that the insurer is of type 1. The net profit of the reinsurer P_S is decreasing in $p \in (0, p^*)$. The welfare gain obtained by the type 2 insurer through buying reinsurance is plotted in Figure 2, from which we know the type 2 insurer will definitely benefit from the reinsurance menu since $WG_{2,S} > 0$. It depends on the probability p via t^*. Taking $p = 0.6$ as an example, we have $f_1(x) \approx \min\{x, 3.88\}$, $x \geq 0$, $\pi_1 \approx 3.58$, $\pi_2 \approx 5.31$, $P_S \approx 3.29$, and $WG_{2,S} \approx 0.30$.

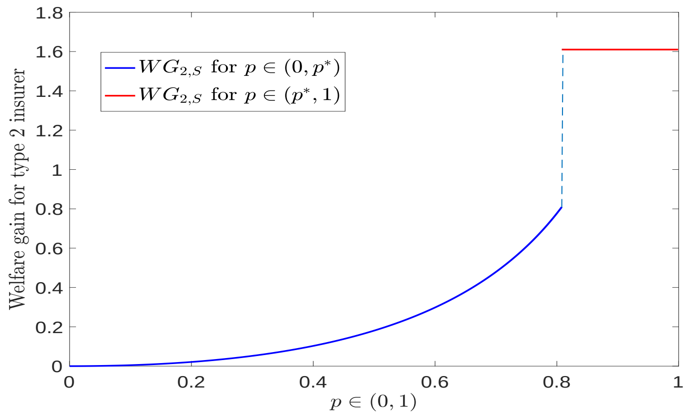

Figure 2: Welfare gain of the type 2 insurer in the optimal reinsurance menu, $WG_{2,S}$, as function of $p \in (0, 1)$, corresponding to Section 5.2.

(b) Case 2: $p = p^*$. It follows that $\psi_1(t) = 0$, for $t \in [0, 1-\alpha_2]$, and $\psi_1(t) > 0$, for $t \in (1-\alpha_2, 1)$. For $z \in \mathcal{B}_{\psi_1}$, the function $a_1(z)$ can be chosen freely between 0 and 1, and we pick $a_1(z) = 0$. Then, we have $f_1(x) = \min\{x, \overline{F}_X^{-1}(1-\alpha_2)\}$, $x \geq 0$.

Now, if X is exponentially distributed with mean λ, then $f_1(x) = \min\{x, -\ln(1-\alpha_2)/\lambda\}$, $x \geq 0$, $\pi_1 = \overline{F}_X^{-1}(1-\alpha_1) - \frac{\alpha_1-\alpha_2}{\lambda(1-\alpha_1)}$, $\pi_2 = \frac{2-\ln(1-\alpha_1)}{\lambda} - \frac{1-\alpha_2}{\lambda(1-\alpha_1)}$, and $P_S = -\frac{\ln(1-\alpha_1)}{\lambda} - \frac{\alpha_1-\alpha_2}{\lambda(1-\alpha_1)} - \frac{p\alpha_2}{\lambda}$. Now, assume that $\lambda = 1$, $\alpha_1 = 0.95$, and $\alpha_2 = 0.99$. Thus, $p = p^* = 0.81$. For this case, we have $f_1(x) \approx \min\{x, 4.61\}$, $x \geq 0$, $\pi_1 \approx 3.80$, $\pi_2 \approx 4.80$, $P_S \approx 3.00$, and the type 2 insurer's benefit from the reinsurance menu is $WG_{2,S} = \rho_{g_2}(X) - \pi_2 \approx 0.81$.

(c) Case 3: $p^* < p < 1$. For this case, it can be seen that $\psi_1(t) > 0$ for all $t \in (0,1)$. Thus, $f_1(x) = x$, $x \geq 0$, that is, the same full reinsurance policy will be ceded by the insurer no matter which type the insurer is. As a result, $\pi_1 = \pi_2 = \rho_{g_1}(X)$ and $P_S = \rho_{g_1}(X) - \mathbb{E}[X]$. Now, let X be exponentially distributed with mean 1, and, moreover, let $\alpha_1 = 0.95$ and $\alpha_2 = 0.99$. We then have $\pi_1 = \pi_2 \approx 4.00$ and $P_S \approx 3.00$ (see the red dashed line in Figure 1). The benefit obtained by the type 2 insurer from the menu is given as $WG_{2,S} = \rho_{g_2}(X) - \pi_2 \approx 1.61$ (see the red line in Figure 2). It is interesting to note that the welfare gain acquired by the type 2 insurer for $p^* < p < 1$ is strictly larger than that for $0 < p \leq p^*$. If the reinsurer believes that it is more likely to face a type 1 insurer (so if p is large), than the optimal reinsurance contract aims to attract maximum welfare from the type 1 insurer, and thus the type 2 insurer benefits more from the asymmetric information.

According to the three cases above, we know that a full reinsurance menu will be provided to the insurer without considering his/her identity when the probability that the insurer is of type 1 exceeds a threshold p^*; otherwise, a dual stop-loss treaty and a full insurance treaty will be ceded by the two types of the insurer.

Next, we discuss the pooling equilibrium contract. We find that the pooling equilibrium reinsurance contract is given by $f_P(x) = x$, $x \geq 0$, and $\pi_P = \rho_{g_1}(X)$. Thus, $P_P = \rho_{g_1}(X) - \mathbb{E}[f_P(X)]$, $WG_{1,P} = 0$, and $WG_{2,P} = \rho_{g_2}(f_P(X)) - \pi_P$. If we assume again that X is exponentially distributed with mean 1, $\alpha_1 = 0.95$, and $\alpha_2 = 0.99$, then $f_P(x) \approx \min\{x, 3.00\}$, $x \geq 0$, $\pi_P \approx 4.00$, $P_P \approx 3.00$, and $WG_{2,P} \approx 1.61$. This means that the type 2 insurer strictly benefits from the pooling reinsurance contract. In fact, if $0 < p < p^*$, then the welfare gain of the type 2 insurer is larger in the pooling equilibrium than in the separating equilibrium. Moreover, the net profit of the reinsurer in the pooling equilibrium contract is not larger than the net profit of the reinsurer in the separating equilibrium contract.

Summary: to conclude this section, let us briefly summarize our findings. First, we summarize the case where the insurer adopts a VaR measure. For the separating equilibrium, both types of insurers buy (possibly different) dual stop-loss treaties, and pay their indifference premiums. This implies that the welfare gains for the types 1 and 2 insurers are both zero. The pooling equilibrium indemnity is also a dual stop-loss treaty.

Next, we summarize the case where the insurer adopts a TVaR measure. For the separating equilibrium, the type 2 insurer buys full insurance, while the type 1 insurer may buy a dual stop-loss or full insurance treaty. The type 2 insurer may strictly benefit from buying reinsurance. The pooling equilibrium indemnity is a full insurance treaty.

6 Conclusion

We have studied a problem of optimal reinsurance design under asymmetric information when the risk preferences of the insurer are unknown to the reinsurer. We propose a framework in which the insurer adopts distortion risk measures and one type of insurer has a larger willingness-to-pay than the other type of insurer for every risk. The optimal reinsurance contract menu is derived in closed-form by maximizing the net profit of the reinsurer under two individual rationality constraints and two incentive compatibility constraints. We also presented two examples to illustrate the reinsurance menu when the insurer uses the VaR or the TVaR. The present results can be easily generalized to the case when the reinsurer minimizes a distortion risk measure as well.

Our results hold under Assumption 1, that states that one type of insurer has a larger willingness-to-pay than the other type of insurer for every risk. Further investigation is needed to find solutions of Problem 3.1 without Assumption 1.

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