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Decision Support

Pricing and hedging in incomplete markets with model uncertainty

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ABSTRACT

We search for a trading strategy and the associated robust price of unhedgeable assets in incomplete markets under the acknowledgement of model uncertainty. Our set-up is that we postulate the management of a firm that wants to maximise the expected surplus by choosing an optimal investment strategy. Furthermore, we assume that the firm is concerned about model misspecification. This robust optimal control problem under model uncertainty leads to (i) risk-neutral pricing for the traded risky assets, and (ii) adjusting the drift of the nontraded risk drivers in a conservative direction. The direction depends on the firm's long or short position, and the adjustment that ensures a robust strategy leads to what is known as "actuarial" or "prudential" pricing. Our results extend to a multivariate setting. We prove existence and uniqueness of the robust price in an incomplete market via the link between the semilinear partial differential equation and backward stochastic differential equations for viscosity and classical solutions.

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1. Introduction

When markets are complete and arbitrage-free then there exists a unique pricing measure such that all contracts have a unique price. However real world processes and situations are not perfectly hedgeable. Therefore there is the risk of a mismatch between optimal policies based on a complete market assumption and the consequence the theoretical strategy has in practice. In this paper, we want to minimise this mismatch by a robust strategy. Pension funds, insurance companies and many other parties need to price liabilities that are not or only partially hedgeable. Multiple equivalent martingale measures exist in an incomplete market. Moreover the underlying model of both hedgeable and unhedgeable risk factors can be wrong. In financial and economic decision making allowing for model uncertainty has an impact on the optimal strategies. Our goal is to find a pricing method for assets in incomplete markets under the acknowledgement of model uncertainty with respect to the noise.

In classical optimisation, the standard optimal strategy is obtained when the firm maximises its surplus based on the assumption that its model description is correct. However, if the firm is uncertain about the model the firm anticipates to a worst-case happening. To ensure a robust strategy the firm pretends that mother nature picks the worst-case model among a set of plausible models, whereafter the firm maximises the surplus given this worst-case model. Without uncertainty, the standard optimal strategy can harm if mother nature selects the worst-case model afterwards. While the robust strategy is less vulnerable to perturbations of the underlying model. Hence mother nature plays the minimising role and the firm the maximising role. This is known as the maxmin method that Gilboa and Schmeidler (1989) introduced to deal with multiple priors in a Bayesian setting.

We focus on uncertainty of the underlying stochastic processes and we adhere to risk neutrality of the firm as is supported by the famous work of Modigliani and Miller (1958). Our postulated firm represents the management board rather than an individual investor. Modigliani and Miller (1958) claim that the shareholders can hedge their risk themselves, while the firms are risk neutral. Moreover, pure profit maximisation is practical in its simplicity and since corporations often use heuristics by setting easy justifiable targets the neutrality is also motivated. Thus the robust price that we obtain from the profit maximisation is unique in the sense that we do not allow for different prices based on different risk attitudes. We show in this paper that profit maximisation with model ambiguity results in a bounded optimisation problem.

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Modelling assumptions on future developments of financial and nonfinancial products are questioned a lot, especially since the latest financial crisis. For the nonfinancial market there is even less validity to claim to know the "true" data generating process. Therefore we focus on the impact of uncertainty on hedging, and we also derive the robust price of nonfinancial contracts. We show that ambiguity about the underlying model leads to a change in the firm's standard delta-hedge position. Hence, the mere objective of maximising the surplus, which is defined as the difference between the investment position and the liabilities, implies an adjusted delta-hedging strategy. In this paper prices of nontraded liabilities that take model uncertainty into account are implied by the robust trading strategies. The robust price, given by a semilinear partial difference equation, is derived via the concept of indifference pricing. Pricing and hedging in incomplete markets is of great interest in financial and economic modelling. In the pricing literature, both in complete and incomplete markets, the focus lies on pricing (complex) contracts where model ambiguity is often ignored. We show in this paper that a profit-maximising objective for a company with an ambiguity averse attitude, leads to a unique time-consistent robust price and to a unique time-consistent robust hedging strategy for both a long and a short position in an incomplete market. We find both the existence and uniqueness of a viscosity and the classical solution that are sufficiently differentiable under the specified assumptions. The main take-away of the robustness adjustment is that the hedgeable risk is priced by the risk-neutral measure without the need to impose this upfront. And the drift of the unhedgeable risk is moved towards the risk-free rate, proportional to the correlation between the financial and nonfinancial risk, which measures the degree of hedgeability. Depending on the position of the owner of the liability, this drift is augmented or reduced by the residual of the degree of hedgeability – which is the part that is not correlated with the financial hedgeable risk – multiplied by the level of uncertainty. Thus the more uncertain, the more conservative the drift of the unhedgeable risk is. Without unhedgeable risk, thus being in a complete market, the optimal hedge position that we obtain coincides with the standard delta-hedge. The presence of nonfinancial risk leads to an increase in the demand for the traded risky assets equal to the delta-hedge of the unhedgeable risk proportional to the correlation and volatilities of both risk types plus a "market confidence term" proportional to the residual of the correlation. This market confidence term equals the market price of risk scaled down by the "net level of uncertainty". Thus the lower the trust in a correct model specification of the investor, the less the firm invests in risky assets while if the set of plausible models shrinks the certainty that the risky assets generates a higher return than the bank account goes up, increasing the demand for the first. We illustrate our approach in the context of pensions and insurances, though it can be applied more generally.

The remainder of this paper is organised as follows: at first, we describe and discuss related literature in Section 2. We introduce the uncertain financial and nonfinancial market in Section 3. In Section 4 we solve the robust optimisation problem by deriving the optimal strategies and indifference price, summarised by our main contribution Theorem 8 and Corollary 11. In Section 5 we show the robust price of the illiquid liability in the extreme setting of a pure incomplete market. And we check the disappearance of the effect of uncertainty on pricing in a pure complete market setting. The economic interpretations of the optimal solutions are explained in detail by the two examples in Section 6. While our approach can be applied more generally, as shown in the first example, the second example is illustrated in the context of asset-liability management which is related to insurance and actuarial problems. This life insurance contract application showcases the approach naturally. Finally, Section 7 concludes.

2. Literature

We first discuss the literature on pricing under uncertainty and robustness. Then we discuss related literature on incomplete markets. To the best of our knowledge we are the first to combine uncertainty with pricing and hedging in incomplete markets. By this we obtain robust prices of (unhedgeable) liabilities.

Portfolio optimisation started with the seminal work of Merton (1969) who solves the investment problem in a complete market setting for power utility. The price process of the asset is assumed to be characterised by a geometric Brownian motion and optimal investment strategy, based on the presence of risk and an aversion to it, is derived. But the large empirical equity premium implies theoretically more risky investments than data shows us. The difference between the optimal and the observed investment strategy is known as the equity premium puzzle. The puzzle might be explained by the presence and aversion to model uncertainty, which is incorporated by Maenhout (2004) and Biagini and Pınar (2017) among others and in Bansal and Yaron (2004)'s long-run risk model, time-varying economic uncertainty justifies the puzzle. The demand for robust strategies is motivated by the fact that "The optimal portfolio based on the inaccurate parameter values may be a solution that is far from optimal in reality" as Lutgens, Sturm, and Kolen (2006) write. Especially the return is difficult to estimate with accuracy. Maximising utility under model uncertainty has been considered in different settings, see e.g. Bordigoni, Matoussi, and Schweizer (2007) or Goldfarb and Iyengar (2003). A literature review on ambiguity in asset pricing and portfolio choice is given by Fabozzi, Huang, and Zhou (2010), Kim, Kim, and Fabozzi (2014) and Guidolin and Rinaldi (2013)

Most investment decisions mentioned above emerge from an individualistic point of view which is common in finance. A branch of the literature contributes to the extension of allowing for ambiguity rather than risk only. Note here the difference between risk and uncertainty[1]. Risk describes the situation that a specific event might happen with a known probability distribution whereas for uncertainty the probability distribution is not precisely known. However, these portfolio problems under uncertainty do not solve pricing issues nor allow for incompleteness. Pricing stems from both large investors as well as sellers of complex products and comes in our paper as a natural by-product of the profit maximisation. In operations research profit maximisation is the viewpoint often considered, and as such a risk neutral attitude is indirectly used to price liabilities in incomplete markets. Without uncertainty this would lead to unboundedness of the problem, whereas the specification of a concave utility function could solve this. Uncertainty however enables us to investigate the well-posedness of profit maximisation. Moreover, the main difference with the listed literature on ambiguity is that we solve a profit maximisation problem under ambiguity in an *incomplete* market. Besides, less liquid assets – leading to incompleteness – are even more prone to an inaccurate description of the risk process. A robust price is obtained when also uncertainty in the underlying risk factors is taken into account.

Baltas, Xepapadeas, and Yannacopoulos (2018a) derive mild solutions of robust control problems for a general class of utility and penalty functions. The preference for robustness parameter needs to be large enough to ensure existence. Baltas and Yannacopoulos (2017) investigate the impact of private information on the optimal investment decisions for several utility parameterisations. Inside information can be interpreted as uncertainty that is revealed to only part of the investors. The market is incomplete due to their stochastic factor model setting. In comparison to our paper,

[1] We use "ambiguity" and "model uncertainty" interchangeably.

we both obtain closed form solutions for the optimal investment decisions though with different objectives. Moreover, we explicitly price the liability product whereas the focus in Baltas and Yannacopoulos (2017) is to find the numerical impact of private information. In Baltas and Yannacopoulos (2016) they also consider an investor who possesses inside information but in a complete market in which the investor is explicitly uncertain about the validity of this private information. Hu, Chen, and Wang (2018) price reinsurance contracts under uncertainty about the intensity of claims. Contrary to representative-agent models, the supply and demand side of the contract determine the price by which the incompleteness of the insurance market is circumvented. Barrieu and El Karoui (2005) consider a buyer and seller perspective in the two player game, where the pricing measure is known as the solution of an inf-convolution problem.

There are several well-known methods to robustify decisions. All these approaches possibly imply different preferences and contribute differently to the optimal strategies. However, they all attribute the similar spirit of safeguarding decisions to the fact that the baseline model might be wrong by allowing for plausible alternative models.

Biagini and Pınar (2017) consider an ellipsoid shape of the uncertainty set, similar as we do, to find the robust strategies. The intuition arises from the constrained optimisation introduced by Gilboa and Schmeidler (1989). Other methods to robustify decisions consist of adding a penalty term to the objective weighted by a preference for robustness. Hansen and Sargent (2008) add a relative entropy penalty to the objective, which measures how far the alternative probability distribution is from the baseline. The penalty that is added is the total relative entropy, i.e. the entropy summed over the whole investment horizon. A translation of this optimisation problem into a forward backward stochastic differential equation framework implies a quadratic driver. In the equivalent constraint approach a constraint is added in which the total relative entropy is bounded. The relation between these approaches is discussed by Hansen, Sargent, Turmuhambetova, and Williams (2006). The main difference is that in the penalty method all equivalent measures are considered – weighted by the divergence measure – whereas in the constraint approach an alternative model is either in the uncertainty set or not – having a binary penalty of infinity or zero. Maccheroni, Marinacci, and Rustichini (2006) show the connection between the multiple priors method from Gilboa and Schmeidler (1989) and the multiplier preferences method from Hansen and Sargent (2001) which are embedded in the variational preferences framework. We however impose a constraint on the market price of risk per instant of time instead of a bound on the total entropy aggregated over all future periods. This makes our solution directly time-consistent, and contrary to the quadratic driver in the penalty approach, our driver is semilinear due to the absolute value dependence.

To apply the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005) that separates ambiguity beliefs from the ambiguity attitude, we would need a distribution for the drifts and an attitude towards ambiguity. If this "prior" distribution over the different models is uniformly distributed on the interval described by the ellipsoid in our constraint and the ambiguity aversion goes to infinity, then the robust maxmin we employ and the smooth ambiguity model coincide. This relates to the Bayesian perspective of model averaging in which a weighted average of several models reduces to a meta model. The solution boils down to the nonrobust strategy based on this meta model.

Allowing for uncertainty via the penalty method in the Merton portfolio problem, in which a geometric Brownian motion with drift is assumed for the stock price process, cannot be solved analytically unless homothetic preferences are imposed as Maenhout (2004) shows. Baltas, Xepapadeas, and Yannacopoulos (2018b) also use the Maenhout parameterisation to obtain closed form expressions for the robust-entropic optimal control problem that arises in the management of financial institutions. In their setting the liability can be transferred to a third party, e.g. a reinsurer. If the decision maker's preference for robustness goes to infinity, i.e., if he has no faith in the model, then existence is not guaranteed as Baltas et al. (2018b) show. Though for the extreme at this end, our problem is well-posed. We however need "enough" uncertainty to guarantee existence. Whereas without uncertainty, Baltas et al. (2018b) result boils down to the classical solutions depending on which concave utility function imposed.

A market is incomplete if there are multiple equivalent martingale measures which is in line with the first fundamental theorem of asset pricing while violating the second (Delbaen & Schachermayer, 1994). In an incomplete market it is not possible to replicate all payoffs. Consequently perfect hedging is not possible. Frictions such as transaction costs, but also illiquidity or asymmetric information cause incompleteness. Examples of pricing in incomplete markets are; pricing and hedging extremely long dated obligations such as pensions that have to be paid on a horizon up to a century (since life expectancy increases the horizon grows even further) or pricing insurance contracts linked to instruments that are not or barely traded, such as the number of survivors (this example is shown in Section 6.2), temperature or natural disasters. We model the incompleteness by allowing liabilities to depend on risk factors that are nontraded.

Caldentey and Haugh (2006) consider a risk averse corporation that hedges in an incomplete market. Moreover, they optimise with respect to the operation policy and they consider different information structures about the degree of completeness. Hernández-Hernández and Schied (2006, 2007) consider HARA and logarithmic utility for robust utility maximisation in incomplete markets and derive the partial differential equation for the classical solution via the duality approach. Mania and Tevzadze (2008) solve optimality of the utility maximisation in incomplete markets via duality too, and derive conditions to ensure optimality arising from the relation between the backward stochastic partial differential equation and the forward stochastic differential equation. Pınar (2006) price and hedge contingent claims in a discrete and incomplete setting in which the objective is to find an optimal balance in a portfolio between the gain and loss due to imperfect hedging. The gain-loss objective is combined with a linear utility which enables him to use linear programming techniques. In our case nonlinearity arises because of the uncertainty which is not present in Pınar (2006).

Gundel and Weber (2007) solve a robust portfolio selection problem with a constraint on the amount of downside risk to generate a robust decision. Here the market is incomplete though not aimed a finding a price of the nontraded asset, nor is this directly affecting the payoff. Incompleteness arises also due to the presence of jumps in processes. Kennedy, Forsyth, and Vetzal (2009) search for the hedge position that eliminates the jump risk at minimised transaction costs. Without a focus on pricing unhedgeable liabilities, stochastic volatility causes markets to be incomplete as well. Marroquı and Moreno (2013) investigate the bounds on security prices such different stochastic volatility models.

The classical solution that we derive ensures the existence of the closed form hedging strategies and the uniqueness of the robust price of the liability. These analytical expressions yield interesting interpretations contrary to viscosity solutions. Cheridito, Horst, Kupper, and Pirvu (2015) and Cheridito and Hu (2011) derive existence and uniqueness conditions of equilibrium prices in incomplete markets with heterogeneous agents or stochastic constraint respectively via the BSDE connection as well.

3. Financial and nonfinancial market with model ambiguity

We consider a firm that has a liability $L(\cdot)$ that must be paid at time T. The firm – representing the board of an (insurance) company – wants to hedge its liability that depends on both hedgeable and unhedgeable risk factors, by finding the optimal hedging portfolio. The hedging portfolio $A(\cdot)$ consists of a portfolio of traded assets, represented by the positions in the hedgeable risk factors. This set-up is similar as in an asset-liability management (ALM) problem.

Moreover, the firm is uncertain about the models underlying the hedgeable and unhedgeable risk factors. Therefore the objective reduces to a robust portfolio optimisation problem in which the firm targets to hedge the liability subject to its uncertainty. Gilboa and Schmeidler (1989) introduced the max min concept that is utilised to obtain a robust solution. In this setting, the firm robustifies its decision by introducing a counterplayer who minimises its objective by selecting the worst-case model from the uncertainty set. Without uncertainty, the objective of the firm is to match the liability as good as possible by the hedging portfolio, which is summarised by maximising the surplus of its position of assets minus liabilities. After introducing the minimising counterplayer, by which the firm anticipates on the effect of uncertainty, the worst-case probability measure is revealed and the firm derives its robust investment strategy. By indifference pricing, we derive the unique robust price of the liability that is driven by the worst-case probability measure. This inherits both an upper and lower bound on the indifference price which can be interpreted as the bid and ask price.

The liabilities may depend on both hedgeable and unhedgeable risk. Hedgeable risk represents the risk that underlies the liquid and traded assets, i.e. the firm can have short and long positions in these assets. If only this type of risk is present, this market is called a complete market. Whereas in an incomplete market, additional risk factors that are not traded are present which are therefore unhedgeable. We consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ that satisfies the usual hypotheses and where $\mathcal{F}_t$ is the natural filtration generated by the standard Brownian motion, i.e. $\mathcal{F}_t = \sigma(W(s), s \le t)$. Throughout, the explicit stochasticity dependence on ω will be suppressed whenever possible. Let there be n tradeable assets $X = (X_1, \dots, X_n)$ and l untradeable risk factors $Y = (Y_1, \dots, Y_l)$, where n and l can be chosen independently, and let there be a bank account X_0 on which one can go short or long for the interest rate $r(t, X_0, X)$. These three processes follow the stochastic differential equation (SDE)

$$d\begin{pmatrix} X_0(t) \\ X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} r(t, X_0, X)X_0(t) \\ \mu^X(t, X, Y) \\ \mu^Y(t, X, Y) \end{pmatrix} dt + \left(\begin{matrix} 0'_{[n+l]} \\ \begin{pmatrix} \Sigma^{XX}(t, X, Y) & \Sigma^{XY}(t, X, Y) \\ \Sigma^{YX}(t, X, Y) & \Sigma^{YY}(t, X, Y) \end{pmatrix}^{1/2} \end{matrix} \right) d\begin{pmatrix} W^X(t) \\ W^Y(t) \end{pmatrix}, \tag{1}$$

with initial conditions $X_0(0) = x_0 \in \mathbb{R}_+, X(0) = x \in \mathbb{R}^n_+, Y(0) = y \in \mathbb{R}^l$ and where $\mu^X(t, X, Y)$ and $\mu^Y(t, X, Y)$ are $\mathbb{R}^n$ and $\mathbb{R}^l$-valued. The dimensions of the covariance matrices are $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$, $\mathbb{R}^{l\times n}$ and $\mathbb{R}^{l\times l}$ for $\Sigma^{XX}(t, X, Y)$, $\Sigma^{YX}(t, X, Y)$, $\Sigma^{XY}(t, X, Y)$ and $\Sigma^{YY}(t, X, Y)$ respectively. Both drift vectors are allowed to depend on the hedgeable risk, i.e. the traded assets and the unhedgeable risk. And W^X and W^Y are the n-dimensional and l-dimensional Brownian motion under $\mathbb{P}$, respectively. In (1) we display each block matrix of the $[(n+l) \times (n+l)]$ covariance matrix $\Sigma(t, X, Y)$ separately. The superscript XX refers to the covariance matrix of the hedgeable process, superscript $XY = (YX)'$ refers to the correlation between the hedgeable and unhedgeable factors and superscript YY refers to the covariance matrix of the unhedgeable process. We assume that the covariance matrix $\Sigma(t, X, Y)$ is positive definite and thus invertible. The subscript attached to $0_{[n+l]}$ indicates the dimension of the zero vector. The interest rate that drives the bank account can either be constant or follow a stochastic process. The interest rate obtained on the bank account is the integral over the short rate $r(t, X_0, X)$ that we assume to be progressively measurable and which maps to $\mathbb{R}$, in other words we do not exclude negative interest rates as is currently the case in the financial market. Without the bank account, the above SDE reduces to the matrix-vector form

$$dZ(t) = \mu(t, Z)dt + \Sigma(t, Z)^{1/2}dW^Z(t), \tag{2}$$

where $Z = (X, Y)'$ and $\Sigma(t, Z)$ is the $[(n+l) \times (n+l)]$ covariance matrix which is decomposed in Eq.(1). Assumptions on the drifts, covariance matrices and terminal condition are extensively discussed in Section 4 where we derive the viscosity and classical solutions.

The amount invested on the bank account is denoted by $\theta_0(t)$ and to later simplify notation we introduce the combing vector $\bar{\theta}(t) = (\theta_0(t), \theta^X(t))'$, $\theta(t) = (\theta^X(t), \theta^Y(t))'$ and $\bar{X}(t) = (X_0(t), X(t))'$. How much the firm goes long or short on the bank account is captured by the position $\theta_0(t)$ on which we do not assume any short selling constraints. Each $\theta_i(t)$ with $i > 0$ represents the number of assets the firm buys of asset $X_i(t)$. Consequently the amount invested is $\theta_i(t) \times X_i(t)$ at time t. The total value of the asset side or wealth A includes the bank account, therefore $A(t) = \bar{\theta}(t)'\bar{X}(t)$. Note that the change in the value of the investment position is denoted by $dA(t)$, while we highlight the explicit dependence on the hedgeable risk factors by $A(t, X(t))$ contrary to the liabilities which might depend on nonhedgeable risk as well. Hence, we use both the notation $A(t)$ and $A(t, X)$ depending on the context, where the latter reminds us explicitly of its dependence structure. At every point in time a change in the value of the assets can only occur due to a change in the underlying values, stated differently gains or losses are not possible by a re-allocation over the different assets. This is incorporated by the self-financing condition $dA(t) = \bar{\theta}(t)'d\bar{X}(t)$.

Before we continue, we rephrase the optimisation problem in terms of a numéraire. The bank account $X_0(t)$ is used as numéraire and we denote all random variables that are expressed in terms of the discounted value with a tilde. Division of the liability by the numéraire gives $\tilde{L}(t, Z) = \frac{L(t,Z)}{X_0(t)}$ and the stochastic process of the asset side is derived in Proposition 1.

The goal of maximising the profit (equivalently payoff or surplus) has a widespread applicability and intuition described in the introduction. The firm's choice variable at every point in time is $\tilde{\theta}^X(t)$ the allocation of its wealth over the different financial assets for the next time step. The investment strategy $\tilde{\theta}^X(t)$ is the n-dimensional progressively measurable process and we assume that $\int_0^T (\tilde{\theta}^X)'(s)\Sigma^{XX}(s, Z)\tilde{\theta}^X(s)ds$ is finite $\mathbb{P}$–a.s. The amount of wealth held in the self-financing portfolio at time t is denoted by the random variable $\tilde{A}(t, X)$ that depends on the investment decision $\tilde{\theta}^X(t)$ and the value of the traded assets $X(t)$ itself and is described by the self-financing SDE given in the following proposition.

Proposition 1. *By the self-financing condition, the total value of wealth evolves by*

$$d\tilde{A}(t) = \tilde{\theta}^X(t)'\big(dX(t) - r(t, X_0, X)X(t)dt\big), \tag{3}$$

where $\tilde{\theta}^X$ is the $[n \times 1]$ vector consisting of the investment strategies in the traded assets $X_1, \dots, X_n$ in terms of units of the bank account, i.e. the numéraire. And $dX(t)$ is the stochastic differential Eq. (1) of the hedgeable risk.

Proof of Proposition 1. The self-financing condition is

$$dA(t) = \bar{\theta}(t)'d\bar{X}(t), \tag{4}$$

where the first term of $\bar{X}(t)$ is represented by the bank account $dX_0(t) = r(t, X_0, X)X_0(t)dt$, accordingly $\bar{X}(t)$ is a vector of size $[(n+1)\times 1]$. By Itō's lemma one can see that a change in asset value may not be caused by a re-allocation of the value over the available assets, this indirectly leads to the second term of $dA(t) = \bar{\theta}(t)'d\bar{X}(t) + \bar{X}(t)'d\bar{\theta}(t)$ to be zero.

The definition of $A(t)$ is

$$A(t) = \bar{\theta}(t)'\bar{X}(t). \tag{5}$$

Rewriting the amount invested in the bank account in terms of the self-financing condition yields

$$\begin{aligned} A(t) &= \theta_0(t)X_0(t) + \theta^X(t)'X(t) \quad (6)\\ dA(t) &= (\theta_0(t)r(t,X_0,X)X_0(t) + \theta^X(t)'\mu^X(t,Z))dt\\ &\quad + \theta^X(t)'\left(\Sigma^{XX}\right)^{1/2}(t,Z)dW^X(t)\\ &\quad + \theta^X(t)'\left(\Sigma^{XY}\right)^{1/2}(t,Z)dW^Y(t). \end{aligned}$$

Let the bank account $X_0(t)$ be the numéraire, then by Itō's lemma the following holds

$$\begin{aligned} d\tilde{A}(t) &= d\left(\frac{A}{X_0}\right)(t) \quad (7)\\ &= \frac{1}{X_0(t)}\Big\{\big(\theta_0(t)r(t,X_0,X)X_0(t) + \theta^X(t)'\mu^X(t,Z) - A(t)r(t,X_0,X)\big)dt\\ &\quad + \theta^X(t)'\left(\Sigma^{XX}\right)^{1/2}(t,Z)dW^X(t) + \theta^X(t)'\left(\Sigma^{XY}\right)^{1/2}(t,Z)dW^Y(t)\Big\}. \quad (8) \end{aligned}$$

Substitution of definition (6) as function for the amount invested in the bank account $\theta_0(t) = \frac{1}{X_0(t)}\left(A(t) - \theta^X(t)'X(t)\right)$ into (8) leads to (3) where

$$\tilde{\theta}^X(t) = \frac{\theta^X(t)}{X_0(t)}. \tag{9}$$

□

The firm chooses the hedge position $\tilde{\theta}^X(t)$ in the financial assets that maximises its expected profit at terminal time T.

Definition 1. Let Θ be the set of all admissible $\tilde{\theta}^X(t)$ from the initial wealth $\tilde{A}(0)$. The investment strategy $\tilde{\theta}^X(t)$ is the n-dimensional progressively measurable process and we assume that $\int_0^T (\tilde{\theta}^X)'(s)\Sigma^{XX}(s,Z)\tilde{\theta}^X(s)ds$ is finite $\mathbb{P}$–a.s. The strategy $\tilde{\theta}^X(t)$ is admissible for the initial wealth $\tilde{A}(0)$ if the solution to the self-financing condition (3), which is defined $\mathbb{P}$–a.s., remains nonnegative at all times $\mathbb{P}$–a.s.

The (nonrobust) objective is to maximise the expected surplus at time T of the investment position minus the liabilities. In other words, the objective is to replicate the contract that depends on both hedgeable and unhedgeable risk by a portfolio of traded assets. Thus the objective of the liable firm is

$$\max_{\tilde{\theta}^X(t)\in\Theta} \mathbb{E}^{\mathbb{P}}[\tilde{A}(T,X) - \tilde{L}(T,X,Y)], \tag{10}$$

with respect to the dynamic processes (1) and where $\mathbb{P}$ denotes the baseline model.

Without the acknowledgement of uncertainty it is ex ante known to the firm whether the *expected* return of the risky asset is larger than the risk-free rate or not. In case the expected return is larger, the firm's optimal strategy is to invest infinity in the stock. Hence, the problem is unbounded. If the risk-free rate is higher, than the optimal strategy is to go short the equivalent. Hence, we need a sufficient level of uncertainty for the well-posedness of the optimisation problem.

We specify model uncertainty by a set of different probability measures, where each measure belongs to an alternative model. By Girsanov (1960)'s theorem the set $\mathcal{L}$ can be identified by

$$\begin{aligned} \mathcal{L} = \Big\{&\mathbb{L}\sim\mathbb{P} : R(t) = \frac{d\mathbb{L}}{d\mathbb{P}}\Big|_t, dR(t) = \lambda(t,X,Y)R(t)dW^Z(t)\\ &\text{with } |\lambda(t,X,Y)| \le k \text{ where } k \text{ satisfies Assumption 2}\Big\}, \end{aligned} \tag{11}$$

where $R(t)$ is the Radon–Nikodym derivative (i.e. likelihood ratio) that describes the change from probability measure $\mathbb{P}$ to $\mathbb{L}$, $\mathbb{P}$ represents the firm's baseline model, $\mathbb{L}$ the alternative model and the probability measures $\mathbb{P}$ and $\mathbb{L}$ are equivalent which implies that they have the same null sets.

Definition 2. The change of measure is driven by the parameter $\lambda(t,X(t),Y(t)) \in \mathbb{R}^{n\times l}$ and is allowed to depend on the stochastic processes X and Y. The uncertain drift adjustment is therefore progressively measurable and is assumed to satisfy Novikov's condition

$\mathbb{E}^{\mathbb{P}}[e^{\frac{1}{2}\int_0^T \lambda^2(t,X,Y)dt}] < \infty$.

We allow for stochastic $\lambda(t, X, Y)$ which implies that the alternative model under $\mathbb{L}$ can have a different distribution than the baseline model, contrary to constant or deterministic $\lambda(t)$'s that take only parameter uncertainty into account. The change in measure yields the transition in Eq. (1) from the process $\{W^Z(t)\}_{0\le t\le T}$ which is a standard Brownian motion (i.e., $\mathcal{N}(0,t)$) under probability measure $\mathbb{P}$ to $\{W^Z(t) + \int_0^t \lambda(s,Z)ds\}_{0\le t\le T}$ which is a standard Brownian motion under the alternative probability measure $\mathbb{L}$. We set $\epsilon(t,X,Y) = (\epsilon^X(t,X,Y), \epsilon^Y(t,X,Y))' := \Sigma^{1/2}(t,X,Y)\lambda(t,X,Y)$ and for ease of analytical exposition. And we suppress the explicit dependence on t and $Z(t)$ as much as possible to enhance readability, i.e., $\mu^X(t, X(t), Y(t)) \equiv \mu^X(t, X, Y) \equiv \mu^X(t, Z(t)) \equiv \mu^X(t, Z) \equiv \mu^X$. The characterisation of the uncertainty set leads to

$$\begin{aligned} \mathcal{L}^\epsilon = \Big\{&\mu(t,Z) + \epsilon(t,Z) | \epsilon(t,Z)'\Sigma^{-1}(t,Z)\epsilon(t,Z)\\ &\le k^2 \text{ where } k \text{ satisfies Assumption 2}\Big\}. \end{aligned} \tag{12}$$

The width of this set of alternatives indicates the amount of ambiguity and is represented by the scalar $k > 0$. The larger k, the larger becomes the set of alternative models and the more uncertain the firm is. In Balter and Pelsser (2019) we focus on the quantification of k and the relation to divergence measures. In one dimension the uncertainty set reduces to an interval, in two dimensions to an ellipse and in multi dimensions to an ellipsoid with the baseline model as middle point.

In order to have a well-defined bounded problem, we assume that the uncertainty around the drift $\mu(t, Z)$ includes the interest rate $r(t, X_0, X)$. If this assumption is violated, implying that even the worst-case drift of the risky assets is larger than the interest rate, then it is optimal to always invest an infinite amount in the risky asset. This leads to an unbounded problem. Hence Assumption 2, which incorporates the interest rate in the uncertainty set such that the worst-case drift on the risky assets is possibly worse than the investing on the bank account, bounds the problem of the risk-neutral firm.

Assumption 2. We assume that the radius on the uncertainty set $\mathcal{L}^\epsilon$ is large enough such that it includes the interest rate, which is obtained by

$$\begin{aligned} k^2 &- (\mu^X(t,Z) - r(t,X_0,X))'(\Sigma^{XX}(t,Z))^{-1}(\mu^X(t,Z)\\ &- r(t,X_0,X)) > 0. \end{aligned}$$

Note that every firm can have a different degree of uncertainty, i.e. a different value k, as long as it fulfils Assumption 2.

The robust equivalent of objective (10) is

$$\max_{\tilde{\theta}^X(t)\in\Theta}\min_{\mathbb{L}\in\mathcal{L}^\epsilon} \mathbb{E}^{\mathbb{L}}[\tilde{A}(T,X) - \tilde{L}(T,X,Y)], \tag{13}$$

with respect to the dynamic process (1) with drifts as specified by (12), i.e., $dZ(t) = (\mu(t,Z)+\epsilon(t,Z))dt + \Sigma(t,Z)^{1/2}dW^Z(t)$. The acknowledgment of uncertainty makes the firm looking for a strategy that is least affected under scenarios that emerge from different models than the baseline model. In anticipation of the worst-case model, the firm solves a max min problem. To robustify the investment strategy, the inner part of the optimisation is played by a so called "mother nature" who acts as a malevolent factor that minimises the surplus by choosing the worst-case measure $\mathbb{L}^*$. Whereas the firm searches for the best hedge strategy that is least affected by mother nature's choice.

Given the initial wealth $\tilde{A}(0) > 0$, the investment strategy $\tilde{\theta}^X(t)$ is admissible with respect to the robust problem if the measurability, integrability and nonnegativity assumptions from Definition 1 hold $\mathbb{L}$-a.s. for all $\mathbb{L} \in \mathcal{L}^{\epsilon}$.

4. Optimisation

We transform the stochastic optimisation into an optimal control problem by the Hamilton–Jacobi–Bellman–Isaacs equation (HJBI). Then we optimise the objective subject to the constraints such that we obtain the robust strategies. By the link between nonlinear partial differential equations (PDEs) and forward backward stochastic differential equations (FBSDEs) with Lipschitz drivers we prove the uniqueness and existence of the viscosity and classical strategy under different assumptions on the drift and diffusion parameters of the price processes.

Proposition 3. *Consider the nonlinear PDE*

$$\begin{aligned} &-\partial_t\tilde{w} - r\partial_X'\tilde{w}X - \partial_Y'\tilde{w}\big(\mu^Y - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X\big) - \tfrac{1}{2}\mathrm{tr}\big(\Sigma\Delta_{(Z)}(\tilde{w})\big) \\ &- c\sqrt{\partial_Y'\tilde{w}\big(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\big)\partial_Y\tilde{w}} = 0, \end{aligned} \tag{14}$$

where $c = \sqrt{k^2 - (q^X)'(\Sigma^{XX})^{-1}q^X}$, *and assume the existence of a sufficiently smooth* $\tilde{w}$. *Then,*

$$v(t,\tilde{A},X,Y) = \tilde{A}(t,X) - \tilde{w}(t,X,Y). \tag{15}$$

is the value function of the robust control problem (13) and

$$\tilde{\theta}^* = \begin{bmatrix} \partial_X\tilde{w} + (\Sigma^{XX})^{-1}\Sigma^{XY}\partial_Y\tilde{w} + \tilde{h}(\Sigma^{XX})^{-1}q^X \\ 0_l \end{bmatrix} \tag{16}$$

$$\epsilon^* = \begin{bmatrix} -q^X \\ \big(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\big)\tilde{h}^{-1}\partial_Y\tilde{w} - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X \end{bmatrix},$$

provide the optimal strategies, where $\tilde{h} = \sqrt{\frac{\partial_Y'\tilde{w}(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY})\partial_Y\tilde{w}}{k^2 - (q^X)'(\Sigma^{XX})^{-1}q^X}}$.

Proof of Proposition 3. As an intermediate step we consider the Feynman–Kaç equation (FK) whereafter we proceed with the optimisation of the associated partial differential equation. The Feynman–Kaç theorem (Kac, 1949) states that the conditional expectation

$$\mathbb{E}^{\mathbb{L}}[\tilde{A}(T,X) - \tilde{L}(T,Z)|\tilde{A}(t) = \tilde{A}, X(t) = X, Y(t) = Y] = \bar{v}(t,\tilde{A},X,Y), \tag{17}$$

where $\mathbb{L}$ represents the measure $(dW^X(t) + \epsilon^X(t,Z)dt, dW^Y(t) + \epsilon^Y(t,Z)dt)'$. The value function $\bar{v}$, which we assume to be sufficiently smooth, satisfies the PDE

$$\begin{aligned} &\partial_t\bar{v} + \tilde{\theta}^X(t)'(q^X(t,Z) + \epsilon^X(t,Z))\partial_{\tilde{A}}\bar{v} + \big(\mu^X(t,Z) + \epsilon^X(t,Z)\big)'\partial_X\bar{v} \\ &+ \big(\mu^Y(t,Z) + \epsilon^Y(t,Z)\big)'\partial_Y\bar{v} \\ &+ \tfrac{1}{2}\mathrm{tr}\big(\Phi\Sigma(t,Z)\Phi'\Delta_{(\tilde{A},Z)}\big(\bar{v}(t,\tilde{A},Z)\big)\big) = 0, \end{aligned} \tag{18}$$

where $q = (q^X, q^Y)' = \mu(t,Z) - r(t,X_0,X)Z(t)$. The operator Δ gives the partial derivatives, and the operator Φ enlarges the covariance matrix $\Sigma(t,Z)$ with the covariance matrices with respect to $\tilde{A}(t,X)$,

$$\Delta_{(\tilde{A},Z)} = \begin{bmatrix} \partial_{\tilde{A}\tilde{A}} & \partial_{\tilde{A}X} & \partial_{\tilde{A}Y} \\ \partial_{X\tilde{A}} & \partial_{XX} & \partial_{XY} \\ \partial_{Y\tilde{A}} & \partial_{YX} & \partial_{YY} \end{bmatrix}, \Phi = \begin{bmatrix} \begin{bmatrix} \tilde{\theta}^X & 0_l \end{bmatrix}' \\ I_{[n+l]} \end{bmatrix}. \tag{19}$$

The PDE remains linear in $\tilde{A}$, because for every $\tilde{\theta}^X(t,Z)$ and every $\epsilon^X(t,Z)$ as fixed stochastics the PDE is linear in all directions of $\bar{v}(\cdot)$ and its derivatives.

Now, when the firm is uncertain and anticipates to this, it optimises the value function

$$\max_{\tilde{\theta}(t)\in\Theta}\min_{\epsilon(t)\in\mathcal{L}^{\epsilon}} \bar{v}(t,\tilde{A},X,Y) = v(t,\tilde{A},X,Y), \tag{20}$$

where

$$\tilde{\theta}(t) = \big(\tilde{\theta}^X(t), \tilde{\theta}^Y(t)\big)' = \left(\frac{\theta^X(t)}{X_0(t)}, \frac{\theta^Y(t)}{X_0(t)}\right)'. \tag{21}$$

The firm can trade in the hedgeable risk, but not in the unhedgeable risk. Therefore the amount the firm invests in the first n assets needs to be chosen such that the firm maximises the surplus while the other l hedge positions with respect to the unhedgeable risk are restricted to zero. Thus the investment strategy equals $\theta(t) := (\overbrace{\theta_1(t),\ldots,\theta_n(t)}^{n}, \overbrace{0,\ldots,0}^{l})' = (\theta^X(t), \theta^Y(t))'$. The restriction on the unhedgeable part can be summarised by the constraint $\Xi\theta(t) = 0_l$, where Ξ is an $[l\times(n+l)]$ matrix equal to $\big[0_{[l\times n]}|I_{[l\times l]}\big]$, with on the left an $[l\times n]$ zero-matrix and next to it an $[l\times l]$ identity matrix.

The robust optimisation problem can be interpreted as a two player game where the firm wants to maximise the surplus whereas the robustness role is played by the counter player "mother nature" who minimises the surplus. The robust optimised value $v(t,\tilde{A},X,Y)$ is given by the HJBI-equation,

$$\begin{aligned} &\partial_t v + \max_{\tilde{\theta}(t)}\min_{\epsilon(t,Z)}\Big\{\tilde{\theta}^X(t)'(q^X(t,Z) + \epsilon^X(t,Z))\partial_{\tilde{A}}v + \partial_Z'v(\mu(t,Z) + \epsilon(t,Z)) \\ &+ \tfrac{1}{2}\partial_{\tilde{A}\tilde{A}}v\tilde{\theta}^X(t)'\Sigma^{XX}(t,Z)\tilde{\theta}^X(t) + \partial_{\tilde{A}X}'v\Sigma^{XX}(t,Z)\tilde{\theta}^X(t) + \partial_{\tilde{A}Y}'v\Sigma^{YX}(t,Z)\tilde{\theta}^X(t)\Big\} \\ &+ \tfrac{1}{2}\mathrm{tr}\big(\Sigma(t,Z)\Delta_{(Z)}\big(v(t,\tilde{A},Z)\big)\big) = 0 \end{aligned} \tag{22}$$

$$\begin{aligned} \text{s.t.}\quad & \epsilon(t,Z)'\Sigma(t,Z)^{-1}\epsilon(t,Z) \le k^2 \\ & \Xi\tilde{\theta} = 0_l \\ \text{with}\quad & v(T,\tilde{A},X,Y) = \tilde{A}(T,X) - \tilde{L}(T,Z), \end{aligned} \tag{23}$$

where (23) is the boundary condition at time T and $\Delta_{(Z)}$ is the lower right second order derivative matrix with respect to the variables Z only. Note that the other terms of $\Delta_{(\tilde{A},Z)}$ include the control variable θ^X and are therefore explicitly written out. The HJBI is intuitively equal to "the FK with optimal $\tilde{\theta}^X(t)$ and $\epsilon(t)$ per dt". The difference between FK and HJBI is that now $\tilde{\theta}(t)$ and $\epsilon(t)$ may depend on the value function and its derivatives which makes the PDE nonlinear. Therefore the value function $v(t,\tilde{A},X,Y)$ can no longer be represented by a linear expectation. We show that the value for $v(t,\tilde{A},X,Y)$ can be represented by a nonlinear expectation $\mathcal{E}[\cdot]$. Since the terminal condition is linear in the random variable $\tilde{A}$, we propose the Ansatz that $v(\cdot)$ is linear in $\tilde{A}$

$$v(t,\tilde{A},X,Y) = \tilde{A}(t,X) - \tilde{w}(t,X,Y). \tag{24}$$

Despite the nonlinearity caused by the max min the linearity in $\tilde{A}$ remains due to the specific boundary condition of the surplus. The function $\tilde{w}(\cdot)$ satisfies

$$
\begin{aligned}
&-\partial_t\tilde{w} + \max_{\tilde{\theta}(t)}\min_{\epsilon(t,Z)}\Big\{\tilde{\theta}^X(t)'(q^X(t,Z)+\epsilon^X(t,Z))\\
&-\partial_Z'\tilde{w}(\mu(t,Z)+\epsilon(t,Z))\Big\} - \tfrac{1}{2}\mathrm{tr}\big(\Sigma(t,Z)\Delta_{(Z)}(\tilde{w}(t,Z))\big) = 0 \qquad (25)\\
&\text{s.t. } \epsilon(t,Z)'\Sigma^{-1}(t,Z)\epsilon(t,Z) \le k^2\\
&\Xi\tilde{\theta} = 0_l\\
&\text{with } \tilde{w}(T,X,Y) = \tilde{L}(T,Z).
\end{aligned}
$$

The firm maximises $-\tilde{w}(t,X,Y)$ which is the difference between the nonlinear expectation of the surplus and the value of the assets. At time T, $\tilde{w}(T,X,Y)$ is exactly the liability value divided by the numéraire. Note that the optimised $\tilde{w}$ may depend on θ and ϵ. Supporting the mindset of the firm it maximises minus $\tilde{w}$ which can be interpreted as an equivalent for the discounted value of the liabilities.Since the only variables that depend on the max min problem are $\theta(t)$ and $\epsilon(t, Z)$, the above objective function can be simplified by eliminating the terms that are independent of the decision variables. First we concentrate on the max min, which we define by

$$
\begin{aligned}
&m(t,X,Y)\\
&= \max_{\tilde{\theta}^X(t)}\min_{\epsilon(t,Z)}\Big\{\tilde{\theta}^X(t)'(q^X(t,Z)+\epsilon^X(t,Z))-\partial_Z'\tilde{w}(\mu(t,Z)+\epsilon(t,Z))\Big\}. \qquad (26)
\end{aligned}
$$

We optimise $m(t, X, Y)$ for every time step dt and plug in the optimal $m^*(t, X, Y)$ in PDE (25) to obtain the optimal nonlinear PDE. For both control variables we obtain a candidate solution that satisfies the first order conditions. In the proof of Corollary 11 together with Verification Lemma 4 we show that the minimisation and maximisation both yield a unique solution. The optimal hedging strategy and drift distortions in terms of the numéraire, are given by (16). Plugging the optimal $m^*(t, X, Y)$ into PDE (25) results in the nonlinear PDE (14). □

The fact that the square root c has to be positive leads to the requirement that the risky asset should not be strictly better than or equal to the bank account. The admissibility of the square root leads to Assumption 2, which implies that the interest rate should be inside the uncertainty set. This yields that the uncertain return on risky assets can possibly be lower than the interest rate. If the firm is confident that even in the worst case a positive excess return can be made by investing in the financial market, then the firm will try to invest a massive amount in the financial market and has a confident expectation to get very rich. However, the assumption ensures that $\max_\theta \min_\epsilon \mathbb{E}[A(T,X(T)) - L(T,X(T),Y(T))] < \infty$. If there is only one hedgeable risk factor in the economy then Assumption 2 states that the market price of risk should be within the firm's set of alternatives and cannot be "too good". This corresponds with the idea of the Good-Deal-Bounds methodology of Cochrane and Saa-Requejo (2000).

Because the financial components X of the risk vector are perfectly replicated by the delta-hedge[2] $\partial_X\tilde{w}$, the ambiguity regarding the drift of the hedgeable risk is eliminated and replaced by the interest rate r. The ambiguity regarding the drift of the unhedgeable processes Y is decomposed into the part consisting of the variance of each component of Y plus terms consisting of the covariances with all risk factors. This is known as the Schur complement $S = \Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}$ which is the conditional variance of the unhedgeable risk given the hedgeable risk. We now derive the optimal pricing rule of the liability by its indifference value, after which we explain the implication and interpretation of the optimal strategies in more detail.

[2] The delta (Hull, 2017) of a derivative is a measure for the rate of change of the derivative's value with respect to small changes, i.e., the first derivative, in the underlying security's price. We apply this terminology to the operator ∂_Z.

The solution of the robust objective (20) characterised by the optimisation problem (22) is proven to be optimal based on the following verification lemma. The lemma is similar as the Davis–Varaiya Martingale Principle of Optimal Control (Rogers, 2013), which Biagini and Pınar (2017) modified to the robust max min setting. We slightly adopt this to the pure terminal wealth case, including the liability in terms of linear utility. Unlike in Baltas et al. (2018b) our optimal solution is not obtained via a saddle point characterisation as the min max equivalent is unbounded here and thus leads to a minimax inequality.

Lemma 4 (Verification)**.** *If*

(i) *There exists a function* $\nu(t,\tilde{A},Z) : [0,T]\times\mathbb{R}_+\times\mathbb{R}^{n+l}\to\mathbb{R}$ *which is* C *on its domain and* C^2 *with respect to* Z*, verifying* $\nu(T,\tilde{A},Z) = \tilde{A}(T,X) - \tilde{L}(T,X,Y)$*;*

(ii) *For any* $\tilde{\theta}\in\Theta$ *there exists an optimal solution* $\tilde{\theta}^*$ *with associated measure* $\mathbb{L}^{(\tilde{\theta})}$ *of the inner minimisation in (22), such that*

$$\Upsilon(t) = \Upsilon(t)^{(\tilde{\theta})} = \nu(t,\tilde{A}(t),X(t),Y(t)), \qquad (27)$$

is a $\mathbb{L}^{(\tilde{\theta})}$*-supermartingale;*

(iii) *There exist some* $\tilde{\theta}^*\in\Theta$ *such that the corresponding* Υ^* *is a* $\mathbb{L}^{(\tilde{\theta}^*)}$*-martingale.*

Then $\tilde{\theta}^*$ *is optimal for the problem (22) and* $\nu(0,\tilde{A},Z)$ *coincides with* $\nu(0,\tilde{A},X,Y)$.

Proof of Lemma 4. The notation C represents the space of those function that are continuous, while C^ℓ is the space of those functions that are ℓ-times continuously differentiable. By the supermartingale property of Υ under $\mathbb{L}^{(\tilde{\theta})}$, and by $\nu(T,\cdot) = \tilde{A}(T,\cdot) - \tilde{L}(T,\cdot)$ we have for any admissible $\tilde{\theta}$,

$$
\begin{aligned}
\mathbb{E}^{\mathbb{L}^{(\tilde{\theta})}}[\Upsilon(T)] &= \mathbb{E}^{\mathbb{L}^{(\tilde{\theta})}}[\nu(T,\tilde{A}(T),Z(T))]\\
&= \mathbb{E}^{\mathbb{L}^{(\tilde{\theta})}}[\tilde{A}(T,X(T)) - \tilde{L}(T,X(T),Y(T))] \qquad (28)\\
&\le \Upsilon(0) = \nu(0,\tilde{A}(0),Z(0)).
\end{aligned}
$$

Thus for any admissible strategy the value is no greater than $\nu(0,\tilde{A}(0),Z(0))$. Taking the supremum over Θ gives

$$
\begin{aligned}
\bar{\nu}(0,\tilde{A}(0),Z(0)) &= \sup_{\tilde{A}}\mathbb{E}^{\mathbb{L}^{(\tilde{\theta})}}[\tilde{A}(T,X(T))\\
&\quad -\tilde{L}(T,X(T),Y(T))] \le \nu(0,\tilde{A}(0),Z(0)). \qquad (29)
\end{aligned}
$$

Since by assumption for some $\tilde{\theta}^*$ the process Υ^* is a martingale under $\mathbb{L}^{(\tilde{\theta}^*)}$, then $\mathbb{E}^{\mathbb{L}^{(\tilde{\theta}^*)}}[\Upsilon(T)] = \Upsilon(0) = \nu(0,\tilde{A}(0),Z(0))$. Thus when we use $\tilde{\theta}^*$, the value is equal to $\nu(0,\tilde{A}(0),Z(0))$ since the (supermartingale) inequality becomes an equality. Hence $\tilde{\theta}^*$ is optimal. This coincides with the intuition that optimality is obtained whenever the complete budget is utilised. □

4.1. Indifference pricing

In order to derive the price π of the liability we express the PDE in terms π by transforming (14) via the concept of indifference pricing. In complete markets, any derivative or claim can be priced uniquely by replicating the payoff with available products. The weighted average of the market prices of these available products determines the fair value of the derivative. However, in reality markets are incomplete, implying that there is risk that cannot be replicated. Not only nontraded assets make the market incomplete, but also the presence of transaction costs or portfolio constraints. The law of one price is violated since many equivalent martingale measures exist and each determines a different no-arbitrage price. Hodges and Neuberger (1989) price options under transaction costs in the Black–Scholes model by the concept of utility indifference pricing. The concept of indifference pricing we employ is based on the price of a claim such that a firm that is maximising its objective is indifferent between having the claim or not. The firm is willing to accept the obligation to fulfil the liability if it receives a certain amount of money today, such that the firm is not worse off in terms of expected payoff than it would be without having the obligation to pay the liability. The indifference price is closely related to the certainty equivalent amount, which is the certain amount of money that makes the firm indifferent between the return from a gamble and receiving a certain cash value.

For an overview on indifference pricing see Henderson and Hobson (2009). For recent work on indifference pricing, and in specific for exponential utility which leads to analytical solutions see Henderson (2002, 2005), Miao and Wang (2007), Hu, Imkeller, and Müller (2005), Musiela and Zariphopoulou (2004), Young and Zariphopoulou (2002) and Zariphopoulou (2001) among others. Mercurio (2001) prices a contingent claim in an incomplete market based on the investor being indifferent. Similar in spirit of the indifference pricing concept, his setting differs in the fact that he considers a quadratic utility and discrete setting without uncertainty. Xanthopoulos and Yannacopoulos (2008) use utility indifference pricing in a trinomial model based on three different scenarios to settle a trade between the buyer and seller of the option. The unique price is based on minimising regret, sharing risk or updating beliefs. Other techniques to price in incomplete markets are super- and subreplication (Cvitanić, Pham, & Touzi, 1999a; 1999b).

Remember the Ansatz of (24) to be $v(t,\tilde{A},X,Y)=\tilde{A}(t,X)-\tilde{w}(t,X,Y)$. The indifference price $\pi(t, X, Y)$ is the extra cash needed to make the firm indifferent between accepting and not accepting the liability. From Eq. (24) the growth of the bank account is known. Without liability contract the initial value of assets generates an expected surplus of $v(t,\tilde{A},X,Y)=\tilde{A}(t,X)$ at time t. Note that without contract the nonlinear expectation becomes linear, while the expected surplus at $0<t<T$ based on the initial assets plus the liability contract is implied by the nonlinear PDE (14). Together with the extra cash, the expected surplus is $v(t,\tilde{A},X,Y)=\tilde{A}(t,X)+\tilde{\pi}(t,X,Y)-\tilde{w}(t,X,Y)$. Being indifferent between these two situations leads to

$$\tilde{A}(t,X)=\tilde{A}(t,X)+\tilde{\pi}(t,X,Y)-\tilde{w}(t,X,Y) \tag{30}$$

$$\tilde{\pi}(t,X,Y)=\tilde{w}(t,X,Y), \tag{31}$$

where $\tilde{w}(t,X,Y)=w(t,X,Y)/X_0(t)=e^{-\int_0^t r(s,X_0,X)ds}w(t,X,Y)$ and $\pi(t,X,Y)=w(t,X,Y)$. Rephrasing the PDE in terms of $\pi(t, X, Y)$ gives

$$\begin{aligned}\partial_t\pi &+ r\partial_X'\pi X+\partial_Y'\pi\left(\mu^Y-\Sigma^{YX}(\Sigma^{XX})^{-1}q^X\right)\\ &+\tfrac{1}{2}\mathrm{tr}\left(\partial_{XX}\pi\,\Sigma^{XX}+2\partial_{XY}\pi\,\Sigma^{XY}+\partial_{YY}\pi\,\Sigma^{YY}\right)\\ &- r\pi+c\sqrt{\partial_Y'\pi\left(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Y\pi}=0.\end{aligned} \tag{32}$$

Note, that the equation is a semilinear PDE that describes the behaviour of the liability $\pi(t, Z)$ as a function of t and Z. The PDE is subject to the boundary condition $\pi(T,Z)=L(T,X,Y)$ which is the value of the insurance contract at time T. Thus we now derived the PDE for which we subsequently prove existence and uniqueness of the solution. The theorem is proven based on the auxiliary representation of the semilinear PDE as a FBSDE problem.

4.2. PDEs and FBSDEs

A viscosity solution[3], which implies that π needs not to be everywhere differentiable, of the FBSDE, is also a viscosity solution of the PDE. The relation between the FBSDE and PDE is weak in the sense that several other viscosity solutions to the PDE might exist as well. On the other hand, a classical solution of the FBSDE which is unique, is also the unique classical solution of the PDE. Hence, the relation between FBSDEs and PDEs is stronger under assumptions by which classical solutions are implied. We connect the optimal PDE to the FBSDE framework in which existence and uniqueness of different solutions are known to hold.

In our context, the connection between nonlinear PDEs and FBSDEs can be summarised by

$$\left.\begin{array}{l}\text{PDE}\\ \mathcal{K}\cdot\pi-\\ g(t,Z,\pi,\partial_Z\pi\,\Sigma)=0\\ \pi(T,Z)=L(T,Z)\end{array}\right\}\Leftrightarrow\left\{\begin{array}{l}\text{BSDE}\\ d\hat{X}(t)=\mu^*(t,\hat{X})dt+\Sigma^*(t,\hat{X})^{1/2}dW(t)\\ d\hat{Y}(t)=-g(t,\hat{X},\hat{Y},\hat{Z})dt+\hat{Z}dW(t)\\ \hat{Y}(T)=f(T,\hat{X}).\end{array}\right. \tag{33}$$

On the left hand side, the PDE is expressed in term of the linear Feynman–Kaç part, denoted by the operator

$$\begin{aligned}\mathcal{K}\cdot\pi=&\Big(\partial_t+rX'\partial_X+(\mu^Y-\Sigma^{YX}(\Sigma^{XX})^{-1}q^X)'\partial_Y\\ &+\tfrac{1}{2}\mathrm{tr}\big(\partial_{XX}'\Sigma^{XX}+2\partial_{XY}'\Sigma^{XY}+\partial_{YY}'\Sigma^{YY}\big)\Big)\cdot\pi,\end{aligned} \tag{34}$$

and the nonlinear part by

$$g(t,Z,\pi,\partial_Z\pi\,\Sigma)=r\pi-c\sqrt{\partial_Y'\pi\left(\Sigma^{YY}-\Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Y\pi}. \tag{35}$$

The first equation on the right hand side of (33) is the forward process where μ^* are the drifts of X and Y under the optimal measure, hence

$$\mu^*=\begin{bmatrix} r(t,X_0,X)X\\ \mu^Y(t,Z)-\Sigma^{YX}(t,Z)(\Sigma^{XX}(t,Z))^{-1}q^X(t,Z)\end{bmatrix}. \tag{36}$$

The covariance matrix Σ^* belongs to the trace terms of the PDE, hence $\Sigma^*=\Sigma(t,Z)$. The second equation is the backward process with the driver $g(\cdot)$ and $\hat{Z}=\partial_{\hat{X}}\hat{Y}\Sigma(t,\hat{X})$. Recall that $\hat{X}=Z=(X,Y)'$ and $\hat{Y}=\pi$. The terminal condition $L(\cdot)=f(\cdot)$.

Viscosity solutions are typically continuous solutions only, whereas classical solutions have well defined derivatives. Mild solutions are somewhere in between and are typically just once continuously differentiable solutions. Baltas et al. (2018a) derive mild solutions of robust control problems for a general class of penalty functions. Our closed-form optimal control decisions given by (16) include delta-hedges, which imply that we need the first derivative of the value function. For the existence of these derivatives we impose necessary conditions such that we obtain a "special" viscosity solutions whose first derivative exists, also known as mild solutions. Thus since the optimal strategies θ^* and ϵ^* depend on $\partial_Z\pi$, and a viscosity solution does not imply differentiability of the solution π we need additional assumptions implying at least differentiability of order one of π. Moreover, the semilinearity of the optimal PDE relates to a nondifferentiable driver of the associated FBSDE. Therefore the assumption that μ, Σ, f and g are three-times continuously differentiable and all derivatives up

[3] A viscosity solution π, as first appeared under this name by Crandall and Lions (1983), needs not to be everywhere differentiable as opposed to a classical solution. In technical terms, a continuous function π is a viscosity solution of the PDE if it is both a supersolution and a subsolution.

to order three are uniformly bounded (El Karoui, Hamadène, & Matoussi, 2008; Ma, Protter, & Yong, 1994; Pardoux & Peng, 1992), which is standard in the FBSDE literature and ensures a classical solution is not applicable since the driver is not differentiable. For the verification theorem we also need second order differentiability for which we impose the conditions implying the classical solution. Zhang (2001) has derived conditions under which both the (once) differentiable viscosity solution and the classical solution exist based on a non-differentiable but Lipschitz driver. The set (μ, Σ, f, g) satisfies Assumption 5 if they fulfil the following

Assumption 5. Let $K > 0$ be the Lipschitz constant

(i) The functions $\mu, \Sigma \in C_b^1$ and

$$\sup_{0\le t\le T}\{|\mu(t,0)| + |\Sigma(t,0)|\} \le K.$$

(ii) We assume that Σ satisfies

$$\Sigma(t,Z) \ge \frac{1}{K} I_{[n+l]},$$

where $I_{[n+l]}$ is the $[(n+l)\times(n+l)]$ identity matrix.

(iii) The terminal condition and driver $f, g \in C_L$ and

$$\sup_{0\le t\le T}|g(t,0,0,0) + f(t,0)| \le K,$$

where C_b^ℓ is the space of those functions that are ℓ-times continuously differentiable and all derivatives up to order ℓ are uniformly bounded and C_L represents Lipschitz continuity. Assumption (i) states that the drift and diffusion of the forward process should be bounded if $Z = 0$. Assumption (ii) states that the diagonal entries of the covariance matrix are all positive. This is implied by the positive definite assumption on Σ. Additionally, the variances should also be bounded away from zero. And assumption (iii) is the Lipschitz condition on the nonlinear driver and terminal condition, as well as the assumption that the driver and terminal condition are bounded in the initial zero states.

Since we need the existence of $\partial_Z\pi$, the viscosity solution as described by Zhang (2001) would be sufficient to prove existence of the investment strategy. Moreover, uniqueness of the viscosity solution is implied here as well. A stronger relation with the FBSDEs is obtained for the classical solution if we assume the following,

Assumption 6. The functions $\mu, \Sigma \in C_b^2$.

We can summarise this by the following lemma.

Lemma 7.

(i) *If* (μ, Σ, f, g) *fulfil Assumption 5 then the viscosity solution* π *is* C_b^1.

(ii) *If* (μ, Σ, f, g) *fulfil Assumption 5 and* (μ, Σ) *fulfil Assumption 6 then the classical solution* π *is unique and* C^2.

Proof of Lemma 7. See Zhang (2001) for the proof.

Brigo, Francischello, and Pallavicini (2016) also make use of not needing the driver to be differentiable, to price nonlinear valuation equations under credit and funding effects. Brigo et al. (2016) initially have a driver that does not satisfy the Lipschitz condition, therefore they move this component from the backward driver to the forward drift to apply Zhang (2001). Pham (2002) extends the Merton investment problem for stochastic volatility and portfolio constraints. He proves existence and uniqueness of the classical solution of the semilinear PDE based on hypothesis (3a) in Pham (2002), which is in the same spirit as Assumption 6 here.

4.3. Optimal pricing result

By combining the PDE of the indifference price (32) with Verification Lemma 4 we derive here the main result of the paper.

Theorem 8 (Existence and uniqueness of classical solution)**.** *Assume that Assumption 2 holds. If* (μ^*, Σ^*, f, g) *fulfils Assumption 5, then the nonlinear PDE (32) admits a unique viscosity solution that is* C_b^1 *which coincides with the value function of the robust control problem. Under additional regularity of* (μ^*, Σ^*) *fulfilling Assumption 6 the viscosity solution is a classical solution.*

Proof of Theorem 8. By applying Itō's lemma to π under the optimal measure $\mathbb{L}^*$ we get

$$\begin{aligned} d\pi(t) = {}& \partial_t\pi\, dt + \partial_X'\pi\left(rXdt + (\Sigma^{XX})^{1/2}dW^X\right)\\ &+ \partial_Y'\pi\left((\mu^Y - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X)dt + (\Sigma^{YY})^{1/2}dW^Y\right)\\ &+ \tfrac{1}{2}\mathrm{tr}\left(\partial_{XX}\pi\,\Sigma^{XX} + 2\partial_{XY}\pi\,\Sigma^{XY} + \partial_{YY}\pi\,\Sigma^{YY}\right)dt. \end{aligned} \quad (37)$$

Furthermore the FBSDE representation gives the relation

$$d\hat{Y}(t) = -g(t,\hat{X},\hat{Y},\hat{Z})dt + \hat{Z}dW(t). \quad (38)$$

The terms related to the Brownian motions are combined into the $\hat{Z}$ term. Thus

$\hat{Z} = (\partial_X'\pi(\Sigma^{XX})^{1/2}, \partial_Y'\pi(\Sigma^{YY})^{1/2})'$. Since $\pi = \hat{Y}$, the definition of $\hat{Z}$ as the derivatives of $\hat{Y}$ is fulfilled. Based on (37) and (38) the driver $g(\cdot)$ equals

$$\begin{aligned} -g(t,\hat{X},\hat{Y},\hat{Z}) = {}& \partial_t\hat{Y} + \partial_X'\hat{Y}rX + \partial_Y'\hat{Y}(\mu^Y - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X)\\ &+ \tfrac{1}{2}\mathrm{tr}\left(\partial_{XX}\hat{Y}\Sigma^{XX} + 2\partial_{XY}\hat{Y}\Sigma^{XY} + \partial_{YY}\hat{Y}\Sigma^{YY}\right). \end{aligned}$$

Recall from Eq. (35) that the driver implied by the forward relation equals

$$\begin{aligned} g(t,Z,\pi,\partial_Z\pi\,\Sigma) = g(t,\hat{X},\hat{Y},\hat{Z}) = {}& r(t,X_0,X)\pi\\ &- c\sqrt{\partial_Y'\pi\left(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Y\pi}, \end{aligned}$$

which satisfies (32). Together with Assumption 2, it follows that the driver g is Lipschitz continuous. In conjunction with Assumptions 5 and 6 we can apply Lemma 7 i.e., Theorem 2.4.1 of Zhang (2001).

Remark 9. The max min problem leads to an upper bound on the set of no-arbitrage prices of the liability for $c\sqrt{\partial_Y'\pi S\partial_Y\pi} > 0$ and to a lower bound for $c\sqrt{\partial_Y'\pi S\partial_Y\pi} < 0$. Hence, these could be interpreted as the bid and ask price. Note that the problem reduces to the linear expectation if $c\sqrt{\partial_Y'\pi S\partial_Y\pi} = 0$.

Remark 10. A shorthand notation for the FBSDE solution and the indifference price is

$$\pi(t,X,Y) = \mathcal{E}^{g^*}\left[e^{-\int_t^T r(s,X_0,X)ds}L(T,X,Y)|t,Z\right],$$

where $\mathcal{E}^{g^*}$ denotes the *nonlinear* expectation with respect to the driver $g^*(t,Z,\pi,\partial_Z\pi\,\Sigma) = r(t,X_0,X)\pi - c\sqrt{\partial_Y'\pi S\partial_Y\pi}$ that is implied by the optimal strategies (θ^*, ϵ^*).

The solution of the firm's problem in which it maximises profit, while being concerned about the underlying model, leads to the explicit hedging demand and worst-case model as stated in the following corollary.

Corollary 11 (Optimal hedging portfolio and robustness adjustments by Theorem 8)**.** *Let* $\pi(t, X, Y)$ *be the indifference price of a firm that maximises the expected surplus*

$\mathbb{E}[A(T,X) - L(T,X,Y)|t,Z]$ *under model ambiguity and in an incomplete market. Under the assumptions of Theorem 8 the firm's robust optimal dynamic hedging portfolio consists of the investment* *strategy*

$$\theta^* = \begin{bmatrix} \partial_X\pi + (\Sigma^{XX})^{-1}\Sigma^{XY}\partial_Y\pi + h(\Sigma^{XX})^{-1}q^X \\ 0_l \end{bmatrix},$$

and the ambiguity ensures the optimal solution to be robust for drift adjustments

$$\epsilon^* = \begin{bmatrix} -q^X \\ \left(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)h^{-1}\partial_Y\pi - \Sigma^{YX}(\Sigma^{XX})^{-1}q^X \end{bmatrix},$$

where $h = \sqrt{\frac{\partial_Y'\pi\left(\Sigma^{YY} - \Sigma^{YX}(\Sigma^{XX})^{-1}\Sigma^{XY}\right)\partial_Y\pi}{k^2 - (q^X)'(\Sigma^{XX})^{-1}q^X}}$.

Proof of Corollary 11. Expression (16) is obtained by solving the first order conditions implied by the Lagrangian of the optimisation problem. Subsequently, $\tilde{w}$ and its derivatives can be expressed in term of π by relation (31).

To prove optimality, first consider the inner minimisation of (26)

$$\min_{\epsilon} \ \tilde{\theta}'q + \epsilon'(\tilde{\theta} - \partial_Z\tilde{w}) \quad \text{s.t.} \quad \epsilon'\Sigma^{-1}\epsilon \le k^2. \tag{39}$$

This is a linear objective with a quadratic constraint, from which we know that it has a unique minimum.

After plugging in the optimal value for ϵ^*, the objective for the firm becomes

$$\max_{\tilde{\theta}} \ \tilde{\theta}'q - k\sqrt{(\tilde{\theta} - \partial_Z\tilde{w})'\Sigma(\tilde{\theta} - \partial_Z\tilde{w})} \quad \text{s.t.} \quad \Xi\tilde{\theta} = 0. \tag{40}$$

This is a quadric objective with linear constraints. The covariance matrix Σ is positive definite and thus the square root is a convex function. Since the optimal ϵ is the negative root, the firm maximises a concave function which results in a unique maximum (to see this, let $\hat{\theta} = \tilde{\theta} - \partial_Z\tilde{w}$, then $\max_{\hat{\theta}}(\hat{\theta} + \partial_Z\tilde{w})'q - k\sqrt{\hat{\theta}'\Sigma\hat{\theta}}$ s.t. $\Xi\hat{\theta} = \Xi\partial_Z\tilde{w}$). The optimal strategy of mother nature is $\epsilon^* = -k\frac{\Sigma(\tilde{\theta} - \partial_Z\tilde{w})}{\sqrt{(\tilde{\theta} - \partial_Z\tilde{w})'\Sigma(\tilde{\theta} - \partial_Z\tilde{w})}}$.

The optimisation problem from (39) yields the Lagrangian

$$L(\tilde{\theta}, \epsilon, \alpha_0, \alpha) = \tilde{\theta}'q + \epsilon'(\tilde{\theta} - \partial_Z\tilde{w}) - \alpha_0\tfrac{1}{2}(\epsilon'\Sigma^{-1}\epsilon - k^2) - \alpha'(\Xi\tilde{\theta} - 0_l),$$

where α has dimension $[l \times 1]$ and α_0 is a scalar.

The FOC are

$$\begin{aligned} \frac{\partial L}{\partial\tilde{\theta}} &= q + \epsilon - \Xi'\alpha = 0_{n+l} \\ \frac{\partial L}{\partial\epsilon} &= -\partial_Z\tilde{w} + \tilde{\theta} - \alpha_0\Sigma^{-1}\epsilon = 0_{n+l} \\ \frac{\partial L}{\partial\alpha} &= -\Xi\tilde{\theta} = 0_l \\ \frac{\partial L}{\partial\alpha_0} &= -\tfrac{1}{2}\left(\epsilon'\Sigma^{-1}\epsilon - k^2\right) = 0. \end{aligned} \tag{41}$$

Note that although Ξ is rank deficient, $\Xi\Sigma^{-1}\Xi'$ has full rank. The first three blocks of equations are linear and yield

$$\begin{aligned} \alpha^* &= (\Xi\Sigma^{-1}\Xi')^{-1}\Xi(\Sigma^{-1}q - \alpha_0^{-1}\partial_Z\tilde{w}) \\ \tilde{\theta}^* &= \partial_Z\tilde{w} + \alpha_0\Sigma^{-1}(\Xi'\alpha^* - q) \\ \epsilon^* &= \Xi'\alpha^* - q. \end{aligned} \tag{42}$$

Now we can plug in α in ϵ^* and solve the constraint $\epsilon'\Sigma^{-1}\epsilon = k^2$ for α_0,

$$\alpha_0^{-1} = \pm\sqrt{\frac{q'\Sigma^{-1}\Xi'(\Xi\Sigma^{-1}\Xi')^{-1}\Xi\Sigma^{-1}q - q'\Sigma^{-1}q + k^2}{\partial_Z'\tilde{w}\Xi'(\Xi\Sigma^{-1}\Xi')^{-1}\Xi\partial_Z\tilde{w}}}. \tag{43}$$

Since mother nature is minimising, the second order derivative with respect to ϵ has to be positive to ensure a minimum. Therefore α_0^{-1} is minus the square root term in Eq. (43). Let $\Psi = \Xi'(\Xi\Sigma^{-1}\Xi')^{-1}\Xi$, which implies Ψ to be the $[(n+l)\times(n+l)]$ null matrix with the Schur complement on the lower right $[l \times l]$ block, hence $\Psi = \begin{bmatrix} 0_{[n\times n]} & 0_{[n\times l]} \\ 0_{[l\times n]} & \Sigma^{YY} - \Sigma^{YX}\left(\Sigma^{XX}\right)^{-1}\Sigma^{XY} \end{bmatrix}$. This results in

$$\begin{aligned} \epsilon^* &= \Psi\left(-\sqrt{\frac{q'\Sigma^{-1}X\Sigma^{-1}q - q'\Sigma^{-1}q + k^2}{\partial_Z'\tilde{w}X\partial_Z\tilde{w}}}\partial_Z\tilde{w} + \Sigma^{-1}q\right) - q \\ &= \begin{bmatrix} -q_x \\ S\sqrt{\frac{k^2 - \left(q^X\right)'\left(\Sigma^{XX}\right)^{-1}q^X}{\partial_Z'\tilde{w}\left(\Sigma^{YY} - \Sigma^{YX}\left(\Sigma^{XX}\right)^{-1}\Sigma^{XY}\right)\partial_Z\tilde{w}}}\partial_Z\tilde{w} - \Sigma^{YX}\left(\Sigma^{XX}\right)^{-1}q^X \end{bmatrix}, \end{aligned}$$

since $\Psi\Sigma^{-1} = \begin{bmatrix} 0_{[n\times n]} & 0_{[n\times l]} \\ -\Sigma^{YX}\left(\Sigma^{XX}\right)^{-1} & I_{[l\times l]} \end{bmatrix}$ and $q'\Sigma^{-1}\Psi\Sigma^{-1}q - q'\Sigma^{-1}q = (-\left(q^X\right)'(\Sigma^{XX})^{-1}q^X, 0)'$ and gives Corollary 11.

If we find a function that satisfies the condition of the Verification Lemma 4, then we know that the corresponding solutions are optimal. The optimal investment and worst-case drift adjustment depend on the first derivative of π. In order to guarantee that ν, which is one-to-one linked with $\tilde{w}$ and the indifference price π, is twice differentiable, the classical solution of the partial differential equation is required. The differentiability is ensured by Assumptions 2 and 6. Hence, the first premise of Lemma 4 is fulfilled.

Using Itō's formula under $\mathbb{L}^{(\tilde{\theta})}$ to the process Υ verifies a stochastic differential equation. Moreover, the supermartingale property under every $\mathbb{L}^{(\tilde{\theta})}$ and the martingale property for some $\mathbb{L}^{(\tilde{\theta}^*)}$ are the other two premises of Lemma 4. These coincide with the drift of Υ to be zero for the supremum over $\tilde{\theta} \in \Theta$ of the infimum over $\epsilon \in \mathcal{L}^{\epsilon}$. Hence, the Hamilton–Jacobi–Bellman–Isaacs equation is derived from the verification requirements.

The connection between PDEs and FBSDEs proves the uniqueness and solvability of the theorem. Algorithms to find numerical solutions of FBSDEs are well-known (Gobet & Labart, 2007; 2010; Gobet, Lemor, & Warin, 2005). See for applications of FBSDEs El Karoui et al. (2008). Time-consistent ambiguity averse preferences including jump processes are priced by Laeven and Stadje (2014). Pelsser and Stadje (2014) arrive at the same class of pricing operators by imposing time- and market-consistency. The time-consistency is in our case directly fulfilled by the initial HJBI formulation of the optimisation problem. Time-consistent coherent risk measures have a Lipschitz driver which coincides with the ellipsoid uncertainty constraint.

The firm that wants to maximise its surplus and acknowledges the ambiguity of the underlying model, acts by the “robust method of pricing” that we derived. A practitioner's methodology to price in incomplete markets is the industry standardised Cost-of-Capital method (Keller & Luder, 2004). This method that insurance companies use quantifies the market value of the replicating portfolio plus a mark-up for the unhedgeable risks, which relies mostly on the subjective quantification of risk. See Filipović and Vogelpoth (2008) for a critical discussion of the Swiss Solvency Test on which the Cost-of-Capital method (CoC) is based. This method leads to a pricing operator that has similar characteristics as our result. The indifference pricing operator from Theorem 8 can be interpreted as a best estimate, which is the conditional expectation (if $g(\cdot) = 0$ then we have a linear PDE), plus a constant (c) times the standard deviation of the unhedgeable component. The standard deviation per dt is a penalty that is added. If $dt \to 0$ this is normally distributed and hence the penalty consists of the quantiles from the normal distribution. The normality assumption is fulfilled since we are in a diffusion setting. The decomposition of the result of Theorem 8 corresponds with the interpretation of the Cost-of-Capital method by $c\sqrt{\partial_Y'\pi S \partial_Y \pi}$ = "CoC" per dt. Pelsser and Ghalehjooghi (2016) show that in a diffusion setting the CoC method and the standard deviation pricing principle have the same limit.

The interpretation of Corollary 11 is discussed in detail in the next two sections. Each case or example is dedicated to a specific part of the optimal robust strategy.

5. Uncertainty in complete and incomplete markets

In this section we show the application of Theorem 8 and Corollary 11 for the boundary cases. On the one hand we consider a market with only hedgeable risk and on the other hand we consider a market with purely unhedgeable risk. The first case yields the complete market setting which is an assumption that is often made in the literature. The characteristics that go along with completeness enlarge the analytically and numerically solvability of many problems, such as option pricing. However, this assumption lacks features that are present in practice. The imperfection of economic and financial markets, due to transaction costs or illiquidity, makes the market incomplete. Financial instruments that are linked to such nontraded underlyings still need to be quantified in order to calculate the present value. This is of high importance to insurance companies and pension funds, whose contracts are based on extremely long-dated interest rates, on mortality rates and on probabilities of natural hazards among others. Moreover, additional to incompleteness, the specification and assumptions of the model that describe the process of the assets contain uncertainty. The two most extreme cases we can encounter is either uncertainty with respect to purely hedgeable risk or uncertainty with respect to purely unhedgeable risk. In Section 6 we consider two applications that incorporate uncertainty in a mixed setting.

5.1. Pure hedgeable risk

Financial theory tells us that pricing in a complete market setting should reduce to pricing under the risk-neutral measure. Hence, uncertainty should vanish under the completeness assumption. This is confirmed for our setting.

In a market where only hedgeable risk is present, all risk is traded. This is what we call a complete market setting. For illustrative purpose of Theorem 8, we assume the bank account to be driven by $dX_0 = rX_0 dt$, where r is constant. If we assume that there is no unhedgeable risk in the market, then $l = 0$. The indifference pricing operator is given by the linear PDE

$$\partial_t \pi + r\partial_X' \pi X + \tfrac{1}{2}\mathrm{tr}\left(\partial_{XX}\pi\, \Sigma^{XX}\right) - r\pi = 0. \tag{44}$$

Note that the drift of the process X is r, the constant interest rate, and not μ^X. Therefore the price of the replicating portfolio is known as the "risk-neutral price". By Corollary 11 the optimal hedging strategy is $\theta_X^* = \partial_X \pi$, which is the delta-hedge that perfectly replicates the derivative contract. The optimal robustness factors are $\epsilon_X^* = -q^X$ for the adjustments on all n traded assets which replaces the uncertain drifts by the risk-free rate. Based on merely two assumptions, (i) a firm that wants to maximise its profit while (ii) the firm is uncertain about the underlying model; it follows that the optimal strategies lead to delta-hedging and risk-neutral pricing. Hence, the complete market setting leads to the elimination of ambiguity and the robust hedge replicates the derivative contract perfectly without specifying any probability measure upfront.

In the Black–Scholes economy, the vector of hedgeable processes is reduced to the one-dimensional traded stock $S(t)$. Examples of liabilities written on this stock are for instance vanilla options such as a call option, $L(T, S(T)) = \max(S(T) - K, 0)$. The familiar Black–Scholes equation is equivalent to Eq. (44) and often expressed in the following notation $\frac{\partial L}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 L}{\partial S^2} + rS\frac{\partial L}{\partial S} - rL = 0$.

5.2. Pure unhedgeable risk

The other extreme is the case when there are no tradeable assets at the firm's disposal, then $n = 0$. This implies that there are no assets available to hedge the risk. Therefore the asset side will grow according to the rate r. The corresponding indifference price is given by the PDE

$$\partial_t \pi + \partial_Y' \pi\, \mu^Y + k\sqrt{\partial_Y' \pi\, \Sigma^{YY} \partial_Y \pi} + \tfrac{1}{2}\mathrm{tr}\left(\partial_{YY}\, \Sigma^{YY}\right) - r\pi = 0. \tag{45}$$

By definition one cannot trade unhedgeable risk. Together with the absence of traded assets, except the presence of the bank account, the only admissible strategy is $\theta_X = 0$ and $\theta_0 = 1$. The response of mother nature is $\epsilon_Y^* = \frac{k}{\sqrt{\partial_Y' \pi' \Sigma^{YY} \partial_Y \pi}} \Sigma^{YY} \partial_Y \pi$. The drift adjustment ϵ_Y^* is the adjustment of the drift of the unhedgeable process (μ^Y) in the "prudent" direction. Mother nature adds a penalty term to the drift proportional to the standard deviation of the unhedgeable risk multiplied by the ambiguity specification k. The adjusted drift is equivalent to the robust pricing method that is known as "actuarial pricing". The more uncertain or the more risky the unhedgeable factor is, the larger the impact on the liability price and the larger the drift adjustment is. In a one-dimensional setting the drift under the robust measure is $\mu^Y \pm k\sqrt{\Sigma^{YY}}$ where the plus and minus depend on the direction of the firm's preference. The sign of the drift adjustment is determined by the minimisation of the objective which is mathematically derived in the proof of Corollary 11. And intuitively, for a long position the worst case is accomplished by a negative relation and for a short position robustness is obtained by an increase in the drift. The firm typically has a short position, since π plays the role of a liability contract. In other words, the firm has the obligation to pay at time T and thus a positive relation determines here the robust price of the liability.

6. Examples

If there is only one risk factor of each type and the partial derivative $\partial_y \pi$ is either monotonically increasing or decreasing, then the nonlinear driver becomes linear because the absolute sign of $\sqrt{\partial_y' \pi S \partial_y \pi}$ can be replaced by $\pm\partial_y \pi$ depending on the sign of $\partial_y \pi$ and the position of the firm. The PDE also becomes linear and we can express the solution, using the Feynman-Kaç formula, by

$$\pi(t, x, y) = e^{-r(T-t)}\mathbb{E}^{\mathbb{L}^*}[L(T, x, y)|t, x, y], \tag{46}$$

where $\mathbb{L}^*$ is the measure with the adjusted means for both risk factors, i.e., rx for the x process and $\mu_y + \epsilon_y$ for the y process. Since $n = 1$ and $l = 1$ we denote the vector X and Y which are one-dimensional random variables by x and y. The terminal condition is $\pi(T, x, y) = L(T, x, y)$.

The first example is based on a contract with an underlying asset that can be quite general, from real estate to weather indicators. This generality goes beyond the asset-liability management of insurance companies or pension funds which is illustrated by the second example of an insurance contract.

6.1. Correlated risk

Consider a nontraded asset that is correlated with a traded asset. Assume there is a risky asset x, the bank account x_0 and a nontraded asset y with the following dynamics

$$d\begin{bmatrix} x(t) \\ y(t) \\ x_0(t) \end{bmatrix} = \begin{bmatrix} \mu_x x(t) \\ \mu_y y(t) \\ r x_0(t) \end{bmatrix} dt + \begin{bmatrix} \begin{bmatrix} \sigma_x^2 x(t)^2 & \rho\sigma_x x(t)\sigma_y y(t) \\ \rho\sigma_x x(t)\sigma_y y(t) & \sigma_y^2 y(t)^2 \end{bmatrix}^{1/2} \\ 0_{[1\times 2]} \end{bmatrix} d\begin{bmatrix} W^x \\ W^y \end{bmatrix}. \tag{47}$$

Both the traded and nontraded asset are assumed to follow a geometric Brownian motion and these processes are correlated by $\rho \in [-1, 1]$. In many financial optimisation problems, such as the Merton portfolio problem, a representative agent instead of a firm is postulated who maximises expected utility from terminal wealth and/or intertemporal consumption. His control variables are how much of his wealth he allocates to risky assets, and depending on the objective, the remainder is invested on the bank account in return for the risk-free rate, or a part of its wealth is consumed. In the classical Merton set-up the agent is assumed to have power utility and the agent is certain about the geometric Brownian motion model specification for the stock price process. The results have been extended in many directions, including alternative preference specifications, more general asset dynamics, the inclusion of parameter or model uncertainty, incorporation of mortality and default risk, and allowing for incomplete markets.

With this financial example we illustrate the effect of uncertainty in the drifts of both the traded and nontraded asset. In a complete market the fair valuation of products is calculated by the risk-neutral probabilities deduced from the available market prices. Contrary to this, in an incomplete market there are products which rely on risk factors that are not liquidly available in the market. Moreover, in an incomplete market the risk-neutral measure is no longer unique. Therefore uncertainty arises naturally among the risk-neutral measures. We have shown that uncertainty about purely hedgeable risk does not survive since the worst-case measure picked by mother nature leads to the risk-neutral pricing measure. Note that the admissibility Assumption 2 needs to be fulfilled to generate this result. If the assumption that the interest rate is inside the uncertainty set is met, then it is upfront never optimal to go extremely long or short in the risky asset since the uncertain return can be lower than the interest rate. Hence, this assumption on the amount and direction of uncertainty confirms a natural reasoning pattern and results in the well-known risk-neutral pricing rule, in which the drift in the Black–Scholes formula is replaced the risk-free rate. More formally, by Theorem 8 we know that the process of the risky asset will be driven by the interest rate, i.e., the drift term changes from μ_x to r due to $\epsilon_x = -(\mu_x x - rx)$. However the ambiguity remains for the uncertain drift of the unhedgeable risk process and is implied by the two dimensional ellipse $\epsilon'\Sigma^{-1}\epsilon \le k^2$. The adjusted drift term of the unhedgeable risk can be expressed in term of the constant k and the market price of risk ϵ_x. We call the remaining uncertainty that reflects the uncertainty related to the unhedgeable process, i.e., that is left after selecting the risk-neutral measure, the *net level of uncertainty* which equals $k^2 - \left(\frac{\mu_x - r}{\sigma_x}\right)^2$.

Due to the uncertainty the pricing operator is nonlinear, which is captured by the driver

$$g(t, z, \pi, \partial_z\pi\,\Sigma) = r\pi(t) \pm c|\partial_y\pi(t)|\sqrt{y^2(t)\sigma_y^2(1-\rho^2)}. \tag{48}$$

Therefore the drift adjustment ϵ_y determines two prices that can be interpreted as either going long or short or as a bid-ask price. Madan and Cherny (2010) introduced the field of "conic finance", in which they extend the law of one price for transaction costs. They model the bid and ask price in an incomplete market by a convex cone that contains all nonnegative variables. Depending on the position of the firm here, i.e., a short or long position, the worst-case driver has either a positive or negative dependence on $\partial_y\pi(t)$. The driver determines the robust price of the nontraded or illiquid asset and the $\pm$-sign is determined by the prudential direction. Or stated differently, we obtain both a robust ask and a robust bid price. By Corollary 11 the optimal drift adjustments are

$$\begin{bmatrix} \epsilon_x^* \\ \epsilon_y^* \end{bmatrix} = \begin{bmatrix} -\mu_x x(t) + r x(t) \\ -\rho\sigma_y y(t)\frac{\mu_x - r}{\sigma_x} \pm \sigma_y y(t)\sqrt{1-\rho^2}\sqrt{k^2 - \left(\frac{\mu_x - r}{\sigma_x}\right)^2} \end{bmatrix}. \tag{49}$$

The optimal hedge position is

$$\theta_x^* = \partial_x\pi + \partial_y\pi\frac{\sigma_y y(t)}{\sigma_x x(t)}\rho \pm \partial_y\pi\frac{\sigma_y y(t)}{\sigma_x x(t)}\sqrt{\frac{1-\rho^2}{k^2 - \left(\frac{\mu_x - r}{\sigma_x}\right)^2}}\frac{\mu_x - r}{\sigma_x}, \tag{50}$$

where the first term $\partial_x\pi$ is the delta-hedging part linked to purely hedgeable risk. The second term $\partial_y\pi\frac{\sigma_y y(t)}{\sigma_x x(t)}\rho$ implies delta-hedging for the unhedgeable risk weighted by the relative standard deviations. And the last term is the product of the residual of the second term, the unhedgeable risk, $\partial_y\pi\frac{\sigma_y y(t)}{\sigma_x x(t)}\sqrt{1-\rho^2}$ and the *market confidence term* $((\mu_x - r)/\sigma_x)/\sqrt{k^2 - ((\mu_x - r)/\sigma_x)^2}$. The market confidence term is market price of risk divided by the square root of the net level of uncertainty. Note that latter term goes to infinity when the market price of risk approaches k, while for small market prices of risk it is approximately the market price of risk scaled down by a factor k. Thus for market prices of risk that are approaching the ones that are "too good to be true" the firm invests huge amounts in the underlying asset.

The robust price of the nontraded asset can be characterised by the stochastic differential equation

$$dy(t) = \left(\mu_y - \rho\sigma_y\frac{\mu_x - r}{\sigma_x} \pm \sigma_y\sqrt{1-\rho^2}\sqrt{k^2 - \left(\frac{\mu_x - r}{\sigma_x}\right)^2}\right)y(t)dt + \tag{51}$$

$$\sigma_y y(t)\left(\rho dW^x + \sqrt{1-\rho^2}dW^y\right). \tag{52}$$

Remark 12. The lowest admissible value that k can have is when k^2 is an infinitesimally small arbitrary number larger than $\left(\frac{\mu_x - r}{\sigma_x}\right)^2$ in order to price the hedgeable part risk neutrally. The remainder, the net level of uncertainty, is the amount of uncertainty on the unhedgeable risk. If this is zero, which is inadmissible, then the speculative part of the optimal hedge position is either an infinite amount long or short as the firm knows with certainty whether μ_x generates a higher return than r or the other way round. The highest admissible value of k is infinity, in that case the firm is so uncertain that the speculative demand shrinks to zero. The drift of the unhedgeable risk is also plus or minus infinity generating robust liability prices that are too extreme. We therefore argue that a reasonable value of k has to be motivated, e.g. based on empirical calibration (Hansen, Lunde, & Nason, 2011), asymptotics (Ben-Tal, Den Hertog, De Waegenaere, Melenberg, & Rennen, 2013) or statistical foundation (Anderson, Hansen, & Sargent, 2003; Balter & Pelsser, 2019).

6.2. Life insurance contract

Consider a life insurance contract with one traded and one nontraded asset. In this unit linked contract, the survivors receive the value of the stock at time T bounded by a minimum guarantee g. Fleten and Lindset (2008) derive the optimal hedging strategies for multi-period guarantees in a discrete setting. They derive the strategy via a binomial tree both with and without transaction costs.

We consider a continuous time set-up and thus rebalance continuously. In this case $n = 1,\ l = 1$ and S is the stock price that follows a lognormal distribution. For ease of exposition, N, the number of survivors in the policy, also follows a lognormal distribution[4]. The stochastic processes are

$$d\begin{bmatrix} S(t) \\ N(t) \end{bmatrix} = \begin{bmatrix} \mu S(t) \\ \nu N(t) \end{bmatrix} dt + \begin{bmatrix} \sigma S(t) & 0 \\ 0 & \beta N(t) \end{bmatrix} \begin{bmatrix} dW^S \\ dW^N \end{bmatrix}.$$

We assume no correlation between the two processes, implying that we assume that there is no causal relation between the price of the stock and the number of survivors. Since the liability the insurer faces is

$$L(T, S(T), N(T)) = \max(S(T), g)N(T),$$

it follows that $\partial_N \pi > 0$ at time T. Therefore π is monotone increasing in N and consequently the PDE is

$$\partial_t \pi \ + r\partial_S \pi S(t) + \partial_N \pi \left(\nu N(t) + \sqrt{k^2 - \frac{(\mu S(t) - rS(t))^2}{\sigma^2 S^2(t)}} \sigma_N N(t) \right)$$
$$+ \tfrac{1}{2}\left(\partial_{SS}\pi \sigma^2 S^2(t) + \partial_{NN}\pi \beta^2 N^2(t) \right) - r\pi = 0.$$

By Theorem 8, the optimal hedging portfolio is

$$\theta_x^* = \partial_S \pi + \partial_N \pi \frac{\beta N(t)}{\sigma S(t)} \frac{(\mu - r)/\sigma}{\sqrt{k^2 - \left(\frac{\mu - r}{\sigma}\right)^2}}. \tag{53}$$

Similar as in the previous example the optimal hedging portfolio consists of a delta-hedging part. And now that there is no correlation between the hedgeable and unhedgeable risk, the investment in the traded asset proportional to the correlation is zero. The second term of the hedge position is the market confidence term, proportional to the relative standard deviations. The larger the uncertainty set, the less aggressive the insurance company hedges the unhedgeable risk by investing in the tradeable assets. And vice versa, if the market price of risk becomes large, investing in the traded asset is more and more profitable.

The robust price of the life insurance contract is $\pi(t, S(t), N(t))$. By solving the PDE we can find the function of $\pi(T - dt, S(t), N(t))$ and its derivatives recursively until time t. The solution can be written as a conditional expectation by the Feynman–Kaç formula

$$\pi(t, S(t), N(t)) = e^{-r(T-t)} \mathbb{E}^{\mathbb{L}}[\pi(T, S(T), N(T)) | S(t) = s, N(t) = n]. \tag{54}$$

Note that the expectation is taken under the probability measure $\mathbb{L}^*$ that belongs to the adjusted risk-neutral drift of the hedgeable component and the additional prudence factor of the unhedgeable part. Measure $\mathbb{L}^*$ belongs to the processes having drifts

$$\begin{cases} rS(t) \\ \nu N(t) + \beta N(t) \sqrt{k^2 - \left(\frac{\mu - r}{\sigma}\right)^2} \end{cases}. \tag{55}$$

The interpretation of risk measure $\mathbb{L}^*$ is that the financial risk is perfectly replicated such that the ambiguity is eliminated and the drift is replaced by the risk-free rate. The ambiguity of the unhedgeable process is now the intersection of the ambiguity set $\mathcal{L}$ and the line r. The intersection of this line and the ellipse has two solutions, corresponding with the sign of the liability function. In this case the zero correlation between the two processes causes the ellipse to be an exact circle. In the general multidimensional case, the optimal measure $\mathbb{L}^*$ can be interpreted as the intersection of the ellipsoid and the risk-neutral measures $\mathcal{L} \cap \mathcal{Q}$, corresponding with the inf-convolution of Barrieu and El Karoui (2005). By specifying the ellipsoid this intersection determines the optimal solution.

For this example, we can solve the conditional expectation analytically

$$\pi(t, S(t), N(t)) = e^{-r(T-t)} \mathbb{E}^{\mathbb{L}^*}[\max(S(T), g)N(T) | \mathscr{F}_t]$$
$$= \left(S(t)\Phi(d_1) - e^{-r(T-t)} g\Phi(d_2) + e^{-r(T-t)} g \right) N(t) e^{\left(\nu + \beta\sqrt{k^2 - ((\mu - r)/\sigma)^2}\right)(T-t)}, \tag{56}$$

where $d_{1,2} = \frac{1}{\sigma\sqrt{T-t}}(\ln(\frac{S(t)}{g}) + (r \pm \frac{\sigma^2}{2})(T-t))$.

7. Conclusion

For a firm (i) that wants to maximise its expected surplus, and (ii) that is uncertain about the modelled economy, we obtain the pricing rule described by the semilinear PDE for the pricing operator.

Firstly, model ambiguity in a complete market leads to risk-neutral pricing. Mother nature minimises the objective of the firm to ensure robustness. When all asset are hedgeable, mother nature's choice eliminates model ambiguity. The firm replicates the derivative contract perfectly by the delta-hedge.

Secondly, when there are only unhedgeable risk factors and no traded assets in an economy, then model ambiguity results in the action of "actuarial pricing". This is a conservative way of pricing where the uncertain drift is adjusted in the "prudent" direction. The penalty term that mother nature adds is proportional to the standard deviation of the unhedgeable risk multiplied by the amount of ambiguity (the size of the ellipsoid). Thus the larger the volatility or the larger the initial uncertainty, the larger becomes the drift adjustment that ensures a robust price of the contract the firm is liable to.

Thirdly, in a model with both hedgeable and unhedgeable risk the firm will price market-consistently and actuarial prudentially. The traded risky assets are priced by the interest rate, whereas the drift adjustments of the nontraded assets are twofold. The drift adjustment can be either negative or positive. The prudent direction depends on the payoff structure of the firm and can be interpreted as the bid or the ask price. Interpreted differently, the worst-case adjustment has either a positive or a negative impact on the price of the liability. The robust relation depends on whether the firm has a short or long position.

The optimal investment strategies that drive the hedging portfolio consist of (i) the delta-hedge linked to hedgeable risk, (ii) the delta-hedge linked to unhedgeable risk proportional to the correlation between the traded and nontraded assets and their standard deviations, plus (iii) the product of the residual of the correlated delta-hedge of the unhedgeable risk and a market confidence term. The market confidence term has as effect that the investment in risky assets goes to infinity when the market prices of risk become "too good to be true", while low market prices of risk lead to an additional term of the market price of risk itself scaled down by the size of the uncertainty set.

For some special cases we can solve the pricing semilinear PDE explicitly, as shown by the given examples. Moreover, we prove existence and uniqueness, i.e., well-posedness, of the robust price of the liabilities by connecting the optimal semilinear PDE to a FBSDE with a Lipschitz driver. The classical solution of the PDE also proves optimality of the robust investment strategies.

[4] This can be unrealistic since there is a non-zero probability of $N(t) < N(t + dt)$. However, if the drift is negative (enough) this probability becomes small. The realistic counterpart with decreasing number of survivors also leads to Lipschitz continuity of the terminal condition, which can be obtained by a change of variables to $\max(\tilde{S}(T), gN(T))$ where $\tilde{S}(t) = S(t)N(t)$.

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Supplementary material

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